J. Math. Kyoto Univ. (JMKYAZ)
26-3 (1986) 403-421

On sojourn times, excursions and spectral measures connected with quasidiffusions

By

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1. Introduction.

Let $X=(X_t)_{t\geqq 0}$ be a quasidiffusion on the real line with natural scale and speed measure m, i.e. a strong Markov process the infinitesimal generator of which is a restriction of the generalized second order differential operator $D_m D_x$.
Examples are diffusions and birth-and death-processes. We shall consider excursions of X from 0 over some level $a>0$ and study the sojourn time T which X spends over $a'(a'\in[0, a])$ during such an excursion.
If $a'=a$, then T has a mixed exponential distribution. The mixing measure can be identified with the normalized spectral-measure of a so-called dual string (see Proposition 3.2).
We consider the process $(P_t(a, a'))_{t\geqq 0}$where $P_t(a, a')$ is equal to the time which X spends over a' during excursions of X from 0 to a occuring before the local time $l(\cdot, 0)$ of X at zero equals t. If $a'>0$ it turns out to be a randomly stopped compounded Poisson process (Theorem 4.4).
If $a'=0$ or $a'=a$ its Laplace transform can be calculated by the spectral measures of some corresponding dual strings (see point 4.2 below). In paritucIar, if ν denotes the Ito excursion law of X at zero, V the "peak" of an excursion, we obtain the Laplace transform of the measure $\nu(T\in dt, V\geqq a)$ (Corollary 4.6). Finally a connection between the trajectories of X and some spectral measures via the local time of X is derived (Proposition 4.10). This extends a result of Ito, McKean [5] for diffusions.

2. Quasidiffusions, strings and spectral measures.

In this chapter we summarize some definitions and facts on quasidiffusions and related topics which are necessary in the sequel. Proofs are omitted, details can be found e.g. in [2, 3, 5, 6, 9–13] or can be easily derived from those.
We denote by R the real line, by $\mathfrak{B}$ the σ-algebra of its Borelian subsets and by K the set of complex numbers. Let $b\mathfrak{B}$ be the set of all bounded measurable real functions on R.

Communicated by S. Watanabe, March 1, 1985

2.1. Quasidiffusions.

Let m be an extended real-valued right-continuous nondecresaing function on R and put

$$l_0: = \inf\{x\in R\,|\,m(x)>-\infty\}\,,\quad l_1: = \sup\{x\in R\,|\,m(x)<\infty\}\,,$$

$$r_0: = \inf\{x>l_0\,|\,m(x)>m(l_0)\}\,,\quad r_1: = \sup\{x<l_1\,|\,m(x)<m((l_1-0)\}\,,$$

$$m_0: = m(\{0\})\,.$$

Assume

$$m(0-0)=0\,,\quad m(x)\not\equiv 0\,,\quad l_0<0<r_1$$

and define

$$I: = (l_0,\, l_1),\ E: = [r_0,\, r_1]\cap I\,.$$

Denote by E_m the points of E in which m increases. The function m and the measure generated by m on I are identified.
Let $W=(W_t,\, \mathfrak{F}_t^W,\, (P_x)_{x\in R})$ be a standard Wiener process and $l^W(t,\, x)(t\geqq 0,\, x\in R)$ its (continuous in $(t,\, x)$) local time with

$$\int_R l(t,\, x)f(x)dx = \int_0^t f(W_s)ds \qquad (t\geqq 0,\ f\in b\mathfrak{B})\,.$$

Put

$$S_t: = \int_R l^W(t,\, x)m(dx) \qquad (t\geqq 0)$$

$$T_t: = \inf\{u\geq 0\,|\,S_u>t\} \qquad (t\geqq 0)$$

with the definition $\inf\phi:=\infty$.
Then $(T_t)_{t\leqq 0}$ is a right-continuous strictly increasng process and T_t is an $\mathfrak{F}^W$-stopping time ($\mathfrak{F}^W:=(\mathfrak{F}_t^W)_{t\geqq 0}$). Define

$$\mathfrak{F}_t: = \mathfrak{F}_{T_t}^W \qquad (t\geqq 0)\,,$$

$$\zeta: = \inf\{t>0\,|\,W_{T_t}\in(l_0,\, l_1)\}\,,$$

$$X_t: = W_{T_t} \qquad (0\leqq t<\zeta)\,.$$

Then $X=(X_t,\, \zeta,\, \mathfrak{F}_t,\, (P_x)_{x\in E_m})$ is a right-continuous strong Markov process with state space E_m killed at the time ζ. We call X the *quasidiffusion with speed measure* m (other notations are: generalized diffusion process [10], gap diffusion [9]). X is skip free in the sense that

$$(X_t\wedge X_{t-},\, X_t\vee X_{t-})\cap E_m = \emptyset \qquad (0<t<\zeta)\,.$$

In particular, starting at $x<y\in E_m$, X enters $[y,\, l_1]$ through y: we have with $\sigma_y:=\inf\{t>0\,|\,X_t\geqq y\}$ the equality

$$X_{\sigma_y} = y \quad \text{on} \quad \{\sigma_y < \infty\} \bmod P_x .$$

The analogous property holds if $x > y$.
If m is strictly increasing, it follows that $E_m = E$ and X is continuous, i.e. X is a diffusion in the sense of [5]. If m is piecewise constant and E_m has no point of accumulation in (r_0, r_1), then X is a birth- and death-process.
By

$$l(t, x) := l^W(T_t, x) \qquad (0 \leqq t < \zeta) \text{ and}$$

$$l(\zeta, x) := l(\zeta - 0, x) \qquad (x \in E_m)$$

the *local time* l *of* X is defined. It is continuous in both variables and for every bounded measurable real function f on E_m we have

$$\int_{E_m} f(x) l(t, x) m(dx) = \int_0^t f(X_s) ds \qquad (0 \leqq t < \zeta), \tag{1}$$

in particular

$$m_0 \cdot l(t, 0) = \int_0^t \mathbf{1}_{\{0\}}(X_s) ds \qquad (0 \leqq t < \zeta). \tag{2}$$

Put

$$\tau_y^W := \inf\{t > 0 \mid W_t = y\}, \quad \tau_y := \inf\{t > 0 \mid X_t = y\} \qquad (y \in E_m).$$

Then it holds

$$T_{\tau_y} = \tau_y^W \ (y \in E_m), \quad T_\zeta = \tau_{l_0}^W \wedge \tau_{l_1}^W =: \zeta^W.$$

Using (1) with $f \equiv 1$ we obtain

$$\tau_y = \int_{E_m} l(\tau_y, x) m(dx) = \int_{E_m} l^W(\tau_y^W, x) m(dx) \qquad (y \in E_m). \tag{3}$$

If $G \in \mathfrak{B}$, we define a measure m^G on I by $dm^G := \mathbf{1}_G dm$ and mark all quantities connected with m^G by G, e.g. X^G, τ_y^G. From (1) and (3) we get

$$\tau_y^G = \int_0^{\tau_y} \mathbf{1}_G(X_s) ds \qquad (y \in E_m \cap G). \tag{4}$$

With m there is connected a generalized second order differential operator $D_m D_x$ defined by[1)]

$$D_m D_x f = g : \Leftrightarrow f(x) = f(0) + x D^- f(0) + \int_0^{\cdot x} (x - s) g(s) m(ds), \qquad (x \in I)$$

on an appropriately chosen domain of continuous functions on I which are linear on the components of the open set $I \backslash E_m$. The restriction of $D_m D_x$ to the Banach space $\boldsymbol{C}_m$ of all such functions that are bounded and satisfy

1) $D^-(D^+)$ denotes the left (right) hand side derivation operator.

$$\lim_{x \to l_i} f(x) = 0 \qquad \text{if} \quad |l_i| < \infty \quad (i = 0, 1)$$

is the *infinitesimal generator of X*.
Let φ and ψ be the (uniquely determined) solution of

$$\varphi(x, \lambda) = 1 - \lambda \int_0^x (x-s)\varphi(s, \lambda) m(ds) \quad \text{and}$$

$$\psi(x, \lambda) = x - \lambda \int_0^x (x-s)\psi(s, \lambda) m(ds), \qquad (\lambda \in K, x \in I)$$

respectively.

It holds the *Lagrange identity*, i.e. for every $\lambda \in K$ we have

$$\varphi(x, \lambda) D^- \psi(x, \lambda) - \psi(x, \lambda) D^- \varphi(x, \lambda) \equiv 1 \qquad (x \in E_m). \tag{5}$$

Suppose $z \in E_m \cup (l_0, l_1)$, $\lambda \in K_0 := K \backslash [0, \infty)$ and put

$$\chi_+^z(x, \lambda) := \varphi(x, \lambda) \int_x^z \varphi^{-2}(s, \lambda) ds \qquad (z \geqq 0, l_0 \leqq x \leqq z)$$

$$\chi_-^z(x, \lambda) := \varphi(x, \lambda) \int_z^x \varphi^{-2}(s, \lambda) ds \qquad (z \leqq 0, z \leqq x \leqq l_1).$$

If $z = l_0$ or $= l_1$ we omit the mark z: $\chi_-^{l_0} = \chi_-$, $\chi_+^{l_1} = \chi_+$. Let z_0, z_1 be given with $z_0 \leqq 0$, $z_1 > 0$, $z_1 \in E_m \cup (l_0, l_1)$ $(i=0, 1)$. Then, for $x \in E_m \cap (z_0, z_1)$ $\sigma := \zeta \wedge \inf\{t > 0 \mid X_t \in (z_0, z_1)\}$, $\lambda < 0$, we have for any $f \in b\mathfrak{B}$

$$E_x \int_0^\sigma e^{\lambda t} f(X_t) dt = \int_{z_0}^{z_1} \chi_-^{z_0}(x \wedge y, \lambda) \chi_+^{z_1}(x \vee y, \lambda) f(y) m(dy).$$

From this formula, the strong Markov property, and the skip freeness of X it follows under the same assumptions on z_0, z_1, x and λ by a known method of [5]:

$$E_x[\exp(\lambda\tau_y) \cdot \mathbf{1}_{\{\tau_{z_1} > \tau_y\}}] = \frac{\chi_+^{z_1}(x, \lambda)}{\chi_+^{z_1}(y, \lambda)} \qquad (y \in E_m, y \leqq x \leqq z_1), \tag{6}$$

$$E_x[\exp(\lambda\tau_y) \cdot \mathbf{1}_{\{\tau_{z_0} > \tau_y\}}] = \frac{\chi_-^{z_0}(x, \lambda)}{\chi_-^{z_0}(y, \lambda)} \qquad (y \in E_m, z_0 \leqq x \leqq y). \tag{7}$$

2.2. Strings and dual strings.

A measure m as considered above is called a *string* if $l_0 = -\infty$ and $r_0 \geqq 0$. Let $\mathfrak{M}$ be the set of all strings and assume $m \in \mathfrak{M}$. The function h defined by

$$h(\lambda) := \int_0^{l_1} \varphi^{-2}(x, \lambda) dx \qquad (\lambda \in K_0)$$

is called the *characteristic function of m* ([10]). We have

$$h(\lambda) = \chi_+(0, \lambda) = \lim_{x \uparrow l_1} \frac{\psi(x, \lambda)}{\varphi(x, \lambda)} \quad \text{and}$$

$$\chi_+(x, \lambda) = h(\lambda)\varphi(x, \lambda) - \psi(x, \lambda) \qquad (\lambda \in K_0) \tag{8}$$

If $m, m' \in \mathfrak{M}$, $a, b, m_0' \geqq 0$, $c > 0$ and

$$m'(x) = m_0' + \frac{b}{c} m\left(\frac{x-a}{c}\right), \qquad (x \in R)$$

then the characteristic functions h_m and $h_{m'}$ of m and m', respectively, are related by

$$\frac{1}{h_{m'}(\lambda)} = -\lambda m_0' + \frac{1}{a + c h_m(b\lambda)} \qquad (\lambda \in K_0). \tag{9}$$

For every $m \in \mathfrak{M}$ there exists a σ-finite measure σ on $[0, \infty)$, called the *spectral measure of m*, with

$$\int_{[0,\infty)} \frac{\sigma(du)}{1+u} < \infty$$

such that

$$h(\lambda) = r_0 + \int_{[0,\infty)} \frac{\sigma(du)}{u - \lambda} \qquad (\lambda \in K_0), \tag{10}$$

$$\sigma([0, u)) = \frac{1}{\pi} \lim_{c \downarrow 0} \int_{[0,u)} \operatorname{Im} h(b+ic) db \qquad (u \geqq 0). \tag{11}$$

Moreover, we have

$$\sigma([0, \infty)) = \frac{1}{m_0} \quad \text{and} \tag{12}$$

$$\sigma(\{0\}) = \frac{1}{m(r_1)} \text{ if } l_1 = \infty, \ = 0 \text{ if } l < \infty. \tag{13}$$

If m is a measure as in 2.1. above and $x \in E_m$, then two strings $m_x^\pm$ are given by

$$m_x^-([0, y)) := m((x-y, x]),$$

$$m_x^+([0, y)) := m([x, x+y)) \qquad (y > 0).$$

Note that $m_x^-(\{0\}) = 0$, $m_x^+(\{0\}) = m(\{x\})$.
For the characteristic functions $h_x^\pm$ of $m_x^\mp$, respectively, we obtain after some calculations

$$h_x^+(\lambda) = \frac{\chi_-(x, \lambda)}{D^-\chi_-(x, \lambda)}, \quad h_x^+(\lambda) = -\frac{\chi_+(x, \lambda)}{D^-\chi_+(x, \lambda)} \qquad (\lambda \in K_0), \tag{14}$$

If $G = [a, \infty)$ $(a > 0)$, (9) and (14) imply

$$h_0^{G,+}(\lambda) = h_a^+(\lambda) + a = \frac{aD^-\chi_+(a, \lambda) - \chi_+(a, \lambda)}{D^-\chi_+(a, \lambda)} \qquad (\lambda \in K_0). \tag{15}$$

Let $m \in \mathfrak{M}$ and put

$$m_d(x) := \inf\{y>0 \mid m(y)>x\} \quad (x \geqq 0), \quad = 0 \quad (x>0).$$

Then it holds that $m_d \in \mathfrak{M}$ and $(m_d)_d = m$. The string m_d is called the *dual string to* m. All quantities connected with m_d will be marked by d, e.g. $\psi_d, \varphi_d, \sigma_d$. With the notation

$$x_+ := \inf\{y>x \mid y \in E_m \cup \{l_1\}\}, \quad x_- = \sup\{y<x \mid y \in E_m \cup \{l_0\}\} \qquad (x \in E_m)$$

we get for $x \in I$, $\lambda \in K$

$$-\lambda\psi(x, \lambda) = D^+\varphi_d(z, \lambda)|_{z=m(x)} + \lambda(x_+ - x)\varphi_d(m(x), \lambda)$$

$$\varphi(x, \lambda) = D^+\psi_d(z, \lambda)|_{z=m(x)} + \lambda(x_+ - x)\psi_d(m(x), \lambda).$$

These equations provide

$$h_d(\lambda) = -(\lambda h(\lambda))^{-1} \qquad (\lambda \in K_0). \tag{16}$$

We conclude this point with several formulas concerning the dual strings m_d. From (10) and (13) it follows

$$h_d(\lambda) = m_0 - (\lambda l_1)^{-1} + \int_{(0,\infty)} \frac{\sigma_d(du)}{u-\lambda} \qquad (\lambda \in K_0). \tag{17}$$

Thus we have by (14)

$$l_1^{-1} - h^{-1}(\lambda) = m_0 + \int_{(0,\infty)} \left(\frac{1}{u} - \frac{1}{u-\lambda}\right) u\sigma_d(du) =$$

$$= m_0 + \int_{(0,\infty)} (1 - e^{\lambda u})\vartheta_d(u)du \qquad (\lambda \in K_0) \tag{18}$$

with

$$\vartheta_d(u) := \int_{(0,\infty)} ve^{-uv}\sigma_d(dv). \qquad (u>0).$$

Note that

$$\int_\varepsilon^\infty \vartheta_d(u)du = \int_{(0,\infty)} e^{-\varepsilon v}\sigma_d(dv) \xrightarrow[\varepsilon \downarrow 0]{} \sigma_d((0, \infty)) = r_0^{-1} - l_1^{-1} \tag{19}$$

$$(r_0^{-1} := \infty \text{ if } r_0 = 0).$$

Furthermore, (14)–(16) imply for $G := [a, \infty)$

$$\frac{D^-\chi_+(a, \lambda)}{\chi_+(a, \lambda) - aD^-\chi_+(a, \lambda)} = \lambda h_d^G(\lambda) \qquad (\lambda \in K_0, a>0) \tag{20}$$

and using (18) we get for $a>0$

$$\frac{\chi_+(a,\lambda)}{\chi_+(a,\lambda)-aD^-\chi_+(a,\lambda)} = 1+a\lambda h_d^G(\lambda) = a\int_{(0,\infty)} \frac{u}{u-\lambda}\,\sigma_d^G(du) \qquad (\lambda \in K_0) \qquad (21)$$

(σ_d^G denotes the spectral measure of the string $(m^G)_d$ which is dual to m^G).

2.3. Examples.

1. If $m(x)=x^\gamma \cdot \mathbf{1}_{[0,\infty)}(x)(\gamma>0$ fixed, $x\in R)$, then ${}'X$ is a diffusion on $[0,\infty)$ instantaneously reflected at zero. We have

$$h(\lambda) = \frac{\Gamma\left(\frac{1}{\gamma+1}\right)}{\Gamma\left(\frac{\gamma}{\gamma+1}\right)} \cdot (\gamma+1)^{(1-\gamma)/(1+\gamma)} \cdot (-\lambda\gamma)^{-1/(\gamma+1)}$$

and

$$\sigma(du) = \left[\Gamma\left(\frac{\gamma}{\gamma+1}\right)\right]^{-2} (\gamma+1)^{(1-\gamma)/(1+\gamma)} (u\gamma)^{-1/(1+\gamma)} du \qquad (u \geqslant 0)\,.$$

The dual quantities m_d, h_d, σ_d are obtained by substituting γ^{-1} for γ.

2. Put $m(dx):=(\mathbf{1}_{[0,v]}(x)+c^{-2}\mathbf{1}_{(v,\infty)}(x))dx$ (c, $v>0$ fixed, $x\in R$)
Then it holds

$$h(\lambda) = (-\lambda)^{-1/2} \cdot \frac{c+\tanh v\sqrt{-\lambda}}{1+c\tanh v\sqrt{-\lambda}} \qquad (\lambda \in K_0)$$

and

$$\sigma(du) = \frac{2c \cdot du}{\pi u^{1/2}(1+c^2+(1-c^2)\cos 2v\sqrt{u})} \qquad (u>0)$$

Replacing c by c^{-1} we obtain m_d, h_d, σ_d.

Define $\qquad p(x):= x\mathbf{1}_{[0,v]}(x)+(v+c(x-v))\mathbf{1}_{[v,\infty)}(x) \qquad (x \geqq 0)\,,$

then $(p^{-1}(X_t))_{t\geqq 0}$ behaves like a skew Brownian motion process reflected at zero with diffusion coefficient 2 and "skew point" v.

3. Assume m is a string consisting of a weightless thread carrying beads with mass $m_i>0$ at p_i ($i=1,\cdots,N$; $0<p_1<\cdots<p_N$). Put $\Delta_0:=p_1$, $\Delta_i:=p_{i+1}-p_i$ ($i=1,\cdots,N-1$) and $l:=\infty$. It holds

$$h_m(\lambda) = \Delta_0+\sum_{k=1}^{N} \frac{\sigma_k}{u_k-\lambda} \qquad (\lambda \in K_0)$$

where u_k ($k=1,\cdots,N$) are the eigenvalues of the matrix $-A$ which is given by $A=(a_{ij})$, $i,j=1,\cdots,N$, $a_{11}=-\lambda_1$, $a_{ii}=-(\lambda_i+\mu_i)$ ($i=2,\cdots,N-1$), $a_{NN}=-\mu_N$, $a_{ii-1}=\mu_i$ ($i=2,\cdots N$), $a_{ii+1}=\lambda_i$ ($i=1,\cdots,N-1$), $a_{ij}=0$ otherwise, $\lambda_i:=(m_i\cdot\Delta_i)^{-1}$ ($i=1,\cdots,N-1$) $\mu_i=(m_i\Delta_{i-1})^{-1}$ ($i=1,\cdots,N$). Furthermore

$$\sigma_k := \left(\sum_{i=0}^{N-1} Q_i^2(u_k)m_{i+1}\right)^{-1},$$

and $(Q_0(u_k), \cdots, Q_{N-1}(u_k))^T$ is the eigenvector of –A belonging to u_k normalized by $Q_0(u_k)\equiv 1$, $(k=1, \cdots, N)$.
The quasidiffusion X with speed measure m is the birth- and death-process on $p_1, \cdots, p_N$ with A as its matrix of intensities. The dual string m_d has the masses $m_{d,i}=\Delta_i$ $(i=0, \cdots, N-1)$ at the points

$$p_{d,0}=0\,, \quad p_{d,k}=\sum_{i=1}^{k} m_i \qquad (k=1, \cdots, N-1)$$

and it holds $(l_1)_d=\sum_{i=1}^{N} m_i$. We have

$$h_d(\lambda)=\sum_{k=1}^{N}\frac{\sigma_{d,k}}{u_{d,k}-\lambda} \text{ where } u_{d,k} \text{ are the eigenvalues of } -A_d \text{ with}$$

$$A_d=((a_d)_{ij})\ (i, j=1, \cdots, N)\,, \quad (a_d)_{ij}=a_{ji}\ (i\neq j)\,,$$

$$a_{11}=-\mu_1\,, \quad a_{ii}=-(\lambda_{i-1}+\mu_i)\ (i=2, \cdots, N)\,,$$

$$\sigma_{d,k}:=\Big(\sum_{i=0}^{N-1}(Q_{d,1}^2(u_{d,k})\Delta_i)^{-1} \tag{22}$$

and $(Q_{d,0}(u_{d,k}), \cdots, Q_{d,N-1}(u_{d,k}))^T$ is the eigenvector of $-A_d$ corresponding to $u_{d,k}$, $Q_{d,0}(u_{d,k})\equiv 1 (k=1, \cdots, N)$.
The quasidiffusion with speed measure m_d is the birth- and death-process on

$$\{0, m_1, m_1+m_2, \cdots, \sum_{1}^{N-1} m_k\} \quad \text{with } A_d \text{ as its matrix of intensities.}$$

3. A sojourn times over positive levels.

Assume X is a quasidiffusion with speed measure m and suppose $r_0\leqq 0$ and $0\in E_m$. Moreover, choose $x, a, a'\in E_m$ with $0<x\leqq a'\leqq a\leqq r_1$ and put $G:=[a', l_1)$. Then, from (6), (7) and the strong Markov property of X it follows for $\lambda<0$:

$$\begin{aligned} &E_x \exp(\lambda\tau_0^G)\mathbf{1}_{\{\tau_a<\tau_0<\zeta\}}= \\ &=(E_x \exp(\lambda\tau_a^G)\mathbf{1}_{\{\tau_a<\tau_0\}})(E_a \exp(\lambda\tau_a^G))\mathbf{1}_{\{\tau_0<\tau_{l_1}\}}) \\ &=\frac{\chi_-^{G,0}(x, \lambda)}{\chi_-^{G,0}(a, \lambda)}\cdot\frac{\chi_+^G(a, \lambda)}{\chi_+^G(0, \lambda)}=:H(a, x, a', \lambda) \end{aligned} \tag{23}$$

(Here $\tau_\cdot^G$ is understood as the last integral in (4).)
Using $D_mD_xf+\lambda f=0$ on G for $f=\varphi^G$ and $f=\psi^G$ it follows that φ^G, ψ^G are linear combinations of φ and ψ on G. Now, from $\varphi^G(y, \lambda)=1$, $\psi^G(y, \lambda)=y$ on $[0, a']$ we obtain

$$\chi_+^G(a, \lambda)=-\frac{\chi_+(a, \lambda)}{D^-\chi_+(a, \lambda)}\,,$$

$$\chi_+^G(0, \lambda)=a'-\frac{\chi_+(a', \lambda)}{D^-\chi_+(a', \lambda)}\,,$$

$$\chi_-^{G,0}(a',\lambda) = D^-\chi_-^{G,0}(a',\lambda)[-\varphi(a,\lambda)\psi(a',\lambda)+\varphi(a',\lambda)\psi(a,\lambda)-$$
$$-a'(D^-\varphi(a',\lambda)\psi(a,\lambda)-D^-\psi(a',\lambda)\varphi(a,\lambda))]$$
$$\chi_-^{G,0}(x,\lambda) = x\cdot D^-\chi_-^{G,0}(a',\lambda) \qquad (\lambda\in K_0)\,.$$

Observe that

$$h_0^{a,+}(\lambda):=\frac{\psi(a,\lambda)}{\varphi(a,\lambda)} \qquad (\lambda\in K_0)$$

is the characteristic function of the string

$$\mathbf{1}_{[0,a]}(x)m(dx)+\infty\cdot\delta_a(dx)$$

(δ_a denotes the measure with unit mass concentrated at a) and that it holds

$$\chi_+^a(a',\lambda) = \varphi(a',\lambda)\int_{a'}^a \varphi^{-2}(s,\lambda)ds =$$
$$=\varphi(a',\lambda)\Big(\frac{\psi(a,\lambda)}{\varphi(a,\lambda)}-\frac{\psi(a',\lambda)}{\varphi(a',\lambda)}\Big) = h_0^{a,+}(\lambda)\varphi(a',\lambda)-\psi(a',\lambda)\,.$$

In the following we write h and h^a instead of h_0^+ and $h_0^{a,+}$, respectively. Then we obtain for $\lambda<0$

$$H(x,a,a'\,\lambda) = \frac{x(h(\lambda)-h^a(\lambda))}{[\chi_+(a',\lambda)-a'D^-\chi_+(a',\lambda))][\chi_+^a(a',\lambda)-a'D^-\chi_+^a(a',\lambda)]}$$

If $\lambda\uparrow 0$ it follows from (23)

$$P_x(\tau_a<\tau_0<\zeta) = \frac{x}{a}\cdot\frac{l_1-a}{l_1} \quad\text{with}\quad \frac{l_1-a}{l_1}=1 \quad\text{if}\quad l_1=\infty\,.$$

Thus we have proved

Proposition 3.1. *If $0<x\leqq a'\leqq a\leqq r_1$ $(x,a,a'\in E_m)$, then for $\lambda<0$*

$$\phi(a,a',\lambda):=E_x[\exp(\lambda\int_0^{\tau_0}\mathbf{1}_{[a',l_1)}(X_s)ds)\,|\,\tau_a<\tau_0<\zeta] =$$
$$=\mu^{-1}\cdot\frac{h(\lambda)-h^a(\lambda)}{(\chi_+(a',\lambda)-a'D^-\chi_+(a',\lambda))(\chi_+^a(a',\lambda)-a'D^-\chi_+^a(a',\lambda))}$$
$$=\mu^{-1}\cdot\frac{h(\lambda)-h^a(\lambda)}{h^G(\lambda)h^{G,a}(\lambda)}\cdot D^-\chi_+(a',\lambda)D^-\chi_+^a(a',\lambda) \tag{24}$$

with the notation $\mu:=a^{-1}-l_1^{-1}$.
(Recall that h^a, h^G, and $h^{G,a}$ are the characteristic functions of

$$\mathbf{1}_{[0,\infty)}dm,\ \mathbf{1}_{[0,a]}dm+\infty\delta_a;\ \mathbf{1}_{[0,\infty)}dm^G \quad\text{and}\quad \mathbf{1}_{[0,a]}dm^G+\infty\delta_a$$

respectively.)

Let $a' \uparrow a$ $(a' \in E_m)$. Then, by using

$$h(\lambda)-h^a(\lambda) = \frac{\chi_+(a, \lambda)}{\varphi(a, \lambda)}, \quad \chi_+^a(a', \lambda) \to 0 .$$

$$D^- \chi_+^a(a', \lambda) \to -\varphi^{-1}(a, \lambda) \ (a' \uparrow a) \text{ and } (21)$$

we obtain

$$\phi(a, \lambda) := \lim_{a' \uparrow a} \phi(a, a', \lambda) = (a\mu)^{-1} \cdot \frac{\chi_+(a, \lambda)}{\chi_+(a, \lambda) - aD^- \chi_+(a, \lambda)} =$$

$$= \mu^{-1} \cdot \int_{(0,\infty)} \frac{u}{u-\lambda} \sigma_d^G(du) = \mu^{-1} \cdot \int_0^\infty e^{u\lambda} \vartheta_d^G(u)\, du \tag{25}$$

with

$$G = [a, \infty), \quad \vartheta_d^G(u) := \int_{(0,\infty)} ve^{-uv} \sigma_d^G(dv) .$$

Because of

$$\sigma_d^G((0, \infty)) = \int_0^\infty \vartheta_d^G(u) du = a^{-1} - l_1^{-1} = \mu < \infty \qquad \text{(see (19))},$$

we can introduce a probability σ on $(0, \infty)$ by

$$\sigma(A) := \mu^{-1} \sigma_d^G(A) . \qquad (A \in \mathfrak{B} \cap (0, \infty)) .$$

Obviously, $\phi(a, \cdot)$ is the Laplace transform of a mixed exponential distribution with mixing measure σ. Thus we have shown the following

Proposition 3.2. *If* $0 < x \leqq a \leqq r_1 (x, a \in E_m)$, *we have*

$$P_x \left(\int_0^{\tau_0} \mathbf{1}_{[a, l_1)} (X_s) ds \in dt \mid \tau_a < \tau_0 < \zeta \right) =$$

$$= dt \int_{(0,\infty)} ue^{-ut} \sigma(du) = \mu^{-1} \vartheta_d^G(t) dt.$$

Let $0 < a < l_1 \leqq \infty$ be fixed, $G = [a, l_1)$ and denote by $\mathfrak{M}_a$ the set of all strings m from $\mathfrak{M}$ with $a \in E_{m^G}$. Then the mapping $m^G \to \sigma$ defined by Proposition 3.2. is a one-onto-one correspondence between the set of all strings of the form

$$dm^G(x) = \mathbf{1}_G(x) dm(x) + \infty \delta_{l_1}(x) dx \qquad (m \in \mathfrak{M}_a)$$

and the set of probabilities σ on $(0, \infty)$.
This is a consequence of M.G. Krein's inverse spectral theorem. (Indeed, given a probability σ on $(0, \infty)$ define $\sigma_1 := \mu\sigma$ with $\mu = a^{-1} - l_1^{-1}$, supplement it by mass l_1^{-1} at zero and choose, by Krein's theorem (see e.g. [2, 6, 10]) the uniquely determined string n having σ_1 as its spectral measure. (12) and (13) imply $n(\{0\}) = a$, $n([0, \infty)) = l_1$. Take the dual string n_d, which turns out to be of the form m^G with the desired properties.)

Examples:

1. Consider the standard Wiener process W, it is a quasidiffusion on R with speed measure $m(x)=2x(x\in R)$, and choose $a>0$. Then, for the characteristic function $h_{0,d}^{G,+}$ of the dual string of $\mathbf{1}_G(x)dm(x)$ $(G=[a,\infty))$ we get

$$h_{0,d}^{G,+}(\lambda) = -\frac{2a\lambda+\sqrt{-2\lambda}}{\lambda+2a^2\lambda^2} \qquad (\lambda\in K_0)$$

and for the normalized measure σ we obtain

$$\sigma(du) = \frac{2a}{\pi\sqrt{2u}}\,\frac{1}{1+2ua^2}\,du \qquad (u>0)\,.$$

Then it follows

$$P_x\Big(\int_0^{\tau_0}\mathbf{1}_{[a,\infty)}(W_s)ds>t\,|\,\tau_a<\tau_0\Big) = \frac{2a}{\pi}\int_0^\infty \frac{e^{-\frac{v^2}{2}t}}{1+v^2a^2}\,dv \qquad (x\in(0,a])\,.$$

2. Let X be a birth- and death-process with finite state space $\{0,1,\cdots,N\}$ and intensities

$$\lambda_i>0 \quad (i=0,\cdots,N-1) \qquad \mu_i>0 \quad (i=1,\cdots,N)$$

Then, starting at 1 the hitting time $\tau_0=\inf\{t>0/X_t=0\}$ has a mixed exponential distribution

$$P_1(\tau_0\in dt/X_0=1) = \sum_{k=1}^N W_k\exp(-W_k t)\,\sigma_k\,dt\,,$$

where $W_k(k=1,\cdots,N)$ are the eigenvalues of the matrix $-A_d$ (see above) and

$$\sigma_k := \sigma_{d,k}\cdot\Big(\sum_1^N \sigma_{d,j}\Big)^{-1} \qquad (k=1,\cdots,N)$$

with $\sigma_{d,k}$ given by (22).

4. Excursions, local time and spectral measure.

4.1. Excursions of the Wiener process.

Assume $W=(W_t,\mathcal{F}_t^W,P_x)$ is a standard Wiener process. In this point we consider W under the measure P_0 only, i.e. W starts at zero. The open set $[0,\infty)\setminus\{t\geqq 0/W_t=0\}$ is the union of its components (a_n^W,b_n^W) $(n\geqq 1)$. For every $n\geqq 1$ the process U_n^W given by

$$U_n^W(t) = W_{t+a_n} \quad (0\leqq t\leqq b_n^W-a_n^W), \qquad =0 \quad (t\geqq b_n^W-a_n^W)$$

is called an *excursion of* W. We introduce the length T_n^W and the peak V_n^W of the excursion U_n^W by defining

$$T_n^W := b_n^W - a_n^W,\ V_n^W := \max_{t\geq 0} U_n^W(t) \operatorname{sgn} U_n^W\left(\frac{b_n^W + a_n^W}{2}\right).$$

If n is not specified we write U^W, V^W, T^W only. K. Ito proved ([4], see also [14]) that the points $(l^W(a_n^W, 0), U_n^W)$ are the atoms of a Poisson measure Q on $((0, \infty)\times U, (\mathfrak{B}(0, \infty))\otimes\mathcal{U})$ where $(U, \mathcal{U})$ denotes the space of all excursions. Moreover, he showed that the σ-finite measure n given by $n(A) := EQ(A)$ satisfies

$$n(dt, df) = dt\nu^W(df)$$

where ν^W is a measure on $(U, \mathcal{U})$, called the Ito excursion law of W at zero. It holds

$$\nu^W(V^W \in dx) = x^{-2}\, dx \qquad (x\in R\setminus\{0\})\,. \tag{26}$$

Before using these properties we shall introduce some notations. If E is a Borel subset of $R\setminus(-\varepsilon, \varepsilon)$ for some $\varepsilon>0$ define $N_t^W(E)$ to be the number of all excursions U_n^W with $V_n^W\in E$ and $l^W(a_n, 0)\leqslant t\ (t\geqslant 0)$. Thus $N_t(E)$ counts the excursions of W from zero with peak in E which are finished before

$$L^W(t, 0) := \inf\{s>0 \mid l^W(s, 0)>t\} \qquad (t\geqslant 0)$$

or, equivalently, up to the moment, when the local time $l^W(\cdot, 0)$ equals t. From the mentioned Ito's results it easily follows

Lemma 4.1. *Let E_1, E_2 be disjoint Borel subsets of*

$$\{x\in R \mid |x|>\varepsilon\} \quad \textit{for some} \quad \varepsilon>0\,.$$

Then $N^W(E_k)$ are mutually independent Poisson processes with intensities

$$\nu^W(V^W\in E_k) \quad (k=1, 2)\,.$$

Denote by $U_{k_j}^V (j\geqslant 1)$ the j-th excursion of W from 0 with peak in E_k and let $s(dx)$ be a σ-finite measure on R. Then for every $A\in\mathfrak{B}$ the variables

$$y_{k_j}(A) := \int_A (l^W(b_{k_j}^W, x) - l^W(a_{k_j}^W, x))\, s(dx) \qquad (k=1, 2; j\geqq 1)$$

are mutually independent, identically distributed and independent of $N^W(E_k)$ $(k=1, 2)$.

Assume a, l_0, l_1 to be real numbers with $l_0<0<a<l_1$ and put $E_1=[a, l_1)$, $E_2=R\setminus(l_0, l_1)$. Then the preceding lemma implies

Corollary 4.2. *The number $N_t^W(E_1)$ of excursions of W with peaks in $[a, l_1)$ which occur before $L^W(t, 0)$ form a Poisson process with intensity $\mu=a^{-1}-l_1^{-1}$ which is independent of*

$$l^W(\zeta^W, 0) = \inf\{t>0 \mid N_t^W(E_2)\geqq 1\}\ ,$$

and this variable is exponentially distributed with the parameter $\kappa := l_1^{-1} - l_0^{-1}$.

4.2. Excursions of quasidiffusions.

Introducing the excursions of a quasidiffusion X from 0 we have to take into consideration the role of the gaps of E_m and the killing time ζ. Let X be a quasidiffusion with speed measure m as constructed in 2.1. above, we also assume here that X starts at zero. Put

$$a_n := S_{a_n^W}, \quad b_n := S_{b_n^W} \qquad (n \in N)$$

It may happen that $a_n = b_n$ (this means the excursion U_n^W does not meet $E_m \setminus \{0\}$) or that $b_n = \infty$(i.e. W hits l_0 or l_1 before b_n^W). Therefore we define

$$\mathfrak{N} := \{n \geqq 1 \mid a_n < b_n < \zeta\} \quad \text{and put}$$

$$U_n(t) := X_{t+a_n} \quad (0 \leqq t < b_n - a_n), \quad = 0 \ (t \geqq b_n - a_n) \ (n \in \mathfrak{N})$$

The processes $U_n (n \in \mathfrak{N})$ are called the excursions of X from 0. This notation is justified by the following properties:

a) $U_n(t) \neq 0$ for $t \in (a_n, b_n)$ $(n \in \mathfrak{N})$

b) If $\tau := \sup \{t < \zeta \mid X_t = 0\}$ and $t \in [0, \tau)$ with $X_t \neq 0$, then $t \in (a_n, b_n)$ for some $n \in \mathfrak{N}$.

If $V_n^W > 0$ then $V_n = (V_n^W)_- = \sup \{y < V_n^W \mid y \in E_m\}$. Now choose $a > 0$, $a \in E_m$ and define $N_t(a)$ to be the number of all excursions $U_n (n \in \mathfrak{N})$ such that $V_n \in (a, l_1)$ and $l(a_n, 0) \leq t$. Then (recall the definition of $\mathfrak{N}$) $N_t(a)$ counts the number of excursions of X from 0 over the level a which occur before $L(t, 0) \wedge \zeta$ where $L(\cdot, 0)$ denotes the inverse local time:

$$L(t, 0) := \inf \{s > 0 \mid l(s, 0) > t\} \qquad (t \geqslant 0).$$

By virtue of $l(\zeta, 0) = l^W(\zeta^W, 0)$ we have

$$N_t(a) = N^W_{t \wedge l^W(\zeta^W, 0)}([a, l_1)) \qquad (t \geqslant 0).$$

Thus from Corollary 4.2. and the second half of Lemma 4.1. with $s(dx) = m(dx)$ we conclude

Proposition 4.3. *For every $a \in E_m$, $a > 0$, the process $N(a)$ forms a Poisson process with intensity $\mu = a^{-1} - l_1^{-1}$ which is stopped at the random time $l(\zeta, 0)$. The variable $l(\zeta, 0)$ is independent of $N(a)$ and exponentially distributed with the parameter $\kappa = l_1^{-1} - l_0^{-1}$.*

For every $A \in \mathfrak{B}$ the variables

$$Y_n(A) := \int_A (l^W(b_n^W, x) - l^W(a_n^W, x))\, m(dx) =$$
$$= \int_A (l(b_n, x) - l(a_n, x))\, m(dx) = \int_{a_n}^{b_n} \mathbf{1}_A(X_s) ds \qquad (n \in \mathfrak{N})$$

are mutually independent, independent of $N(a)$ and identically distributed.

Choose an $a' \in (0, a) \cap E_m$ and put

$$Y_n(a') := Y_n([a', l_1)) \qquad (n \in \mathcal{N}).$$

The strong Markov property of X implies that the Laplace transform

$$E_0(\exp(\lambda\, Y_n(a')) \mid V_n \geqslant a)$$

of $Y_n(a')$ under $\{V_n \geqslant a\}$ equals
$E_{a'}(\exp(\lambda \int_0^{\tau_0} \mathbf{1}_{[a', l_1)}(X_s)ds) \mid \tau_a < \tau_0 < \zeta)$, i.e. it is given by $\Phi(a, a', \lambda)$ from formula (24) ($n \in \mathcal{N}$, $\lambda < 0$). Put

$$P_t(a, a') := \sum Y_n(a') \qquad (t \geqslant 0)$$

where the sum is going over all $n \in \mathcal{N}$ such that $l(a_n, 0) \leqslant t$ and $V_n \in [a, l_1)$, and define

$$P(a, a') := (P_t(a, a'))_{t \geqslant 0}.$$

The variable $P_t(a, a')$ is equal to the time which X spends over a' during excursions of X from zero to a occuring before $L(t, 0) \wedge \zeta$, i.e. before the local time $l(\cdot, 0)$ equals $t \wedge l(\zeta, 0)$.
From the Proposition 4.3. and the remarks after this proposition it follows now

Theorem 4.4. *Assume $0 < a' < a$ with $a, a' \in E_m$. Then the process $P(a, a')$ is an increasing compounded Poisson process with intensity $\mu = a^{-1} - l_1^{-1}$ stopped at the random time $l(\zeta, 0)$ which is independent of $P(a, a')$ and exponentially distributed with the parameter $\kappa = l_1^{-1} - l_0^{-1}$.*
In particular

$$\begin{aligned} \Phi_t(a, a', \lambda) &:= E_0(\exp(\lambda P_t(a, a')) \mid l(\zeta, 0) > t) = \\ &= \exp(-\mu t(1 - \Phi(a, a', \lambda))) \qquad (t \geqslant 0, \lambda < 0) \end{aligned} \tag{27}$$

with $\Phi(a, a', \lambda)$ as given in (24).

Proposition 4.5. *The process $P(a, 0) := (P_t(a, 0))_{t \geq 0}$ is an increasing compounded Poisson process with intensity μ stopped at $l(\zeta, 0)$ which is independent of $P(a, 0)$ and exponentially distributed with the parameter κ. It holds*

$$\begin{aligned} E_p(\exp(\lambda P_t(a, 0)) \mid l(\zeta, 0) > t) &= \exp\left(-t\left[\left(\frac{1}{a} - \frac{1}{h^a(\lambda)}\right) - \left(\frac{1}{l_1} - \frac{1}{h(\lambda)}\right)\right]\right) \\ &= \exp(-t(\int_0^\infty (1 - e^{\lambda u})(\vartheta_d(u) - \vartheta_d^{(a)}(u))du) \qquad (t \geqq 0, \lambda < 0), \end{aligned} \tag{28}$$

where

$$\vartheta_d^{(a)}(u) = \int_{(0,\infty)} v e^{-vu}\, \sigma_d^{(a)}(dv), \qquad (u > 0)$$

and $\sigma_d^{(a)}$ denotes the spectral measure of the string dual to

$$\mathbf{1}_{[0,a)}(x) m(dx) + \delta_a(dx) \cdot \infty.$$

Note that the integral

$$\int_0^\infty (\vartheta_d(u)-\vartheta_d^{(a)}(u))\,du = \lim_{\lambda\downarrow-\infty}\int_0^\infty (1-e^{\lambda u})(\vartheta_d(u)-\vartheta_d^{(a)}(u))du$$

equals μ (this also follows from

$$\frac{1}{h^a(\lambda)}-\frac{1}{h(\lambda)} = \frac{\varphi(a,\lambda)}{\psi(a,\lambda)} - \lim_{x\uparrow l_1}\frac{\varphi(x,\lambda)}{\psi(x,\lambda)} = \int_a^{l_1}\psi^{-2}(s\,\lambda)\,ds \to 0$$

for $\lambda\downarrow -\infty$) but the integrals over ϑ_d and over $\vartheta_d^{(a)}$ must not be finite. (We supposed $a'\downarrow 0$, $a'\in E_m$, which demands $0_+=0$. However the results are valid for $0_+>0$, too. This becomes obvious by using $\chi_+(0_+,\lambda)-0_+\cdot D^-\chi_+(0_+,\lambda)=h(\lambda)$.) In particular, the preceding Proposition implies that the distribution of the length

$$T_n = y_n(0) = \int_{a_n}^{b_n}\mathbf{1}_{[0,l_1)}(x_s)\,ds$$

under $\{V_n\geqq a\}$ is given by

$$P_0(T_n\in du\,|\,V_n\geqslant a) = \mu^{-1}(\vartheta_d(u)-\vartheta_d^{(a)}(u))\,du \qquad (u\geqslant 0) \tag{29}$$

With these remarks Proposition 4.5. provides a connection between Ito's excursion law of X and the spectral measures of two dual strings:

Corollary 4.6. *Assume* 0 *is a recurrent state, i.e.* $l_1=-l_0=\infty$ *and therefore* $\zeta=\infty$. *Denote by* ν *the Ito excursion law of* X *at zero, by* T *the length and by* V *the peak of the excursion* U *of* X *from* 0. *Then it holds*

$$\nu(V\geqslant a) = \nu^W(V^W\geqslant a) = a^{-1} \tag{30}$$

$$\begin{aligned}\nu(T\in du,\, V\geqslant a) &= \nu(V\geqslant a)\,P_0(T\in du\,|\,V\geqslant a)\\ &= (\vartheta_d(u)-\vartheta_d^{(a)}(u))\,du\end{aligned} \tag{31}$$

$$\nu(T\in du) = \vartheta_d(u)\,du \quad \textit{and}$$

$$\int_0^\infty (1-e^{\lambda t})\,\nu(T\in dt,\, V\geqslant a) = \frac{1}{h(\lambda)}-\left(\frac{1}{h^a(\lambda)}-\frac{1}{a}\right).$$

In particular (30) implies now

$$\int_0^\infty e^{\lambda t}\,\nu(T\in dt,\, V\geqslant a) = \frac{1}{h^a(\lambda)}-\frac{1}{h(\lambda)} = \int_a^\infty \psi^{-2}(x,\lambda)\,dx \qquad (a\in E_m)$$

This formula extends a result for the standard Wiener process ($m(x)=2x$, $x\in R$) for which $\psi(x,\lambda)=\dfrac{\sinh\sqrt{-2\lambda}x}{\sqrt{-2\lambda}}$ holds (see Williams [14], p. 99). In this case we get

$$\nu(T\in dt,\, V\geqslant a) = ((2\pi t^3)^{-1/2} - \frac{1}{a}\sum_{k=1}^\infty \frac{k^2\pi^2}{a^2}\exp\left(-\frac{k^2\pi^2}{2a^2}\right))\,dt \quad (t>0,\, a>0)\,.$$

Now let us consider the second special case: $a' \uparrow a$ where $a>0$, $a \in E_m$ is fixed. From (25) and (27) it follows

$$\begin{aligned}\Phi_t(a, \lambda) :&= \lim_{a' \uparrow a} \Phi_t(a, a', \lambda) = \exp(-\mu\, t(1-\Phi(a, \lambda))) = \\ &= \exp(-\mu\, t(1-\int_{(0,\infty)} \frac{u}{u-\lambda}\, \sigma(du))) = \\ &= \exp(-t \int_0^\infty (1-e^{u\lambda})\, \vartheta_d^G(u)\, du) \qquad (\lambda<0)\,, \end{aligned} \tag{32}$$

where $G=[a, l_1)$, $\sigma=\mu^{-1}\sigma_d^G$, $\vartheta_d^G(u)=\int_{(0,\infty)} ve^{-uv}\, \sigma_d^G(dv)$ with $\int_0^\infty \vartheta_d^G(u)\, du=\mu$ and σ_d^G is the spectral measure of the string dual to m^G. Then we have shown

Proposition 4.8. *The process $P(a, a)=(P_t(a, a))_{t\geq 0}$ is an increasing compounded Poisson process with intensity μ stopped at the random time $l(\zeta, 0)$ which is independent of $P(a, a)$ and exponentially distributed with the parameter κ. It holds*

$$E_0(\exp(\lambda P_t(a, a)) \mid l(\zeta, 0)>t) = \Phi_t(a, \lambda)$$

where $\Phi_t(a, \lambda)$ is given by (32).

Note that

$$P_t(a, a) = \int_0^{l(t,0)\wedge\zeta} \mathbf{1}_{[a,l_1)}(X_s)ds \qquad (t \geqq 0)\,.$$

(These properties also hold, if a is not an accumulation point of E_m from the left.)

In both particular cases $a' \downarrow 0$ and $a' \uparrow a$ we have explicit representations of the Lévy-measure of the increasing compounded Poisson process $P=(P_t)_{t\geq 0}$ by spectral measures of some corresponding dual strings. Moreover, in these cases the distribution of P_t under $\{l(\zeta, 0)>t\}$ belongs to the so-called *Bondesson-class* of distributions (see [1]). (It should be remarked, that every distribution of the *Bondesson-class* occur if m runs through all strings, this is also a consequence of M.G. Krein's theorem mentioned above.)

It remains an oepn problem to derive similar statements for the general case given in (27).

4.3. Local times and spectral measures.

We consider (32) again and let $a \downarrow 0$. Then, from (25) we have

$$\begin{aligned}\Phi_t(0, 0, \lambda) :&= E_0(\exp(\lambda \int_0^{L(t,0)} \mathbf{1}_{(0,\infty)}(X_s)\, ds) \mid l(\zeta, 0)>t) = \\ &= \lim_{a \downarrow 0} \Phi_t(a, a, \lambda) = \lim_{a \downarrow 0} \exp[t\Big(l_1^{-1} + \frac{D^-\chi_+(a, \lambda)}{\chi_+(a, \lambda)-aD^-\chi_+(a, \lambda)}\Big)] = \\ &= \exp[t\Big(l_1^{-1} + \frac{D^+\chi_+(0, \lambda)}{\chi_+(0, \lambda)}\Big)] = \exp[t\Big(\frac{1}{l_1} - \frac{1}{h(\lambda)} - \lambda m_0\Big)] \qquad (\lambda<0, t \geqq 0)\end{aligned}$$

(Note that $\left(\frac{1}{h(\lambda)}+\lambda m_0\right)^{-1}$ is the characteristic function of the string $\mathbf{1}_{(0,\infty)}(x)dm(x)$, see (9).)
From (2) and $l(L(t,0),0)=t$ on $\{l(\zeta,0)>t\}$ it follows

$$E_0(\exp(\lambda\int_0^{L(t,0)}\mathbf{1}_{\{0\}}(X_s)ds\,|\,l(\zeta,0)>t)=\exp(\lambda\, tm_0)\qquad(\lambda<0,\ t\geqq 0)$$

Define $A_1:=(l_0,0)$, $A_2:=[0,l_1]$, $P_t(A_i):=\int_0^{L(t,0)\wedge\zeta}\mathbf{1}_{A_i}(X_s)ds$ $(t\geqq 0,\ i=1,2)$. Then we have shown

Proposition 4.9. *The two processes $P(A_i):=(P_t(A_i))_{t\geqq 0}$ $(i=1,2)$ are independent increasing processes with independent stationary increments killed at the random time $l(\zeta,0)$ which is independent of $P(A_i)$ $(i=1,2)$ and exponentially distributed with the parameter κ. Furthermore it holds*

$$E_0(\exp(\lambda P_t(A_i))\,|\,l(\zeta,0)>t)=$$
$$=\exp\left[-t\left(\frac{1}{l_0}+\frac{1}{h_0^-(\lambda)}\right)\right]\qquad \textit{for}\quad i=1\,,$$
$$=\exp\left[-t\left(-\frac{1}{l_1}+\frac{1}{h_0^+(\lambda)}\right)\right]\qquad \textit{for}\quad i=2\quad(\lambda<0)\,.$$

Applying (18) we obtain

$$\frac{1}{l_0}+\frac{1}{h_0^-(\lambda)}=\int_{(0,\infty)}(1-e^{\lambda u})\vartheta_{0,d}^-(u)\,du$$
$$-\frac{1}{l_1}+\frac{1}{h_0^+(\lambda)}=\int_{(0,\infty)}(1-e^{\lambda u})\vartheta_{p,d}^+(u)du+\lambda m_0$$

with

$$\vartheta_{0,d}^{\pm}(u)=\int_{(0,\infty)}ve^{-uv}\sigma_{0,d}^{\pm}(dv)\qquad(u>0)$$

where $\sigma_{0,d}^+$ and $\sigma_{0,d}^-$ denote the spectral measure of the string dual to $\mathbf{1}_{(0,\infty)}m_0^+$ and m_0^- respectively. In particular we have

$$E_0(\exp(\lambda L(t,0))\,|\,l(\zeta,0)>t)=$$
$$=\exp[-t(\lambda m_0+\int_{(0,\infty)}(1-e^{\lambda u})\vartheta_{0,d}(u)du]\qquad(\lambda<0)\tag{33}$$

with $\vartheta_{0,d}:=\zeta_{0,d}^++\vartheta_{0,d}^-$.
Proposition 4.9. was proved for $r_0=0$ and $l_1=\infty$ in [9], see also [5] for the case of diffusions. Define

$$G(\varepsilon):=\int_\varepsilon^\infty\vartheta_{0,d}(u)du=\int_{(0,\infty)}e^{-v\varepsilon}\sigma_{0,d}(dv)$$

with $\sigma_{0,d}:=\sigma_{0,d}^++\sigma_{d,0}^-$.

The following proposition generalizes a result for diffusions from [5], Chapter 6.3.

Proposition 4.10. *Assume $m_0=0$, $0\in E_m$ and $l_0=\infty$, $l_1=-\infty$. Then it holds*

$$l(t,0)=\lim_{\varepsilon\downarrow 0}\frac{\textit{number of all excursions of } X \textit{ from } 0 \textit{ before } t \textit{ with length} \geqq \varepsilon}{G(\varepsilon)}$$

Proof. The proof is similar to those for the Wiener process ([5], Chapter 2.2). We sketch it only. Firstly observe that $\lim_{\varepsilon\downarrow 0} G(\varepsilon)=\infty$. This comes from the assumption $m_0=0$, $0\in E_m$ which implise that 0 is a point of accumulation of E_m. Therefore at least one of the functions $\vartheta_{0,d}^{+}$ is not integrable. Furthermore we have $\zeta=\infty$ a.s. because of $|l_i|=\infty$ $(i=0, 1)$. We know from Proposition 4.9. that $(L(t,0))_{t\geq 0}$ is an increasing process with independent stationary increments having the Lévy measure $\vartheta_{0,d}(u)du$ $(u>0)$. Thus by a strong law of large numbers it follows

$$\lim G_{(\varepsilon)}^{-1}.\ (\text{Number of all jumps of } L(\cdot,0) \text{ with magnitude} \geqq \varepsilon \text{ up to } s)=s \text{ a.s.}$$

Recall that the jumps of $L(\cdot,0)$ up to s correspond to the flat stretches of $l(\cdot,0)$ up to $L(s,0)$, i.e. to the excursions of X from 0 up to $L(s,0)$. Applying this, substituting $s=l(t,0)$ and noting that $L(l(t,0))-t$ is finite by virtue of $\lim_{u\to\infty} l(u,0)=\infty$ we get the assertion.

It should be remarked that by applying Tauberian type theorems the limit behaviour of $G(\varepsilon)$ for $\varepsilon\downarrow 0$ can be obtained from the limits of

$$\int_0^\infty e^{\lambda\varepsilon}G(\varepsilon)d\varepsilon=-\frac{1}{\lambda}\left(\frac{1}{h_0^+(\lambda)}+\frac{1}{h_0^-(\lambda)}\right)\qquad \text{for}\quad \lambda\downarrow\infty.$$

For instance, for the diffusion on $[0,\infty)$ with speed measure $m(x)=x^\gamma$ $(x\geqq 0)$ (see example 1. in 2.3) we have

$$G(\varepsilon)\sim\frac{1}{\Gamma\left(\frac{1}{\gamma+1}\right)}(\gamma+1)^{\frac{\gamma-1}{\gamma+1}}\gamma^{\frac{1}{\gamma+1}}\varepsilon^{-\frac{1}{\gamma+1}}\qquad(\varepsilon\downarrow 0)$$

and for the example 2 from 2.3 we obtain

$$G(\varepsilon)\sim\frac{1}{\Gamma\left(\frac{1}{2}\right)}\varepsilon^{-\frac{1}{2}}\frac{1+c\tanh v}{c+\tanh v}=\frac{1}{\sqrt{\pi\varepsilon}}\frac{1+c\tanh v}{c+\tanh v}\qquad(\varepsilon\downarrow 0)\,.$$

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