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Magnetic Schrödinger operators with compact resolvent

By

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1. Introduction.

In this paper we shall consider the magnetic Schrödinger operators

$$L_V(\boldsymbol{a}) = -\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j}-ia_j\right)^2+V \tag{1.1}$$

where a_j and V are the operators of multiplication by real-valued functions $a_j(x)$ and $V(x)$, respectively. We assume

$$\begin{cases} a_j(x)\in L^2_{\mathrm{loc}}(\boldsymbol{R}^n) & \text{for } j=1,\cdots,n\,, \\ V(x)\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n) & \text{and } \ V(x)\geqq 0\,, \end{cases} \tag{1.2}$$

where, for $p\geqq 1$ and an open set Ω in $\boldsymbol{R}^n$, $L^p_{\mathrm{loc}}(\Omega)=\{f\,|\,\zeta f\in L^p(\Omega)$ for all $\zeta\in C_0^\infty(\Omega)\}$, $L^p(\Omega)$ being the space of complex-valued measurable functions f on Ω with $\|f\|_{L^p(\Omega)}=[\int_\Omega |f|^p]^{1/p}<\infty$ and $C_0^\infty(\Omega)$=the space of C^∞ complex-valued functions with compact support in Ω. Consider the form in the Hilbert space $L^2(\boldsymbol{R}^n)$

$$\begin{aligned} h_{\boldsymbol{a},V}(\phi,\psi) &= (L_V(\boldsymbol{a})\,\phi,\psi) \\ &= \sum_{j=1}^{n}(\Pi_j(\boldsymbol{a})\,\phi,\Pi_j(\boldsymbol{a})\,\psi)+(V\phi,\psi) \end{aligned} \tag{1.3}$$

for ϕ, $\psi\in Q(h_{\boldsymbol{a},V})\equiv$" the form domain of $h_{\boldsymbol{a},V}$ "$\equiv C_0^\infty(\boldsymbol{R}^n)$, where $(u,v)=\int_{\boldsymbol{R}^n}u\bar{v}$ and

$$\Pi_j(\boldsymbol{a})=\frac{1}{i}\frac{\partial}{\partial x_j}-a_j\,.$$

Then it is known (see, e.g., Leinfelder and Simader [5]) that $h_{\boldsymbol{a},V}$ is closable and its form closure $\bar{h}_{\boldsymbol{a},V}$ is a non-negative symmetric form such that:

$$\begin{cases} Q(\bar{h}_{\boldsymbol{a},V}) = \{u\in L^2(\boldsymbol{R}^n)\,|\,\Pi_j(\boldsymbol{a})\,u\in L^2(\boldsymbol{R}^n) \ \text{ for } j=1,\cdots,n \\ \quad \text{and } \ V^{1/2}u\in L^2(\boldsymbol{R}^n)\}\,, \\ \bar{h}_{\boldsymbol{a},V}(u,v) = \sum_{j=1}^{n}(\Pi_j(\boldsymbol{a})\,u,\Pi_j(\boldsymbol{a})\,v)+(V^{1/2}u,V^{1/2}v)\,, \end{cases} \tag{1.4}$$

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where and in the sequel differentiation is understood in the distribution sense. Denote the self-adjoint operator in $L^2(\boldsymbol{R}^n)$ associated with $\bar{h}_{a,V}$ by $H_V(\boldsymbol{a})$. $\boldsymbol{a}=(a_1, \cdots, a_n)$ and V are called a (magnetic) vector potential and a scalar potential, respectively, and the corresponding magnetic field is the skew-symmetric matrix-valued distribution $B \equiv \operatorname{curl} \boldsymbol{a}$ with the (j, k) component

$$B_{jk} = \frac{\partial a_k}{\partial x_j} - \frac{\partial a_j}{\partial x_k} \quad \text{for } j, k = 1, \cdots, n . \tag{1.5}$$

It is known that all $H_V(\boldsymbol{a})$ with common $B=\operatorname{curl} \boldsymbol{a}$ are unitarily equivalent to each other (gauge invariance: see, e.g., [4]).

In the present paper we shall study the following condition:

(Cpt) $H_V(\boldsymbol{a})$ has compact resolvent (i.e., the resolvent $(H_V(\boldsymbol{a})-i)^{-1}$ is a compact operator in $L^2(\boldsymbol{R}^n)$),

which is equivalent to the discreteness of the spectrum of $H_V(\boldsymbol{a})$ (see [7, Chap. XIII. 14] for details about operators with compact resolvent).

The aim of the present paper is to offer a simple criterion for (Cpt) and to give its several applications.

For an open set Ω in $\boldsymbol{R}^n$, define

$$e_{a,V}(\Omega) \equiv \inf \left\{ \frac{h_{a,V}(\phi, \phi)}{(\phi, \phi)} ; \phi \in C_0^\infty(\Omega), \ \phi \not\equiv 0 \right\} . \tag{1.6}$$

Then we have $0 \leqq e_{a,V}(\Omega) < +\infty$ for non-empty Ω by the assumption (1.2). Our main theorem is the following:

Main Theorem. *The following four conditions are equivalent to each other*:

(a) $H_V(\boldsymbol{a})$ *has compact resolvent.*

(b) $e_{a,V}(\Omega_R) \to \infty$ *as* $R \to \infty$, *where* $\Omega_R = \{x \mid |x| > R\}$.

(c) $e_{a,V}(Q_x) \to \infty$ *as* $|x| \to \infty$, *where* $Q_x = \{y \mid |x-y| < 1\}$.

(d) *There exists a real-valued continuous function* $\lambda(x)$ *on* $\boldsymbol{R}^n$ *such that*

$$\lambda(x) \to \infty \quad as \quad |x| \to \infty ,$$

$$h_{a,V}(\phi, \phi) \geqq \int \lambda(x) |\phi(x)|^2 \, dx \quad for\ all \quad \phi \in C_0^\infty(\boldsymbol{R}^n) .$$

Note that $e_{a,V}(\Omega_R)$ is increasing in $R > 0$, since we have by the definition (1.6)

$$e_{a,V}(\Omega) \geqq e_{a,V}(\Omega') \quad \text{if} \quad \Omega \subset \Omega' . \tag{1.7}$$

Part of the Main Theorem is already known: In the case where $\boldsymbol{a}=\boldsymbol{0}$, the sufficiency of (d) for (Cpt) is a well-known fact, which was extended to the case where $\boldsymbol{a} \neq \boldsymbol{0}$ by Avron, Herbst and Simon [2, Theorem 2.8]; it is known (see, e.g., Agmon [1]; see also the remark after Lemma 2.1 below) that $\sum \equiv \lim_{R \to \infty} e_{0,V}(\Omega_R)$ equals the infimum of the essential spectrum of $-\Delta + V$, and the particular case $\sum = \infty$ means the equivalence of (a) and (b) of the theorem; Molčanov [6] has obtained a necessary and sufficient condition on V for the discreteness of the spectrum of $-\Delta+V$, where he exploited (c) of the theorem as a criterion for (Cpt). However, in the case where $\boldsymbol{a}\neq 0$, it seems to us that there is few literature which points out the equivalence of the four conditions of the theorem except the sufficiency of (d) for (a) obtained by [2].

Next, we shall treat the case where $V=0$ and a_j are smooth and investigate the relationship between the property (Cpt) and the asymptotic behavior at infinity of the magnetic field B (rather than the vector potential $\boldsymbol{a}$ according to gauge invariance) by applying the Main Theorem. In connection with (Cpt), it may be natural to consider the following property by analogy with the case of $-\Delta+V$:

(Div) $$|B(x)|\equiv(\sum_{j<k}|B_{jk}(x)|^2)^{1/2}\to\infty \quad \text{as} \quad |x|\to\infty .$$

Avron, Herbst and Simon [2, Corollary 2.10] have shown that (Cpt) follows from (Div) if the direction of B is supposed not to vary too wildly in the following sense:

(P1) Let Δ_α be a covering of $\boldsymbol{R}^n$ by cubes of size L about the points $L\alpha$ ($\alpha\in\boldsymbol{Z}^n$). If there exist two unit vectors e_α and $f_\alpha\in\boldsymbol{R}^n$ for each $\alpha\in\boldsymbol{Z}^n$ such that $\inf_{x\in\Delta_\alpha} B(x)(e_\alpha, f_\alpha)\to\infty$ as $|\alpha|\to\infty$ (where $B(x)(u, v)=\sum_{j,k=1}^n B_{jk}(x)u_j v_k$), then (Cpt) holds.

As noted by Dufresnoy [3], it is not difficult to verify that from (P1) follows

(P2) In $\boldsymbol{R}^2$, (Div) implies (Cpt).

Contrary to this, [3] has given an example showing

(P3) In $\boldsymbol{R}^n$ ($n\geqq 3$), (Div) does not imply (Cpt),

and given also a result similar to (P1):

(P4) If, in addition to (Div), $\nabla\beta_{jk}(x)=o(|B(x)|^{1/2})$ as $|x|\to\infty$, where $\beta_{jk}(x)=B_{jk}(x)/|B(x)|$ and $\nabla=\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$, then (Cpt) holds.

In view of (P4), we shall consider in the present paper the following condition to be combined with (Div):

(A_δ) $$\nabla B_{jk}(x) = o(|B(x)|^\delta) \quad \text{for} \quad j, k = 1, \cdots, n ,$$

where $\delta>0$ (($A_{3/2}$) implies $\nabla\beta_{jk}=o(|B|^{1/2})$). Then, we have the following:

(P5) (Div) and (A_δ) imply (Cpt) if $0<\delta\leqq 2$ (Theorem 6.1).
(P6) In $\boldsymbol{R}^n$ ($n\geqq 3$), (Div) and (A_δ) do not imply (Cpt) if $\delta>2$ (Assertion 7.1).

Thus, $\delta=2$ is the largest number such that (Div) and (A_δ) imply (Cpt) in $\boldsymbol{R}^n$ ($n\geqq 3$). However, this does not mean that (A_2) is the weakest condition with which (Div) implies (Cpt), for (P1), (P4) and (P5) give different sufficient conditions for (Cpt). As for the necessary condition for (Cpt), we obtain the following results:

(P7) (Cpt) does not imply (Div) (Assertion 7.2).

(P8) (Cpt) implies that $\int_{|y-x|<1} |B(y)|^2 dy \to \infty$ as $|x| \to \infty$ (Theorem 5.2).

In Section 2, a proof of the Main Theorem will be given. In Section 3, we shall show two propositions for later use; one relates to gauge invariance, and the other reconstructs a vector potential $\boldsymbol{a}$ satisfying some local L^p estimate when a magnetic field $B=\text{curl } \boldsymbol{a}$ is given. Section 4 is concerned with perturbation of the magnetic fields, including a theorem (Theorem 4.2) which asserts that $H_V(\boldsymbol{a}^1)$ has compact resolvent if and only if $H_V(\boldsymbol{a}^2)$ does, provided that $|B^1-B^2|$ is bounded on $\boldsymbol{R}^n$ (where $B^j=\text{curl } \boldsymbol{a}^j$ for $j=1, 2$). This result is not so obvious as one might think at first sight, for, even if $|B^1-B^2|$ is bounded, it may not be possible to choose $\boldsymbol{a}^1$ and $\boldsymbol{a}^2$ so that $|\boldsymbol{a}^1-\boldsymbol{a}^2|$ is bounded. In Section 5, we shall offer a necessary condition for (Cpt) (Theorem 5.2), which shows (P8) when $V=0$. In the last two sections, we shall treat the case where $V=0$ and a_j are smooth. Section 6 is devoted to proving (P5) (Theorem 6.1), where we show that (Div) and (A_2) imply (Cpt), since (A_δ) implies (A_2) if $\delta \leqq 2$. In Section 7, we give an example of the vector potential to show (P6) (this is naturally also an example for (P3)), which is of the form given in (7.1), much simpler and easier to manipulate than the example given by [3]. Finally, we construct an example for (P7) with the use of this vector potential.

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2. Proof of the Main Theorem.

In this section, we assume (1.2) only. Let E be the spectral measure associated with $H_V(\boldsymbol{a})$. Then $\sigma_{ess}(H_V(\boldsymbol{a}))$ is defined by

$$\sigma_{ess}(H_V(\boldsymbol{a})) \equiv \{\mu \in \boldsymbol{R} \mid \dim \operatorname{Ran}(E(\mu-\varepsilon, \mu+\varepsilon)) = \infty \quad \text{for any} \quad \varepsilon > 0\} ,$$

where Ran(·) denotes the range of an operator. Note that it is known that

$$(2.1) \qquad \sigma_{ess}(H_V(\boldsymbol{a})) = \phi \quad \text{if and only if} \quad H_V(\boldsymbol{a}) \text{ has compact resolvent.}$$

We need a lemma for the proof of the Main Theorem:

Lemma 2.1. *Let s be a real number. Then the following conditions are equivalent to each other*:

(a) $\inf \sigma_{ess}(H_V(\boldsymbol{a})) \geqq s$.

(b) $\lim_{R\to\infty} e_{a,V}(\Omega_R) \geqq s$, *where* $\Omega_R = \{x \mid |x| > R\}$.

(c) *There exists a continuous function* $\lambda(x)$ *on* $\boldsymbol{R}^n$ *such that*:

$$\liminf_{|x|\to\infty} \lambda(x) \geqq s ,$$

$$h_{a,V}(\phi, \phi) \geqq \int \lambda(x) |\phi(x)|^2 dx \quad \text{for all} \quad \phi \in C_0^\infty(\boldsymbol{R}^n) .$$

Remark. By (a) and (b), we obtain the following equality:

$$\lim_{R\to\infty} e_{a,V}(\Omega_R) = \inf \sigma_{ess}(H_V(\boldsymbol{a})) ,$$

which is known in the case where $\boldsymbol{a}=0$ (see, e.g., [1]).

Proof. (a)⇒(b): First note that $e_{a,V}(\Omega_R)$ is increasing in R by (1.7) and hence the limit exists. Suppose that (a) holds and (b) does not hold. Then $s>0$ by $e_{a,V}(\Omega)\geqq 0$, and there exists some $s'<s$ such that $e_{a,V}(\Omega_R)<s'$ for all $R>0$. Then one can choose successively a sequence $\{\phi_k\}_{k=1}^{\infty}\subset C_0^\infty(\boldsymbol{R}^n)$ such that

$$\text{(2.2)} \qquad \begin{cases} ||\phi_k|| = 1 , \\ h_{a,V}(\phi_k, \phi_k)<s' , \\ \operatorname{supp} \phi_k\subset \{x\,|\,a_k<|x|<a_{k+1}\} , \end{cases}$$

for some $\{a_k\}_{k=1}^{\infty}$ such that $a_k \uparrow \infty$ as $k\to\infty$. On the other hand, $E(t)\equiv E((-\infty, t])$ is compact for $t<s$ by (a), and hence $E(t)\phi_k$ converges strongly to 0 as $k\to\infty$ since ϕ_k converges weakly to 0 in $L^2(\boldsymbol{R}^n)$ by (2.2). Therefore we have, for $t\in[0, s)$,

$$\begin{aligned} h_{a,V}(\phi_k, \phi_k) &= ||H_V(\boldsymbol{a})^{1/2}\, \phi_k||^2 \\ &= \int_0^\infty \mu \mathrm{d}||E(\mu)\, \phi_k||^2 \geqq \int_t^\infty t\mathrm{d}||E(\mu)\, \phi_k||^2 \\ &= t(||\phi_k||^2-||E(t)\, \phi_k||^2)\to t \quad \text{as} \quad k\to\infty . \end{aligned}$$

Hence we obtain $\liminf_{k\to\infty} h_{a,V}(\phi_k, \phi_k)\geqq t$ for any $t\in[0, s)$, which contradicts $\limsup_{k\to\infty} h_{a,V}(\phi_k, \phi_k)\leqq s'$ obtainable from (2.2) since $s'<s$.
(b)⇒(c): Let $\{\zeta_k\}_{k=0,1,2\cdots}$ be a sequence of real-valued functions $\subset C_0^\infty(\boldsymbol{R}^n)$ such that

$$\text{(2.3)} \qquad \begin{cases} \sum_{k=0}^{\infty} \zeta_k^2 = 1 , \\ \zeta_k(x) = \zeta_1(x/2^{k-1}) \quad \text{for} \quad k\geqq 1 , \\ \operatorname{supp} \zeta_0\subset \{x\,|\,|x|<2\} , \\ \operatorname{supp} \zeta_k\subset \{x\,|\,2^{k-1}<|x|<2^{k+1}\} \quad \text{for} \quad k\geqq 1 . \end{cases}$$

(Such a sequence can be constructed as follows: Let ϕ_0, ϕ_1 be real-valued smooth functions on $\boldsymbol{R}$ such that $\operatorname{supp} \phi_0\subset(-\infty, 2)$, $\operatorname{supp} \phi_1\subset(1, \infty)$ and $\phi_0(r)^2+\phi_1(r)^2=1$ for all $r\in\boldsymbol{R}$. Define $\zeta_0(x)=\phi_0(|x|)$, $\zeta_1(x)=\phi_1(|x|)$ if $|x|\leqq 2$ and $=\phi_0(|x|/2)$ if $|x|\geqq 2$, and $\zeta_k(x)=\zeta_1(x/2^{k-1})$ for $x\in\boldsymbol{R}^n$ and for $k\geqq 2$.) Since we have by direct computation

$$\text{(2.4)} \qquad \begin{aligned} &\mathrm{Re}(\Pi_j(\boldsymbol{a})\, \phi\, \overline{\Pi_j(\boldsymbol{a})\, (\zeta^2 \phi)}) \\ &= \zeta^2|\Pi_j(\boldsymbol{a})\, \phi|^2+2\zeta(\partial\zeta/\partial x_j)\, \mathrm{Re}(\Pi_j(\boldsymbol{a})\, \phi i \bar{\phi}) \\ &= |\Pi_j(\boldsymbol{a})\, (\zeta\phi)|^2-|(\partial\zeta/\partial x_j)\, \phi|^2 \end{aligned}$$

for $\phi\in C_0^\infty(\boldsymbol{R}^n)$ and for a real-valued C^∞ function ζ, where Re=the real part of a complex number, we obtain, by using (2.3),

$$h_{a,V}(\phi, \phi) = \mathrm{Re}(\textstyle\sum_{k=0}^{\infty} h_{a,V}(\phi, \zeta_k^2 \phi)) \tag{2.5}$$
$$= \textstyle\sum_{k=0}^{\infty} h_{a,V}(\zeta_k\phi, \zeta_k\phi) - \sum_{k=0}^{\infty} \|(\nabla\zeta_k)\,\phi\|^2 .$$

We have by (1.6), (2.3) and (2.5)

$$h_{a,V}(\phi, \phi) \geqq \textstyle\sum_{k=0}^{\infty} e_k\|\zeta_k\phi\|^2 - \sum_{k=0}^{\infty}\|(\nabla\zeta_k)\,\phi\|^2 \tag{2.6}$$
$$= \int \lambda(x)\,|\phi(x)|^2 dx ,$$

where $e_0 = e_{a,V}(\{x \mid |x| < 2\})$, $e_k = e_{a,V}(\Omega_{2^{k-1}})$ for $k \geqq 1$ and

$$\lambda(x) = \textstyle\sum_{k=0}^{\infty} e_k\, \zeta_k(x)^2 - \sum_{k=0}^{\infty} |\nabla\zeta_k(x)|^2 . \tag{2.7}$$

Let $\varepsilon > 0$. Then there is some $k_0 \geqq 2$ such that $e_k \geqq s - \varepsilon$ for $k \geqq k_0$ by (b). Hence, if $2^k \leqq |x| < 2^{k+1}$ for $k \geqq k_0$, we have by (2.3) and (2.7)

$$\lambda(x) = e_k\, \zeta_k(x)^2 + e_{k+1}\, \zeta_{k+1}(x)^2 - (|\nabla\zeta_k(x)|^2 + |\nabla\zeta_{k+1}(x)|^2)$$
$$\geqq (s-\varepsilon) - \frac{5}{2^{2k}} \sup_{x \in R^n} |\nabla\zeta_1(x)|^2$$

Therefore by letting $|x| \to \infty$ we have $\liminf_{|x|\to\infty} \lambda(x) \geqq s - \varepsilon$. Since ε was arbitrary, we obtain $\liminf_{|x|\to\infty} \lambda(x) \geqq s$ and thus (c) by (2.6).

(c)⇒(a): Let $\tilde{\lambda}(x) = \min(\lambda(x), s)$, where $\min(a, b) = a$ if $a \leqq b$ and $= b$ if $a > b$. Then we have by (c)

$$\tilde{\lambda}(x) \to s \quad \text{as} \quad |x| \to \infty , \tag{2.8}$$
$$h_{a,V}(\phi, \phi) \geqq \int \tilde{\lambda}(x)\,|\phi(x)|^2 dx .$$

Hence we have

$$H_V(\boldsymbol{a}) \geqq s + K , \tag{2.9}$$

where K denotes the operator of multiplication by $\tilde{\lambda}(x) - s$, which is a relatively compact operator with respect to $H_V(\boldsymbol{a})$ by (2.8) as we shall see in the next lemma. Let $\sigma \in \sigma_{ess}(H_V(\boldsymbol{a}))$. Then it is well known that there exists a sequence $\{u_k\}_{k=1}^{\infty}$ such that $u_k \in D(H_V(\boldsymbol{a}))$ ($D(\cdot)$ = the operator domain of an operator), $\|u_k\| = 1$, $u_k \to 0$ weakly as $k \to \infty$ and $\|H_V(\boldsymbol{a})\, u_k - \sigma u_k\| \to 0$ as $k \to \infty$. Consequently, $Ku_k \to 0$ strongly as $k \to \infty$ by the relative compactness of K with respect to $H_V(\boldsymbol{a})$ and by the boundedness of $\{u_k\}$ and $\{H_V(\boldsymbol{a})\, u_k\}$. On the other hand, by (2.9), we have

$$(H_V(\boldsymbol{a})\, u_k, u_k) \geqq s + (Ku_k, u_k) ,$$

whose left-hand side converges to σ as $k \to \infty$, while the right-hand side converges to s as $k \to \infty$. Thus we have $\sigma \geqq s$ for any $\sigma \in \sigma_{ess}(H_V(\boldsymbol{a}))$ and thus (a). □

Now, we state a lemma used in the above proof, which is obtainable in the same manner as [2, Theorem 2.6]. Since this lemma is essential in our argument and [2] does not include scalar potentials V, we sketch here a proof.

Lemma 2.2. *Let K be an operator of multiplication by a bounded measurable function $k(x)$ on $\boldsymbol{R}^n$ tending to 0 at infinity. Let $\gamma>0$ and $E>0$. Then $K(H_V(\boldsymbol{a})+E)^{-\gamma}$ is a compact operator in $L^2(\boldsymbol{R}^n)$.*

Proof. It is known [5, Lemma 6] that, under the assumption (1.2)

$$|(H_V(\boldsymbol{a})+\mu)^{-1}f|\leqq(-\Delta+\mu)^{-1}|f| \tag{2.10}$$

for $f\in L^2(\boldsymbol{R}^n)$ and for any $\mu>0$, where $|g|\leqq h$ means $|g(x)|\leqq h(x)$ for almost all $x\in\boldsymbol{R}^n$. By iterating (2.10) and using the formula $e^{-tA}f=\text{s}-\lim_{n\to\infty}\left(1+\frac{t}{n}A\right)^{-n}f$, we obtain

$$|e^{-tH_V(a)}f|\leqq e^{t\Delta}|f| \quad \text{for} \quad t>0 \quad \text{and for} \quad f\in L^2(\boldsymbol{R}^n). \tag{2.11}$$

Since by the Laplace transformation we have

$$(A+E)^{-\gamma}=\frac{1}{\Gamma(\gamma)}\int_0^\infty t^{\gamma-1}e^{-tE}e^{-tA}dt\,,$$

we obtain by (2.11)

$$|(H_V(\boldsymbol{a})+E)^{-\gamma}f|\leqq(-\Delta+E)^{-\gamma}|f| \quad \text{for} \quad f\in L^2(\boldsymbol{R}^n)\,. \tag{2.12}$$

Hence, we have the compactness of $K(H_V(\boldsymbol{a})+E)^{-\gamma}$ with the use of [2, Theorem 2.2] and the estimate

$$|K(H_V(\boldsymbol{a})+E)^{-\gamma}f|\leqq|K|(-\Delta+E)^{-\gamma}|f| \quad \text{for} \quad f\in L^2(\boldsymbol{R}^n)$$

obtainable from (2.12), where $|K|$ is the operator of multiplication by $|k(x)|$, since it is well known (see, e.g., the proof of [7, Theorem XIII. 65]) that $|K|(-\Delta+E)^{-\gamma}$ is compact if $k(x)$ satisfies the assumption of the lemma. □

Proof of the Main Theorem. First note that, by (2.1), (a) is equivalent to that (a) of Lemma 2.1 holds for any $s\in\boldsymbol{R}$. Thus the equivalence of (a) and (b) follows from that of (a) and (b) of Lemma 2.1.
(b)⇒(c): Obvious by (1.7).
(c)⇒(d): Let $L=\{(l_1,\cdots,l_n)\,|\,l_j=k_j/\sqrt{n}\,;\ k_j$ is an integer for $j=1,\cdots,n\}$ and let $\{\zeta_l\}_{l\in L}$ be a sequence of real-valued functions $\subset C_0^\infty(\boldsymbol{R}^n)$ such that:

$$\zeta_l(x)=\zeta_{(0,\cdots,0)}(x-l) \quad \text{for} \quad l\in L\,, \tag{2.13}$$

$$\begin{cases}\sum_{l\in L}\zeta_l(x)^2=1\,,\\ \text{supp}\,\zeta_l\subset Q_l\,.\end{cases} \tag{2.14}$$

(Such a sequence can be constructed as follows: Let $\phi_0(x)$ be a real-valued C^∞

function on $\boldsymbol{R}^n$ such that $\phi_0(x)=1$ if $|x|\leqq 1/2$ and supp $\phi_0\subset Q_{(0,\cdots,0)}$. Then, if we define $\Phi(x)=\sum_{l\in L}\phi_0(x-l)^2$, $\Phi(x)\geqq 1$ for all $x\in\boldsymbol{R}^n$ since, for all $x\in\boldsymbol{R}^n$, there exists a point $l\in L$ such that $|x-l|\leqq 1/2$. Thus, let $\zeta_l(x)=\phi_0(x-l)/\sqrt{\Phi(x)}$.) In the same manner as we obtained (2.6) and (2.7) from (2.3), we have with the use of (2.14)

$$h_{a,V}(\phi,\phi)\geqq\int\lambda(x)|\phi(x)|^2dx \quad \text{for all} \quad \phi\in C_0^\infty(\boldsymbol{R}^n)\,, \tag{2.15}$$

$$\lambda(x)=\sum_{l\in L}e_l\,\zeta_l(x)^2-\sum_{l\in L}|\nabla\zeta_l(x)|^2\,, \tag{2.16}$$

where $e_l=e_{a,V}(Q_l)$. It is not difficult to check by using (2.14) that $\sum_{l\in L}e_l\,\zeta_l(x)^2\to\infty$ as $|x|\to\infty$, since $e_l\to\infty$ as $|l|\to\infty$ by (c), while $M(x)\equiv\sum_{l\in L}|\nabla\zeta_l(x)|^2$ is bounded on $\boldsymbol{R}^n$ since $M(x+l)=M(x)$ for $l\in L$ by (2.13). Hence $\lambda(x)\to\infty$ as $|x|\to\infty$ by (2.16). Thus we have (d) by (2.15).

(d)⇒(a): Since $\lambda(x)\to\infty$ as $|x|\to\infty$, $\liminf_{|x|\to\infty}\lambda(x)\geqq s$ holds for any $s\in\boldsymbol{R}$. Therefore, (c) of Lemma 2.1 holds and thus, by Lemma 2.1, (a) of Lemma 2.1 holds for any $s\in\boldsymbol{R}$, which implies (a) by (2.1). □

3. Gauge invariance and the reconstruction of vector potentials.

In this section, we give two technical propositions which are needed in later sections. First we shall assume (1.2) and prove a proposition in connection with gauge invariance (see [4]). Let Ω be an open set in $\boldsymbol{R}^n$ and define

$$A_a(\Omega)=\{\boldsymbol{b}\in(L^2_{\mathrm{loc}}(\Omega))^n\,|\,\mathrm{curl}\,\boldsymbol{a}=\mathrm{curl}\,\boldsymbol{b}\}\,. \tag{3.1}$$

Proposition 3.1. *Suppose that Ω is simply connected. Then $e_{b,V}(\Omega)=e_{a,V}(\Omega)$ for $\boldsymbol{b}\in A_a(\Omega)$.*

Proof. If Ω is simply connected, it is known [4, Lemma 1.1] that, for $\boldsymbol{b}\in A_a(\Omega)$, there exists a real-valued scalar function $g\in W^1_{\mathrm{loc}}(\Omega)=\{g\in L^2_{\mathrm{loc}}(\Omega)\,|\,\partial g/\partial x_j\in L^2_{\mathrm{loc}}(\Omega)$ for $j=1,\cdots,n\}$ such that

$$\boldsymbol{b}=\boldsymbol{a}+\nabla g\,. \tag{3.2}$$

As in the proof of [4, Theorem 1.2], take a sequence $\{g_k\}$ of C^∞ functions on Ω such that $g_k\to g$ in $W^1_{\mathrm{loc}}(\Omega)$ as $k\to\infty$. Then we have by (3.2)

$$\boldsymbol{b}_k\equiv\boldsymbol{a}+\nabla g_k\to\boldsymbol{b} \text{ in } L^2_{\mathrm{loc}}(\Omega) \text{ as } k\to\infty\,. \tag{3.3}$$

Let $\phi\in C_0^\infty(\Omega)$. Then we have by (3.3)

$$\Pi_j(\boldsymbol{a})(e^{-ig_k}\phi)=e^{-ig_k}\,\Pi_j(\boldsymbol{b}_k)\phi \quad \text{for} \quad j=1,\cdots,n\,.$$

Hence we have by (1.3)

$$h_{a,V}(e^{-ig_k}\phi,e^{-ig_k}\phi)=h_{b_k,V}(\phi,\phi)\,.$$

Therefore, since $\prod_j(\boldsymbol{b}_k)\phi \to \prod_j(\boldsymbol{b})\phi$ as $k\to\infty$ in $L^2(\Omega)$ by (3.3), we obtain

$$\begin{aligned} h_{\boldsymbol{b},V}(\phi, \phi) &= \lim_{k\to\infty} h_{\boldsymbol{b}_k,V}(\phi, \phi) \\ &= \lim_{k\to\infty} h_{\boldsymbol{a},V}(e^{-ig_k}\phi, e^{-ig_k}\phi) \\ &\geqq e_{\boldsymbol{a},V}(\Omega) \liminf_{k\to\infty} \|e^{-ig_k}\phi\|^2 = e_{\boldsymbol{a},V}(\Omega)\|\phi\|^2 \end{aligned}$$

Thus, since ϕ was arbitrary, we have

$$e_{\boldsymbol{b},V}(\Omega) \geqq e_{\boldsymbol{a},V}(\Omega)\,. \tag{3.4}$$

Similarly, we have $e_{\boldsymbol{a},V}(\Omega) \geqq e_{\boldsymbol{b},V}(\Omega)$, which, with (3.4), shows the equality of $e_{\boldsymbol{a},V}(\Omega)$ and $e_{\boldsymbol{b},V}(\Omega)$. □

Next, we assume that a_j are smooth. A vector potential $\boldsymbol{a}$ is considered as a 1-form $\boldsymbol{a} = \sum_j a_j \mathrm{d}x_j$ and the corresponding magnetic field B as the exterior differential of $\boldsymbol{a}$: B is the 2-form $\mathrm{d}\boldsymbol{a} = \sum_{j<k} B_{jk}\,\mathrm{d}x_j \wedge \mathrm{d}x_k$, where B_{jk} is given by (1.5). Hence, by the formula dd$=0$, the magnetic field B is closed, i.e.,

$$\frac{\partial B_{jk}}{\partial x_i} + \frac{\partial B_{ki}}{\partial x_j} + \frac{\partial B_{ij}}{\partial x_k} = 0 \quad \text{for} \quad i, j, k = 1, \cdots, n\,. \tag{3.5}$$

Conversely, it is a well known fact that, when a C^∞ 2-form B on $\boldsymbol{R}^n$ is closed, there exists a C^∞ 1-form $\boldsymbol{a}$ such that $B = \mathrm{d}\boldsymbol{a}$. We shall prove a proposition concerning this fact:

Proposition 3.2. *Let B be a C^∞ skew-symmetric matrix-valued function on $\boldsymbol{R}^n$ satisfying* (3.5). *Moreover, let $p=2$ or ∞ and Ω be a bounded convex open set in $\boldsymbol{R}^n$. Then there exist a constant C dependent only on p, n and* $\mathrm{diam}(\Omega) \equiv \sup_{x,y\in\Omega} |x-y|$, *and a C^∞ vector potential $\boldsymbol{b}$ on $\boldsymbol{R}^n$ such that:*

$$\mathrm{curl}\, \boldsymbol{b} = B \text{ on } \boldsymbol{R}^n\,, \tag{3.6}$$

$$\|\boldsymbol{b}\|_{p,\Omega} \leqq C\|B\|_{p,\Omega}\,, \tag{3.7}$$

where $\|u\|_{2,\Omega} = [\int_\Omega |u|^2]^{1/2}$, $\|u\|_{\infty,\Omega} = \sup_{x\in\Omega} |u(x)|$, $|\boldsymbol{b}| = (\sum_j |b_j|^2)^{1/2}$

and $|B| = (\sum_{j<k} |B_{jk}|^2)^{1/2}$.

Remark. It is not difficult to verify by a similar proof that the proposition holds also in the case where $1 \leqq p < \infty$ with $\|u\|_{p,\Omega} = [\int_\Omega |u|^p]^{1/p}$.

Proof. First, define for $x, y \in \boldsymbol{R}^n$

$$b_j^0(x, y) = \sum_{k=1}^n (x_k - y_k) \int_0^1 B_{kj}(y + t(x-y))\, t\mathrm{d}t\,. \tag{3.8}$$

Then $b_j^0(x, y) \in C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ and it is not difficult to check with the use of (3.5) that

$$\operatorname{curl}_x \boldsymbol{b}^0(x, y) = B(x)\,, \tag{3.9}$$

where curl_x means that curl is taken as a function of x with y fixed. Moreover, we have, by (3.8) and by using the Schwarz inequality repeatedly,

$$\begin{aligned} |b_j^0(x, y)|^2 &\leqq \left(\sum_{k=1}^n |x_k - y_k|^2\right) \left(\sum_{k=1}^n \int_0^1 |B_{jk}(y+t(x-y))|^2\, \mathrm{d}t \int_0^1 t^2\, \mathrm{d}t\right) \\ &= \frac{1}{3} |x-y|^2 \sum_{k=1}^n \int_0^1 |B_{jk}(y+t(x-y))|^2\, \mathrm{d}t\,. \end{aligned}$$

Therefore we have

$$|\boldsymbol{b}^0(x, y)| \leqq |x-y| \left[\int_0^1 |B(y+t(x-y))|^2\, \mathrm{d}t\right]^{1/2}. \tag{3.10}$$

In the case where $p=\infty$, we let $\boldsymbol{b}(x)=\boldsymbol{b}^0(x, y_0)$ for some $y_0 \in \Omega$. We have (3.7) with $C=\operatorname{diam}(\Omega)$ by (3.10), since Ω is convex, and (3.6) by (3.9), which proves the proposition.

In the case where $p=2$, define

$$b_j(x) = \frac{1}{m(\Omega)} \int_\Omega b_j^0(x, y)\, \mathrm{d}y\,, \tag{3.11}$$

where $m(\Omega) = \int_\Omega \mathrm{d}x$. Then $b_j(x) \in C^\infty(\boldsymbol{R}^n)$ and (3.6) holds by (3.9). Thus, it remains to show (3.7). By (3.11) we have with the use of the Schwarz inequality

$$\begin{aligned} |\boldsymbol{b}(x)|^2 &= \frac{1}{m(\Omega)^2} \sum_{j=1}^n \left|\int_\Omega b_j^0(x, y)\, \mathrm{d}y\right|^2, \\ &\leqq \frac{1}{m(\Omega)} \int_\Omega |\boldsymbol{b}^0(x, y)|^2\, \mathrm{d}y\,. \end{aligned}$$

Hence we have by (3.10)

$$\begin{aligned} \|\boldsymbol{b}\|_{2,\Omega}^2 &\leqq \frac{\operatorname{diam}(\Omega)^2}{m(\Omega)} \int_{\Omega\times\Omega} \int_0^1 |B(y+t(x-y))|^2\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y \\ &= \frac{\operatorname{diam}(\Omega)^2}{m(\Omega)} \int_0^1 I(t)\, \mathrm{d}t\,, \end{aligned} \tag{3.12}$$

where we have put

$$I(t) = \int_{\Omega\times\Omega} |B(y+t(x-y))|^2\, \mathrm{d}x\, \mathrm{d}y\,. \tag{3.13}$$

First, let $1/2 \leqq t \leqq 1$. Put $z=tx+(1-t)y$ for y fixed. Then

$$\begin{aligned} I(t) &= \int_\Omega \left(\int_{t\Omega+(1-t)y} |B(z)|^2\, t^{-n}\, \mathrm{d}z\right) \mathrm{d}y \\ &\leqq 2^n \int_{\Omega\times\Omega} |B(z)|^2\, \mathrm{d}z\, \mathrm{d}y = 2^n m(\Omega)\, \|B\|_{2,\Omega}^2\,, \end{aligned}$$

where we have used that Ω is convex. In the second place, let $0\leqq t<1/2$. Then, we have the same estimate

$$I(t)\leqq 2^n m(\Omega)\,||B||_{2,\Omega}^2$$

as the case where $1/2\leqq t\leqq 1$, since $I(t)=I(1-t)$ by (3.13). Therefore we have by (3.12)

$$||\boldsymbol{b}||_{2,\Omega}^2\leqq \mathrm{diam}(\Omega)^2\, 2^n\, ||B||_{2,\Omega}^2 .$$

Thus we have (3.7) with $C=2^{n/2}\,\mathrm{diam}(\Omega)$. □

4. Perturbation of the magnetic fields.

In this section, we shall consider perturbation of the vector potentials, or the magnetic fields, with a scalar potential V fixed. We shall assume that V satisfies (1.2). We begin with

Theorem 4.1. *Let $\boldsymbol{a}^k$ be L_{loc}^2 vector potentials for $k=1, 2$. Suppose that there exist constants $C\geqq 0$ and $4>\delta\geqq 0$ such that, for any $x\in\boldsymbol{R}^n$, there is some $\boldsymbol{b}\in A_{\boldsymbol{a}^2-\boldsymbol{a}^1}(Q_x)$ satisfying*

$$|\boldsymbol{b}(y)|^2\leqq\delta V(y)+C \quad \textit{for all} \quad y\in Q_x , \tag{4.1}$$

where $|\boldsymbol{b}|=(\sum_{j=1}^n|b_j|^2)^{1/2}$ and $A_{\boldsymbol{a}^2-\boldsymbol{a}^1}(Q_x)$ is as in (3.1). *Then $H_V(\boldsymbol{a}^2)$ has compact resolvent if and only if $H_V(\boldsymbol{a}^1)$ has compact resolvent.*

Proof. First fix x. Let $\boldsymbol{b}\in A_{\boldsymbol{a}^2-\boldsymbol{a}^1}(Q_x)$ satisfy (4.1) and set

$$\tilde{\boldsymbol{a}}^2=\boldsymbol{a}^1+\boldsymbol{b} . \tag{4.2}$$

Then we have, for $\varepsilon>0$ and for $\phi\in C_0^\infty(Q_x)$,

$$\begin{aligned} &|h_{\tilde{\boldsymbol{a}}^2,V}(\phi, \phi)-h_{\boldsymbol{a}^1,V}(\phi, \phi)| \\ &=|\sum_{j=1}^n(\Pi_j(\boldsymbol{a}^1)\,\phi, b_j\,\phi)+\sum_{j=1}^n(b_j\,\phi, \Pi_j(\tilde{\boldsymbol{a}}^2)\,\phi)| \\ &\leqq\varepsilon\sum_{j=1}^n(||\Pi_j(\boldsymbol{a}^1)\,\phi||^2+||\Pi_j(\tilde{\boldsymbol{a}}^2)\,\phi||^2)+\frac{1}{2\varepsilon}(|\boldsymbol{b}|^2\,\phi, \phi) \\ &=\varepsilon h_{\boldsymbol{a}^1,V}(\phi, \phi)+\varepsilon h_{\tilde{\boldsymbol{a}}^2,V}(\phi, \phi)+((\frac{1}{2\varepsilon}|\boldsymbol{b}|^2-2\varepsilon V)\,\phi, \phi) , \end{aligned} \tag{4.3}$$

where we have used the Schwarz inequality and $|pq|\leqq\varepsilon p^2+\frac{1}{4\varepsilon}q^2$ for $p, q\in\boldsymbol{R}$. Let $\varepsilon=\sqrt{\delta}/2$ if $\delta>0$ and $=1/2$ if $\delta=0$. Then $\frac{1}{2\varepsilon}|\boldsymbol{b}|^2-2\varepsilon V\leqq C/2\varepsilon$ by (4.1) if $\delta>0$ and by (4.1) and (1.2) if $\delta=0$. Hence by (4.3) we have

$$\begin{aligned} h_{\tilde{\boldsymbol{a}}^2,V}(\phi, \phi)&\geqq h_{\boldsymbol{a}^1,V}(\phi, \phi)-|h_{\tilde{\boldsymbol{a}}^2,V}(\phi, \phi)-h_{\boldsymbol{a}^1,V}(\phi, \phi)| \\ &\geqq h_{\boldsymbol{a}^1,V}(\phi, \phi)-\varepsilon h_{\tilde{\boldsymbol{a}}^2,V}(\phi, \phi) \\ &\qquad -\varepsilon h_{\boldsymbol{a}^1,V}(\phi, \phi)-C||\phi||^2/2\varepsilon . \end{aligned}$$

Therefore we have

$$(4.4)\qquad h_{\tilde{a}^2,V}(\phi,\phi)\geqq\frac{1-\varepsilon}{1+\varepsilon}h_{a^1,V}(\phi,\phi)-\frac{C}{2\varepsilon(1+\varepsilon)}\|\phi\|^2 .$$

Hence we obtain by taking the infimum over $\phi\in C_0^\infty(Q_x)$, $\phi\neq 0$ after deviding the both sides of (4.4) by $\|\phi\|^2$,

$$e_{\tilde{a}^2,V}(Q_x)\geqq\frac{1-\varepsilon}{1+\varepsilon}e_{a^1,V}(Q_x)-\frac{C}{2\varepsilon(1+\varepsilon)},$$

where we have used $0<\varepsilon<1$ obtainable from the assumption $0\leqq\delta<4$. Consequently, by applying Proposition 3.1 since $\tilde{\boldsymbol{a}}^2\in A_{a^2}(Q_x)$ by (4.2) and by $\boldsymbol{b}\in A_{a^2-a^1}(Q_x)$, we obtain

$$(4.5)\qquad e_{a^2,V}(Q_x)\geqq\frac{1-\varepsilon}{1+\varepsilon}e_{a^1,V}(Q_x)-\frac{C}{2\varepsilon(1+\varepsilon)}.$$

Since the constants $\varepsilon=\sqrt{\delta}/2$ or $1/2$ and C in (4.5) are independent of x, (4.5) shows that $e_{a^2,V}(Q_x)\to\infty$ as $|x|\to\infty$ if $e_{a^1,V}(Q_x)\to\infty$ as $|x|\to\infty$. Thus we obtain by applying the Main Theorem ((a)$\Leftrightarrow$(c)) that $H_V(\boldsymbol{a}^2)$ has compact resolvent if $H_V(\boldsymbol{a}^1)$ has compact resolvent. By interchanging the superscripts 1 and 2 in the above argument, we have the conclusion of the theorem. □

Theorem 4.2. *Let $\boldsymbol{a}^k$ be C^∞ vector potentials for $k=1, 2$. Suppose that $B^2(x)-B^1(x)$ is bounded on $\boldsymbol{R}^n$ where $B^k=\operatorname{curl}\boldsymbol{a}^k$ for $k=1, 2$. Then $H_V(\boldsymbol{a}^2)$ has compact resolvent if and only if $H_V(\boldsymbol{a}^1)$ has compact resolvent.*

Proof. As noted before Proposition 3.2, $B=B^2-B^1$ satisfies (3.5). Thus, we can apply Proposition 3.2 to B with $p=\infty$ and $\Omega=Q_x$. Let $M=\sup_{x\in\boldsymbol{R}^n}|B(x)|$. Then, for each $x\in\boldsymbol{R}^n$, we have a vector potential $\boldsymbol{b}_x$ on $\boldsymbol{R}^n$ such that curl $\boldsymbol{b}_x=B$ and $|\boldsymbol{b}_x(y)|\leqq CM$ for $y\in Q_x$ by (3.7), where C is independent of x since diam$(Q_x)=2$. Therefore, the assumption of Theorem 4.1 is satisfied with C replaced by CM and with $\delta=0$, since curl $\boldsymbol{b}_x=$curl $\boldsymbol{a}^2-$curl $\boldsymbol{a}^1$ and thus $\boldsymbol{b}_x\in A_{a^2-a^1}(Q_x)$. Thus the conclusion of the theorem holds by Theorem 4.1. □

5. A necessary condition for (Cpt).

In this section, we assume that V satisfies (1.2). The purpose of this section is to show that, in the case where a_j are smooth, (Cpt) implies that the integral of $|B|^2+V$ over the ball Q_x centered at x with radius 1 tends to ∞ as $|x|\to\infty$. We begin with a lemma which gives a necessary condition for (Cpt) in the case where a_j are locally L^2:

Lemma 5.1. *Suppose that $H_V(\boldsymbol{a})$ has compact resolvent. Then we have*

$$(5.1)\qquad \inf_{\boldsymbol{b}\in A_{\boldsymbol{a}}(Q_x)}\int_{Q_x}\{|\boldsymbol{b}(y)|^2+V(y)\}\,dy\to\infty \text{ as } |x|\to\infty ,$$

where $|\boldsymbol{b}|=(\sum_{j=1}^{n}|b_j|^2)^{1/2}$ *and* $A_a(Q_x)$ *is as in* (3.1).

Proof. Let ϕ be a real-valued function $\in C_0^\infty(Q_0)$ such that $\phi\not\equiv 0$ and $|\phi(y)|\leqq 1$ for all y and let $\phi_x(y)=\phi(y-x)$. Since we have, for real-valued functions f,

$$|\Pi_j(\boldsymbol{b})f(y)|^2=|\frac{\partial f}{\partial x_j}(y)|^2+|b_j(y)f(y)|^2,$$

we obtain

$$\int\{|\nabla\phi_x(y)|^2+(|\boldsymbol{b}(y)|^2+V(y))\,\phi_x(y)^2\}\,\mathrm{d}y=h_{\boldsymbol{b},V}(\phi_x,\phi_x)\,.$$

Therefore, since $|\boldsymbol{b}(y)|^2+V(y)\geqq(|\boldsymbol{b}(y)|^2+V(y))\,\phi_x(y)^2$ by (1.2),

$$\begin{aligned}\int_{Q_x}\{|\boldsymbol{b}(y)|^2+V(y)\}\,\mathrm{d}y&\geqq h_{\boldsymbol{b},V}(\phi_x,\phi_x)-\|\nabla\phi\|^2\\&\geqq e_{\boldsymbol{b},V}(Q_x)\,\|\phi\|^2-\|\nabla\phi\|^2,\end{aligned}$$

whose last member equals $e_{a,V}(Q_x)\,\|\phi\|^2-\|\nabla\phi\|^2$ for $\boldsymbol{b}\in A_a(Q_x)$ by Proposition 3.1. Thus we have

$$\inf_{\boldsymbol{b}\in A_a(Q_x)}\int_{Q_x}\{|\boldsymbol{b}(y)|^2+V(y)\}\,\mathrm{d}y\geqq e_{a,V}(Q_x)\,\|\phi\|^2-\|\nabla\phi\|^2\,.$$

Therefore we have (5.1) since $e_{a,V}(Q_x)\to\infty$ as $|x|\to\infty$ by the Main Theorem ((a) $\Leftrightarrow$(c)). □

Theorem 5.2. *Suppose that* $H_V(\boldsymbol{a})$ *has compact resolvent and that* a_j *are smooth. Then we have*

$$\int_{Q_x}(|B(y)|^2+V(y))\,\mathrm{d}y\to\infty \quad as \quad |x|\to\infty\,, \tag{5.2}$$

where $B=\mathrm{curl}\,\boldsymbol{a}$. *In particular, we have by letting* $V=0$ *that, if* $H_0(\boldsymbol{a})$ *has compact resolvent,* $\int_{Q_x}|B(y)|^2\,\mathrm{d}y\to\infty$ *as* $|x|\to\infty$.

Proof. As noted before Proposition 3.2, B satisfies (3.5). Thus, we can apply Proposition 3.2 with $p=2$ and $\Omega=Q_x$. Then, for each x, there exists a C^∞ vector potential $\boldsymbol{b}_x$ on $\boldsymbol{R}^n$ such that $\int_{Q_x}|\boldsymbol{b}_x(y)|^2\,\mathrm{d}y\leq C\int_{Q_x}|B(y)|^2\,\mathrm{d}y$ and curl $\boldsymbol{b}_x=B$, where C is independent of x since $\mathrm{diam}(Q_x)=2$. Thus we have

$$\inf_{\boldsymbol{b}\in A_a(Q_x)}\int_{Q_x}|\boldsymbol{b}(y)|^2\,\mathrm{d}y\leqq C\int_{Q_x}|B(y)|^2\,\mathrm{d}y\,.$$

Therefore, by applying Lemma 5.1, we have (5.2) since

$$\int_{Q_x}(|B(y)|^2+V(y))\,\mathrm{d}y\geqq\frac{1}{C+1}\int_{Q_x}(C|B(y)|^2+V(y))\,\mathrm{d}y$$

by (1.2). □

6. A sufficient condition for (Cpt).

In this section, we assume that $V=0$ and a_j are smooth. This section is devoted to the proof of Theorem 6.1 below, where the commutation relation

$$(6.1)\qquad B_{kj} = i[\Pi_k(\boldsymbol{a}), \Pi_j(\boldsymbol{a})] \quad \text{for} \quad k, j = 1, \cdots, n$$
$$(B_{kj} = (\text{curl } \boldsymbol{a})_{kj})$$

plays an important role as in the proof of (P1), (P4) in [2, 3].

Theorem 6.1. *Suppose that* $B=\text{curl } \boldsymbol{a}$ *satisfies* (Div) *and* (A_2). *Then* $H_0(\boldsymbol{a})$ *has compact resolvent.*

Proof. Let A_{kj} be real-valued smooth functions on an open set $\Omega\subset \boldsymbol{R}^n$ such that $A_{kj}=-A_{jk}$ for $k, j=1, \cdots, n$. Then we have by partial integration and by using (6.1)

$$(6.2)\qquad 2\sum_{k<j} \text{Im}(A_{kj}\, \Pi_k(\boldsymbol{a})\, \phi, \Pi_j(\boldsymbol{a})\, \phi)$$
$$= \frac{1}{i} \sum_{k<j} \{(A_{kj}\, \Pi_k(\boldsymbol{a})\, \phi, \Pi_j(\boldsymbol{a})\, \phi)-(A_{kj}\, \Pi_j(\boldsymbol{a})\, \phi, \Pi_k(\boldsymbol{a})\, \phi)\}$$
$$= ((\sum_{k<j} A_{kj}\, B_{kj})\, \phi, \phi)+ \sum_{k,j=1}^{n} ((\partial A_{kj}/\partial x_k)\, \Pi_j(\boldsymbol{a})\, \phi, \phi)$$

for $\phi\in C_0^\infty(\Omega)$, where Im denotes the imaginary part of a complex number. Now, by (Div), we can take a constant $R>0$ such that $|B(x)|\neq 0$ for $|x|>R$. Let $A_{kj}(x) =B_{kj}(x)/|B(x)|^2$ for $|x|>R$. Then we have

$$(6.3)\qquad \sum_{k<j} A_{kj}\, B_{kj} = 1 \quad \text{for} \quad |x|>R\,.$$

Moreover, we have by the assumption (Div) and (A_2)

$$(6.4)\qquad \sum_{k<j} (|A_{kj}(y)|+|\nabla A_{kj}(y)|)\to 0 \quad \text{as} \quad |y|\to\infty\,,$$

in view of $\nabla A_{kj}=\nabla B_{kj}/|B|^2-2B_{kj}(\sum_{i<m} B_{im}\, \nabla B_{im})/|B|^4$. Let

$$\varepsilon_x\equiv \sup_{y\in Q_x} (\sum_{k<j} (|A_{kj}(y)|+|\nabla A_{kj}(y)|))\,,$$

for $|x|>R+1$. Then we have by (6.2) and (6.3)

$$||\phi||^2\leqq C\varepsilon_x(\sum_{j=1}^{n} ||\Pi_j(\boldsymbol{a})\, \phi||^2+\sum_{j=1}^{n} ||\Pi_j(\boldsymbol{a})\, \phi||\, ||\phi||)$$
$$\leqq C'\varepsilon_x(\sum_{j=1}^{n} ||\Pi_j(\boldsymbol{a})\, \phi||^2+||\phi||^2)$$

for $\phi\in C_0^\infty(Q_x)$, where C and C' are constants independent of x. Hence we have for $\phi\in C_0^\infty(Q_x)$

$$h_{a,0}(\phi,\phi)=\sum_{j=1}^{n}\|\Pi_j(\boldsymbol{a})\,\phi\|^2\geqq(1-C'\varepsilon_x)\,\|\phi\|^2/C'\varepsilon_x\,.$$

Therefore, we have by (1.6) that $e_{a,0}(Q_x)\geqq(1-C'\varepsilon_x)/C'\varepsilon_x$, which tends to ∞ as $|x|\to\infty$ since $\varepsilon_x\to 0$ as $|x|\to\infty$ by (6.4). Consequently, we have the conclusion of the theorem by the Main Theorem ((c)⇔(a)). □

Remark. One can obtain also a proof of (P1) and (P4) by an argument similar to the above by using (6.2) with a suitable choice of A_{kj} (A_{kj}=constant for (P1) and $A_{kj}=\beta_{kj}$ for (P4)).

7. Examples.

In this section, we restrict ourselves to the case of the space dimension $n\geqq 3$ (for the case $n=2$, see Remark 1 at the end of this section).

Let g be a real-valued C^∞ function on $\boldsymbol{R}^n$ and define the vector potential $\boldsymbol{a}=\boldsymbol{a}(g)$ by

$$\begin{cases} a_1(x)=\cos g(x), & a_2(x)=\sin g(x)\,, \\ a_k(x)=0 \quad \text{for} \quad k\geqq 3\,. \end{cases} \tag{7.1}$$

Put $A(x)=a_1(x)+ia_2(x)=e^{ig(x)}$. Then, according to (1.5) and (7.1), we obtain the magnetic field $B(g)=\operatorname{curl}\boldsymbol{a}(g)$:

$$\begin{cases} B_{12}=\operatorname{Im}(\partial_1-i\partial_2)\,A=\operatorname{Im}\{i(\partial_1 g-i\partial_2 g)\,e^{ig}\}\,, \\ B_{1k}+iB_{2k}=-\partial_k A=-i(\partial_k g)\,e^{ig} \quad \text{for} \quad k\geqq 3\,, \\ B_{jk}=0 \quad \text{for} \quad j,k\geqq 3\,, \end{cases} \tag{7.2}$$

where ∂_k denotes $\partial/\partial x_k$ for $k=1,\cdots,n$. For this field we obtain

$$|B(g)|\geqq(B_{1n}^2+B_{2n}^2)^{1/2}=|\partial_n g|\,. \tag{7.3}$$

On the other hand, let

$$\Phi(x)=x_n\,\Psi(x),\quad \Psi(x)=\log(\log(|x|^2+2))\,. \tag{7.4}$$

Then we have

$$\partial_n\Phi(x)\to\infty \quad \text{as} \quad |x|\to\infty\,, \tag{7.5}$$

$$\partial^\alpha\Phi(x)\to 0 \quad \text{as} \quad |x|\to\infty \quad \text{for} \quad \alpha\neq 0, e\,, \tag{7.6}$$

where α is a multi-index$=(\alpha_1,\cdots,\alpha_n)\in\{k\,|\,k \text{ is an integer}\geqq 0\}^n$, $\partial^\alpha=\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$, $e=(0,\cdots,0,1)$. In fact, we have first that $(|x|+1)^{|\alpha|}\,\partial^\alpha\Psi\to 0$ as $|x|\to\infty$ for $\alpha\neq 0$, where $|\alpha|=\alpha_1+\cdots+\alpha_n$, since $\partial^\alpha\Psi$ is a sum of terms of the form

$C_{\beta,\gamma,\delta}^\alpha\, x_1^{\beta_1}\cdots x_n^{\beta_n}(|x|^2+2)^{-\gamma}\,(\log(|x|^2+2))^{-\delta}$
(β is a multi-index, γ, δ are integers$\geqq 1$, $|\alpha|+|\beta|=2\gamma$)

for $\alpha \neq 0$, where $C^{\alpha}_{\beta,\gamma,\delta}$ is a constant. Hence, (7.5) and (7.6) follow from $\partial^{\alpha}\Phi = x_n\partial^{\alpha}\Psi + \alpha_n\partial^{\alpha-e}\Psi$ and $\Psi(x)\to\infty$ as $|x|\to\infty$.

Example 1. Let $\boldsymbol{a}=\boldsymbol{a}(\Phi)$ given by (7.1) with $g=\Phi$ in (7.4) and let $B=B(\Phi)$. Then there exists a constant C such that

$$(7.7) \qquad |\nabla B_{jk}(x)| \leqq C|B(x)|^2 \quad \text{for} \quad x\in\boldsymbol{R}^n \quad \text{and for} \quad j, k = 1, \cdots, n\,.$$

In fact, we have by (7.2)

$$\nabla B_{12} = \mathrm{Im}\{(i(\partial_1 - i\partial_2)\nabla\Phi - (\partial_1\Phi - i\partial_2\Phi)\nabla\Phi)\, e^{i\Phi}\}\,,$$

$$\nabla B_{1k} + i\nabla B_{2k} = (-i\partial_k\nabla\Phi + (\partial_k\Phi)\nabla\Phi)\, e^{i\Phi} \quad \text{for} \quad k\geqq 3\,,$$

$$\nabla B_{jk} = 0 \quad \text{for} \quad j, k\geqq 3\,,$$

and thus we have (7.7) by (7.5) and (7.6) and by noting that

$$(7.8) \qquad |B(x)| \geqq |\partial_n\Phi(x)|$$

by (7.3). Now we have the following

Assertion 7.1. *Let $B=$*curl $\boldsymbol{a}$. *Suppose that $\delta>2$ and $n\geqq 3$. Then* (Div) *and* (A_8) *are not sufficient for $H_0(\boldsymbol{a})$ to have compact resolvent.*

Proof. Take the above example $\boldsymbol{a}=\boldsymbol{a}(\Phi)$. B satisfies (Div) by (7.8) and (7.5), and (A_8) by (7.7). On the other hand, since $|\boldsymbol{a}(x)|=1$ for all $x\in\boldsymbol{R}^n$ by (7.1), $H_0(\boldsymbol{a})$ cannot have compact resolvent as is known from Lemma 5.1. □

Example 2. Let $\boldsymbol{a}'$ be the vector potential

$$(7.8) \qquad \begin{cases} a_1'(x) = \partial_n\Phi(x)\,\cos\, x_n,\ a_2'(x) = \partial_n\Phi(x)\,\sin\, x_n\,, \\ a_k'(x) = 0 \quad \text{for} \quad k\geqq 3\,, \end{cases}$$

where Φ is given by (7.4). Then, as we have obtained (7.2), we have the magnetic field $B'=$curl $\boldsymbol{a}'$:

$$(7.9) \qquad \begin{cases} B_{12}' = \mathrm{Im}\{(\partial_1\,\partial_n\Phi - i\partial_2\,\partial_n\Phi)\, e^{iq}\}\,, \\ B_{1k}' + iB_{2k}' = -(\partial_k\,\partial_n\Phi)\, e^{iq} \quad \text{for} \quad k = 3, \cdots, n-1\,, \\ B_{1n}' + iB_{2n}' = -(\partial_n^2\Phi + i\partial_n\Phi)\, e^{iq}\,, \\ B_{jk}' = 0 \quad \text{for} \quad j, k\geqq 3\,, \end{cases}$$

where we have put $q(x)=x_n$. Then we have

$$(7.10) \qquad |B'| \geqq (B_{1n}'^2 + B_{2n}'^2)^{1/2} \geqq |\partial_n\Phi|\,.$$

Assertion 7.2. (Cpt) *for $H_0(\boldsymbol{a})$ does not imply* (Div).

Proof. First, we shall show that

$$(7.11) \qquad H_0(\boldsymbol{a}') \quad \text{has compact resolvent.}$$

B' satisfies (Div) by (7.10) and (7.5), and (A_2) by (7.10) and by the fact that

$$\nabla B'_{1n}+i\nabla B'_{2n} = (\partial_n\Phi)\, e^{iq}(0, \cdots, 0, 1)+o(1)\,,$$
$$\nabla B'_{jk} = o(1) \quad \text{otherwise},$$

which follows from (7.9) and (7.6). Thus B' satisfies the assumption of Theorem 6.1, which implies (7.11). Next, let $\boldsymbol{a}=\boldsymbol{a}(\Phi)$ in Example 1 and let $\boldsymbol{a}''=\boldsymbol{a}'+\boldsymbol{a}$, $B''=\operatorname{curl}\boldsymbol{a}''$. Then, in view of (7.11) and $|\boldsymbol{a}''(x)-\boldsymbol{a}'(x)|=|\boldsymbol{a}(x)|=1$ for all $x\in\boldsymbol{R}^n$, we have, by applying Theorem 4.1 with $\boldsymbol{a}^1=\boldsymbol{a}'$, $\boldsymbol{a}^2=\boldsymbol{a}''$,

$$H_0(\boldsymbol{a}'') \quad \text{has compact resolvent.}$$

Therefore, for obtaining the assertion, it suffices to show that B'' does not satisfy (Div). For this purpose, we shall show that there exists a sequence $\{x^l\}_{l=1}^{\infty}$ of points in $\boldsymbol{R}^n$ such that $|x^l|\to\infty$ as $l\to\infty$ and $|B''(x^l)|\to 0$ as $l\to\infty$. By (7.9), (7.2) with $g=\Phi$ and (7.6) we have

$$\text{(7.12)} \qquad \begin{cases} B''_{1n}+iB''_{2n} = -i(\partial_n\Phi)\,(e^{i\Phi}+e^{iq})+o(1)\,, \\ B''_{jk} = o(1) \quad \text{otherwise.} \end{cases}$$

Since $\Phi(x)-q(x)=x_n(\log(\log(|x|^2+2))-1)$ is a continuous function of x and $\to\infty$ as $|x|\to\infty$ with $x_n=1$, there exists a sequence $\{x^l\}_{l=1}^{\infty}$ such that $|x^l|\to\infty$ as $l\to\infty$ and $(\Phi(x^l)-q(x^l))/2-\pi/2\equiv 0 \pmod{\pi}$. Hence, since $e^{i\Phi}+e^{iq}=2e^{i(\Phi+q)/2}\cos((\Phi-q)/2)$, this sequence $\{x^l\}$ has the required property by (7.12). □

Remark 1. In $\boldsymbol{R}^2$, as we saw in the introduction, Assertion 7.1 does not hold (see (P2)). But Assertion 7.2 holds also in $\boldsymbol{R}^2$. In fact, one can argue in a manner similar to the above using Theorem 4.1 (e.g., let $\boldsymbol{a}'(x, y)=(0, x^3+3xy^2)$, $\boldsymbol{a}(x, y)=(0, \cos(x^3+3xy^2))$, and $\boldsymbol{a}''=\boldsymbol{a}'+\boldsymbol{a}$).

Remark 2. Consider the condition in $\boldsymbol{R}^n$ ($n\geqq 3$)

$$(\mathrm{Div}_\rho) \qquad |B(x)|\geqq\rho(x)\,,$$

where ρ is a real-valued C^∞ function on $\boldsymbol{R}^n$ tending to ∞ at infinity. Then, (Div$_\rho$) cannot be a sufficient condition for (Cpt) for any choice of ρ. (In fact, if we define $g(x)=\int_0^{x_n}\rho(x)\,dx_n$ and $\boldsymbol{a}(g)$ by (7.1), $\boldsymbol{a}(g)$ satisfies (Div$_\rho$) by (7.3) but $H_0(\boldsymbol{a}(g))$ does not have compact resolvent since $|\boldsymbol{a}(g)|=1$.) [3] has given a remark on this fact but not a precise formulation.

Remark 3. There is an example of a vector potential $\boldsymbol{a}$ in $\boldsymbol{R}^n$ ($n\geqq 3$) such that (Div) holds and

$$\sigma(H_0(\boldsymbol{a})) = [0, \infty)\,.$$

In fact, let $\boldsymbol{a}=f\,\boldsymbol{a}(g)$, where $f(x)=\langle x\rangle^{-\gamma}$, $g(x)=\int_0^{x_n}\langle x\rangle^{\gamma+\varepsilon}\,dx_n$ (γ, $\varepsilon>0$, $\langle x\rangle=(1+|x|^2)^{1/2}$) and $\boldsymbol{a}(g)$ is as in (7.1). Then, in the same manner as we have obtained

(7.2), we have that the corresponding magnetic field B satisfies

$$B_{1n}+iB_{2n} = -\partial_n(fe^{ig}) = -(\partial_n f+if\partial_n g)\, e^{ig}\,,$$

and thus $|B(x)| \geqq (B_{1n}(x)^2+B_{2n}(x)^2)^{1/2} \geqq |f(x)\,\partial_n g(x)| = \langle x\rangle^{\varepsilon}$. Therefore (Div) holds. On the other hand, since $|\boldsymbol{a}(x)| = \langle x\rangle^{-\gamma} \to 0$ as $|x| \to \infty$, we have $\sigma_{ess}(H_0(\boldsymbol{a})) = [0, \infty)$ according to [4, Theorem 2.5]. Hence $\sigma(H_0(\boldsymbol{a})) = [0, \infty)$ since $H_0(\boldsymbol{a}) \geqq 0$.

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