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Differential closure of differential field of positive characteristic

By

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0. Introduction.

Let I be a set of indices and K a differential field of positive characteristic p with aset of (commutative and iterative higher) derivation oparators $\Delta=\{\delta_i ; i\in I\}$. We denote an algebraic closure of K by K_a. Every derivation $\delta_i=(\delta_{i\nu} ; \nu\in N)$ $(i\in I)$, N being the set of all natural numbers including zero, has a unique extension derivation to the separably algebraic closure K_s of K in K_a which we denote also by δ_i; moreover, since these extension derivations δ_i $(i\in I)$ are commutative, K_s is uniquely regarded as a differential extension of K (see [1]). By the paper [2] of myself, we get easily the following two theorems about the extensions of the derivations.

Let x be an element of K_a and δ_i any element of Δ. We say that δ_i can be *extended to* x, if δ_i has an extension derivation to some extension of K_s that contains x. For convenience, we shall denote the e-th power of the characteristic p by $p(e)$.

Theorem A. *An element δ_i of Δ can be extended to x if and only if the condition*

$$(1) \qquad \delta_{i\nu}(x^{p(e)})=0 \qquad (0<\nu<p(e))$$

is satisfied for some element $e\in N$ with $x^{p(e)}\in K_s$. When that is so, setting $y=x^{p(e)}$, the subfield

$$K_{s,x}=K_s((\delta_{i,\nu p(e)}y)^{p(-e)} ; \nu\in N)$$

of K_a has a unique extension derivation $\delta'_i=(\delta'_{i\nu} ; \nu\in N)$ of δ_i which is defined by the fermula

$$(2) \qquad \delta'_{i\nu}z=(\delta_{i,\nu p(e)}(z^{p(e)}))^{p(-e)} \qquad (\nu\in N,\ z\in K_{s,x});$$

the equality $K_{s,x}=K_s(\delta'_{i\nu}x ; \nu\in N)$ holds true, and $K_{s,x}$ is the smallest extension of K_s containing x that has an extension derivation of δ_i.

Remark. We see by [1] that the condition (1) is equivalent to the condition

$$\delta_{i\nu}(x^{p(e)})=0 \quad (\nu\in N-\{0\} \ \text{ with } p(e)\nmid\nu).$$

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Theorem B. *The set M of all those elements $x \in K_a$ such that every $\delta_i \in \Delta$ can be extended to x is an extension of K_s which has a unique extension derivation for every $\delta_i \in \Delta$. Associated with the extension derivations of δ_i $(i \in I)$, M is the largest differential extension of K (and of K_s) in K_a.*

The largest differential extension of K in K_a, determined by Th. B, is called *differential closure* of K in K_a and denoted by K_Δ. Applying this notion, the universal differential extension of differential fields of positive characteristic was established (see [2]): a differential extenison U of K is called *universal differential extension* of K, if $U=U_\Delta$ and, for every finitely generated differential extension L of K in U, every $n \in N-\{0\}$ and every prime differential ideal $\mathfrak{p}$ of the differential polynomial algebra $L\{X_1, \cdots, X_n\}$ having a generic zero (z) over L such that $L\langle z\rangle$ is separable over L, there exists a generic zero (x) over L of $\mathfrak{p}$ with $x_1, \cdots, x_n \in U$.

Professor Kôtaro Okugawa and the author reported at the meeting of Mathematical Society of Japan in the autumn of 1981 that if the field of constants C of K is algebraically closed, Galois theory of *Picard-Vessiot extensions* of K and, more generally, of *strongly normal extensions* of K can be developed as a whole. Recently, the Galois theory was established also when C is not necessarily algebraically closed (see [3]). Throughout these works, it became certain that the differential closure plays an important role in the theory of differential fields. The purpose of the present paper is to show some basic properties, newly obtained and applied, of the differential closure.

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1. The differential closure K_Δ of K in K_a.

Throughout this paper, U denotes a fixed universal differential extension of K, Every differential field considered is supposed to be a differential subfield of U. Let C, K_a, K_s and K_Δ be as above in the preceding section.

Thoerem 1. *The field of constants of K_s is the separably algebraic closure of C in K_a and the field of constants of K_Δ is the algebraic closure of C in K_a.*

The proof is easy.

Theorem 2. *Let M be an algebraic differential extension of K and σ a differential isomorphism of K into U. Then, any field-isomorphism σ' into U of M that extends σ is a differential isomorphism of M into $(\sigma K)_\Delta$ in U.*

Proof. We may suppose that $M \subset K_a$. Let x be an element of M.

(I) Suppose that x is separably algebraic over K. Let

$$F(X)=a_nX^n+\cdots+a_1X+a_0 \qquad (a_k \in K,\ 0 \leqq k \leqq n\ ;\ a_n=1)$$

be the minimal polynomial of x over K; then, for each $i \in I$, $\delta_{i\nu}x$ $(\nu \in N)$ is defined inductively by the formula

$$(3)\qquad 0=(\mathrm{d}F/\mathrm{d}X)(x)\cdot\delta_{i\nu}x+\sum_{k=0}^{n}\sum{}'\delta_{i\lambda(0)}a_k\delta_{i\lambda(1)}x\cdots\delta_{i\lambda(k)}x$$

where the summation $\sum'$ ranges all over the $(k+1)$-tuples $(\lambda(0),\cdots,\lambda(k))\in N^{k+0}$ with $\lambda(0)+\cdots+\lambda(k)=\nu$, $\lambda(1)\neq\nu,\cdots,\lambda(k)\neq\nu$. Since $0=\sigma'F(x)=F^{\sigma}(\sigma'x)$ where

$$F^{\sigma}(X)=(\sigma a_n)X^n+\cdots+(\sigma a_1)X+\sigma a_0\,,$$

we can get inductively $\delta_{i\nu}(\sigma'x)$ $(\nu\in N)$ by the formula

$$(4)\qquad 0=(\mathrm{d}F^{\sigma}/\mathrm{d}X)(\sigma'x)\cdot\delta_{i\nu}(\sigma'x)+\sum_k\sum{}'\delta_{i\lambda(0)}(\sigma a_k)\delta_{i\lambda(1)}(\sigma'x)\cdots\delta_{i\lambda(k)}(\sigma'x).$$

By induction assumption, for each $\lambda<\nu$,

$$\delta_{i\lambda}\sigma'x=\sigma'\delta_{i\lambda}x$$

and σ is a differential one; therefore, applying σ' to (3), we get

$$(5)\qquad 0=(\mathrm{d}F^{\sigma}/\mathrm{d}X)(\sigma'x)\cdot\sigma'(\delta_{i\nu}x)+\sum_k\sum{}'\delta_{i\lambda(0)}(\sigma a_k)\delta_{i\lambda(1)}(\sigma'x)\cdots\delta_{i\lambda(k)}(\sigma'x).$$

Formulas (4) and (5) imply that

$$\sigma'\delta_{i\nu}x=\delta_{i\nu}\sigma'x$$

Hence, σ' is a differential isomorphism of $K\langle x\rangle$ into $(\sigma K)_{\Delta}\subset U$.

(II) On the contrary, suppose that x is inseparably algebraic over K. By (I), we may assume that K is separably algebraically closed in M; then, x is purely inseparably algebraic over K. Since x is in K_{Δ}, by Th. A,

$$\delta_{i\nu}x^{p(e)}=0\qquad(i\in I,\ 0<\nu<p(e))$$

where $p(e)$ is the degree of x over K. Then, we have

$$\delta_{i\nu}(\sigma'x)^{p(e)}=\delta_{i\nu}\sigma(x^{p(e)})=\sigma\delta_{i\nu}x^{p(e)}=0\qquad(i\in I,\ 0<\nu<p(e)),$$

and $\sigma'x$ is in $(\sigma K)_{\Delta}$. By the definition,

$$\begin{aligned}\delta_{i\nu}\sigma'x&=(\delta_{i,\nu p(e)}(\sigma'x)^{p(e)})^{p(-e)}\\&=(\delta_{i,\nu p(e)}\sigma(x^{p(e)}))^{p(-e)}=(\sigma\delta_{i,\nu p(e)}(x^{p(e)}))^{p(-e)}\\&=(\sigma((\delta_{i\nu}x)^{p(e)}))^{p(-e)}=((\sigma'\delta_{i\nu}x)^{p(e)})^{p(-e)}\\&=\sigma'\delta_{i\nu}x\qquad(i\in I,\ \nu\in N).\end{aligned}$$

Therefore, σ' is a differential isomorphism of K into $(\sigma K)_{\Delta}$ in U. q. e. d.

The following theorem corresponds to the fact that an extension of K_a is always regular over K_a.

Theorem 3. *If L is a differential extension of K_{Δ}, then L is regular over K_{Δ}.*

Proof. K_{Δ} is algebraically closed in L by Th. B. We claim that L^p and K_{Δ} are linearly disjoint over K_{Δ}^p; let elements $x_1^p,\cdots x_n^p$ of L^p (with $x_1,\cdots,x_n$

$\in L$) be linearly dependent over K_Δ then we must show that they are linearly dependent over K_Δ^p, and in doing this we may suppose that $n>1$ and that no $n-1$ of them are linearly dependent over K_Δ. Then, there exist nonzero elements $y_1, \cdots, y_{n-1}$ of K_Δ with

$$(6) \qquad x_n^p = y_1 x_1^p + \cdots + y_{n-1} x_{n-1}^p .$$

There exist nonzero elements $z_1, \cdots, z_{n-1}$ of K_a such that $z_k^p = y_k$ $(1 \leqq k \leqq n-1)$. Assume that z_k's are not all in K_Δ. Changing the order if necessary, we may assume that z_1 is not in K_Δ. Th. A implies that $\delta_{i\nu} y_1 \neq 0$ for some $i \in I$ and some positive integer ν with $1<\nu<p$. Applying $\delta_{i\nu}$ to (6), we get

$$0 = (\delta_{i\nu} y_1) x_1^p + \cdots + (\delta_{i\nu} y_{n-1}) x_{n-1}^p .$$

All $\delta_{i\nu} y_k$'s are in K_Δ and $\delta_{i\nu} y_1 \neq 0$, therefore $x_1^p, \cdots, x_{n-1}^p$ are linearly dependent over K_Δ. This contradicts the above. Hence, z_k's are in K_Δ and then y_k's are in K_Δ^p $(1 \leqq k \leqq n-1)$. q. e. d.

2. The purely inseparable closure K_∞ of K in K_Δ.

We denote the purely inseparable closure of K in K_a by K_i and the purely inseparable closure of K in K_Δ by K_∞, hence $K_\infty = K_i \cap K_\Delta$. If $K_i = K_\infty$, then $K_a = K_\Delta$ and every element of K is constant, because every derivation of a perfect field of positive characteristic is trivial. Therefore, if K has a non-constant, $K_i \neq K_\infty$ and $K_a \neq K_\Delta$.

K_a is separably algebraic over K_i. Correspondingly, we get the following proposition in the differential case.

Proposition. *K_∞ is a differential extension of K and K_Δ is separably algebraic over K_∞.*

Proof. Let x be an element of K_∞, then there is a positive integer e such that $x^{p(e)}$ is in K. Since x is in K_Δ, by Th. A, $\delta_{i\nu} x$ is defined by $(\delta_{i,\nu p(e)}(x^{p(e)}))^{p(-e)}$ and contained in $K_\Delta \cap K_i = K_\infty$ $(i \in I, \nu \in N)$, therefore K_∞ is a differential extension of K.

We show that K_Δ is separably algebraic over K_∞. Let x be an element of K_Δ and

$$F(X) = (X^{p(e)})^n + a_{n-1}(X^{p(e)})^{n-1} + \cdots + a_0$$

the minimal polynomial of x over K_∞ where $p(e)$ is the inseparable factor of the degree. For each k $(0 \leqq k \leqq n-1)$, the $p(e)$-th root b_k of a_k is in K_i, but it may be not in K_Δ. Assume that b_r is not in K_Δ for some r. By Th. A, for some $j \in I$, there exists a positive integer ν with $p(e) \nmid \nu$ such that $\delta_{j\nu} a_r \neq 0$; therefore, $\lambda = \min\{\nu ; \delta_{j\nu} a_k \neq 0$, for some k with $p(e) \nmid \nu$, $0 \leqq k \leqq n-1\}$ is not zero. Applying $\delta_{j\lambda}$ to $F(x)$, we get

$$0 = \delta_{j\lambda} a_{n-1} (x^{n-1})^{p(e)} + \cdots + \delta_{j\lambda} a_0 .$$

The coefficients are not all zero and the degree of this equation on x is less than $np(e)$. These contradict the minimality of $F(X)$. Therefore, all b_k's are in $K_\Delta \cap K_i = K_\infty$ and the polynomial

$$G(X) = X^n + b_{n-1}X^{n-1} + \cdots + b_0$$

in X over K_∞ vanishes at x. The minimality of $F(X)$ implies that $n = \deg G \geqq \deg F = np(e)$, and $p(e) = 1$, therefore, x is separably algebraic over K.

q. e. d.

Let L be an extension of K, then, there is an extension M of K such that the compositum LM is separable over M and that M is purely inseparably algebraic over K, for example, $M = K_i$. Th. 4 states that if L is differential one, then K_∞ is a differential extension of K which has above properties.

Theorem 4. *Let L be a differential extension of K. Then, the compositium LK_∞ is a differential extension of K_∞ which is separable over K_∞.*

Proof. By Prop., LK_∞ is a differential extension of K_∞. Let $H = K_\Delta \cap LK_\infty$, then K_Δ is separably algebraic over H because K_Δ is separably algebraic over K_∞ by Prop.. As H is algebraically closed in LK_∞, LK_∞ and K_Δ are linearly disjoint over H (see [4]). On the other hand, LK_Δ is regular over K_Δ by Th. 3, and LK_∞ is regular over H (see [4]). Since H is contained in K_Δ, H is separably algebraic over K_∞. Hence, LK_∞ is separable over K_∞. q. e. d.

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References

[1] K. Okugawa, Basic properties of differential fields of an arbitrary characteristic and Picard-Vessiot theory, Journ. Math. Kyoto Univ., **2** (1963), 295-322.

[2] K. Shikishima, Maximal differential field and its applications, Acta Hum. Sci., Univ. Sangio Kyotiensis, Vol. IX Nar. Sci. Ser. No. 9 (1980), 1-10.

[3] K. Tsuji, Galois theory of differential fields of positive characteristic, in preparation.

[4] A. Weil, Foundations of algebraic geometry, Revised and enlarged ed., Amer. Math. Soc. Col. Publ., Vol. XXIX, 1962.