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On the lowest index for semi-elliptic operators to be Gevrey hypoelliptic

Dedicated to Professor SIGERU MIZOHATA on his sixtieth birthday

By

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1. Introduction

Let $m=(m_1,\dots, m_n)$, $(m_j\in N\backslash 0)$, and for multi-index $\alpha\in N^n$

$$|\alpha:\ m|=\alpha_1/m_1+\cdots+\alpha_n/m_n.$$

We consider the partial differential operator P given by

$$P(x, D_x)=\sum_{|\alpha:m|<1} a_\alpha(x)D_x^\alpha. \tag{1}$$

where $a_\alpha(x)$ is analytic in an open set $\Omega\subset R^n$. We assume that

$$P_0(x, \xi)=\sum_{|\alpha:m|=1} a_\alpha(x)\xi^\alpha\neq 0 \qquad \text{for any} \quad \xi\in R^n\backslash 0. \tag{2}$$

Then we call this operator a semi-elliptic operator. We are concerned with Gevrey hypoellipticity for semi-elliptic operators.

We note $\gamma^{\{s\}}(\omega)$ the space of functions u of class C^∞ such that for every compact set K of ω, there are constants C and h such that

$$\sup_K |D_x^\alpha u(x)|\leqslant Ch^{|\alpha|}(|\alpha|!)^s \qquad \text{for any} \quad \alpha\in N^n.$$

We shall say that P is $\gamma^{\{s\}}$-hypoelliptic in a neighborhood of x_0 iff there exists a neighborhood ω of x_0 such that for any open subset $\omega'\subset\omega$, the following implication holds;

$$u\in\mathscr{D}'(\omega),\ Pu\in\gamma^{\{s\}}(\omega')\Longrightarrow u\in\gamma^{\{s\}}(\omega').$$

Let $P(x, \xi)$ be the symbol of $P(x, D_x)$. Then, there are some constants C_0, C_1 and B such that

$$|D_x^\beta D_\xi^\alpha P(x, \xi)/P(x, \xi)|\leqslant C_0C_1{}^{|\alpha+\beta|}\alpha!\beta!(1+|\xi|)^{-\rho|\alpha|} \qquad \text{for} \quad x\in K\Subset\Omega,\ |\xi|\geqslant B>0, \tag{3}$$

where $\rho=\min\{m_i/m_j\}$. Combining this estimate with (2), by [7], [1], [2], we know that our operator P is $\gamma^{\{s\}}$-hypoelliptic in a neighborhood of every point in

Ω if $s \geqslant s_0 = \max\{m_i/m_j\}$.

Our purpose is to show that s_0 is the smallest index for P to be $\gamma^{\{s\}}$-hypoelliptic in a neighborhood of points in Ω. Namely, let x_0 be any point of Ω. Then we have

Theorem. *Under the condition* (2), *for* $1 \leqslant s < s_0$, *P is not $\gamma^{\{s\}}$-hypoelliptic in a neighborhood of* x_0.

Remark. For the operator with constant coefficients, more general results have been obtained by L. Hörmander. ([3]). For $s=1$, our result follows from the result of O. A. Oleinik and E. V. Radkevič ([6]). When for any j, m_j is either 1 or 2, our result has been shown implicitly by G. Metivier ([4]).

In the next section, we shall give the proof of theorem. We shall do this by contradiction. Namely, we shall construct the asymptotic soluton which violate a priori estimate. This is inspired by [4].

2. Proof of theorem

Lemma 1. *Suppose that there are a neighborhood $\tilde{\omega}$ of x_0 and constants $\varepsilon > 0$, $C > 0$ such that for any $\phi \in C_0^\infty(\tilde{\omega})$,*

(4) $$\|\phi\|_{H^\varepsilon(\tilde{\omega})} \leqslant C(\|P^*\phi\|_{L^2(\tilde{\omega})} + \|\phi\|_{L^2(\tilde{\omega})}), \quad \text{and}$$

P is $\gamma^{\{s\}}$-hypoelliptic in a neighborhood of x_0. Then, there is a neighborhood ω_0 of x_0 such that the following fact holds; for any neighborhood $\omega' \Subset \omega \Subset \omega_0$, there are constants L and C' such that the following inequality holds; for any $k \in N$ and $u \in \mathscr{D}'(\omega)$,

(5) $$\|u\|_{k,\omega'} \leqslant C' L^k (|||Pu|||_{k,\omega,s} + (k!)^s \|u\|_{0,\omega}).$$

Here, $|||v|||_{k,\omega,s} = \sum_{|\alpha|<k} k^{s(k-|\alpha|)} \|D_x^\alpha u\|_{L^2(\omega)}$, *and* $\|v\|_{k,\omega}^2 = \sum_{|\alpha|<k} \|D_x^\alpha u\|_{L^2(\omega)}^2$.

This result was obtained by G. Metivier. (Remark 3.2 in [4]) But there is a little change in the definition of the norm $|||\cdot|||$, in comparison with his original form; i.e., we introduce 's' in it. So, we shall give the proof of this lemma in the appendix.

Without loss of generality, we may assume that

$$m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n \quad \text{and} \quad a_{(m_1,0,\dots,0)}(x) \equiv 1.$$

It is classical for P to satisfy the inequality (4). So, in view of lemma 1, in order to prove theorem, it is sufficient to construct the function u_ρ such that the following conditions hold; there are the constants C, L, ε, ε', k_0 and σ independent of ρ such that

$$\begin{cases} \|D_x^\alpha P u_\rho\|_{L^2(\omega)} \leqslant C(\rho L)^{s_0|\alpha|} e^{-\varepsilon\rho} & \text{for } \forall|\alpha| \leqslant \sigma\rho, \\ \|u_\rho\|_{k,\omega'} \geqslant C^{-1} |D_{x_n}^{k-k_0} u_\rho(x_0)| \geqslant C^{-2} \rho^{s_0(k-k_0)} & \text{for } \forall k \leqslant \sigma\rho, \quad \text{and} \\ \|u_\rho\|_{L^2(\omega)} \leqslant C e^{\varepsilon'\rho}. \end{cases}$$

In fact, let $k=[\sigma'\rho]$ (σ' is sufficiently small), then for $s<s'<s_0$, (5) is not hold.

Let $s_j=m_1/m_j$ and $M=m_1$. We shall seek u_ρ in the following form;

$$u_\rho(x)=e^{iw_\rho(x)}U(x,\ \rho x_1),$$

where, $w_\rho(x)=\rho^{s_n}\cdot x_n+\cdots+\rho^{s_2}\cdot x_2$. Let $x=z$, $\rho x_1=t$. We shall work in the set $\{(z,\ t)\in\tilde{\omega}_d\times R\}$. Here,

$$\tilde{\omega}_d=\{z\in C^n;\ \mathrm{dist}\,(z,\ \omega)<d\}.$$

Then, we have

$$Pu_\rho=\rho^M e^{iw_\rho}(\mathscr{P}_\rho U)(z,\ t),$$

where $\mathscr{P}_\rho=\mathscr{P}_0+\mathscr{P}'$,

$$\mathscr{P}_0=\sum_{|\alpha:m|=1}a_\alpha(x_0)D_t^{\alpha_1},\quad \text{and}$$

$$\mathscr{P}'=\sum_{j=1}^{M+1}\rho^{-(j-1)}\mathscr{P}_j.$$

Here, $\mathscr{P}_j=\mathscr{P}_j(z,\ \rho,\ D_z,\ D_t)$ is the partial differential operator of order $j-1$ whose coefficients are analytic functions in z and polynomial in ρ^{-1}; especially,

$$\mathscr{P}_1=\sum_{|\alpha:m|=1}(a_\alpha(x)-a_\alpha(x_0))D_t^{\alpha_1},\quad \text{and}$$

$$\mathscr{P}_j=\sum_{|\alpha+\beta|<j-1}b_{\alpha,\beta}(z,\ \rho)D_z^\alpha D_t^\beta,$$

where $b_{\alpha,\beta}(z,\ \rho)$ is holomorphic and bounded in $\tilde{\omega}_d\times\{\rho>1\}$.

Let $Q(\tau)=\sum_{|\alpha:m|=1}a_\alpha(x_0)\cdot\tau^{\alpha_1}$. Then by the assumption (2), $Q(\tau)=0$ has the nonreal root $\delta_1,\dots,\ \delta_M\in C$. We take a positive number δ_0 such that $\delta_0>\max\cdot(|\mathrm{Im}\delta_j|)$. We introduce some function spaces;

$$W_r=\{u;\ e^{-\delta_0|t|}D_t^ju\in L^2(R),\ j=0,\ 1,\dots,\ r\}.\quad \text{and}$$

$$V_r=\{u\in W_r;\ \sup_{j>0}L_0^{-j}\|D_t^ju\|_{W_r}<+\infty\}.\quad L_0=\max_j\{2Ma_j,\ 1\}.$$

Then we have

Lemma 2. *$\mathscr{P}_0$ is surjective from W_M to W_0 and has a right inverse which is continuous from W_0 to W_M, and from V_0 to V_M.*

Proof. We may write

$$\mathscr{P}_0=D_t^M+\sum_{j=1}^M a_jD_t^{M-j},\quad (a_j\in C).$$

Let $v_0=v$, $v_1=D_tv,\dots,\ v_{M-1}=D_t^{M-1}v$. Then, the equation $\mathscr{P}_0v=f$ is transformed into

$$D_tV=AV+F,$$

where $V={}^t(v_0,\dots,\ v_{M-1})$, $F={}^t(0,\dots,\ 0,f)$, and

$$A=\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & 0 \\ 0 & & \ddots & \ddots & \\ & & & 0 & 1 \\ -a_M & \cdots & \cdots & \cdots & -a_1 \end{bmatrix}.$$

By Petrowsky's lemma (ex. see [5]), there is a non-singular constant matrix C such that

$$CA=DC,$$

where

$$D=\begin{bmatrix} d_{11} & & 0 \\ & \ddots & \\ * & & d_{MM} \end{bmatrix}, \quad \{d_{ii}\}=\{\delta_i\},$$

$$|\det C|=1\ (|c_{ij}|\leqslant 1), \quad \text{and}$$

$$|d_{ij}|\leqslant (M-1)!2^M \max\{1, |a_j|\}.$$

Let $CV=W$. Then, W satisfies

$$D_tW=DW+CF.$$

Namely, denoting $W={}^t(w_1,\dots, w_M)$, we have

$$w_j(t)=i\int_0^t e^{i\delta_j(t-s)}g_j(s)ds.$$

Here, $g_j(t)=\sum_k d_{jk}w_k+c_{jM}f$. Let $\tilde{w}_j(t)=w_j(t)e^{-\delta_0|t|}$, and $\tilde{g}_j(t)=e^{-\delta_0|t|}g_j(t)$. Then,

$$\tilde{w}_j(t)=i\int_0^t e^{(i\delta_j-\delta_-)(t-s)}e^{-(\delta_0-\delta_-)(t-s)}\tilde{g}_j(s)ds \text{ if } t>0, \text{ and}$$

$$\tilde{w}_j(t)=i\int_0^t e^{-(\delta_+-i\delta_j)(t-s)}e^{-(\delta_0-\delta_+)|t-s|}\tilde{g}_j(s)ds \quad \text{if} \quad t<0,$$

where $\delta_-=\min_j\{\mathrm{Re}\,(i\delta_j)\}$ and $\delta_+=\max_j\{\mathrm{Re}\,(i\delta_j)\}$. So we have

$$\|w_j(t)\|_{L^2(R)}\leqslant C_0\|g_j(t)\|_{L^2(R)}.$$

Returning to v, we have

$$\|e^{-\delta_0|t|}D_t^j\, v(t)\|_{L^2(R)}\leqslant C_1\|e^{-\delta_0|t|}f(t)\|_{L^2(R)}, \quad j=0, 1,\dots, M.$$

Here, C_1 is a constant depending only on $\{a_j\}$.

Since $D_t^{j+1}V=\sum_{0<l<j} A^lD_t^{j-l}F$, we obtain

$$\|e^{-\delta_0|t|}D_t^{j+1}V\|_{L^2(R)}\leqslant C\sum_{l=0}^{j}(L_0/2)^l\|e^{-\delta_0|t|}D_t^{j-l}F\|_{L^2(R)}.$$

From this, we conclude that

$\|v\|_{V_M} \leqslant C\|f\|_{V_0}$. ($C$ is a constant depending only on a_j). Q. E. D.

Let $E_d(\omega, V_r)$ be the space of functions analytic in z with values in V_r which can be prolonged as bounded holomorphic functions on $\tilde{\omega}_d$. Let $\|u\|_{E_d(\omega, V_r)} = \max_{z \in \omega_d} \|u(z)\|_{V_r}$. Since $\partial/\partial z_j$ is a bounded operator from E_d to $E_{d'}$ $(d > d')$ with norm less than $1/(d-d')$, taking the form of $\mathscr{P}_j$ into considerations, we have

Lemma 3. *There are a neighborhood ω of x_0, a positive constant d_0 and the constants C_j $(j=1,\dots, M+1)$ such that for $0 < d' < d \leqslant d_0$, and $u \in E_d(\omega, V_M)$, the following inequalities hold;*

$$\|\mathscr{P}_1 u\|_{E_d(\omega, V_0)} \leqslant C_1(d + diam\ \omega)\|u\|_{E_d(\omega, V_M)}$$

$$\|\mathscr{P}_j u\|_{E_{d'}(\omega, V_0)} \leqslant \{C_j/(d-d')\}^{j-1}\|u\|_{E_d(\omega, V_M)}, \quad j=2,\dots, M+1.$$

Now, we define $\{u_k\}$ as follows;

$$\mathscr{P}_0 u_k = -\mathscr{P}'_\rho u_{k-1},\ k > 1, \quad \text{and} \quad u_0 = e^{\delta t}, \quad (\delta \in \{i\delta_j\}_{j=1}^M)$$

Then, by lemma 3, we have

Lemma 4. *There exists constant C such that for $0 < d' < d \leqslant d_0$ and $\rho \geqslant 1$, $\|u_k\|_{E_{d'}(\omega, V_M)} \leqslant \|u_0\|_{V_M}\{C(d + diam\omega + k/\rho(d-d') + \cdots + (k/\rho(d-d'))^{M+1})\}^k$.*

Summing up, we conclude that

Proposition. *Let $U_\rho = \sum_{k < \rho d^2} u_k(z, t)$, and $\Omega_d = \{x \in R^n;\ |x - x_0| < d\}$. Then there are the positive constants d_0 and C such that for $0 < d < d_0$ and $\rho \geqslant 1/d^2$,*

$$\|\mathscr{P}_\rho U_\rho\|_{F_d} \leqslant Ce^{-\rho d^2},\ \|U_\rho\|_{F_d} \leqslant C,\ \|U_\rho - U_0\|_{F_d} \leqslant Cd, \text{ and } U_0 = e^{\delta t}.$$

Here, $\|v\|_{F_d} = \sup_{\Omega_d \times R, j, \gamma} \dfrac{d^{\gamma+j}}{|\gamma|!} e^{-\delta_0|t|}|D_t^j D_z^\gamma v(z, t)|$.

Proof. In lemma 4, let $\omega = \Omega_d$ and $k \leqslant \rho d^2$. Then, we have

$$\|u_k\|_{E_d(\Omega_d, V_M)} \leqslant \|u_0\|_{V_M}(Cd)^k.$$

Let d_0 sufficiently small such that $\delta_0 \leqslant d^{-1}$ if $d \leqslant d_0$. Then, it is easy to see that

$$\|u_k\|_{F_d} \leqslant C_1\|u_k\|_{E_d(\Omega_d, V_M)}, \quad \text{and}$$

$$\|\mathscr{P}_0 u_k\|_{F_d} \leqslant C_1\|\mathscr{P}_0 u_k\|_{E_d(\Omega_d, V_0)} \leqslant C_1 C_2\|u\|_{E_d(\Omega_d, V_M)}.$$

Also let d_0 small such that $Cd_0 \leqslant 1/e$. Then, using the above inequalities, we have

$$\|U_\rho\|_{F_d} \leqslant \sum_{k < \rho d^2} \|u_k\|_{F_d} \leqslant 2C_1\|u_0\|_{V_M},$$

$$\|U_\rho - U_0\|_{F_d} \leqslant 2C_1\|U_0\|_{V_M}(Cd), \quad \text{and}$$

$$\|\mathscr{P}_\rho U_\rho\|_{F_d} = \|\mathscr{P}'_\rho u_{k_0}\|_{F_d} = \|\mathscr{P}_0 u_{k_0+1}\|_{F_d}$$

$$\leqslant C_1 C_2\|U\|_{V_M}(Cd)^{k_0+1} \leqslant C_1 C_2\|U_0\|_{V_M} e^{-\rho d^2}.$$

Here, we take $k_0=[\rho d^2]$. Q.E.D.

Let $U_\rho(z, t)$ be in the proposition, and

$$u(x)=e^{iw_\rho(x)}U_\rho(x, \rho x_1).$$

Then, we have

$$|D_x^\gamma u(x)|\leqslant\sum\left|\binom{j}{h}\binom{\alpha_1}{\beta_1}\cdots\binom{\alpha_n}{\beta_n}\rho^{j-h-(s_j,\beta)}D_t^h D_{x_1}^{j-h}D_{z'}^{\alpha-\beta}U_\rho\right|,$$

where $z'=(z_2,\dots, z_n)$. Since for $|\gamma|\leqslant d\rho$

$$(|\gamma|-|\beta|-h)!d^{-(|\gamma|-|\beta|-h)}\leqslant\rho^{|\gamma|-|\beta|-h},$$

$$|D_t^h D_{x_1}^{j-h}\ D_{z'}^{\alpha-\beta}\ U_\rho|\leqslant\|U_\rho\|_{F_d}e^{\delta_0|t|}(|\gamma|-|\beta|-h)!d^{-(|\gamma|-|\beta|-h)}$$
$$\leqslant\|U\|_{F_d}e^{\delta_0|t|}\rho^{|\gamma|-|\beta|-h}.$$

From these two inequalities, we conclude that

$$|D_x^\gamma u(x)|\leqslant\|U_\rho\|_{F_d}e^{\delta_0|t|}\rho^{|\gamma|}\sum\binom{j}{h}\binom{\alpha_1}{\beta_1}\cdots\binom{\alpha_n}{\beta_n}\rho^{(s_j-1,\beta)}$$
$$\leqslant\|U_\rho\|_{F_d}e^{\rho\delta_0|x_1|}\rho^{|\gamma|}(1+\rho^{s_2-1}+\cdots+\rho^{s_n-1})^{|\gamma|}$$
$$\leqslant\|U_\rho\|_{F_d}e^{\rho\delta_0|x_1|}\rho^{s_n|\gamma|}.$$

Similarly, we see that for $\rho\geqslant 1/d^2$ and $x\in\Omega_d$ $(d\leqslant d_0)$,

$$|D_x^\gamma Pu(x)|\leqslant Ce^{-\rho d^2}e^{\rho\delta_0|x_1|}\rho^{s_n(|\gamma|+M)},$$

$$|D_x^\gamma(u-u^0)(x)|\leqslant Cde^{\rho\delta_0|x_1|}\rho^{s_n|\gamma|},\quad\text{for}\quad|\gamma|\leqslant\rho d,$$

where $u^0(x)=e^{iw_\rho(x)}e^{\delta\rho x_1}$.
Therefore, let $d=1/2C$, then we have

$$|D_{x_n}^k u(x_0)|\geqslant(1/2)\rho^{s_n k},\qquad\text{for}\quad k\leqslant d\rho.$$

Moreover, let $\Omega=\Omega_d\cap\{|x_1|<d^2/2\delta_0\}$, we have

$$\begin{cases}|D_x^\gamma Pu|_\infty\leqslant Ce^{-\rho d^2/2}\rho^{s_n(|\gamma|+M)},\ \text{and}\\ |u|_\infty\leqslant Ce^{\rho d^2/2}.\end{cases}$$

Remarking that $s_n=s_0$, this proves theorem. Q.E.D.

3. Appendix (Proof of lemma 1).

First, we recall the following well-known result;

For any open set $\omega'\Subset\omega\subset R^n$, there are the functions χ_k $(k\in N)$ of $C_0^\infty(\omega)$ which take values 1 on ω' and satisfy the following inequalities;

(A–1) $\qquad \forall k\in N,\ \forall\alpha\in N^n,\ |\alpha|\leqslant k,\ |D_x^\alpha\chi_k|_\infty\leqslant(r_0k/r)^{|\alpha|}.$

where $r=\operatorname*{Inf}_{x\in\omega'}\operatorname{dist}(x, \omega^c)$ and r_0 is a constant depending only on n. Then we have

Lemma A–1. *Let Ω be a neighborhood of $x_0\in R$ and B be a Banach space which is imbeded continuously into $L^2(\Omega)$. We suppose that there exists a nighborhood $\omega\Subset\Omega$ of x_0 such that for any $u\in B$, $u|_\omega\in\gamma^{\{s\}}(\omega)$. Then, for any neighborhood $\omega'\Subset\omega$ of x_0, and χ_k satisfying* (A–1), *there are the constants C and C' such that for $\forall k\in N$ and $\forall u\in B$,*

$$|\xi|^k\widehat{\chi_k u}\in L^2(R^n) \quad \textit{and}$$

$$\||\xi|^k\widehat{\chi_k u}\|_{L^2(R^n)}\leqslant C(C'k)^{sk}\cdot\|u\|_B.$$

Proof. For a compact set $K\subset R^n$, we denote by $\gamma_h^{\{s\}}(K)$ the space of functions of class C^∞ such that there is a constant C such that

$$\forall\alpha\in N^n,\ \sup_K|D_x^\alpha u|\leqslant Ch^{|\alpha|}(|\alpha|!)^s.$$

Let $\gamma^{\{s\}}(K)=\varinjlim_{h\to\infty}\operatorname{ind}\gamma_h^{\{s\}}(K)$. Then, $\gamma^{\{s\}}(K)$ is a space of type $\mathscr{L}\mathscr{F}$ in the sense of A. Grothendieck. ([8])

So, by the closed graph theorem, the mapping $u\mapsto u|_{\bar\omega}$ is continuous from B to $\gamma^{\{s\}}(\bar\omega)$ and a Banach space B is in some $\gamma_h^{\{s\}}(\bar\omega)$;

$$\|u\|_{s,h,\omega'}=\sup_\alpha\|D_x^\alpha u\|_{0,\omega'}/|\alpha|!^s h^{|\alpha|}\leqslant C\|u\|_B.$$

Let χ_k be the functions satisfying (A–1). Then

$$\|D_x^\alpha\chi_k u\|_{L^2(R^n)}\leqslant\sum_{\beta<\alpha}\binom{\alpha}{\beta}(r_0k/r)^{|\beta|}|\alpha-\beta|!^s h^{|\alpha-\beta|}\|u\|_{s,h,\omega'}$$

$$\leqslant C(h+(r_0/r))^{|\alpha|}k^{s|\alpha|}\|u\|_B \quad\text{for}\quad |\alpha|\leqslant k.$$

So, we have

$$\||\xi|^k\widehat{\chi_k u}\|_{L^2(R_n)}\leqslant n^{k/2}C(h+(r_0/r))^k k^{sk}\|u\|_B. \qquad \text{Q.E.D.}$$

Let $G_s=\{u\in L^2(R^n);\ e^{|\xi|^{1/s}}\hat u\in L^2(R^n)\}$. Then, we obtain

Lemma A–2. *Let k be an integer$\geqslant 1$. Then, for any $u\in H^k(R^n)$, we can write u in the following form;*

$u=\sum u_j$, u_j being in G_1 and satisfying: $\forall s\geqslant 1$,

$$\Phi_{k,s,R^n,G_s}^2(\{u_j\})=\sum_j N_j^{2sk}(\|u_j\|_{0,R^n}^2+e^{-2N_j}\|u_j\|_{G_s}^2)$$

$$\leqslant 2(2C)^{sk}|||u|||_{k,R^n,s}^2,$$

where $N_j=k2^j$ $(j=0, 1,\dots)$ and C is a constant depending only on n.

Proof. Let $N_{-1}=0$ and set

$$u_j(x)=(2\pi)^{-2n}\int_{N_{j-1}<|\xi|^{1/s}<N_j}e^{ix\cdot\xi}\hat u(\xi)d\xi.$$

Then, Remarking that for $|\xi|^{1/s} \geqslant N_{j-1}$, $N_j \leqslant 2^s(|\xi|+k^s)^{2k}$, we have the desired inequality. Q.E.D.

Let B be in Lemma A–1; especially, there is a neighborhood ω of x_0 such that for $u \in B$, $u|_{\bar{\omega}} \in \gamma^{\{s\}}(\bar{\omega})$. Then, for $\omega' \Subset \omega$, we have

Lemma A–3. *There is a constant C such that if $u_j \in B$ satisfy $\Phi_{k,s,\Omega,B}(\{u_j\}) < +\infty$, then $u = \sum u_j$ converges in $L^2(\Omega)$,* and

$$u|_{\omega'} \in H^k(\omega') \quad \textit{with} \quad \|u\|_{H^k(\omega')} \leqslant C^{k+1}\Phi_{k,s,\Omega,B}(\{u_j\}). \quad (^\forall k \in N)$$

Proof. By Lemma A–1,

$$\|(|\xi|/C'N^s)^N \widehat{\chi_N u}\|_{L^2(R_n)} \leqslant C\|u\|_B . \tag{A–2}$$

By the hypothesis, $\sum u_j$ converges to $u \in L^2(\Omega)$. Let $v = \sum \chi_{N_j} u_j$. Then,

$$v|_{\omega'} = u.$$

So, it is sufficient to show

$$\|v\|_{H^k(R^n)} \leqslant C^{k+1}\Phi_{k,s,\Omega,B}(\{u_j\}).$$

Let $\Theta(j, \xi, s) = e^{-N_j}(|\xi|/C'N_j^s)^{N_j}$ and $g_j(\xi) = (1+\Theta(j, \xi, s))\widehat{\chi_{N_j} u_j}(\xi)$. Then,

$$|\xi|^k v(\xi) = \sum (1+\Theta(j, \xi, s))^{-1} g_j(\xi)|\xi|^k, \text{ and}$$

$$|\xi|^{2k}|v(\xi)|^2 \leqslant (\sum |g_j(\xi)|^2 N_j^{2s\,k})\Theta(\xi),$$

where $\Theta(\xi) = \sum (|\xi|/N_j^s)^{2k}(1+\Theta(j, \xi, s))^{-2}$. By (A–2), we have

$$\sum \|g_j(\xi)\|^2_{L^2(R^n)} N_j^{2s\,k} \leqslant (1+C^2)\Phi^2_{k,s,\Omega,B}.$$

Considering two cases: $C'e^2N_j^s \leqslant |\xi|$ and $C'e^2N_j^s > |\xi|$, we have

$$|\Theta(\xi)|_{L^\infty(R^n)} \leqslant C^{k+1} \quad \text{with} \quad C = \max(e/(e^2-1), 2, (C'e^2)^2). \qquad \text{Q.E.D.}$$

*Proof of lemma*1. By hypothesis, there is a neighborhood Ω of x_0 such that P has a right inverse R which is continuous from $L^2(\Omega)$ to $L^2(\Omega)$ and for $\omega \Subset \Omega$, $u \in \gamma^{\{s\}}(\omega) \Rightarrow Ru \in \gamma^{\{s\}}(\omega)$. Let $\omega' \Subset \omega \Subset \Omega$ and χ_k satisfy (A–1G). Also, let

$$G' = \{u \in L^2(\Omega);\ ^\exists v \in G_s \text{ such that } v|_\Omega = u\},$$

$$\|u\|_{G'} = \inf_{v \in V} \|v\|_{G_s}, \quad V = \{v \in G_s;\ v|_\Omega = u\}.$$

Finally, let $B = R(G')$ with norm $\|u\|_B = \|R^{-1}u\|_{G'}$. Then, by hypothesis, the Banach space B satisfies the assumption of lemma A–1. Let

$$u \in \mathscr{D}'(\omega) \quad \text{such that} \quad Pu \in H^k(\omega).$$

Put $f = \chi_k Pu$. Then we have

$$|||f|||_{k,R^n,s} \leqslant L^k |||Pu|||_{k,\omega,s} \qquad \text{with a constant } L \text{ independent of } k.$$

By lemma A–2,

$$f = \sum f_j \quad \text{with} \quad f_j \in G_1 \quad \text{and}$$

$$\sum N_j^{2sk}(\|f_j\|^2_{L^2(R^n)}+e^{-2N_j}\|f_j\|^2_{G_s})\leqslant 2(2C)^k|||f|||^2_{k,R^n,s}.$$

Put $v_j=R(f_j|_\Omega)$. Then, we have

$$\sum N_j^{2sk}(\|v_j\|^2_{0,\Omega}+e^{-2N_j}\|v_j\|^2_{G_s})\leqslant \tilde{C}|||f|||_{k,R^n,s}$$

Therefore, by lemma A–3, $\sum v_j$ converges to $v\in L^2(\Omega)$ and

$$\text{(A–3)}\qquad \|v|_{\omega'}\|_{H^k(\omega')}\leqslant C^{k+1}|||f|||_{k,R^n,s}.\qquad (^\forall k\in N)$$

Since $(u-v)|_{\omega'}=0$, we have $P(u-v)|_{\omega'}=0$. Let $\mathscr{N}=\{u\in\mathscr{D}('\omega');\ Pu=0\}$ with the topology induced by $L^2_{loc}(\omega')$. Then, $\mathscr{N}$ is a Frechet space. So, by Baire's theorem, for $\omega''\Subset\omega'$, we have for some constant C_0

$$\text{(A–4)}\qquad \|(u-v)|_{\omega''}\|_{H^k(\omega'')}\leqslant(k!)^sC_0^{k+1}\|(u-v)|_{\omega'}\|_{L^2(\omega')}$$
$$\leqslant(k!)^sC_0^{k+1}(\|u\|_{0,\omega'}+\|R\|\cdot\|Pu\|_{0,\omega'}).\qquad (^\forall k\in N)$$

By (A–3) and (A–4), we have the inequality (5). Q. E. D.

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