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Existence, uniqueness and analyticity of solutions to boundary value problems for equations of mixed type in a half space

Dedicated to Professor SIGERU MIZOHATA on his sixtieth birthday

By

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§1. Introduction and statements of results.

We consider the following boundary value problem

$$(P)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial x^2}+q(x)\dfrac{\partial^2 u}{\partial t^2}=0\,, & (x,\,t)\in(0,\,\infty)\times R\,, \\ \lim\limits_{x\to 0} u(x,\,t)=g(t) \quad \text{and} \quad \lim\limits_{x\to\infty} u(x,\,t)=0\,, & t\in R\,. \end{cases}$$

The coefficient $q(x)$ is assumed to be a real-valued bounded smooth function on $[0, \infty)$ throughout this paper. We assume

$$(C)\qquad \overline{\lim_{x\to\infty}}\, q(x)<0\,.$$

In our previous work [4] we showed some existence theorems for the above problem (P) with $\dfrac{\partial^2}{\partial^2 x}$ replaced by Δ, the Laplacian in a higher dimensional space, where $q(x)$ satisfies (C) or $\underline{\lim}_{x\to\infty} q(x)>0$. Here confining ourselves to the condition (C) we gain an insight into the problem (P) to obtain the existence and uniqueness theorem of the solution in a fairly distinct and self-contained way and exhibit the analyticity in t of the solution $u(x, t)$ for any fixed x larger than $\inf\{x; q(x)>0\}$. We mention also to the existence of solutions satisfying zero boundary data.

Notation. $g(t)$ is said to belong to H^k_γ if $e^{-\gamma t}g(t)\in H^k$, $(k>-\infty)$. We note $\|g\|^2_{H^k_\gamma}\equiv\|g\|^2_{\gamma,k}=\sum_{j=0}^{k}\int_{-\infty}^{\infty}\left|\dfrac{d^j}{dt^j}(e^{-\gamma t}g(t))\right|^2 dt$ for integer $k\geq 0$ and $H_\gamma=H^\infty_\gamma=\bigcap_{k=0}^{\infty}H^k_\gamma$. Denote $g(t)\in B^k_\gamma$ if $e^{-\gamma t}g(t)\in B^k$, and $|g|_{B^k_\gamma}=|g|_{\gamma,k}=\sum_{j=0}^{k}\sup_{t\in R}\left|\dfrac{d^j}{dt^j}(e^{-\gamma t}g(t))\right|$ and $B_\gamma=\bigcap_{k=0}^{\infty}B^k_\gamma$. $f(x, t)\in C^k([0, \infty); H)$ means that $f(x, t)$ is k-times continuously differentiable on $[0, \infty)$ and $\lim_{x\to\infty} f(x, t)=0$ with values in H.

Theorem 1. *Suppose that $q(x)$ satisfies (C). Then there exists a set*

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$$S=\{r=\pm r_k;\ k=0,1,2,\cdots,\ r_0=0<r_1<r_2<\cdots,\ \lim_{k\to\infty} r_k=\infty\}$$

in the case of $\{x;\ q(x)>0\}\neq\phi$ *and* $S=\{r=0\}=\{0\}$ *in the case of* $\{x;\ q(x)>0\}=\phi$, *such that for any* $r\in R-S$ *the problem* (P) *with* $g(t)\in H_\gamma$ *has a unique solution* $u(x,t)$ *belonging to* $C^2([0,\infty);H_\gamma)$.

Theorem 1°. *We have the same result as in Theorem* 1 *replacing* H_γ *by* B_γ.

Remrak. If $q(x)\equiv -1$, then $u(x,t)=g(t-x)$ for $r>0$ and $u(x,t)=g(t+x)$ for $r<0$. Remark that it holds $\lim_{x\to\infty}||u(x,\cdot)||_{\gamma,0}=0$ in spite of $||u(x,\cdot)||_{0,0}=||g||_{0,0}$.

If $q(x)<0$ in $(0,\infty)$, it is well-known that $\operatorname{supp} g(t)\subset[a,\infty)$ implies $\operatorname{supp}_t u(x,t)\subset[a,\infty)$ for all $x>0$. Now it is remarkable that in the case of $\{x;q(x)>0\}\neq\phi$ the solution $u(x,t)$ with $r\in(0,\infty)-S$ never has this property. Moreover we have

Theorem 2. *Suppose* (C) *and* $\{x;\ q(x)>0\}\neq\phi$. *For any* $r\in R-S$, *the solution* $u(x,t)$ *given by Theorem* 1 *is real-analytic with respect to* t *for any fixed* $x>\inf\{x;\ q(x)>0\}$. *More precisely* $u(x,t)$ *is analytic in* $\{t\in C;\ |\operatorname{Im} t|<\delta(x)\}$, *where* $\delta(x)=\int_0^x q(s)_+ds$. *Here* $q(x)_+=\max\{q(x),0\}$.

Remark. In the case of $q(x)\equiv 1$, the solution of (P) is given by $u(x,t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{xg(s)}{x^2+(t-s)^2}ds$. Suppose its integrability. Then $u(x,t)$ is analytic in $\{t\in C;\ |\operatorname{Im} t|<x=\int_0^x 1^{1/2}ds\}$, where $x^2+(t-s)^2\neq 0$ for all $s\in R$.

Theorem 3. *Suppose* (C) *and* $\{x;q(x)>0\}\neq\phi$. *Suppose* $r\in S-\{0\}$. *Then there exists a soluiton* $u(x,t)\not\equiv 0$ *of* (P) *with* $g(t)\equiv 0$ *belonging to* $C^2([0,\infty):B_\gamma)$.

Now we state the estimate of the solution in Theorem 1 more precisely.

Theorem 1′. *In Theorem* 1 *we can replace* $g(t)\in H_\gamma$ *and* $u(x,t)\in C^2([0,\infty):H_\gamma)$ *by* $g(t)\in H_\gamma^{k+(3/2)}$ *and* $u(x,t)\in C^2([0,\infty):H_\gamma^k)$, $(k>-\infty)$ *respectively. Moreover there exists a constant* $C_{k,\gamma}$ *for* $r\in R-S$, *such that the following estimate holds:*

$$\sup_{(0,\infty)}||u(x,\cdot)||_{\gamma,k}^2+\int_0^\infty\Big(||u(x,\cdot)||_{\gamma,k}^2+\Big\|\frac{\partial}{\partial x}u(x,\cdot)\Big\|_{\gamma,k}^2\Big)dx\le C_{k,\gamma}||g||_{\gamma,k+(3/2)}^2\,.$$

Theorem 2′. *We have the same result as in Theorem* 2 *replacing* $g(t)\in H_\gamma$ *by* $g(t)\in H_\gamma^k$, $(k>-\infty)$.

To understand the estimate in Theorem 1′, it is convenient to state

Remark. Notice that the estimate in Theorem 1′ is inhomogeneous in the sense that $u(x,t)$ and $\frac{\partial}{\partial x}u(x,t)$ have the same index. This is derived from the fact that we use a non-linear method associated with the Riccati equation. If $0<m\leq q(x)\leq M<\infty$ in $(0, \infty)$, then the equation is elliptic and we know,

$$\sup_{(0,\infty)}\|u(x, \cdot)\|^2_{k+(3/2)}+\int_0^\infty(\|u(x, \cdot)\|^2_{k+2}+\left\|\frac{\partial}{\partial x}u(x, \cdot)\right\|^2_{k+1})dx$$
$$\leq C_k(\|g\|^2_{k+(3/2)}+\int_0^\infty\|u(x, \cdot)\|^2_0 dx)\,,$$

where $\|g\|_k$ means $\|g\|_{0,k}=\|g\|_{H^k}$, (cf. for example [5] and [8]). If $-\infty<-M<q(x)<-m<0$, then the equation is hyperbolic and we can verify the following estimate in the course of our proof: For $k>-\infty$,

$$\sup_{(0,\infty)}\|u(x, \cdot)\|^2_{\gamma,k+(1/2)}+\int_0^\infty(\|u(x, \cdot)\|^2_{\gamma,k+1}+\left\|\frac{\partial}{\partial x}u(x, \cdot)\right\|^2_{\gamma,k})dx$$
$$\leq C_k\frac{1}{\gamma^2}\|g\|^2_{\gamma,k+3(/2)}\,,\qquad \gamma\in R-\{0\}\,.$$

Now let us explain our method in short. If the equation is elliptic or hyperbolic, a priori estimate follows from the integration by parts combined with some suitable partitions of unity. However for equations of mixed type such a direct method is not effective. Besides we do not know in advance the function spaces to be considered, which we will rather determine through our reasoning. Let us rely upon the Fourier-Laplace inversion formula $g(t)=\int_\Gamma e^{i\tau t}\hat{g}(\tau)d\tau$. Since the problem (P) is linear, we seek the solution $u(x, t)$ described in the form

$$u(x, t)=\int_\Gamma e^{i\tau t}v(x, \tau)\hat{g}(\tau)d\tau\,. \tag{1.1}$$

Here $v(x, \tau)$ is the solution of

$$(P)_0\qquad \begin{cases} \dfrac{d^2}{dx^2}v(x, \tau)=\tau^2 q(x)v(x, \tau)\,, & x\in(0, \infty)\,,\\ v(0, \tau)=1 \quad\text{and}\quad \lim\limits_{x\to\infty}v(x, \tau)=0\,, \end{cases}$$

and Γ is a path in C such that $(P)_0$ has a unique solution for $\tau\in\Gamma$. So we study the existence and uniqueness theorem for $(P)_0$ with a complex parameter τ. Then the existence of the solution of (P) follows from the integrability of (1.1), (cf. (2.1) in Section 2). Moreover the uniqueness of the solution can be reduced to that of $(P)_0$. We will see that a serious argument in the theory of the Lebesgue integral plays an important role in this reduction. To obtain Theorem 2, we need a sharper estimate (E) stated in Section 6. Theorem 3 follows from a residue caliculus at each simple pole of $v(x, \tau)$. We prove Theorems 1, 2 and 3 through the following devices (a) and (b);

(a) Construction to some fundamental lemmas concerning the behavior of solutions to Riccati equation $w'=\tau^2 q(x)-w^2$,

(b) Introduction of an oblique coordinate in C depending on a parameter for obtaining the estimate of $v(x, \tau)$.

These methods will be useful in some researches for other problems. To justify the interest of the problem studied here the author consulted to [1], [2], [6], [9] and [10].

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§2. Relation between (P) and $(P)_0$.

In this section we reduce Theorems 1, 2 and 3 to some theorems on the problem $(P)_0$. We begin our argument with the uniqueness in Theorem 1.

Proposition 2.1. *Let $g(t)$ belong to H^2_γ for a certain $\gamma \in R$. Suppose that the problem (P) has a solution $u(x, t) \in C^0([0, \infty); H^2_\gamma) \cap C^2([0, \infty); L^2_\gamma)$. Then $\hat{u}(x, \tau) = \frac{1}{2\pi} \lim_{A \to \infty} \int_{-A}^{A} e^{-i\tau t} u(x, t) dt$ is a solution of the following problem $\underline{(P)}_0$ for almost everywhere $\tau \in \{\tau = \sigma - i\gamma,\ \sigma \in R\}$,*

$$\underline{(P)}_0 \qquad \begin{cases} \dfrac{d^2}{dx^2} \hat{u}(x, \tau) = \tau^2 q(x) \hat{u}(x, \tau), & x \in (0, \infty), \\ \hat{u}(0, \tau) = \hat{g}(\tau) \quad \text{and} \quad \varliminf_{x \to \infty} |\hat{u}(x, \tau)| = 0. \end{cases}$$

Proof. Multiply $e^{-\gamma t}$ to (P) and take the Fourier transform. Then we have a boundary value problem $(P)_0$ for an ordinary differential equation with values in $L^2(\sigma)$. Now apply Radon-Nikodym theorem, then for almost everywhere $\hat{u}(x, \tau)$ and $\frac{d}{dx} \hat{u}(x, \tau)$ are absolutely continuous in x and $\frac{d^2}{dx^2} \hat{u}(x, \tau) = \tau^2 q(x) \hat{u}(x, \tau)$ holds. From $\lim_{x \to \infty} e^{-\gamma t} u(x, t) = 0$ follows $\lim_{x \to \infty} \hat{u}(x, \tau) = 0$ in $L^2(\sigma)$, which implies that $\hat{u}(x, \tau)$ converges to zero in measure as x tends to ∞. Hence from the theory of the Lebesgue integral there exists a sequence $\{x_n\}$ tending to ∞ such that $\lim_{n \to \infty} \hat{u}(x_n, \tau) = 0$ for almost everywhere τ.

As we can imagine, the problem $(P)_0$ is equivalent to $\underline{(P)}_0$. Namely we have

Lemma 2.1. *Suppose* (C) *and* $\operatorname{Im} \tau \neq 0$. *Let $v(x, \tau)$ be a solution of $\frac{d^2}{dx^2} v = \tau^2 q(x) v$ in $(0, \infty)$. Then $\varliminf_{x \to \infty} |v(x, \tau)| = 0$ implies $\lim_{x \to \infty} v(x, \tau) = 0$.*

The exact proof of Lemma 2.1 is given later. Now from Proposition 2.1 we have

Proposition 2.2. *Let γ be a real number such that the solution of $(P)_0$ exists uniquely for almost everywhere $\tau \in \{\tau = \sigma - i\gamma,\ \sigma \in R\}$. Then the solution of (P)*

belonging to $C^0([0, \infty); H_\gamma^2) \cap C^2([0, \infty); L_\gamma^2)$ *is unique.*

By virtue of Proposition 2.2 the uniqueness in Theorem 1 follows from

Theorem 2.1. *Suppose* $q(x)$ *satisfy* (C). *Then there exists the set* S *described in Theorem* 1 *such that the problem* $(P)_0$ *has the unique solution* $v(x, \tau)$ *if and only if* τ *belongs to*

$$D = C - \{\{\tau = \pm i\gamma_k;\ \pm\gamma_k \in S\} \cup \{\tau \in C;\ \operatorname{Im}\tau = 0\}\}.$$

To show the existence in Theorem 1 we use

Theorem 2.2. *Suppose* (C). *For* $\gamma \in R-S$, *there exists a positive constant* C_γ *such that thv solution* $v(x, \tau)$ *of* $(P)_0$ *satisfies*

$$\left|\frac{v(x, \tau)}{v(x', \tau)}\right| \leq C_\gamma(|\tau|+1)^{3/2} \qquad \text{for} \quad \tau \in \{\tau = \sigma - i\gamma;\ \sigma \in R\},\ 0 \leq x' < x. \tag{2.1}$$

Since $v(0, \tau)=1$, from (2.1) we have $|v(x, \tau)| \leqq C(|\tau|+1)^{3/2}$ for all $x>0$. In order to verify Theorem 2, we improve the estimate (2.1) partially in the case of $\{x;\ q(x)>0\} \neq \phi$ as follows.

Theorem 2.3. *Suppose* (C) *and* $\gamma \in R-S$. *Assume* $q(x) \geq \delta > 0$ *in an interval* $[x_0, x_0']$, $x_0' = x_0 + d$. *Then for any* $\varepsilon \in (0, 1)$ *there exists a positive constant* $C_{\varepsilon,\gamma}$ *such that the solution* $v(x, \tau)$ *of* $(P)_0$ *satisfies*

$$|v(x_0+d, \tau)/v(x_0, \tau)|\,| \leq C_{\varepsilon,\gamma} \exp(-(1-\varepsilon)\sqrt{\delta}\, d\, |\tau|), \qquad |\operatorname{Im}\tau| = |\gamma|. \tag{2.2}$$

To obtain Theorem 3 we show

Theorem 2.4. *Suppose* (C) *and* $\{x;\ q(x)>0\} \neq \phi$. *Then for all* $x>0$, $v(x, \tau)$ *is analytic with respect to* τ *in* D *and has simple poles at* $\tau \in \{\tau = \pm i\gamma_k,\ \gamma_k \in S - \{0\}\}$.

§3. Strum- Liouville equations and Riccati equations.

The problem $(P)_0$ is equivalent to

$$\frac{d}{dx} w = \tau^2 q(x) - w^2, \quad \lim_{x\to\infty} \exp\int_0^x w(s, \tau) ds = 0, \tag{3.1}$$

if $v(x, \tau) \neq 0$ for all $x \in [0, \infty)$. Here we have put

$$w = w(x, \tau) = \frac{d}{dx} v(x, \tau)/v(x, \tau), \tag{3.2}$$

$$v(x, \tau) = v(x', \tau) \exp\int_{x'}^x w(s, \tau) ds, \qquad 0 \leq x' < x. \tag{3.3}$$

Considering the problem (3.1) in detail, we will be able to prove naturally the following Proposition 3.1, which implies Lemma 2.1.

Proposition 3.1. *Suppose (C) and* $\operatorname{Im}\tau \neq 0$. *Then any solution* $v(x, \tau)$ *of* $\frac{d^2}{dx^2} v=\tau^2 q(x)v$ *satisfies* $\lim_{x\to\infty} v(x, \tau)=0$ *or* $\lim_{x\to\infty} |v(x, \tau)|=\infty$.

Now let us rewrite the condition (C) as follows. There exist positive numbers M, m and x_1 such that

$$-M<q(x)<-m<0 \quad \text{in} \quad (x_1, \infty). \tag{3.3}$$

We note incidentally

$$-M<q(x)<M \quad \text{in} \quad (0, \infty). \tag{3.4}$$

Evidently $\lim_{x\to\infty} v(x, \tau)=0$ follows from, for example,

$$\overline{\lim_{s\to\infty}} \operatorname{Re} w(s, \tau)<0. \tag{3.5}$$

In the case of $\operatorname{Re}\tau^2<0$ we can prove (3.5), (Lemma 3.1). However (3.5) is not necessary for obtaining $\lim_{x\to\infty} v(x, \tau)=0$. Now we describe

Proposition 3.2. *Suppose* $\psi\in(0, \pi)$. *Put* $z=e^{i\psi}$. *Assume that* $\tilde{q}(x)$ *is a continuous function satisfying* $|\tilde{q}(x)|<M'$ *and* $\operatorname{Im}(\tilde{q}(x)\bar{z})<-m'<0$ *for all* $x\in[a, \infty)$. *Let* $w(x)$, (*respectively* $w_1(x)$) *be a bounded continuous solution of* $\frac{d}{dx} w=\tilde{q}(x)-w^2$ *satisfying* $0<m''<\operatorname{Im}(w(x)\bar{z})<M''$ *in* (a, ∞), (*respectively* $-M''<\operatorname{Im}(w_1(x)\bar{z})<-m''<0$ *in* (a, ∞)). *Then* $v(x)=\exp\int_a^x w(s)ds$ *satisfies* $\lim_{x\to\infty} v(x)=0$ *and* $\sup_{a<x}|v(x)|^2+\int_a^\infty |(v(s)|^2+|v'(s)|^2)ds\leq C$, *where C is a contant depending one* m', M'', m'' *and* z, ($v_1(x)=\exp\int_a^x w_1(s)ds$ *satisfies* $\lim_{s\to\infty}|v_1(s)|=\infty$ respectively).

We postpone this proof and state

Lemma 3.1. *Let* $\tilde{q}(x)$ *be a bounded continuous function defined on* $[a, \infty)$. *Suppose that the value* $\tilde{q}(x)$ *belongs to*

$$G_1(M_1, m_1) = \{\xi\in C;\ 0<m_1<\operatorname{Re}\xi,\ |\xi|<M_1\} \tag{3.6}$$

for all $x\in[a, \infty)$. *Put*

$$\Omega_1(M_1, m_1) = \left\{w\in C;\ \operatorname{Re} w<-m_1^{1/2},\quad \operatorname{Im}\left(\frac{e^{-\frac{\pi}{4}i}}{w}\right)>\frac{1}{2M^{1/2}} \quad \text{or} \quad \operatorname{Im}\left(\frac{e^{\frac{\pi}{4}i}}{w}\right)<\frac{-1}{2M^{1/2}}\right\}. \tag{3.7}$$

Let x' *belong to* $[a, \infty)$. *Suppose* $w_0\in\Omega_1$, (*respectively* $w_0\in -\Omega_1$), *then the solution* $w(x)$ *of* $w'=\tilde{q}(x)-w^2$ *and* $w(x')=w_0$ *satisfies* $w(x)\in\Omega_1$ *for all* $x\in[a, x')$, (*respectively* $w(x)\in -\Omega_1=\{w\in C;\ -w\in\Omega_1\}$ *for all* $x\in(x', \infty)$). *Moreover there exists a unique solution* $w(x)$ *of* $w'=\tilde{q}(x)-w^2$ *satisfying* $w(x)\in\Omega_1$ *for all* $x\in[a, \infty)$.

Lemma 3.2. *Suppose the same condition as in Lemma* 3.1 *replaced* G_1 *by*

$$(3.8)\qquad G_2(\theta, M_2, m_2) = \{\xi\in C;\ -\theta\leq \arg\xi<0,\ |\xi|<M_2,\ \mathrm{Im}\,(\xi e^{\frac{\theta-\pi}{2}i})<-m_2\},$$

where M_2, m_2 *and* θ *satisfy* $0<m_2<M_2$ *and* $0<\theta<\pi$. *Then we have the same results as in Lemma* 3.1 *replacing* Ω_1 *by*

$$(3.9)\qquad \Omega_2 = \Omega_2(\theta, M_2, m_2)$$

$$= \left\{w\in C;\ \pi-\theta<\arg w<\pi,\ \mathrm{Im}\,(we^{\frac{\theta-\pi}{2}i})>\left(m_2\cos\frac{\theta}{2}\right)^{1/2}<\frac{\mathrm{Im}(we^{(\theta-\pi)i})}{\sin\frac{\theta}{2}},\right.$$

$$\left.\mathrm{Im}\left(\frac{e^{\frac{\pi-\theta}{2}i}}{w}\right)<\frac{-\cos\frac{\theta}{2}}{3M_2^{1/2}}\right\}.$$

Replacing G_2 *and* Ω_2 *by* $\bar{G}_2=\{\xi\in C;\ \bar{\xi}\in G_2\}$ *and* $\bar{\Omega}_2$ *respectively, we have the same results.*

We prove the above lemmas in Section 7. Using Lemmas 3.1, 3.2 and Proposition 3.2 we can prove the following Propositino 3.3 which implies Proposition 3.1 and Lemma 2.1.

Proposition 3.3. *Suppose* (3.3) *and* Im $\tau\neq 0$. *Then there exists a unique solution* $\tilde{v}(x, \tau)$ *of*

$$\tilde{v}'' = \tau^2 q(x)\tilde{v}\ \textit{ in } (x_1, \infty),\quad \tilde{v}(x_1, \tau) = 1\quad \textit{and}\quad \lim_{x\to\infty}\tilde{v}(x, \tau) = 0.$$

We have also $\int_{x_1}^{\infty}(|\tilde{v}(s,\tau)|^2+|\tilde{v}'(s,\tau)|^2)ds<\infty$. *If* $\tilde{v}_1(x,\tau)$ *satisfies* $\tilde{v}_1''=\tau^2 q(x)\tilde{v}_1$ *and* $\tilde{v}_1(x_1,\tau)=1$, *then it holds* $\lim_{x\to\infty}\tilde{v}_1(x,\tau)|=\infty$ *or* $\tilde{v}_1\equiv\tilde{v}$.

Proof of Proposition 3.3. Put $\tilde{q}(x)=\tau^2 q(x)$ and $a=x_1$. If Re $\tau^2<0$, we can apply Lemma 3.1 with $M_1=M|\tau|^2$ and $m_1=m|\mathrm{Re}\,\tau^2|$. Then $w'=\tau^2 q(x)-w^2$ has two solutions $w(x,\tau)$ and $w_1(x,\tau)$ satisfying $w(x,\tau)\in\Omega_1$ for all $x\in[x_1,\infty)$ and $w_1(x,\tau)\in-\Omega_1$ for all $x\in[x_1,\infty)$. Put

$$(3.10)\qquad \tilde{v}(x,\tau) = \exp\int_{x_1}^{x} w(s,\tau)ds,\quad \tilde{v}_1(x,\tau) = \exp\int_{x_1}^{x} w_1(s,\tau)ds.$$

Then by virtue of Proposition 3.2 we have Proposition 3.3 putting $z=e^{i\pi/2}$ in this case. If Im $\tau^2>0$, we apply Lemma 3.2 with

$$(3.11)\qquad \theta=\pi-\frac{1}{2}\arg\tau^2,\quad M_2=M|\tau|^2,\quad m_2=m|\tau|^2\sin\frac{\pi-\theta}{\theta}.$$

Similarly to the above case we have $\tilde{v}(x,\tau)$ and $\tilde{v}_1(x,\tau)$ satisfying the desired conditions. Since $v=\overline{\tilde{v}(x,\tau)}$ satisfies $v''=\bar{\tau}^2 q(x)v$ if $\tilde{v}$ is a solution of $v''=\tau^2 q(x)v$, we can reduce the case of Im $\tau^2<0$ to the case of Im $\tau^2>0$. Namely $\tilde{v}(x,\bar{\tau})=\overline{\tilde{v}(x,\tau)}$. Thus we have Proposition 3.3.

Proof of Proposition 3.2. Integrate $-v''\bar{v}=-\tilde{q}(x)|v|^2$ by parts in (a, x). Then we have

$$w(a) = w(x)|v(x)|^2-\int_a^x |v'(s)|^2 ds-\int_a^x \tilde{q}(s)|v(s)|^2 ds . \tag{3.21}$$

Take the imaginary part of (3.12) multiplied by $\bar{z}$ and make x tend to ∞, then we have

$$\int_a^\infty |v'(s)|^2 ds<\frac{M''}{\sin\psi} \quad\text{and}\quad \int_a^\infty |v(s)|^2 ds<\frac{M''}{m'} . \tag{3.13$_1$}$$

Therefore it holds

$$\sup_{a<x}|v(x)|^2<\frac{M''}{m''} \quad\text{and}\quad \lim_{x\to\infty} v(x) = 0 . \tag{3.13$_2$}$$

Replacing $w(x)$ and $v(x)$ by $w_1(x)$ and $v_1(x)$ respectively in (3.12), we can verify $\lim_{x\to\infty}|v_1(x)|=\infty$.

§4. Oblique coordinate and calculus for basic estimate.

We extend $\tilde{v}(x, \tau)$ defined on $[x_1, \infty)$ in Proposition 3.3 to $[0, \infty)$ as a solution of $\tilde{v}''=\tau^2 q(x)\tilde{v}$. Let us denote

$$N = \{\tau\in C-R;\ \tilde{v}(0, \tau) = 0\} . \tag{4.1}$$

Concerning N and $\tilde{v}(x, \tau)$ we have

Proposition 4.1. *Suppose* (C). *Then for any* $x\in[0, \infty)$, $\tilde{v}(x, \tau)$ *is analytic with respect to* τ *in* $C-R$. *If we define* $\tilde{v}(x, 0)\equiv 1$, $\tilde{v}(x, \tau)$ *is continuous with respect to* τ *on* $I=\{\tau;\ \mathrm{Re}\,\tau=0\}$.

Proposition 4.2. *Suppose* (C). *Then there exists a sequence of positive numbers* $\{r_k\}_{k=1}^\infty$ *tending to* ∞ *such that*

$$N = \{\tau:\ \tau = \pm i r_k,\quad k = 1, 2, 3, \cdots\} ,$$

if $\{x;\ q(x)>0\} \neq\phi$. *We have* $N=\phi$ *if* $\{x;\ q(x)>0\} =\phi$. *Moreover* $\tilde{v}(0, \tau)$ *has a simple pole at each* $\tau\in N$.

The proof will be given in Section 5. Here we define

$$v(x, \tau) = \tilde{v}(x, \tau)/\tilde{v}(0, \tau) ,\quad \tau\in C-N\cup R . \tag{4.2}$$

Then $v(x, \tau)$ is the unique solution of $(P)_0$. Our purpose in this section is to obtain the following proposition which yields Theorem 1 and Theorem 1′.

Proposition 4.3. *Suppose* (3.3) *and* (3.4). *Then there exists a positive constant* C *such that the following inequalities hold for* $\mathrm{Im}\,\tau^2\neq 0$

$$\sup_{0<x'\leq x} |v(x, \tau)/v(x', \tau)| \leq CE(\tau) , \tag{4.3}$$

$$\int_0^\infty \left|\frac{d}{ds}v(s,\tau)\right|^2 ds \le C\frac{|\tau|}{|\tau|+1}E(\tau)^2\,, \tag{4.4}$$

$$\int_0^\infty |v(s,\tau)|^2 ds \le CE(\tau)^2\left(1+\frac{1}{|\mathrm{Im}\,\tau|^2}\right), \quad \textit{where} \tag{4.5}$$

$$E(\tau)^2 = \{|\tau|^3(|\tau|+1)\}/\{|\mathrm{Re}\,\tau|\,|\mathrm{Im}\,\tau|\min\{|\mathrm{Re}\,\tau|,\,|\mathrm{Im}\,\tau|\}\}.$$

Remark that (2.1) follows from (4.3) with $x'=0$ and $v(0,\tau)=1$. In order to prove Proposition 4.3 we introduce a fairly general

Notation. Let ψ be a real number belonging to $[0,\pi)$. Put

$$z = e^{i\varphi}, \quad \sqrt{-z} = e^{i(\varphi+\pi)/2}\,. \tag{4.6}$$

We see $\mathrm{Im}\sqrt{-z}<0$ and $\mathrm{Re}\sqrt{-z}<0$ for $\varphi\in(0,\pi)$. Any complex number w is decribed as

$$w = P_z(w)z+Q_z(w)\sqrt{-z}\,, \quad \text{where} \tag{4.7}$$

$$P_z(w) = \frac{\mathrm{Im}(w\overline{\sqrt{-z}})}{\mathrm{Im}(z\overline{\sqrt{-z}})}, \quad Q_z(w) = \frac{\mathrm{Im}(w\bar z)}{\mathrm{Im}(\sqrt{-z}\,\bar z)} = \frac{\mathrm{Im}(w\bar z)}{\mathrm{Im}\sqrt{-z}}\,, \tag{4.7$'$}$$

which we call P_z and Q_z component of w.

Remark. For $z=1$, $(P_1(w), Q_1(w))$ is the usual Gauss coordinate. Relating to Lemma 3.2 we remark

$$\mathrm{Im}\,(we^{-i(\pi-\theta)})/\sin\frac{\theta}{2} = Q_z(w)\,, \qquad \text{if} \quad z = e^{i(\pi-\theta)}\,.$$

Notice that $Q_z(w)<0$ means $-\theta<\arg\xi<\pi-\theta$.

Application to energy identity. Similarly to (3.12) it holds

$$w(x')|v(x')|^2 = w(x)|v(x)|^2-\int_{x'}^x |v'(s)|^2 ds-\tau^2\int_{x'}^x q(s)|v(s)|^2 ds\,, \tag{4.8}$$

for $0\le x'<x$. Here $v(x)$ stands for $v(x,\tau)$ and $w(x)=v'(x,\tau)/v(x,\tau)$. Let us put, for $\mathrm{Im}\,\tau^2>0$,

$$z = \frac{\tau^2}{|\tau|^2} = e^{i(\pi-\theta)}\,, \quad Q_z(-1) = 2\frac{|\mathrm{Im}\,\tau|}{|\tau|}\,, \quad Q_z(\tau^2) = 0\,, \tag{4.9$_1$}$$

$$z_1 = \sqrt{z}\,, \quad |\mathrm{Im}\,\tau|/|\tau| < Q_{z_1}(-1) < 2\frac{|\mathrm{Im}\,\tau|}{|\tau|}\,, \quad Q_{z_1}(\tau^2) = |\tau|\,|\mathrm{Im}\,\tau|\,, \tag{4.9$_2$}$$

and take Q_z or Q_{z_1} component of (4.8), then

$$Q_z(w(x'))|v(x')|^2 = Q_z(w(x))|v(x)|^2+\frac{2|\mathrm{Im}\,\tau|}{|\tau|}\int_{x'}^x |v'(s)|^2 ds\,, \tag{4.10$_1$}$$

$$(4.10)_2 \qquad Q_{z_1}(w(x'))|v(x')|^2 = Q_{z_1}(w(x))|v(x)|^2 + Q_{z_1}(-1)\int_{x'}^{x}|v'(s)|^2 ds - |\tau|\,|\mathrm{Im}\,\tau|\int_{x'}^{x} q(s)|v(s)|^2 ds .$$

Now as for the estimates of $Q_z(w(x, \tau))$ and $Q_{z_1}(w(x, \tau))$ in (x_1, ∞), we have

Proposition 4.4. *Suppose* (3.3) *and* (3.4). *Then there exist positive constants* C *and* C_1 *such that for* $\mathrm{Im}\,\tau^2>0$ *we have*

$$(4.11)_1 \qquad \frac{|\mathrm{Im}\,\tau|}{C} < Q_z(w(x, \tau)) < C\frac{|\tau|^2}{|\mathrm{Im}\,\tau|}, \qquad x \geq x_1 ,$$

$$(4.11)_2 \qquad \frac{|\mathrm{Im}\,\tau|}{C_1} < Q_{z_1}(w(x, \tau)) < C_1\frac{|\tau|^2}{|\mathrm{Im}\,\tau|}, \qquad x \geq x_1 .$$

Proof of Proposition 4.4. Apply Lemma 3.2 with $a=x_1$, $\tilde{q}(x)=\tau^2 q(x)$, $\theta=\pi-\arg\tau^2$, $M_2=M|\tau|^2$ and $m_2=m|\tau|^2\cos\frac{\theta}{2}$. Then $w(x, \tau)$ belongs to $\Omega_2(\theta, M_2, m_2)\equiv\Omega_2(\tau)$ for $x\geq x_1$. Therefore we have from $\cos\frac{\theta}{2}=\frac{|\mathrm{Im}\,\tau|}{|\tau|}$

$$Q_z(w(x, \tau)) \geq \left(m_2\cos\frac{\theta}{2}\right)^{1/2} \geq m^{1/2}|\mathrm{Im}\,\tau| ,$$

$$Q_z(w(x, \tau)) \leq \min\left\{2, \frac{1}{\sin\frac{\theta}{2}}\right\}|w(x, \tau)| \leq \frac{6M_2^{1/2}}{\cos\frac{\theta}{2}} = \frac{6M_2^{1/2}|\tau|^2}{|\mathrm{Im}\,\tau|} .$$

Therefore we have $(4.11)_1$. Replacing θ by $\theta_1=\pi-\frac{1}{2}\arg\tau^2$ in the above proof we have $(4.11)_2$.

In order to show the estimate of $Q_z(w(x, \tau))$ on $[0, x_1]$ we prepare

Lemma 4.1. *Suppose that* $\tilde{q}(x)$ *satisfies all the conditions as in Lemma* 3.1 *replaced* G_1 *by*

$$(4.12) \qquad G_3 = \{\xi\in C;\ Q_z(\xi)\leq 0, |\xi|<M_3\} ,$$

where $z=e^{i\psi}$, $0<\psi<\pi$. *Then we have the same results as in Lemma* 3.1 *replacing* Ω_1 *by*

$$(4.13) \qquad \Omega_3=\left\{w\in C;\ Q_z(w)>0,\ \ \mathrm{Im}\left(\frac{e^{(\psi/2)i}}{w}\right)>\frac{\sin\frac{\psi}{2}}{3M_3^{1/2}}\ \ or\ \ \mathrm{Re}\left(\frac{e^{(\psi/2)i}}{w}\right)>\frac{\cos\frac{\psi}{2}}{3M_3^{1/2}}\right\} .$$

Proposition 4.5. *Suppose* (3.3) *and* (3.4). *Then in it holds for Im* $\tau^2>0$

$$(4.14) \qquad |w(x, \tau)| \leq 3M^{1/2}|\tau|^2/\min\{|\mathrm{Re}\,\tau|, |\mathrm{Im}\,\tau|\} ,$$

$$(4.15) \qquad \frac{|\mathrm{Im}\,\tau|}{C(|\tau|+1)} \leq Q_z(w(x, \tau)) \leq C\left\{\frac{|\tau|^3}{|\mathrm{Re}\,\tau|\min\{|\mathrm{Re}\,\tau|, |\mathrm{Im}\,\tau|\}}\right\} ,$$

for all $x\in[0, \infty)$, *where* C *depends only on* M, m *and* x_1.

Proof of Proposition 4.5. Since $w(x_1, \tau)$ belongs to $\Omega_2\subset\Omega_3$, $w(x, \tau)$ belongs to Ω_3 for all $x\in[0, x_1)$ by virtue of Lemma 4.1 with $\tilde{q}(x)=q\tau^2(x)$, $z=\tau^2/|\tau|^2$ and $M_3=M|\tau|^2$. Thus we have (4.14). Since it holds

$$Q_z(w(x, \tau))\leq\frac{|w(z, \tau)|}{\cos\frac{\psi}{2}}=\frac{|w(x, \tau)||\tau|}{|\mathrm{Re}\,\tau|},$$

we have the upper estimate in (4.15). For the lower estimate we consider a differential inequality as follosw. The linear property of Q_z yields

$$\frac{d}{dx}Q_z(w)=Q_z\left(\frac{d}{dx}w\right)=Q_z(q(x)|\tau|^2z-w^2)=-Q_z(w^2)$$

Substituting $w^2=P_z(w)^2z^2+2P_z(w)Q_z(w)z\sqrt{-z}-Q_z(w)^2z$, $Q_z(z^2)=2|\mathrm{Im}\,\tau|/|\tau|$, $Q_z(z\sqrt{-z})=1$ and $Q_z(z)=0$ to the above identity, we have

$$f'=-2(|\mathrm{Im}\,\tau)|/|\tau|)P_z(w)^2-2P_z(w)f,$$

where $f=f(x)=Q_z(w(x, \tau))$. Let us recall $(4.11)_1$. Then we have

$$f'\leq(|\tau|/2|\mathrm{Im}\,\tau|)f^2, \qquad f(x_1)>|\mathrm{Im}\,\tau|/C. \tag{4.16}$$

Now consider $g'=(|\tau|/2|\mathrm{Im}\,\tau|)\,g^2$ and $g(x_1)=|\mathrm{Im}\,\tau|/C$, and compare its solution with $f(x)$. Then

$$f(x)>g(x)=\frac{1}{\frac{2|\tau|}{|\mathrm{Im}\,\tau|}(x_1-x)+\frac{C}{|\mathrm{Im}\tau|}}, \qquad \text{for all} \quad x\in[0, x_1).$$

Thus we have (4.15) for $x\in[0, x_1]$. From $(4.11)_1$ follows (4.15) for $x\in[0, \infty)$.

Proof of Proposition 4.3. (4.3) and (4.4) follows from $(4.10)_1$ with $x'=0$ and (4.15). Since (4.3) yields $\int_0^{x_1}|v(s, \tau)|^2ds\leq Cx_1E(\tau)^2$, it suffices to consider

$$\int_{x_1}^{\infty}|v(s, \tau)|^2ds=|v(x_1, \tau)|^2\int_{x_1}^{\infty}|\tilde{v}(s, \tau)|^2ds. \tag{4.17}$$

On the other hand $(4.10)_2$ with $x'=x_1$ and $v(x)=\tilde{v}(x, \tau)$ makes

$$Q_{z_1}(w(x_1, \tau))\geq m|\tau||\mathrm{Im}\,\tau|\int_{x_1}^{\infty}|\tilde{v}(s, \tau)|^2ds. \tag{4.18}$$

Remark that $Q_z(w(0, \tau))\geq Q_z(w(x_1, \tau))|v(x_1, \tau)|^2$ holds from $(4.10)_1$. Then from (4.17) and (4.18) we have

$$\int_{x_1}^{\infty}|v(s, \tau)|^2ds\leq(m|\tau||\mathrm{Im}\,\tau|)^{-1}Q_z(w(0, \tau))\frac{Q_{z_1}(w(x_1, \tau))}{Q_z(w(x_1, \tau))}.$$

Therefore (4.5) holds from $(4.11)_1$, $(4.11)_2$ and (4.15). Thus the proof of Proposition 4.3 is complet.

Remark to Proposition 4.3. We can make (4.5) more precise as follows.

$$(4.5)' \qquad \int_0^{x_1} |v(s, \tau)|^2 ds \leq C x_1 E(\tau)^2, \quad \int_{x_1}^{\infty} |v(s, \tau)|^2 ds \leq C \frac{E(\tau)^2}{(|\tau|+1)|\mathrm{Im}\,\tau|}.$$

In order to prove (4.5)′ we use the following (4.19) instead of $(4.11)_1$ and $(4.11)_2$.

$$(4.19) \qquad |Q_z(w(x_1, \tau))/Q_{z_1}(w(x_1, \tau))| \leq \frac{1}{C} \frac{|\mathrm{Im}\,\tau|}{|\tau|}$$

We can prove (4.19) using (4.7)′, $(4.9)_1$, $(4.9)_2$ and $w(x_1, \tau) \in \Omega_2(\theta, M_2, m_2)$, where $\theta = \pi - \arg \tau^2$, $M_2 = M|\tau|^2$ and $m_2 = m|\tau|^2 \cos\frac{\theta}{2}$.

§5. The analyticity of the solution.

In this section we consider the analyticity of $v(x, \tau)$ in τ and some properties of its poles. For this purpose first we show that $\tilde{v}(x, \tau)$ is analytic with respect to τ in $C-R$ and has infinite number of simple zeros on pure imaginary line. Now we state

Lemma 5.1. *Let $q(x, \tau)$ be a bounded continuous function defined on $[a, \infty) \times D$, where D is a domain in C. Suppose that $q(x, \tau)$ satisfies all the conditions in Lemma* 3.1, (*respectively Lemma* 3.2) *for all $\tau \in D$. Moreover $q(x, \tau)$ is assumed to be analytic in D for each fixed $x \in [a, \infty)$. Then the unique solution $\tilde{w}(x, \tau)$ of $w' = q(x, \tau) - w^2$ belonging to Ω_1, (respectively Ω_2) for all $(x, \tau) \in [a, \infty) \times D$ is analytic with respect to τ in D.*

The proof of the above lemma is given in Section 7. Now we give

Proof of Proposition 4.1. Fix an arbitrary $\tau \in C-R$ and take a small neighbourhood D of τ in $C-R$ such that we can apply Lemma 5.1 with $q(x, \tau) = \tau^2 q(x)$ and $a = x_1$. Then the solution $w(x, \tau)$ of (3.1) is analytic with respect to τ for each fixed $x \in [x_1, \infty)$. $\tilde{v}(x, \tau)$ defined by (3.10) is also analytic for each $x \in [x_1, \infty)$. Hence $\tilde{v}(x, \tau)$ is analytic for each fixed $x \in [0, \infty)$ by virtue of the well-known theorem on the analyticity of the solution with respect to a parameter. Now confine τ to $I = \{\tau : \mathrm{Re}\,\tau = 0\}$ and make τ tends to zero. Then Lemma 3.1 says that $w(x_1, \tau) = \tilde{v}'(x_1, \tau)$ tends to zero. Notice that $1 \equiv \tilde{v}(x, 0)$ satisfies $\tilde{v}'' = 0$, $\tilde{v}(x_1, 0) = 1$ and $\tilde{v}'(x_1, 0) = 0$. Therefore $\tilde{v}(x, \tau)$ tends to $1 \equiv \tilde{v}(x, 0)$, because $\tilde{v}(x, \tau)$ is the solution of $\tilde{v}'' = \tau^2 q(x) \tilde{v}$ with $\tilde{v}(x_1, \tau) = 1$ and $\tilde{v}'(x, \tau) = w(x_1, \tau)$.

Proof of Proposition 4.2. As we saw in Proposition 4.5, for $\mathrm{Im}\,\tau^2 \neq 0$, $w(x, \tau)$ is bounded for $x \in [0, \infty)$. Therefore $\tilde{v}(x, \tau)$ does not vanish on $[0, \infty)$ if $(\mathrm{Re}\,\tau)(\mathrm{Im}\,\tau) \neq 0$. In order to clarify the set N defined by (4.1), it suffices to consider the case where τ belongs to $I=\{\tau\in C-R;\ \mathrm{Re}\,\tau=0\}$. From $\tilde{v}(x,\bar{\tau})=\overline{\tilde{v}(x,\tau)}$, we have $N=-N=\{\tau:\ -\tau\in N\}$. Now let us prove that N is a set of infinite points without any accumulating point. Suppose that $q(x)\geq\delta>0$ holds in (x_2, x_3), where $0<x_2<x_3<x_1$. Put

$$q_1(x)=\begin{cases} M\,, & 0\leq x\leq x_1\,,\\ -m\,, & x_1<x\,,\end{cases}$$

$$q_2(x)=\begin{cases} \delta\,, & x_2<x<x_3\,,\\ -M & 0\leq x\leq x_2,\ x_3\leq x\,.\end{cases}$$

Then $q_1(x)\tau^2\leq q(x)\tau^2\leq q_2(x)\tau^2$ for $\tau\in I$. Let $n(\tau)$, $n_1(\tau)$ and $n_2(\tau)$ be numbers of zeros of solutions of $v''=\tau^2q(x)v$, $v_1''=\tau^2q_1(x)v_1$ and $v_2'=\tau^2q_2(x)v_2$, satisfying $\lim_{x\to\infty}v(x,\tau)=0$, $\lim_{x\to\infty}v_1(x,\tau)=0$ and $\lim_{x\to\infty}v_2(x,\tau)=0$ respectively. Sturm's separation theorem says

$$(5.1)_1\qquad n_2(\tau)-1<n(\tau)<n_1(\tau)+1\,,$$

$$(5.1)_2\qquad \left[\frac{\delta^{1/2}|\tau|(x_3-x_2)}{\pi}\right]\leq n_2(\tau)\,,\quad n_1(\tau)\leq\left[\frac{M^{1/2}|\tau|x_1}{\pi}\right]+1\,,$$

where $[\alpha]$ is the largest integer less than or equal to α. On the other hand we remark that $\tilde{v}(x,\tau)$ is continuous in $(x,\tau)\in[0,\infty)\times\{C-R\}$ and has no zero of order two with respect to x. Remark also $\tilde{v}(x,0)\equiv 1$ and that for $\tau\in I$, $\tilde{v}(x,\tau)$ is positive valued on $[x_1,\infty)$. Then we see that the number of elements of $\{i\gamma:\ v(0,i\gamma)=0,\ |\gamma|\leq|\tau|\}$ is not less than $n(\tau)$. Thus from $(5.1)_1$ and $(5.1)_2$ the number of elements of N is infinite. Furthermore we can prove later

$$(5.2)\qquad \tau\frac{\partial}{\partial\tau}\tilde{v}(0,\tau)=-2\tilde{v}'(0,\tau)^{-1}\int_0^\infty\tilde{v}'(s,\tau)^2ds\,,\qquad \text{if}\quad \tilde{v}(0,\tau)=0\,.$$

Since $\tilde{v}(0,\tau)=0$ implies $\tilde{v}'(0,\tau)\neq 0$, we have from (5.2)

$$\tau\frac{\partial}{\partial\tau}\tilde{v}(0,\tau)\neq 0\,,\qquad \text{if}\quad \tau\in N\,.$$

Therefore τ is a simple zero of $\tilde{v}(0,\tau)$ for $\tau\in N$. Putting

$$N=\{\tau=\pm i\gamma_k;\ 0<\gamma_1<\gamma_2<\cdots,\ \lim_{k+\infty}\gamma_k=\infty\}\,,$$

we have Proposition 4.2 if (5.2) is verified. Denote

$$g_h=(g(x,\tau))_h=\frac{g(x,\tau+h)-g(x,\tau)}{h}$$

and $g'(x,\tau)=\dfrac{\partial}{\partial x}g(x,\tau)$. From (3.1) follows

$$w_h'=(\tau^2)_hq(x)-(w(x,\tau+h)+w(x,\tau))w_h\,.$$

Hence we have

$$(5.3)\qquad (\tilde v(x,\tau+h)\tilde v(x,\tau)w_h(x,\tau))' = \tilde v(x,\tau+h)\tilde v(x,\tau)(\tau^2)_h q(x)\,.$$

Integrate (5.3) from 0 to ∞ and make h tend to zero, then

$$\tilde v(0,\tau)^2\frac{\partial}{\partial\tau}w(0,\tau) = -2\tau\int_0^\infty q(s)\tilde v(s,\tau)^2 ds$$

follows from Fatou theorem. Since it holds

$$\tilde v(0,\tau)^2\frac{\partial}{\partial\tau}w(0,\tau) = \tilde v(0,\tau)\frac{\partial}{\partial\tau}\tilde v'(0,\tau)-\tilde v'(0,\tau)\frac{\partial}{\partial\tau}\tilde v(0,\tau)\,,$$

we have

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\tilde v(0,\tau) = 2\tilde v'(0,\tau)^{-1}\int_0^\infty q(s)\tilde v(s,\tau)^2 ds\,,\qquad \text{if}\quad \tilde v(0,\tau)=0\,.$$

On the other hand the integration by parts of $\tilde v''\tilde v=\tau^2 q(x)\tilde v^2$ yields

$$\int_0^\infty \tilde v'(s,\tau)^2 ds = -\tau^2\int_0^\infty q(s)\tilde v(s,\tau)^2 ds\,,\qquad \text{if}\quad \tilde v(0,\tau)=0\,.$$

Hence we have (5.2) and complete the proof of Proposition 4.2.

Proof of Theorem 3. Let Γ_k be a closed path in $C-R$ such that only one point $\tau_\gamma=i\gamma_k$ belongs to the closure of the interior of Γ_k. Then

$$u_k(x,t) = c_k^{-1}\int_{\Gamma_k} e^{i\tau t}v(x,\tau)d\tau$$

is a solution of (P) with boundary condition $g(t)\equiv 0$ for any constant $c_k\neq 0$. Then the residue calculus gives

$$c_k u_k(x,t) = 2\pi i e^{-\gamma_k t}\tilde v(x,i\gamma_k)\lim_{\tau\to i\gamma_k}\frac{\tau-i\gamma_k}{\tilde v(0,\tau)}\,.$$

From $\tilde v(0,i\gamma_k)=0$ and (5.2), $u_k=e^{-\gamma_k t}\tilde v(x,i\gamma_k)$ if

$$c_k = 2\pi i\Big/\frac{\partial\tilde v}{\partial\tau}(0,i\gamma_k) = \pi\gamma_k\tilde v'(0,i\gamma_k)\Big(\int_0^\infty \tilde v'(s,i\gamma_k)^2 ds\Big)^{-1}.$$

§6. Proof of Theorem 2.

In this section we prove Theorem 2.3, Theorem 2 and Theorem 2′. First we state

Proposition 6.1. *Suppose the same conditions as in Theorem* 2.3. *Assume* $|\mathrm{Re}\,\tau|>|\mathrm{Im}\,\tau|>0$. *Denote by* $w(x,\tau)$ *the unique solution of* (3.1). *Then there exist two solutions* $w_1(x,\tau)$ *and* $w_2(x,\tau)$ *of* $w'=\tau^2 q(x)-w^2$ *in an interval* (x_0,x_0') *involved in* $(0,\infty)$, *such that we have*

(6.1) $$|w_1(x_0', \tau)-w(x_0', \tau)| \leq C \frac{|\tau|^2}{|\operatorname{Im} \tau|},$$

(6.2) $$|w_2(x_0', \tau)-w(x_0', \tau)| \geq \frac{1}{C}|\operatorname{Im} \tau|,$$

(6.3) $$\operatorname{Re} w_1(x, \tau) \leq -\sqrt{\delta}\, \frac{|\operatorname{Re} \tau|^2}{|\tau|}, \quad \sqrt{\delta}\, \frac{|\operatorname{Re} \tau|^2}{|\tau|} \leq \operatorname{Re} w_2(x, \tau),$$

(6.4) $$|w_1(x, \tau)| \leq C|\tau|, \quad |w_2(x, \tau)| \leq C|\tau|,$$

for all $x \in [x_0, x_0']$, *where* C *is a positive constant.*

Proposition 6.2. *Suppose the same conditions as in Theorem* 2.3. *Assume* $|\operatorname{Re} \tau| > |\operatorname{Im} \tau| > 0$. *Let* $v(x, \tau)$ *be the unique solution of* $(P)_0$. *Then* $v'' = \tau^2 q(x) v$ *has two solutions* $v_1(x, \tau)$ *and* $v_2(x, \tau)$ *satisfying*

$(6.5)_1$ $$|v_1(x, \tau)| \geq e^{-\sqrt{\delta}\left(\frac{|\operatorname{Re} \tau|^2}{|\tau|}\right)(x-x_0')}, \quad x \in (x_0, x_0'),$$

$(6.5)_2$ $$|v_2(x, \tau)| \leq e^{\sqrt{\delta}\left(\frac{|\operatorname{Re} \tau|^2}{|\tau|}\right)(x-x_0')}, \quad x \in (x_0, x_0').$$

And we have

(6.6) $$v(x, \tau) = c_1(\tau) v_1(x, \tau) + c_2(\tau) v_2(x, \tau),$$

(6.7) $$|c_2(\tau)/c_1(\tau)| \leq C(|\tau|/|\operatorname{Im} \tau|)^2.$$

Proposition 6.3. *Suppose the same conditions as in Theorem* 2.3. *Then the unique solution* $w(x, \tau)$ *of* (3.1) *satisfies*

(6.8) $$\overline{\lim_{|\operatorname{Re} \tau| \to \infty}} \sup_{x_0 < x < x_0 + (1-\varepsilon)d} \frac{\operatorname{Re} w(x, \tau)}{\sqrt{\delta}\, |\tau|} \leq -1,$$

for any fixed $\gamma = -\operatorname{Im} \tau \neq 0$ *and all* $\varepsilon \in (0, 1)$.

Using Proposition 6.3, now we can verify Theorem 2.3 and Theorem 2′.

Proof of Theorem 2.3. For $\gamma \in S$, $w(x, \tau)$ is continuous on $(x, \tau) \in [0, \infty) \times \{\tau;\ \operatorname{Im} \tau = -\gamma\}$. By virtue of (6.8), for any $\varepsilon \in (0, 1)$ there exists a positive constant $C'_{\varepsilon,\gamma}$ such that it holds for $\tau \in \{\tau;\ \operatorname{Im} \tau = \pm\gamma\}$

$$\sup_{x_0 < x < x_0 + (1-\varepsilon)d} \operatorname{Re} w(x, \tau) \leq -(1-\varepsilon)\sqrt{\delta}\, |\tau| + C'_{\varepsilon,\gamma}.$$

Therefore we have from $v(x, \tau) = v(x_0, \tau) \exp \int_{x_0}^{x} w(s, \tau) ds$

(6.9) $$\left| \frac{v(x_0 + (1-\varepsilon)d, \tau)}{v(x_0, \tau)} \right| \leq C''_{\varepsilon,\gamma} \exp\left(-(1-\varepsilon)^2 \sqrt{\delta}\, d\, |\tau|\right).$$

From (2.1) and (6.9) we have (2.2) replacing ε suitably.

Proof of Theorem 2'. Since x is larger than $\inf\{x; q(x)>0\}$, there exist positive numbers x_0, d and δ satisfying $q(x)\geq\delta$ in (x_0, x_0+d). Hence (6.9) holds. On the other hand from $(4.10)_1$ and (4.15) there exist positive constants C and C_γ such that

$$\sup_{0\leq x'<x}\left|\frac{v(x,\tau)}{v(x',\tau)}\right|\leq C\frac{|\tau|^{3/2}}{|\operatorname{Im}\tau|}+C_\gamma, \qquad \tau\in\{\tau; \operatorname{Im}\tau=-\gamma\}. \tag{6.10}$$

Decomposing $v(x,\tau)=\dfrac{v(x_0,\tau)}{v(0,\tau)}\dfrac{v(x_0+(1-\varepsilon)d,\tau)}{v(x_0,\tau)}\dfrac{v(x,\tau)}{v(x_0+(1-\varepsilon)d,\tau)}$, we can prove that $u(x,t)$ is real-analytic with respect to t. Moreover taking account of the definition of Riemann integral and the finer decomposition

$$v(x,\tau)=\prod_{j=0}^{k}\frac{v(x_j,\tau)}{v(x_{j-1},\tau)}\frac{v(x,\tau)}{v(x_k,\tau)}, \qquad x_{-1}=0,$$

for sufficiently large k, we can obtain the estimate (E) below from (6.9) and (6.10). For any fixed $x>\inf\{x: q(x)>0\}$, $\gamma\in S$ and $\varepsilon\in(0,1)$, there exists a positive constant $C_{\varepsilon,\gamma,x}$ such that it holds

$$|v(x,\tau)|\leq C_{\varepsilon,\gamma,x}\exp(-(\delta(x)-\varepsilon)|\tau|), \quad \tau\in\{\tau: |\operatorname{Im}\tau|=|\gamma|\}. \tag{E}$$

Denoting $\Gamma_\gamma=\{\tau: \operatorname{Im}\tau=-\gamma, \operatorname{Re}\tau=\sigma\in R\}$, we define $u(x,t)$ by (1.1) with $\Gamma=\Gamma_\gamma$. Suppose $g\in H^k_\gamma$, i.e. $e^{-\gamma t}g(t)\in H^k$, which is equal to saying $(|\tau|+1)^k\hat{g}(\tau)\in L^2$ on Γ_γ. Notice

$$e^{-\gamma t}u(x,t)=\int_{\Gamma_\gamma}e^{it\operatorname{Re}\tau}v(x,\tau)\hat{g}(\tau)d\tau.$$

From (E) follows

$$|e^{it\operatorname{Re}\tau}v(x,\tau)|\leq C_{\varepsilon,\gamma,x}\exp(-(\delta(x)-\varepsilon)+|\operatorname{Im}t|)|\tau|.$$

Therefore if $|\operatorname{Im}t|\leq\delta(x)-2\varepsilon$, we have

$$|e^{it\operatorname{Re}\tau}v(x,\tau)\hat{g}(\tau)|\leq C_{\varepsilon,\gamma,x}e^{-\varepsilon|\tau|}|\hat{g}(\tau)|.$$

This implies that $e^{-\gamma t}u(x,t)$ is analytic with respect to t in the compex domain $\{t: |\operatorname{Im}t|<\delta(x)\}$ in view of well-known theorems due to Cauchy, Morera, Lebesgue and Fubini.

Proof of Proposition 6.1. Notice that $\bar{w}=\overline{w(x,\tau)}$ is a solution of $w'=\bar{\tau}^2q(x)-w^2$ if $w(x,\tau)$ satisfies $w'=\tau^2q(x)-w^2$. Therefore it suffices to prove (6.1)~(6.4) in the case of $\operatorname{Im}\tau^2>0$. Put $\theta=\arg\tau^2$, $M_2=M|\tau|^2$ and $m_2=\delta|\tau|^2\cos\dfrac{\theta}{2}$ and apply Lemma 3.2 to the equation $\dfrac{d}{dx}w(x,\bar{\tau})=\bar{\tau}^2q(x)-w(x,\bar{\tau})^2$. Since $\tilde{q}(x)=\bar{\tau}^2q(x)$ belongs to $G_2(\theta, M_2, m_2)$ for all $x\in[x_0, x_0']$, there exist two solutions $w_1(x,\bar{\tau})$ and $w_2(x,\bar{\tau})$ belonging to $\Omega_2(\theta, M_2, m_2)$ and $-\Omega_2(\theta, M_2, m_1)$ respectively. Therefore $w_1(x,\tau)=\overline{w_1(x,\bar{\tau})}$ and $w_2(x,\tau)=\overline{w_2(x,\bar{\tau})}$ satisfy $w'=\tau^2q-w^2$ and belong to $\overline{\Omega_2(\theta, M_2, m_2)}$ and

$-\overline{\Omega_2(\theta, M_2, m_2)}$ respectively. Hence w we have

$$(6.11)\qquad |w_k(x,\tau)|\leq 3M_2^{1/2}\left(\cos\frac{\theta}{2}\right)^{-1}=3M^{1/2}\frac{|\tau|^2}{|\operatorname{Re}\tau|}\,,\qquad k=1,2\,,$$

$$(6.12)\qquad \frac{\operatorname{Im}(\overline{w_1(x,\tau)}e^{-i(\pi-\theta)})}{\sin\frac{\theta}{2}}\geq\left(m_2\cos\frac{\theta}{2}\right)^{1/2}\leq\frac{\operatorname{Im}(-\overline{w_2(x,\tau)}e^{-i(\pi-\theta)})}{\sin\frac{\theta}{2}}\,,$$

$$(6.13)\qquad \operatorname{Im}(\overline{w_1(x,\tau)}\,e^{-i(\pi-\theta)/2})\geq\left(m_2\cos\frac{\theta}{2}\right)^{1/2}\leq\operatorname{Im}(-\overline{w_2(x,\tau)})^{-i(\pi-\theta)/2})\,,$$

for all $x\in[x_0, x_0']$. From (6.11) follows (6.4). From (6.12) and (6.13) it holds $|\operatorname{Re} w_1|\geq\left(m_2\cos\frac{\theta}{2}\right)^{1/2}\frac{|\operatorname{Re}\tau|}{|\tau|}$. Thus (6.3) holds. Now let us put $z=\tau^2/|\tau|^2$. Then we have $Q_z(w(x,\tau))\geq 0$ and $|w(x,\tau)|\leq 3M^{1/2}|\tau|^2/|\operatorname{Im}\tau|$, for all $x\in[0,\infty)$, by virtue of Proposition 4.5. Thus we have (6.1). To verify (6.2) we remark first

$$Q_z(w_2(x,\tau))\equiv\frac{\operatorname{Im}(w_2e^{-i\theta})}{\cos\frac{\theta}{2}}=\frac{-\operatorname{Im}(-\bar{w}_2e^{-i(\pi-\theta)})}{\cos\frac{\theta}{2}}\leq\frac{-\left(m_2\cos\frac{\theta}{2}\right)^{1/2}}{1\Big/\tan\frac{\theta}{2}}<0$$

Therefore $|w_2(x_0',\tau)-w(x_0',\tau)|\geq|Q_z(w_2(x_0',\tau))\cos\frac{\theta}{2}|=\left(m_2\cos\frac{\theta}{2}\right)^{1/2}\sin\frac{\theta}{2}$ holds. The right hand side is equal to $\delta^{1/2}|\tau|\cos\frac{\theta}{2}\sin\frac{\theta}{2}=\delta^{1/2}|\operatorname{Re}\tau|\,|\operatorname{Im}\tau|/|\tau|$. Hence we have (6.2).

Proof of Proposition 6.2. We put $v_1(x,\tau)=\exp\int_{x_0'}^{x}w_1(s,\tau)ds$ and $v_2(x,\tau)=\exp\int_{x_0'}^{x}w_2(s,\tau)ds$. Then (6.5) follows from (6.3). (6.6) and $w(x,\tau)v(x,\tau)=c_1(\tau)w_1(x,\tau)v_1(x,\tau)+c_2(\tau)w_2(x,\tau)v_2(x,\tau)$ with $x=x_0'$ make

$$\frac{c_1(\tau)}{v(x_0',\tau)}=\frac{w(x_0',\tau)-w_2(x_0',\tau)}{w_1(x_0',\tau)-w_2(x_0',\tau)}\,,\quad \frac{c_2(\tau)}{v(x_0',\tau)}=\frac{w(x_0',\tau)-w_1(x_0',\tau)}{w_2(x_0',\tau)-w_1(x_0',\tau)}\,,$$

where we have used $v_1(x_0',\tau)=v_2(x_0',\tau)=1$. Therefore (6.7) follows from (6.1) and (6.2).

Proof of Proposition 6.3. Since $w(x,\tau)$ is represented by

$$w(x,\tau)=\frac{v'(x,\tau)}{v(x,\tau)}=\left(w_1(x,\tau)+\frac{c_2(\tau)v_2(x,\tau)}{c_1(\tau)v_1(x,\tau)}w_2(x,\tau)\right)\Big/\left(1+\frac{c_2(\tau)v_2(x,\tau)}{c_1(\tau)v_1(x,\tau)}\right),$$

we have (6.8) from (6.3), $(6.5)_1$, $(6.5)_2$ and (6.7).

§7. Proof of Lemmas and final comments.

Here we prove Lemmas 3.1, 3.2, 4.1 and 5.1. To make our proof short we prepare

Lemma 7.1. *Let $Q(x, U)$ be a continuous function defined on $[a, \infty)\times R^n$ with values in R^n. Suppose the local uniqueness of the solution to $U'=Q(x, U)$. Let Ω be a bounded domain in R^n and S_i, $(i=1, 2, \cdots, k)$ be hypersufaces in R^n such that it holds $\partial\Omega=\bigcup_{i=1}^{k}(\bar{\Omega}\cap S_i)$. Assume $Q(x, U)\cdot n_i(U)>0$ for all $x\in[a, \infty)$ and all $U\in\partial\Omega\cap S_i$, $i=1, 2, \cdots, k$, where $n_i(U)$ is the unit outer normal to S_i. Then there exists at least one solution $U(x)$ of $U'=Q(x, U)$ satisfying $U(x)\in\Omega$ for all $x\in[a, \infty)$.*

Proof of Lemma 7.1. Take a sequence $\{x_n\}$ satisfying $a<x_1<x_2<\cdots$, $\lim_{n\to\infty} x_n=\infty$. Denote by $U(x; x_0, U_0)$ the solution of $U'=Q(x, U)$ and $U(x_0)=U_0$. Put $\bar{\Omega}_n=\{U(a; x_n, U_0); U_0\in\bar{\Omega}\}$. Then follows $\Omega\supset\bar{\Omega}_n\supset\bar{\Omega}_{n+1}\neq\phi$, $n=1, 2, \cdots$. Put $F=\lim_{n\to\infty}\bar{\Omega}_n$, then $F\neq\phi$. If U_0 belongs to F then $U(x; a, U_0)$ belongs to Ω for all $x\in[a, \infty)$.

Lemma 7.1′. *Suppose the same conditions as in Lemma* 7.1 *replaced $Q(x, U)\cdot n_i(U)>0$ by $Q(x, U)\cdot n_i(U)\geq 0$. Then we have the same results as in lemma* 7.1 *replacing Ω by $\bar{\Omega}$.*

Proof of Lemma 7.1′. We remark only that $U_0\in\bar{\Omega}$ implies $U(x; x_0, U_0)\in\bar{\Omega}$ for all $x\in[a, x_0)$. The same proof as that of Lemma 7.1 is valid if we replace Ω by $\bar{\Omega}$.

Proof of Lemmas 3.1, 3.2 *and* 4.1. Put $U={}^t(\text{Re}\, w, \text{Im}\, w)$, $\Omega=\Omega_k$, $k=1, 2, 3$, and $Q(x, U)={}^t(\text{Re}(\tilde{q}(x)-w^2), \text{Im}(\tilde{q}(x)-w^2))$. In order to prove Lemmas 3.1 and 3.2 we may apply Lemma 7.1. In fact $Q(x, U)$ and Ω_k, $k=1, 2$, satisfy all the conditions in Lemma 7.1, if $\tilde{q}(x)$ belongs to G_k, $k=1, 2$ for all $x\in[a, \infty)$, (cf. the proof of Lemmas 2.1, and 2.4 in [4]). The uniqueness follows from Proposition 3.2. By virtue of Lemma 7.1′ we have Lemma 4.1.

Proof of Lemma 5.1. At first we show the continuity of $\tilde{w}(x, \tau)$ with respect to τ as follows. Suppose that τ_j tends to $\tau_0\in D$. Taking a subsequence, if necessary, we can suppose $\lim_{j\to\infty} w(0, \tau_j)=w_0\in\Omega_k$, $(k=1$ or $2)$. Then the solution $\tilde{\tilde{w}}(x, \tau_0)$ of $w'=q(x, \tau_0)-w^2$ and $w(0)=w_0$, belongs to Ω_k by virtue of the continuity of solutions with respect to initial data and coefficient. Therefore the uniqueness implies $\tilde{w}(x, \tau_0)=\tilde{\tilde{w}}(x, \tau_0)$. Hence $\tilde{w}(x, \tau)$ is continuous in τ for any fixed $x\in[a, \infty)$. Now we proceed to prove the analyticity. Put $v(x, \tau)=\exp\int_a^x\tilde{w}(s, \tau)ds$. Then from Lemma 3.1, 3.2 and Proposition 3.2, there exists a positive constant M suhc that

$$(7.1)\qquad \sup_{a<x}|v(x, \tau)|^2+\int_a^\infty(|v(s, \tau)|^2+|v'(s, \tau)|^2)ds\leq M\,, \qquad \text{for all}\quad \tau\in D\,.$$

Let us fix τ in D. We have (5.3) replaced $q(x)\tau^2$ by $q(x, \tau)$ for sufficiently small complex number h such that $\tau+h$ belongs to D. Integrate it from x to ∞ and make h tend to zero, then from (7.1) we have

$$\frac{\partial}{\partial\tau}\tilde{w}(x,\tau)=\frac{-1}{v(x,\tau)^2}\int_x^\infty v(s,\tau)^2\frac{\partial}{\partial\tau}q(s,\tau)ds\,. \tag{7.2}$$

Since the right hand side is continuous in τ, $\tilde{w}(x,\tau)$ is analytic with respect to τ in D for any fixed $x\in[a,\infty)$.

Proof of Theorem 1°. Take a C^∞-function $\alpha(t)$ with compact support satisfying $\alpha(t)\equiv 1$ for $|t|<1$. Put $\alpha_1(t)=\alpha(t)$, $\alpha_2(t)=1-\alpha_1(t)$, $g_k(t)=\alpha_k(t)g(t)$ and $u_k(x,t)=\int e^{i\tau t}v(x,\tau)\hat{g}_k(\tau)d\tau$. By virtue of Theorem 1 and Sobolev Lemma $u_1(x,t)$ has the desired properties. Since

$$u_2(x,t)=(-1)^{j+k}\int e^{i\tau(t-s)}\frac{(i\tau)^{-j}}{(i(t-s))^k}\frac{\partial^k v(x,\tau)}{\partial\tau^k}\frac{d^j g_2(s)}{ds^j}dsd\tau \tag{7.3}$$

holds, if t is restricted to $\left\{t\in R;\ |t|<\frac{1}{2}\right\}$. The analyticity of $v(x,\tau)$ gives

$$\left(\frac{\partial}{\partial\tau}\right)^k v(x,\tau)=\frac{k!}{2\pi i}\int_{\Gamma_\tau}\frac{v(x,\tau')}{(\tau'-\tau)^{k+1}}d\tau',\quad \tau\in\{\tau;\ -\mathrm{Im}\,\tau=r\}, \tag{7.4}$$

where $\Gamma_\tau=\{\tau';\ |\tau-\tau'|=\frac{1}{2}\min\{|r\pm r_k|,\ k=0,1,2,\cdots\}\}$. Now let us use the estimate (4.3) with $\tau=\tau'$ and $v(x',\tau)=v(0,\tau')=1$. Then (7.3) and (7.4) yield

$$\sup_{|t|<1/2}\left|\frac{d^j}{dt^j}(e^{-\gamma t}u_2(x,t))\right|\le C_{\gamma,j}\sup_{t\in R}\left|\frac{d^{j+3}}{dt^{j+3}}(e^{-\gamma t}g_2(t))\right|, \tag{7.5}$$

for all $x\in[0,\infty)$ and $j=1,2,\cdots$. If we use $\alpha_1(t)=\alpha(t-t_0)$ and $\alpha_2(t)=1-\alpha_1(t)$, we can replace $|t|<\frac{1}{2}$ by $|t-t_0|<\frac{1}{2}$ in (7.5). Thus the proof of Theorem 1° is complet.

Conclusion. Now we notice that the problem for equations of mixed type involves essentially that for elliptic or hyperbolic equations. It is known that equations of mixed type are indispensable in the description of some real phenomena such as subsonic and transsonic waves. Furthermore they surve many important subjects to pure mathematics. Let us point out related topics to our problems. (1) Boundary value problems for elliptic or hyperbolic equations in a half space, (2) Local boundary value problems such as Tricomi and Frankl problems, (cf. for example [2], [6] and [9]). (3) Asymptotic benhaviors of solutions to ordinary differential equations of second order with a large parameter, (cf. for example [7]). In the course of solving our problem we encountered some intimate relations among them, and resolved them making use of the devices (a) and (b) stated in Introduction. Conversely (a) and (b) will give some new viewpoints and results to problems (1), (2) and (3). We have limited ourselves to the case where the coefficient depends only on x, because of our method. And the generalization remains difficult. However now we can feel that there exist various problems behind.

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References

[1] L. Bers, Mathematical aspects of subsonic and transsonic gas dynamics, Wiley, New York, (1958).
[2] A. R. Manwell, The Tricomi equation with application to the theory of plane transsonic flow, Pitman Adv. Publ. (1979).
[3] S. Miyatake, Construction of eignefunctions for Sturm-Liouville operator in $(-\infty, \infty)$, J. Math. Kyoto Univ., **23** (1983).
[4] S. Miyatake, Boundary value problems for some second order equations of variable type in a half space, J. Math. Kyoto Univ., **25** (1985)
[5] S. Mizohata, Theory of partial differential equations, Cambridze Univ. Press, (1973).
[6] C. S. Morawetz, Note on a maximum principle and a uniqueness theorem for an elliptic-hyperbolic equation, Proc. Roy. Soc. London Ser. A., **236**, (1956).
[7] F. W. Olver, Asymptotics and special functions, Academic Press, New York, (1973).
[8] M. Schechter, General boundary value problems for elliptic differentical equations, Comm. Pure and Appl. Math., **19** (1959).
[9] M. M. Smirnov, Equations of mixed type, American Math. Monographs, **51** (1978).
[10] Mathematical Reviews, 35M, 1979~1984.