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Some remarks on Euclid rings

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In the present note, we show that the notion of a Euclid ring may be defined as below, and then it is quite easy to prove that the direct sum of a finite number of Euclid rings is again a Euclid ring.

Definition 1. A commutative ring R with identity is called a Euclid ring if there is a mapping φ of $R-\{0\}$ into some ordered set M with minimum condition satifying the condition:

If $a, b \in R$ and if $a \neq 0$, then there are $q, r \in R$ such that $b=aq+r$ with either $r=a$ or $\varphi r<\varphi a$.

In the circumstances, we say that (R, M, φ) is a Euclid ring.

Definition 2. r as above is called a right residue at the division of b by a; for the case where $a=0$, we say b is the right residue at the division of b by 0.

We show in this note also that if R is a Euclid ring, then we can choose a well-ordered set W and a mapping ψ of $R-\{0\}$ into W so that (R, W, ψ) is a Euclid ring in the sense of Nagata [1]. This means that, by defining $\psi(0)$ to be smaller than any element of W, our Euclid ring becomes a Euclid ring in the sense of Samuel [2].

Proposition 1. *If (R, M, φ) is Euclid ring, then R is a principal ideal ring.*

Proof. Let I be a non-zero ideal in R. Take a minimal element $\varphi a (a \in I)$ among $\{\varphi x \mid x \in I, x \neq 0\}$. For an arbitrary b in I, let r be a right residue at the division of b by a. By the minimality of φa, we have $r=a$. This means that b is divisible by a. Thus $I=aR$. Q.E.D.

Assume now that (R, M, φ) and (S, N, ψ) are Euclid rings. Let $M'=M \cup \{t\}$, $N'=N \cup \{u\}$ with t and u bigger than any element of M and N, respectively. We extend φ and ψ so that $\varphi 0=t$ and $\psi 0=u$. Let $M' \times N'$ be an ordered set by defining that $(m', n') \geqq (m, n)$ if and only if $m' \geqq m$ and $n' \geqq n$. Then $M' \times N'$ satisfies the minimum condition. A mapping (φ, ψ) of the direct sum $R+S$ into $M' \times N'$ is naturally defined by $(a, b) \rightarrow (\varphi a, \psi b)$. Then we have

Theorem 2. *The direct sum $(R+S, M' \times N', (\varphi, \psi))$ is a Euclid ring.*

Proof. Let (a, b) and (c, d) be elements of $R+S$ with $(a, b)\neq(0, 0)$. Let r, s be right residues at the division of c, d by a, b, respectively. Then we have one of the following cases:

$$(1)\quad (r, s)=(a, b) \qquad (2)\quad r=a,\ \psi s<\psi b$$

$$(3)\quad \varphi r<\varphi a,\ s=c \qquad (4)\quad \varphi r<\varphi a,\ \psi s<\psi b$$

$$(5)\quad a=0,\ b\neq 0,\ r=c \qquad (6)\quad a\neq 0,\ b=0,\ s=d$$

It is obvious that the cases (1)-(4) are good. As for the case (5), if $c\neq 0$, then $\varphi c<\varphi 0$ (by definition) and $\psi s\leqq\psi b$, and this is a good case. If $c=0$, then either $s=b$ or $\psi s<\psi b$, and this is also a good case. The case (6) is similar.

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Theorem 3. *If R is a Euclid ring, then there is a mapping ρ of $R-\{0\}$ into a suitable well-ordered set W so that (R, W, ρ) is a Euclid ring.*

For the proof of this, it suffices to prove the following:

Proposition 4. *If M is an ordered set with minimum condition, then there is a mapping f of M into a suitable well-ordered set W so that if $a, b\in M$ and if $a>b$, then $fa>fb$.*

Proof. Take a well-ordered set W which is big enough (if we need later, we are allowed to enlarge W by adding new elements which are bigger than any element of the original W). We define f inductively. Namely, consider an element w of W, and assume that for all $y<w$, $f^{-1}(y)$ are defined. Let M_w be the complement of $T_w=\bigcup_{y<w} f^{-1}(y)$ with respect to M. Then we define $f^{-1}(w)$ to be the set of minimal elements in M_w. Thus we defined f on the union of all of $f^{-1}(w)$. If the union is not M, then we can go on further, because of the minimum condition. Therefore f is a mapping of M into W. If $a>b$ $(a, b\in M)$, and if $fb=w$, then a is in M_y for $y\leqq w$ $(y\in W)$ because $a>b\in M_y$ and a is not a minimal element in any such M_y. Therefore, we have $fa>fb$.

Q. E. D.

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References

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