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# Theory of $Y$-extremal and minimal hypersurfaces in a Finsler space

On Wegener's and Barthel's theories

By

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The theory of hypersurfaces in a Finsler space has been first considered by E. Cartan [2] from two points of view. One is to regard a hypersurface as the whole of tangent line-elements and then it is also a Finsler space [9]. The other is to regard it as the whole of normal line-elements and then it is a Riemannian space. J. M. Wegener ([11], [12]) has treated hypersurfaces from the latter viewpoint and dealt in particular with minimal hypersurfaces. E. T. Davies [3] has considered subspaces from the former viewpoint mainly, but referred a little to minimal subspaces. Both of them have pointed out a weak point of their theories that the minimal subspaces are characterized only by the vanishing of the mean curvature provided Cartan's torsion vector vanishes. To overcome this weak point, W. Barthel [1] has proposed a new Finsler connection with surviving torsion tensor (Postulate 5) and obtained a satisfactory result for the time being. B. Su [10] has further developed the theory of minimal subspaces based on Barthel's standpoint.

There is, however, a strange circumstances; Barthel's characteristic equation of minimal hypersurface does not coincide with Wegener's even if Cartan's connection is treated. Moreover the present author is dissatisfied with Barthel's Postulate 6 "The connection is uniquely determined." from the standpoint of the theory of Finsler connections which has been recently developed.

The purpose of the present paper is to give Wegener's and Barthel's theories respective precise formulations. It is indicated here that the symmetry property of Finsler metric really concerns Barthel's theory. It is the most noteworthy result from the viewpoint of recent theory of Finsler connections that every formulation gives rise to the most suitable connection which is different from the well-known connections; in § 4 is the Cartan $Y$-connection defined from the fundamental function and a non-zero vector field, and the Cartan $C$-connection is determined in § 8 from the fundamental function alone.

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## Chapter I. Y-extremal hypersurface

### § 1. Linear Y-connection.

Let $F^n=(M^n, L, F\Gamma)$ be a Finsler space on an $n$-dimensional underlying manifold $M^n$, equipped with a fundamental function $L=L(x, y)$ and a Finsler connection $F\Gamma=(\Gamma, N)$ ([7], [8]), where $\Gamma$ is a connection in the Finsler bundle $\pi_1: F(M)\to T_0(M)$ and $N$ is a nonlinear connection in the bundle $\pi_T: T_0(M)\to M$. $T_0(M)$ is a subbundle of the tangent bundle $\pi_T$: $T(M)\to M$ consisting of nonzero tangent vectors and $\pi_1$: $F(M)\to T_0(M)$ is a principal bundle induced from the linear frame bundle $\pi_L$: $L(M)\to M$ by the projection $\pi_T$. Let $\omega$ be the connection form associated to $\Gamma$ and let $\theta^v$ be the $v$-basic form associated to $N$. In the canonical coordinate system $(x^i, y^i, z^i_a)$ of $F(M)$ induced from a coordinate system $(x^i)$ of the base manifold $M^n$, respective components $\omega^a_b$ and $(\theta^v)^a$ of $\omega$ and $\theta^v$ are written

$$\omega^a_b=(z^{-1})^a_i\{dz^i_b+z^j_b(\Gamma^i_{jk}dx^k+C^i_{jk}dy^k)\},$$

$$(\theta^v)^a=(z^{-1})^a_i(dy^i+N^i_jdx^j),$$

where $\Gamma^i_{jk}(x, y)$ and $N^i_j(x, y)$ are functions of $2n$ variables $x^i$ and $y^i$ and $C^i_{jk}(x, y)$ are components of a Finsler tensor field.

Throughout the present paper we shall be concerned with a special class of Finsler connections as follows:

**Definition.** A Finsler connection is called a *generalized Cartan connection* and denoted by $C\Gamma(T)$, if the following four conditions are satisfied:

(C1) $h$-metrical, (C2) deflection tensor$=0$,
(C3) $v$-metrical, (C4) $(v)v$-torsion tensor$=0$.

The fundamental (metric) tensor is defined by $g_{ij}(x, y)=\dot\partial_i\dot\partial_jL^2(x, y)/2$. Then the conditions (C1) and (C3) are respectively written as

$$\text{(C1)} \qquad g_{ij|k}=\delta_kg_{ij}-F_{ijk}-F_{jik}=0,$$

$$\text{(C3)} \qquad g_{ij}|_k=\dot\partial_kg_{ij}-C_{ijk}-C_{jik}=0,$$

where $F_{ijk}=g_{jr}F^r_{ik}$, $C_{ijk}=g_{jr}C^r_{ik}$, $F^i_{jk}=\Gamma^i_{jk}-C^i_{jr}N^r_k$ and $\delta_k=\partial_k-(\dot\partial_r)N^r_k$. (C2) is written $y^rF^i_{rj}=N^i_j$ and (C4) is $C^i_{jk}=C^i_{kj}$. Thus (C3) and (C4) lead to $C_{ijk}=\dot\partial_kg_{ij}/2$ (Cartan's $C$-tensor). It is well-known ([4], [5]) that if the $(h)h$-torsion tensor $T^i_{jk}=F^i_{jk}-F^i_{kj}$ is given for $C\Gamma(T)$ as a known tensor, we obtain such a $C\Gamma(T)$ uniquely. In fact, if $T^i_{jk}=0$ specially, we get the Cartan connection $C\Gamma$. In general, putting

$$\text{(1.1)} \qquad 2A_{ijk}=T_{ijk}-T_{jki}+T_{kij}, \qquad (T_{ijk}=g_{jr}T^r_{ik}),$$

(C1) gives immediately

$$\text{(1.2)} \qquad F_{ijk}=\gamma_{ijk}-C_{ijr}N^r_k-C_{jkr}N^r_i+C_{kir}N^r_j+A_{ijk},$$

where $\gamma_{ijk}$ are Christoffel symbols $\gamma_{ijk}=(\partial_k g_{ij}+\partial_i g_{jk}-\partial_j g_{ki})/2$. Then (1.2) yields

$$F_{0jk}=\gamma_{0jk}-C_{jkr}N^r_0+A_{0jk}, \qquad F_{0j0}=\gamma_{0j0}+A_{0j0}. \tag{1.3}$$

Thus (C2) and (1.3) lead to

$$F^j_{0k}=N^j_k=\gamma^j_{0k}-C^j_{kr}(\gamma^r_{00}+A^r_{00})+g^{jr}A_{0rk}, \tag{1.4}$$

and (1.2) determines $F_{ijk}$.

For instance, the so-called Wagner connection $W\Gamma$ has played an important role in Hashiguchi-Ichijyō's theories [5] of generalized Berwald spaces and conformally Minkowski spaces. The $(h)h$-torsion tensor of $W\Gamma$ is given by a semi-symmetric type $T^i_{jk}=\delta^i_j s_k-\delta^i_k s_j$, where $s=(s_j(x))$ is a known covariant vector field. Therefore $W\Gamma$ may be called a connection of the $(L, s)$-structure, not of the (Finslerian) $L$-structure.

We shall propose a generalized Cartan connection, denoted by $CY\Gamma$, in §4; it may be called of the $(L, Y)$-structure. On the other hand, the $T$-tensor of Barthel's connection [1] is given by

$$T^i_{jk}=L(l_j C^i_{|k}-l_k C^i_{|j}), \tag{1.5}$$

where $C^i_{|k}$ is the $h$-covariant derivative of contracted $C$-tensor $C^i_{jk}g^{jk}$ with respect to Barthel's connection. Thus this connection is certainly of the $L$-structure, although the above $T^i_{jk}$ is given in form but not known. We shall propose such a Finsler connection $C\Gamma(T_c)$ of the $L$-structure in §8.

Now, throughout Chapter I we shall restrict our consideration to a domain $D$ of the underlying manifold $M^n$ where a non-zero tangent vector field $Y=(Y^i(x))$ is given. Then we have a mapping $\eta:\pi_L^{-1}(D)\to F(M)$ such that $\eta(x^i, z^i_a)=(x^i, y^i=Y^i(x), z^i_a)$ and the dual mapping $\eta^*$ of the differential $\eta'$ of $\eta$. Put $\bar\omega=\eta^*\omega$ and $\bar\theta^v=\eta^*\theta^v$. It is easily verified that $\bar\omega$ is a connection form and $\bar\theta^v$ is a tensorial form on $\pi_L^{-1}(D)\subset L(M)$, and we get a linear connection $\Gamma(Y)$ associated to $\bar\omega$ and a tensor field $Y_2$ of (1, 1)-type on $D$ the components of which are coefficients of $\bar\theta^v$. It is easily seen that the connection coefficients $\bar\Gamma^i_{jk}(x)$ of $\Gamma(Y)$ are

$$\begin{aligned}\bar\Gamma^i_{jk}(x)&=\Gamma^i_{jk}(x, Y)+C^i_{jr}(x, Y)\partial_k Y^r\\ &=F^i_{jk}(x, Y)+C^i_{jr}(x, Y)Y^r_k(x),\end{aligned} \tag{1.6}$$

where

$$Y^i_k(x)=\partial_k Y^i+N^i_k(x, Y) \tag{1.7}$$

are components of the tensor $Y_2$. Since the components $Y^i_{|k}$ of $h$-covariant derivative $\nabla^h Y$ of $Y$ are given by $\partial_k Y^i+Y^r F^i_{rk}$, the condition (C2) leads to

$$Y^i_k(x)=Y^i_{|k}(x, Y). \tag{1.7'}$$

**Definition.** The linear connection $\Gamma(Y)$ as above obtained from a generalized Cartan connection $C\Gamma(T)$ and a non-zero tangent vector field $Y(x)$ is called the *linear Y-connection* associated to $C\Gamma(T)$ by $Y$.

It follows from (1.6) that the torsion tensor $\bar{T}^i_{jk}(x)$ of $\Gamma(Y)$ is

$$\bar{T}^i_{jk}(x)=T^i_{jk}(x, Y)+C^i_{jr}(x, Y)Y^r_k-C^i_{kr}(x, Y)Y^r_j. \tag{1.8}$$

Next we introduce the Riemannian metric

$$\bar{g}_{ij}(x)=g_{ij}(x, Y) \tag{1.9}$$

on $D$, which will be called the *Riemannian Y-metric*. In general we get an ordinary tensor field $\bar{K}^i_j(x)=K^i_j(x, Y)$ from a Finsler tensor field $K^i_j(x, y)$. Then it is easily verified that the covariant derivative $\bar{K}^i_{j;k}$ of $\bar{K}^i_j$ with respect to $\Gamma(Y)$ is written

$$\bar{K}^i_{j;k}=K^i_{j|k}(x, Y)+K^i_j|_r(x, Y)Y^r_k, \tag{1.10}$$

where $K^i_{j|k}$ and $K^i_j|_k$ are components of $h$- and $v$-covariant derivatives $\nabla^h K$, $\nabla^v K$ of $K$ respectively.

**Proposition 1.1.** *The linear Y-connection $\Gamma(Y)$ associated to a generalized Cartan connection $C\Gamma(T)$ by $Y$ is metrical with respect to the Riemannian Y-metric and its torsion tensor is given by* (1.8).

## § 2. Transversal hypersurface.

If there exists a family of hypersurfaces $M^{n-1}(c)$: $x^i=x^i(u^1, \cdots, u^{n-1}; c)$ with a parameter $c$ which are transversal to integral curves of the non-zero tangent vector field $Y$ on the domain $D$, then $Y$ will be called a *transversal vector field*. The transversality is that $Y$ is the normal vector field of $M^{n-1}$ with respect to itself, that is,

$$g_{ij}(x, Y)Y^iB^j_\alpha=0, \qquad \alpha=1, \cdots, n-1, \tag{2.1}$$

where $B_\alpha=(B^j_\alpha=\partial x^j/\partial u^\alpha)$ are $n-1$ independent tangent vectors of $M^{n-1}(c)$. If $M^{n-1}(c)$ is expressed as a set of zero-points of a function $S(x^1, \cdots, x^n; c)$, we have $(\partial S/\partial x^j)B^j_\alpha=0$, and (2.1) is equivalent to the equations

$$\partial S/\partial x^j=e^{p(x)}g_{ij}(x, Y)Y^i, \tag{2.2}$$

where $p(x)$ is a function. The condition for $Y$ to be transversal is clearly $\partial^2S/\partial x^i\partial x^j=\partial^2S/\partial x^j\partial x^i$, which is written

$$\partial_jY_i-\partial_iY_j+Y_i\partial_jp-Y_j\partial_ip=0, \qquad (Y_i=g_{ij}(x, Y)Y^j). \tag{2.3}$$

To write (2.3) in other form, we shall refer to the linear $Y$-connection $\Gamma(Y)$ as introduced in the last section. From Proposition 1.1 we observe $Y_{i;j}=\bar{g}_{ir}Y^r_{;j}$. Because of $Y^i|_j(x, Y)=Y^rC^i_{rj}(x, Y)=0$, (1.10) and (1.7′) yield $Y_{i;j}=\bar{g}_{ir}Y^r_j$. Further from (1.8) we have $Y_r\bar{T}^r_{ij}=Y_rT^r_{ij}(x, Y)$. Consequently (2.3) is rewritten in the form

$$\{\bar{g}_{ir}(Y^r_j+Y^r\partial_j p)-i/j\}+Y_r T^r_{ij}(x, Y)=0, \tag{2.4}$$

where $i/j$ denotes the term obtained from the previous terms by interchange of $i$ and $j$ [6].

We may be concerned with the *normalized* $Y$, that is, $L(x, Y)=1$. In the notation of (1.10) we have $\bar{L}(x)=1$ and $\bar{L}_{;i}=Y_r Y^r_i$, so that we get

$$Y_r Y^r_i=0. \tag{2.5}$$

In this case, contracting (2.4) by $Y^j$, we get

$$\bar{g}_{ir}Y^r_j Y^j+Y_i Y^j\partial_j p+T_{irj}(x, Y)Y^r Y^j=\partial_i p,$$

and (2.4) is again rewritten as

$$\{\bar{g}_{ir}(Y^r_j-Y^r_s Y^s Y_j)+Y_i T_{jrs}(x, Y)Y^r Y^s-i/j\}+Y_r T^r_{ij}(x, Y)=0. \tag{2.6}$$

Suppose that the normalized $Y$ is *autoparallel*, that is, each integral curve of $Y$ satisfies the differential equations

$$d^2x^i/ds^2+N^i_j(x, dx/ds)dx^j/ds=0.$$

Then $Y^i=dx^i/ds$ satisfies

$$Y^i_r Y^r=0, \tag{2.7}$$

and (2.6) reduces to

$$\{\bar{g}_{ir}Y^r_j+Y_i T_{jrs}(x, Y)Y^r Y^s-i/j\}+Y_r T^r_{ij}(x, Y)=0. \tag{2.8}$$

**Proposition 2.1.** *If the normalized vector field $Y$ is transversal, it satisfies* (2.5) *and* (2.6). *Further, if $Y$ is autoparallel, it satisfies* (2.5), (2.7) *and* (2.8).

We are concerned with the Cartan connection $C\Gamma(T^i_{jk}=0)$. Then (2.6) reduces to

$$\bar{g}_{ir}(Y^r_j-Y^r_s Y^s Y_j)=\bar{g}_{jr}(Y^r_i-Y^r_s Y^s Y_i). \tag{2.6$'$}$$

In this case, if $Y$ is autoparallel, each integral curve of $Y$ is a geodesic curve and $Y$ may be called *geodesic*. Then (2.8) is simplified as

$$\bar{g}_{ir}Y^r_j=\bar{g}_{jr}Y^r_i. \tag{2.8$'$}$$

Now we consider the geometry of a transversal hypersurface $M^{n-1}$: $x^i=x^i(u)$. The vector field $Y$, normalized by the Finsler metric $L$, i.e.,

$$g_{ij}(x, Y)Y^i Y^j=1, \tag{2.9}$$

is the unit vector field orthogonal to $M^{n-1}$ in the sense of (2.1). Thus we get a field of frame $(B^i_\alpha, Y^i)$ along $M^{n-1}$. Further from (1.9) we get the *induced Riemannian Y-metric* on $M^{n-1}$:

$$g_{\alpha\beta}(u)=g_{ij}(x, Y)B^i_\alpha B^j_\beta, \tag{2.10}$$

and have a linear connection $\underline{\Gamma}(Y)$ induced from the linear $Y$-connection $Y(\Gamma)$ by the well-known way. That is, denoting by $(B^\alpha_i, Y_i)$ the dual coframe of $(B^i_\alpha, Y^i)$, the connection coefficients $\Gamma^\alpha_{\beta\gamma}(u)$ of $\underline{\Gamma}(Y)$ are

$$\Gamma^\alpha_{\beta\gamma}(u)=B^\alpha_i(B^i_{\beta\gamma}+B^j_\beta\bar{\Gamma}^i_{jk}B^k_\gamma), \qquad (B^i_{\beta\gamma}=\partial B^i_\beta/\partial u^\gamma).$$

Therefore we get the so-called Gauss equation

$$B^i_{\alpha;\beta}=H_{\alpha\beta}Y^i, \tag{2.11}$$

where ; denotes the relative covariant differentiation along $M^{n-1}$, i.e.,

$$B^i_{\alpha;\beta}=B^i_{\alpha\beta}+B^j_\alpha\bar{\Gamma}^i_{jk}B^k_\beta-B^i_\gamma\Gamma^\gamma_{\alpha\beta}, \tag{2.12}$$

and $H_{\alpha\beta}$ is the second fundamental tensor of $M^{n-1}$. Further we obtain the so-called Weingarten equation

$$Y^i_{;\beta}=-H^\alpha_\beta B^i_\alpha, \qquad (H^\alpha_\beta=g^{\alpha\gamma}H_{\gamma\beta}). \tag{2.13}$$

It is obvious from Proposition 1.1 that $\underline{\Gamma}(Y)$ is also metrical with respect to the induced Riemannian $Y$-metric. From (2.11) and (2.12) the torsion tensor $T^\alpha_{\beta\gamma}$ of $\underline{\Gamma}(Y)$ and $H_{\alpha\beta}-H_{\beta\alpha}$ are given by

$$T^\alpha_{\beta\gamma}=B^\alpha_i\bar{T}^i_{jk}B^j_\beta B^k_\gamma, \tag{2.14}$$

$$H_{\alpha\beta}-H_{\beta\alpha}=Y_i\bar{T}^i_{jk}B^j_\alpha B^k_\beta. \tag{2.15}$$

We consider the $Y_2$-tensor field $Y^i_j$ along $M^{n-1}$. The relative covariant derivative $Y^i_{;\beta}$ of $Y^i$ in (2.13) is defined as $Y^i_{;\beta}=\partial Y^i/\partial u^\beta+Y^j\bar{\Gamma}^i_{jk}B^k_\beta=Y^i_kB^k_\beta$. Therefore, if we write $Y^i_j$ with respect to the frame $(B^i_\alpha, Y^i)$, we have $Y^i_j=-H^\alpha_\beta B^i_\alpha B^\beta_j+(Y^\alpha B^i_\alpha+YY^i)Y_j$ for some functions $Y^\alpha$ and $Y$. Further (2.5) gives $Y=0$. Consequently we have

$$Y^i_j=B^i_\alpha(-H^\alpha_\beta B^\beta_j+Y^\alpha Y_j). \tag{2.16}$$

It is remarked that (2.6) is satisfied as a consequence of (2.15) and $Y_i\bar{T}^i_{jk}=Y_iT^i_{jk}(x, Y)$.

In particular we shall deal with the Cartan connection $C\Gamma(T=0)$. Then (1.8) reduces to $\bar{T}^i_{jk}=C^i_{jr}(x, Y)Y^r_k-C^i_{kr}(x, Y)Y^r_j$, and (2.15) gives $H_{\alpha\beta}=H_{\beta\alpha}$ because of $Y_iC^i_{jr}(x, Y)=0$. On the other hand, (2.14) gives

$$T_{\alpha\beta\gamma}=H_{\alpha\rho}C^\rho{}_{\beta\gamma}-H_{\gamma\rho}C^\rho{}_{\beta\alpha}, \tag{2.17}$$

where $C^\alpha_{\beta\gamma}=B^\alpha_iC^i_{jk}(x, Y)B^j_\beta B^k_\gamma$. It is noted that $C^i_{jk}(x, Y)$ has no components in the normal direction $Y$.

**Proposition 2.2.** *In case of the Cartan connection $C\Gamma$ the induced linear $Y$-connection $\underline{\Gamma}(Y)$ is metrical with respect to the induced Riemannian $Y$-metric and the torsion tensor of $\underline{\Gamma}(Y)$ is given by* (2.17). *The second fundamental tensor is symmetric.*

Compare Proposition 2.2 with Theorem 4.2 in case of the Cartan $Y$-connection $CY\Gamma$.

## § 3. $Y$-extremal hypersurface.

The transversal hypersurfaces $M^{n-1}(c)$ as considered in the last section has the induced Riemannian $Y$-metric (2.10), which yields the volume element $\sqrt{g}$, $g=\det(g_{\alpha\beta})$. We observe that this $g$ is directly written as a function of $x^i$ and $B_\alpha^i$.

In general we shall be concerned with an integral

$$I=\int_{\underline{D}} f(u)du^1 \cdots du^{n-1}$$

of a function $f(u)$ on a domain $\underline{D}$ of $M^{n-1}(c)$, where $f$ is assumed to be directly written as a function of $x^i$ and $B_\alpha^i$. To find the variation of $I$, we consider the integral $I'$ of $f(u)$ on such a domain of $M^{n-1}(c')$ near to $M^{n-1}(c)$ that corresponds to $\underline{D}$ by an equation $\bar{x}^i=x^i(u)+\varepsilon Y^i(u)$ for an infinitesimal $\varepsilon$. Then we get the first variation

$$I'(0)=\int_{\underline{D}}\{(\partial f/\partial x^i)Y^i+(\partial f/\partial B_\alpha^i)Y_\alpha^i\}du^1 \cdots du^{n-1},$$

where $Y_\alpha^i=\partial Y^i/\partial u^\alpha$. Stokes' theorem leads to

$$I'(0)=\int_{\delta\underline{D}}(-1)^{\alpha-1}(\partial f/\partial B_\alpha^i)Y^i du^1 \cdots d\hat{u}^\alpha \cdots du^{n-1}$$

$$+\int_{\underline{D}}\{\partial f/\partial x^i-\partial(\partial f/\partial B_\alpha^i)/\partial u^\alpha\}Y^i du^1 \cdots du^{n-1},$$

where $\delta\underline{D}$ is the boundary of $\underline{D}$ and $\hat{}$ of $d\hat{u}^\alpha$ shows omission of $du^\alpha$.

We now deal with $f=\sqrt{g}$. Then (2.10) gives

$$\partial g/\partial B_\alpha^i=(\partial g_{\beta\gamma}/\partial B_\alpha^i)g g^{\beta\gamma}=g_{jk}(x, Y)\{\partial(B_\beta^j B_\gamma^k)/\partial B_\alpha^i\}g g^{\beta\gamma}=2gB_i^\alpha,$$

so that we have

$$\partial\sqrt{g}/\partial B_\alpha^i=\sqrt{g}B_i^\alpha. \tag{3.1}$$

Consequently the first integral over $\delta\underline{D}$ vanishes, although the variation vector $Y^i$ does not vanish at the boundary. Therefore $I$ has a stationary value, if

$$\{\partial\sqrt{g}/\partial x^i-\partial(\partial\sqrt{g}/\partial B_\alpha^i)/\partial u^\alpha\}Y^i=0. \tag{3.2}$$

**Definition.** A transversal hypersurface $M^{n-1}(c)$ is called a *Y-extremal hypersurface*, if it satisfies the equation (3.2).

**Remark.** (1) Wegener's way to arrive at the following (3.6) did not start from the equation (3.2). (2) If $Y^i$ may be dropped in (3.2), it becomes a generalized Euler-Lagrange differential equation in the theory of variations, but $Y^i$ in (3.2) can not be dropped, because $g_{\alpha\beta}$ is induced from the Riemannian $Y$-metric and $g$ depends on $Y$ certainly.

We shall rewrite the equation (3.2). From (2.10) and the metrical property of $\Gamma(Y)$ we observe

$$\begin{aligned}\partial\sqrt{g}/\partial x^i &= (\sqrt{g}/2)(\partial g_{\alpha\beta}/\partial x^i)g^{\alpha\beta} \\ &= (\sqrt{g}/2)(\partial \bar{g}_{jk}/\partial x^i)B^j_\alpha B^k_\beta g^{\alpha\beta} = \sqrt{g}\,\bar{\Gamma}_{jki}(\bar{g}^{jk}-Y^jY^k),\end{aligned}$$

which is written in the form

$$\partial\sqrt{g}/\partial x^i = \sqrt{g}\,(\bar{\Gamma}^r_{ri} - \bar{\Gamma}_{rsi}Y^rY^s). \tag{3.3}$$

Next, from (3.1) we have $\partial(\partial\sqrt{g}/\partial B^i{}_\alpha)/\partial u^\alpha = (\partial\sqrt{g}/\partial u^\alpha)B^\alpha_i + \sqrt{g}\,(\partial B^\alpha_i/\partial u^\alpha)$. Since $\underline{\Gamma}(Y)$ is metrical, we have $\partial\sqrt{g}/\partial u^\alpha = (\sqrt{g}/2)(\partial g_{\beta\gamma}/\partial u^\alpha)g^{\beta\gamma} = \sqrt{g}\,\Gamma_{\beta\gamma\alpha}g^{\beta\gamma} = \sqrt{g}\,\Gamma^\rho_{\rho\alpha}$, and (2.11) leads to $\partial B^\alpha_i/\partial u^\alpha = H^\alpha_\alpha Y_i - B^\gamma_i\Gamma^\alpha_{\gamma\alpha} + (\delta^k_j - Y_jY^k)\bar{\Gamma}^j_{ik}$. Therefore we obtain

$$\partial(\partial\sqrt{g}/\partial B^i_\alpha)/\partial u^\alpha = \sqrt{g}\,(H^\rho_\rho Y_i + T^\rho_{\rho\alpha}B^\alpha_i + \bar{\Gamma}^r_{ir} - \bar{\Gamma}_{irs}Y^rY^s). \tag{3.4}$$

As a consequence of (3.3) and (3.4) the inside term of $\{\cdots\}$ in the equation (3.2) is written in the form

$$\sqrt{g}\,(\bar{T}^r_{ri} + \bar{T}_{irs}Y^rY^s - H^\rho_\rho Y_i - T^\rho_{\rho\alpha}B^\alpha_i). \tag{3.5}$$

Further from (2.14) and (1.8) it is rewritten as

$$= \sqrt{g}\,\{T^j_{jk}(x, Y)Y^k + C_j(x, Y)Y^j_kY^k - H^\rho_\rho\}Y_i.$$

Finally the equation (3.2) is written in the form

$$M = \{T^j_{jk}(x, Y) + C_j(x, Y)Y^j_k\}Y^k, \tag{3.6}$$

where $M = g^{\alpha\beta}H_{\alpha\beta}$ is called the *mean curvature*.

**Theorem 3.1.** *With respect to a generalized Cartan connection $C\Gamma(T)$ a transversal hypersurface is Y-extremal, if and only if the equation* (3.6) *is satisfied.*

**Remark.** (1) The equation (3.6) is not a consequence of the ordinary variation theory such that the variation vector is arbitrarily taken except that it vanishes at the boundary. The hypersurface may vary to a near transversal hypersurface only. (2) If we consider the Cartan connection $C\Gamma$, (3.6) reduces to Wegener's equation $M = C_j(x, Y)Y^j_kY^k$ [12]. Further, if $Y$ is a geodesic transversal vector field, (2.7) shows that the equation (3.6) asserts vanishing of the mean curvature $M$.

## § 4. Cartan Y-connection.

It is noteworthy to observe that if we consider the Cartan connection $C\Gamma$, the second term of the right-hand side of (3.6) does not vanish in general, as Wegener [12], Davies [3] and Barthel [2] have indicated. Moreover it is seen from (1.8) and (3.5) that in case of $C\Gamma$ the linear $Y$-connection $\Gamma(Y)$ has surviving torsion tensor $\bar{T}^i_{jk} = C^i_{jr}(x, Y)Y^r_k - C^i_{kr}(x, Y)Y^r_j$ which plays a role in (3.5).

These observations enable us to notice the following idea and lead to the Riemannian connection:

**Proposition 4.1.** *We consider a domain $D$ of the underlying manifold $M^n$ of a Finsler space $F^n=(M^n, L)$ where a field of unit tangent vector $Y=(Y^i(x))$ is defined. Then a Finsler connection $F\Gamma=(F^i_{jk}, N^i_j, C^i_{jk})$ on $D$ is uniquely determined from $(L, Y)$ by the following five axioms:*

(Y1) *h-metrical,* (Y2) *deflection tensor*$=0$,

(Y3) *(h)h-torsion tensor $T$ is given by*

$$T^i_{jk}=L(Y^r_jC^i_{rk}-Y^r_kC^i_{rj}), \tag{4.1}$$

*where* $Y^r_j=\partial Y^r/\partial x^j+N^r_j(x, Y)$,

(Y4) *v-metrical,* (Y5) *(v)v-torsion tensor*$=0$.

**Remark.** This connection is a kind of generalized Cartan connection; $T$ is given by (4.1), but unknown, because $N^i_j$ are unknown.

*Proof.* From (Y4) and (Y5) we have $C^i_{jk}$=Cartan's $C$-tensor. Next $A_{ijk}$ defined by (1.1) is now $A_{ijk}=L(Y^r_iC_{rjk}-Y^r_jC_{rik})$, so (1.3) and (1.4) are written in the form

$$F_{ijk}=\gamma_{ijk}-C_{ijr}N^r_k+C_{jkr}(LY^r_i-N^r_i)-C_{kir}(LY^r_j-N^r_j), \tag{4.2}$$

$$F_{0jk}=\gamma_{0jk}+C_{jkr}(LY^r_0-N^r_0), \qquad F_{0j0}=\gamma_{0j0}. \tag{4.3}$$

Therefore we have $F^j_{00}(=N^j_0)=\gamma^j_{00}$ from (Y2) and

$$N^j_k=\gamma^j_{0k}+C^j_{kr}(LY^r_0-\gamma^r_{00}). \tag{4.3$'$}$$

We shall find $N^j_k(x, Y)$. Since $Y^r_0(x, Y)$ is equal to $Y^r_i(x)Y^i=Y^i\partial Y^r/\partial x^i+N^r_0(x, Y)$, (4.3$'$) and $L(x, Y)=1$ lead to

$$N^j_k(x, Y)=\gamma^j_{0k}(x, Y)+C^j_{kr}(x, Y)(\partial Y^r/\partial x^i)Y^i. \tag{4.4}$$

Thus $N^j_k(x, Y)$ and so $Y^j_k(x)$ are determined. Then (4.3$'$) gives $N^j_k(x, y)$ and finally (4.2) determines $F_{ijk}(x, y)$.

**Definition.** The Finsler connection which is uniquely determined from $(L, Y)$ as in Proposition 4.1 is called the *Cartan Y-connection* and denoted by $CY\Gamma$.

From (4.1) and (1.8) we obtain

**Theorem 4.1.** *The linear Y-connection $\Gamma(Y)$ associated to the Cartan Y-connection $CY\Gamma$ by $Y$ is the Riemannian connection with respect to the Riemannian Y-metric.*

Therefore, in case of $CY\Gamma$ the linear $Y$-connection $\underline{\Gamma}(Y)$ induced on a hypersurface which is transversal to $Y$ is also the Riemannian connection with respect to the induced Riemannian $Y$-metric and $H_{\alpha\beta}$ is symmetric. In particular it is remarkable that (4.1) shows vanishing of the right-hand side of (3.6). Consequently we have

**Theorem 4.2.** *In case of the Cartan Y-connection $CY\Gamma$ the induced linear Y-connection $\underline{\Gamma}(Y)$ is the Riemannian connection with respect to the induced Riemannian Y-metric and the second fundamental tensor is symmetric. A Y-extremal hypersurface is characterized by vanishing of the mean curvature.*

## Chapter II. Minimal hypersurface

### § 5. The unit normal vector.

Let a hypersurface $M^{n-1}$: $x^i = x^i(u^1, \cdots, u^{n-1})$ be given in the underlying manifold $M^n$ of an $n$-dimensional Finsler space $F^n = (M^n, L, C\Gamma(T))$, equipped with a Finsler metric $L$ and a generalized Cartan connection $C\Gamma(T)$. The $n-1$ tangent vectors $B_\alpha = (B^i_\alpha = \partial x^i/\partial u^\alpha)$ are assumed to be linearly independent. A set $(x, B_\alpha)$ of a point $x(u)$ of $M^{n-1}$ and $n-1$ tangent vectors $B_\alpha(u)$ at $x(u)$ is called a *hypersurface element* in $F^n$.

The unit normal vector $N = (N^i)$ of a hypersurface element $(x, B_\alpha)$ is defined by the equations

$$L(x, N) = 1 \quad \text{or} \quad g_{ij}(x, N) N^i N^j = 1, \tag{5.1}$$

$$g_{ij}(x, N) B^i_\alpha N^j = 0, \qquad \alpha = 1, \cdots, n-1. \tag{5.2}$$

That is, $N$ has unit absolute length and is orthogonal to each $B_\alpha$ with respect to itself. We shall construct such a vector $N$ in the following.

First we take $n$ constants $d^i$ such that the square matrix $(B^i_\alpha, d^i)$ has nonzero determinant $D = \det(B^i_\alpha, d^i)$. Let $q_i$ be the cofactor of $d^i$ in $(B^i_\alpha, d^i)$. Then $q_i = q_i(B)$ are functions of $B^i_\alpha$, independent of the choice of $d^i$ and satisfying

$$q_i B^i_\alpha = 0, \qquad q_i d^i = D. \tag{5.3}$$

Next the $n$ equations

$$g_{ij}(x, p) p^j = q_i \tag{5.4}$$

give $p^j$ uniquely, because the Jacobian

$$\det\{\partial(g_{ij}(x, p) p^j - q_i)/\partial p^k\} = \det\{2C_{ijk}(x, p) p^j + g_{ik}(x, p)\},$$

which is equal to $\det\{g_{ik}(x, p)\} \neq 0$. Thus we get $n$ functions $p^j = p^j(x, q(B))$. Finally we put

$$N^i = p^i / L(x, p), \tag{5.5}$$

which are components of $N$ we wished to construct.

To verify this fact, we must show that $N$ given by (5.5) satisfies (5.1) and (5.2) and that it has the property of contravariant vector.

First we have $L(x, p/L(x, p)) = L(x, p)/L(x, p) = 1$, because of $L(x, p) > 0$ and positive homogeneity of $L(x, y)$ in $y^i$. Secondly we have $g_{ij}(x, p/L(x, p))$

$B_\alpha^i\{p^j/L(x, p)\}=\{g_{ij}(x, p)B_\alpha^i p^j\}/L(x, p)=(q_i B_\alpha^i)/L(x, p)=0$ from (5.4) and (5.3) and positive homogeneity of $g_{ij}(x, y)$ in $y^j$. Consequently $N^i$ satisfies (5.1) and (5.2).

To show the property of contravariant vecor of $N^i$, we shall deal with a coordinate transformation $(x^i)\to(\bar{x}^a)$. Put $\bar{X}_i^a=\partial\bar{x}^a/\partial x^i$, $\bar{X}=\det(\bar{X}_i^a)$, $\underline{X}_a^i=\partial x^i/\partial\bar{x}^a$, $\underline{X}=\det(\underline{X}_a^i)$. We get $\bar{B}_\alpha^a(=\partial\bar{x}^a/\partial u^\alpha)=\bar{X}_i^a B_\alpha^i$ immediately. If we take $n$ constants $\bar{d}^a=\bar{X}_i^a d^i$ in $(\bar{x}^a)$, we have $\bar{D}\{=\det(\bar{B}_\alpha^a, \bar{d}^a)\}=\bar{X}D$. Then it is easy to show that the cofactor $\bar{q}_a$ of $\bar{d}^a$ in $(\bar{B}_\alpha^a, \bar{d}^a)$ is equal to $\bar{X}\underline{X}_a^i q_i$. Since $g_{ij}$ has the covariant property, i. e., $\bar{g}_{ab}(\bar{x}^c, \bar{X}_h^c p^h)=g_{ij}(x^k, p^k)\underline{X}_a^i\underline{X}_b^j$, we have

$$\{\bar{g}_{ab}(\bar{x}^c, \bar{X}_h^c p^h)\}(\bar{X}_l^a p^l)=g_{ij}(x^k, p^k)\underline{X}_b^j p^i=q_j\underline{X}_b^j=\underline{X}\bar{q}_b.$$

Then, provided $\underline{X}>0$, we get $\{\bar{g}_{ab}(\bar{x}^c, \bar{X}\bar{X}_h^c p^h)\}(\bar{X}\bar{X}_l^a p^l)=\bar{q}_b$. Therefore (5.4) in the new coordinates $(\bar{x}^a)$ leads to $\bar{p}^a=\bar{X}\bar{X}_l^a p^l$. Then (5.5) in $(\bar{x}^a)$ and the scalar property of $L$, i. e., $L(x^i, p^i)=\bar{L}(\bar{x}^a, \bar{X}_h^a p^h)$ yield

$$\bar{N}^a=\bar{X}\bar{X}_i^a p^i/\bar{L}(\bar{x}^b, \bar{X}\bar{X}_h^b p^h)=\bar{X}_i^a p^i/L(x^k, p^k)=\bar{X}_i^a N^i$$

from $\bar{X}>0$ and positive homogeneity of $L$. Consequently we have proved the contravariant property of $N^i$ by means of the well-known properties of $L$ and $g_{ij}$, *provided that* $\bar{X}>0$, that is, *the transformation* $(x^i)\to(\bar{x}^a)$ *has positive Jacobian.*

**Remark.** To construct the unit normal vector $N$, Barthel made use of a way which is little different from the above; his way is based on the fact that $\sqrt{g}$, $g=\det\{g_{ij}(x, y)\}$, is a relative scalar of weight $+1$. But it should be remarked that this property of $\sqrt{g}$ is true under the assumption $\bar{X}>0$, because $\bar{g}(\bar{X})^2=g$.

It is well-known that the algebraic sign of $\bar{X}$ is related to the orientation of coordinate neighborhoods of $(x^i)$ and $(\bar{x}^a)$. Further we have to pay attention to the algebraic sign of $N^i$ in (5.1) and (5.2); it is not sure whether $(-N^i)$ satisfy these equations or not, even if $(N^i)$ satisfy these equations. If we restrict our consideration to *symmetric Finsler metrics*, i. e., $L(x, -y)=L(x, y)$, it is obvious that $(N^i)$ given by (5.5) have the properties we wished to verify, independently of the sign of Jacobian $\bar{X}$ and that $(-N^i)$ also satisfy (5.1) and (5.2). If the metric is not symmetric, it will be obvious intuitively that the opposite orientation of the normal vector field may induce a different geometrical structure in the same hypersurface.

As our consideration is only local, it may be assumed in the following that the unit normal vector $N$ of $(x, B_\alpha)$ is locally oriented in some standpoint.

Now it follows from (5.5) that components $N^i$ are functions of the form $N^i(x, B)=p^i(x, q(B))/L(x, p(x, q(B)))$. To consider $\partial N^i/\partial B^j{}_\alpha$, we shall first find $\partial q_i(B)/\partial B^j{}_\alpha$.

In the above we had two square matrices $F_d=(B_\alpha^i, d^i)$ and $F_N=(B_\alpha^i, N^i)$ which have non-zero determinant. Let $F_d^*=(C_i^\alpha, c_i)$ and $F_N^*=(B_i^\alpha, N_i)$ be the respective inverse matrices. Further we define on $M^{n-1}$ the induced Riemannian $N$-metric

$$g_{\alpha\beta}(u)=g_{ij}(x, N)B^i_\alpha B^j{}_\beta. \tag{5.6}$$

Then we have $B^\alpha_i=g^{\alpha\beta}g_{ij}(x, N)B^j{}_\beta$ and $N_i=g_{ij}(x, N)N^j$, as will be easily verified. From (5.4), (5.5) and the definitions of $q_i$ and $c_i$ we have

$$q_i=Dc_i=L(x, p)N_i. \tag{5.7}$$

Next, put $d^i=d^\alpha B^i_\alpha+dN^i$ with respect to $F_N$. Then (5.3) shows $D=q_i(d^\alpha B^i_\alpha+dN^i)=dq_iN^i$ and (5.7) gives

$$d^i=d^\alpha B^i_\alpha+DN^i/L(x, p), \tag{5.8}$$

(It is obvious from the definition of $d^i$ that the tangential parts $d^\alpha$ of $d^i$ are arbitrarily chosen.) Also, put $C^\alpha_i=C^\alpha_\beta B^\beta_i+C^\alpha N_i$ with respect to $F^*_N$. Then $\delta^\alpha_\beta=C^\alpha_iB^i_\beta=C^\alpha_\beta$ and (5.8) leads to $0=C^\alpha_id^i=d^\alpha+C^\alpha D/L(x, p)$. Therefore from (5.7) we get

$$C^\alpha_i=B^\alpha_i-d^\alpha q_i/D. \tag{5.9}$$

Now, differentiating (5.1) by $B^j{}_\beta$, we get

$$0=(\partial q_i/\partial B^j{}_\beta)B^i_\alpha+q_j\delta^\beta_\alpha, \qquad \partial D/\partial B^j{}_\beta=(\partial q_i/\partial B^j{}_\beta)d^i.$$

From the definition of $C^\alpha_i$ we have $\partial D/\partial B^j{}_\beta=(\partial B^k_\gamma/\partial B^j{}_\beta)DC^\gamma_k=DC^\beta_j$ and the second of the above yields $(\partial q_i/\partial B^j{}_\beta)d^i=DC^\beta_j$. Then, contracting the first of the above by $C^\alpha_k$, we get $0=(\partial q_i/\partial B^j{}_\beta)(\delta^i_k-d^ic_k)+q_jC^\beta_k=\partial q_k/\partial B^j{}_\beta-DC^\beta_jc_k+q_jC^\beta_k$. Thus (5.9) and (5.7) lead to

$$\partial q_k/\partial B^j{}_\beta=q_kB^\beta_j-q_jB^\beta_k. \tag{5.10}$$

We differentiate (5.4) by $B^j{}_\alpha$. Paying attention to $(\partial g_{ik}(x, p)/\partial p^h)p^k=0$, we get $g_{ik}(x, p)(\partial p^k/\partial B^j{}_\alpha)=\partial q_i/\partial B^j{}_\alpha$ and (5.10) gives

$$\partial p^i/\partial B^j{}_\alpha=p^iB^\alpha_j-q_jB^{\alpha i}, \tag{5.11}$$

where $B^{\alpha i}=g^{ij}(x, N)B^\alpha_j=g^{\alpha\beta}B^i_\beta$.

Secondly, from (5.11) and the well-know equation $\partial L(x, p)/\partial p^i=g_{ij}(x, p)p^j/L(x, p)=q_i/L(x, p)$ we have

$$\partial L(x, p(x, B))/\partial B^i_\alpha=L(x, p)B^\alpha_i. \tag{5.12}$$

By means of (5.11) and (5.12) we finally obtain

$$\partial N^i(x, B)/\partial B^j{}_\alpha=-B^{\alpha i}N_j. \tag{5.13}$$

## § 6. The induced connection.

In the last section we get the field of frame $F_N=(B^i_\alpha, N^i)$ and the dual coframe $F^*_N=(B^\alpha_i, N_i)$ along a hypersurface $M^{n-1}$: $x^i=x^i(u)$ of a Finsler space $F^n=(M^n, L, C\Gamma(T))$. Therefore we get a linear connection $\underline{\Gamma}$ on $M^{n-1}$ which is induced from $C\Gamma(T)$ by the ordinary way [9]. That is, the absolute differential $DX^\alpha$ of a tangent vector field $X^i=X^\alpha B^i_\alpha$ of $M^{n-1}$ is defined by $DX^\alpha=B^\alpha_iDX^i$, where $DX^i$ is the absolute differential of $X^i$ with respect to $C\Gamma(T)$ in which the supporting element $y^i$ is specified as the normal vector $N^i$. Thus we have

$$DX^i = dX^i + X^j\omega_j^i(N),$$

$$\omega_j^i(N) = \Gamma^{*i}_{jk}(x, N)dx^k + C^i_{jk}(x, N)dN^k$$

$$= [F^i_{jk}(x, N)B^k_\gamma + C^i_{jk}(x, N)\{\partial N^k/\partial u^\gamma + N^k_h(x, N)B^h_\gamma\}]du^\gamma.$$

Putting

(6.1) $$N^k_\gamma(u) = \partial N^k/\partial u^\gamma + N^k_h(x, N)B^h_\gamma,$$

(6.2) $$F^i_{j\gamma}(u) = F^i_{jk}(x, N)B^k_\gamma + C^i_{jk}(x, N)N^k_\gamma,$$

and $DX^\alpha = dX^\alpha + X^\beta\Gamma^\alpha_{\beta\gamma}du^\gamma$, the connection coefficients $\Gamma^\alpha_{\beta\gamma}(u)$ of $\underline{\Gamma}$ are given by

(6.3) $$\Gamma^\alpha_{\beta\gamma} = B^\alpha_i(B^i_{\beta\gamma} + B^j_\beta F^i_{j\gamma}).$$

Therefore we obtain the Gauss equation

(6.4) $$B^i_{\beta;\gamma} = H_{\beta\gamma}N^i,$$

where $B^i_{\beta;\gamma}$ is the relative covariant derivative of $B^i_\beta$ with respect to $\underline{\Gamma}$, i.e.,

(6.5) $$B^i_{\beta;\gamma} = B^i_{\beta\gamma} + B^j_\beta F^i_{j\gamma} - B^i_\alpha\Gamma^\alpha_{\beta\gamma},$$

and $H_{\beta\gamma}$ is the second fundamental tensor.

The torsion tensor $T^\alpha_{\beta\gamma}$ of $\underline{\Gamma}$ is

(6.6) $$T^\alpha_{\beta\gamma} = B^\alpha_i(B^j_\beta F^i_{j\gamma} - B^j_\gamma F^i_{j\beta}),$$

and we get

(6.7) $$H_{\beta\gamma} - H_{\gamma\beta} = N_i(B^j_\beta F^i_{j\gamma} - B^j_\gamma F^i_{j\beta}).$$

To consider the relative covariant derivative of a tensor field of $F^n$ along $M^{n-1}$, we shall deal with a Finsler vector field $X^i(x, y)$. From (6.1) we first have

$$\partial X^i(x, N)/\partial u^\alpha = (\delta X^i/\delta x^j)B^j_\alpha + \{\partial X^i(x, N)/\partial N^j\}N^j_\alpha,$$

where $\delta/\delta x^j = \delta/\delta x^j - (\partial/\partial y^r)N^r_j(x, N)$. Therefore, in terms of $h$- and $v$-covariant derivatives in $F^n$, we get

(6.8) $$X^i_{;\alpha} = X^i_{|j}(x, N)B^j_\alpha + X^i|_j(x, N)N^j_\alpha,$$

where we put

(6.9) $$X^i_{;\alpha} = \partial X^i(x, N)/\partial u^\alpha + X^h(x, N)F^i_{h\alpha},$$

which is the relative covariant derivative of $X^i$.

Next we are concerned with the relative covariant derivative $N^i_{;\alpha}$ of the unit normal vector $N^i$ of $M^{n-1}$. It is not a Finsler vector field in $F^n$, but from (6.9) and (6.2) is $N^i_{;\alpha}$ formally written as

(6.10) $$N^i_{;\alpha} = \partial N^i/\partial u^\alpha + N^i_j(x, N)B^j_\alpha,$$

which is nothing but $N_{\alpha}^{i}(u)$ given by (6.1), although the righthand side of (6.8) can not be applicable to $N^{i}$. Really we shall consider the right-hand side of (6.10). $\partial N^{i}(x, B)/\partial B^{j}_{\alpha}$ are already found as (5.13). We shall find $\partial N^{i}(x, B)/\partial x^{j}$ in the following. Paying attention to $q_{i}=q_{i}(b)$, differentiation of (5.4) by $x^{k}$ gives

$$\{\partial g_{ij}/\partial x^{k}+(\partial g_{ij}/\partial p^{h})(\partial p^{h}/\partial x^{k})\}p^{j}+g_{ij}\{\partial p^{j}(x, B)/\partial x^{k}\}=0.$$

The term $\{\partial g_{ij}(x, p)/\partial p^{h}\}p^{j}$ vanishes. Then, contracting by $g^{ih}(x, p)$, we have

$$\begin{aligned}\partial p^{h}(x, B)/\partial x^{k}&=-g^{ih}p^{j}(\partial g_{ij}/\partial x^{k})\\&=-g^{ih}p^{j}(2C_{ijl}N_{k}^{l}+F_{ijk}+F_{jik}),\end{aligned}$$

which gives

$$\partial p^{h}(x, B)/\partial x^{k}=-F^{h}_{\cdot 0k}(x, p)-N_{k}^{h}(x, p). \tag{6.11}$$

Next we have

$$\partial L(x, p(x, B))/\partial x^{j}=\{\partial L(x, p)/\partial p^{h}\}\{N_{j}^{h}(x, p)+\partial p^{h}/\partial x^{j}\}.$$

Then the identity $\partial L(x, p)/\partial p^{h}=q_{h}/L(x, p)$ and (6.11) lead to

$$\partial L(x, p(x, B))/\partial x^{j}=-N_{j}^{0}(x, p)/L(x, p). \tag{6.12}$$

Finally, differentiating (5.5) by $x^{j}$, substituting from (6.11) and (6.12) and paying attention to the homogeneity property of those quantities, we obtain

$$\partial N^{i}(x, B)/\partial x^{j}=(-F^{i}_{\cdot 0j}-N_{j}^{i}+N^{i}N_{j}^{0})_{y=N}. \tag{6.13}$$

Therefore (6.13) and (5.13) yield

$$\partial N^{i}/\partial u^{\alpha}+N_{j}^{i}(x, N)B^{j}_{\alpha}=(-F^{i}_{\cdot 0j}+N^{i}N_{j}^{0})B^{j}_{\alpha}-B^{\beta i}N_{j}B^{j}_{\beta\alpha}.$$

Then (6.4) and (6.5) show that the right-hand side is equal to $-H_{\alpha}^{\beta}B_{\beta}^{i}$ ($H_{\alpha}^{\beta}=g^{\beta\gamma}H_{\gamma\alpha}$) and, as a consequence, we get the so-called Weingarten equation

$$N^{i}_{;\alpha}(=N_{\alpha}^{i})=-H_{\alpha}^{\beta}B_{\beta}^{i}. \tag{6.14}$$

It is remarked that $H_{\alpha}^{\beta}$ is different from $H_{\alpha\gamma}g^{\gamma\beta}$ in general.

## § 7. Minimal hypersurface.

From the induced Riemannian $N$-metric (5.6) of the hypersurface $M^{n-1}$ we have the volume element $\sqrt{\underline{g}(u)}$, $\underline{g}(u)=\det\{g_{\alpha\beta}(u)\}$. If we put $g(x, N)=\det\{g_{ij}(x, N)\}$, (5.6), (5.1) and (5.2) give

$$\underline{g}=\begin{vmatrix} g_{ij}(x, N)B_{\alpha}^{i}B^{j}_{\beta} & g_{ij}(x, N)B_{\alpha}^{i}N^{j} \\ g_{ij}(x, N)N^{i}B^{j}_{\beta} & g_{ij}(x, N)N^{i}N^{j}\end{vmatrix}=g\{\det(B_{\alpha}^{i}, N^{i})\}^{2}.$$

From the definition of $q_{i}$ and (5.7) we have $\det(B_{\alpha}^{i}, N^{i})=q_{i}N^{i}=L(x, p)$. Thus we get

$$\underline{g}(u)=g(x, N)L^{2}(x, p). \tag{7.1}$$

Here it is noted that $g(x, N)$ may be replaced by $g(x, p)$ by the homogeneity property of $g_{ij}(x, y)$.

Now we consider the volume $I=\int\sqrt{\underline{g}(u)}\, du^1 \cdots du^{n-1}$ over a domain $\underline{D}$ of $M^{n-1}$. It is seen from (7.1) and the construction of $N$ in §5 that $\underline{g}(u)=g(x, p(x, B))L^2(x, p(x, B))$ is directly a function of $x^i$ and $B^i_\alpha$ as $f(u)$ which was treated in §3. But our $M^{n-1}$ is not necessarily a transversal hypersurface and the variation may be taken arbitrarily.

**Definition.** If the volume integral $I=\int\sqrt{\underline{g}(u)}\, du^1 \cdots du^{n-1}$ over a compact hypersurface $M^{n-1}$ has a stationary value, i.e., vanishing first variation, $M^{n-1}$ is called a *minimal hypersurface.*

It is well-known that the generalized Euler-Lagrange equation

$$\partial\sqrt{\underline{g}}/\partial x^i-\partial(\partial\sqrt{\underline{g}}/\partial B^i_\alpha)/\partial u^\alpha=0 \tag{7.2}$$

characterizes a minimal hypersurface. We shall write (7.2) in terms of quantites of $M^{n-1}$ in the following.

The equation $\partial g(x, y)/\partial y^i=2gC_i$ is well-known. Therefore (5.13) shows

$$\partial g(x, p)/\partial B^i_\alpha=-2g(x, N)C^\alpha(u)N_i, \tag{7.3}$$

where $C^\alpha(u)=C^i(x, N)B^\alpha_i$. Then (7.3) and (5.12) give

$$\partial\sqrt{\underline{g}}/\partial B^i_\alpha=\sqrt{\underline{g}}(B^\alpha_i-C^\alpha N_i). \tag{7.4}$$

Next we shall find $\partial\sqrt{\underline{g}}/\partial x^i$. From (6.11) we have

$$\partial g(x, p(x, B))/\partial x^i=(\delta g/\delta x^i-2gC_jF^j_{\cdot 0i})_{y=p}.$$

Then, from $\delta g/\delta x^i=(\delta g_{jk}/\delta x^i)gg^{jk}=2gF_j{}^j{}_i$ we get

$$\partial g(x, p(x, B))/\partial x^i=2g(x, N)(F_j{}^j{}_i-C_jF^j_{\cdot 0i})_{y=N}. \tag{7.5}$$

Therefore (7.5) and (6.12) lead to

$$\partial\sqrt{\underline{g}}/\partial x^i=\sqrt{\underline{g}}(F_j{}^j{}_i-C_jF^j_{\cdot 0i}-N^0_i)_{y=N}. \tag{7.6}$$

Next, to find $\partial(\partial\sqrt{\underline{g}}/\partial B^i_\alpha)/\partial u^\alpha$, we consider $\partial\sqrt{\underline{g}}/\partial u^\alpha$, $\partial B^\alpha_i/\partial u^\alpha$, $\partial C^\alpha/\partial u^\alpha$ and $\partial N_i/\partial u^\alpha$.

First we have $\partial\sqrt{\underline{g}}/\partial u^\alpha=(\partial\sqrt{\underline{g}}/\partial x^i)B^i_\alpha+(\partial\sqrt{\underline{g}}/\partial B^i_\beta)B^i_{\beta\alpha}$. Then (6.5), (7.4) and (7.6) lead to

$$\partial\sqrt{\underline{g}}/\partial u^\alpha=\sqrt{\underline{g}}\underline{\Gamma}_\beta{}^\beta{}_\alpha, \tag{7.7}$$

which may be rather well-known equation in viewpoint of the induced metric $g_{\alpha\beta}$ and induced connection $\underline{\Gamma}$.

Secondly (6.4) gives $B^\alpha_{i;\alpha}=MN_i$, where $M=g^{\alpha\beta}H_{\alpha\beta}$ is the mean curvature, so that $\partial B^\alpha_i/\partial u^\alpha=MN_i+B^\alpha_jF_i{}^j{}_\alpha-B^\beta_i\underline{\Gamma}_\beta{}^\alpha{}_\alpha$. (6.2) and (6.14) show $B^\alpha_jF_i{}^j{}_\alpha=F_i{}^j{}_j-F_{i00}-B^\gamma_iC^{\alpha\beta}_\gamma H_{\alpha\beta}$. Therefore we obtain

(7.8) $$\partial B^a_i/\partial u^\alpha = MN_i + (F_i{}^j{}_j - F_{i00})_{y=N} - B^\gamma_i(C^{\alpha\beta}_\gamma H_{\alpha\beta} + \Gamma_\gamma{}^\alpha{}_\alpha).$$

Thirdly we have $\partial C^\alpha/\partial u^\alpha = C^\alpha_{;\alpha} - C^\beta \Gamma^\alpha_{\beta\alpha}$. (6.8), (6.4) and (6.14) show

$$C^a_{;\alpha} = (C^i B^a_i)_{;\alpha} = (C^i_{|j} B^j_\alpha + C^i|_j N^j_\alpha) B^a_i + C^i(x, N) N_i M,$$

which is equal to $C^i_{|i} - C^i|_j B^j_\beta H^\beta_\alpha B^a_i$ from $C^i(x, N)N_i = 0$ and $C^i_{|j}(x, N)N_i = 0$. Therefore we get

(7.9) $$\partial C^\alpha/\partial u^\alpha = C^i|_i(x, N) - C^i|_j(x, N) B^j_\beta H^\beta_\alpha B^a_i - C^\beta \Gamma_\beta{}^\alpha{}_\alpha.$$

Fourthly (6.14) yields $N_{i;\alpha} = -H^\beta_\alpha B_{\beta i}$, so we get

(7.10) $$\partial N_i/\partial u^\alpha = F_{i0j}(x, N) B^j_\alpha - H_{\beta\alpha} B^\beta_i.$$

Consequently (7.7), (7.8), (7.9) and (7.10) lead to

(7.11) $$\begin{aligned}\partial(\partial\sqrt{g}/\partial B^i_\alpha)/\partial u^\alpha = \sqrt{g}\{ & B^a_i(T_\beta{}^\beta{}_\alpha + H_{\alpha\beta}C^\beta - C^{\beta\gamma}_\alpha H_{\beta\gamma}) \\ & + N_i(M - T_\beta{}^\beta{}_\alpha C^\alpha - C^j_{|j} + C^j|_k B^k_\beta H^\beta_\alpha B^a_j) \\ & + (F_i{}^j{}_j - F_{i00} - F_{i0j}C^j)\}_{y=N}.\end{aligned}$$

Therefore the left-hand side of the equation (7.2) divided by $\sqrt{g}$ is written

$$\begin{aligned}\{ & T_j{}^j{}_i + T_{i00} + T_{i0j}C^j - B^a_i(T_\beta{}^\beta{}_\alpha + H_{\alpha\beta}C^\beta - C^{\beta\gamma}_\alpha H_{\beta\gamma}) \\ & - N_i(M - C^\alpha T_\beta{}^\beta{}_\alpha - C^j_{|j} + C^j|_k B^k_\beta H^\beta_\alpha B^a_j))\}_{y=N}.\end{aligned}$$

Moreover the equations given in §6 lead to

$$B^a_i T_\beta{}^\beta{}_\alpha = T_j{}^j{}_i + T_{i00} - T_j{}^j{}_0 N_i + B^a_i(C^{\beta\gamma}_\alpha H_{\beta\gamma} - H_{\beta\alpha}C^\beta),$$

$$B^a_i H_{\alpha\beta} C^\beta = C^\beta H_{\beta\alpha} B^a_i + T_{i0j}C^j - N_i T_{00j} C^j,$$

$$C^\alpha T_\beta{}^\beta{}_\alpha = C^i T_j{}^j{}_i - T_{00j}C^j - H_{\alpha\beta}C^\alpha C^\beta + C^\gamma C^{\alpha\beta}_\gamma H_{\alpha\beta},$$

and the definition of $C^j|_k$ shows $C^j|_k B^k_\beta H^\beta_\alpha B^a_j = (\dot\partial_k C^j) H^\beta_\alpha B^a_j B^k_\beta + C^\gamma C^{\alpha\beta}_\gamma H_{\alpha\beta}$. Finally it is seen that the left-hand side of (7.2) has the normal component alone, hence we have the equation (7.2) of the scalar form

(7.12) $$\begin{aligned}&(T^i{}_{i0} + T^i{}_{ij}C^j + C^i_{|i})_{y=N} \\ &\qquad = H_{\alpha\beta} B^a_i B^\beta_j (g^{ij} + C^i C^j + g^{ik}\dot\partial_k C^j)_{y=N}.\end{aligned}$$

It is noted that the first term of the right-hand side, $H_{\alpha\beta} B^a_i B^\beta_j g^{ij}$ is the mean curvature $M$ of $M^{n-1}$ and that (7.12) is quite different from (3.6).

## § 8. Cartan $C$-connection.

We shall observe the equation (7.12) characterizing a minimal hypersurface with respect to a generalized Cartan connection $C\Gamma(T)$. If we are concerned with the Cartan connection ($T=0$), (7.12) becomes

(8.1) $$C^i_{|i}(x, N) = H_{\alpha\beta} B^a_i B^\beta_j (g^{ij} + C^i C^j + g^{ik}\dot\partial_k C^j)_{y=N}.$$

Therefore a hyperplane ($H_{\alpha\beta}=0$) is minimal if and only if the ambient space $F^n$ satisfies the condition $C^i_{|i}=0$; this may be a little strange situation.

If we are concerned with Barthel's connection [1], $T^i_{jk}$ is given by (1.5) and $T^i_{ik}(x, N)=-N_k C^i_{|i}(x, N)$, $T^i_{i0}(x, N)=-C^i_{|i}(x, N)$ and $T^i_{ik}(x, N)C^k(x, N)=0$. Thus (7.12) reduces to

$$H_{\alpha\beta}B^\alpha_i B^\beta_j (g^{ij}+C^iC^j+g^{ik}\dot{\partial}_k C^j)_{y=N}=0. \tag{8.2}$$

As a consequence a hyperplane is necessarily minimal.

**Remark.** If we consider a $C\Gamma(T)$, $T$ being of the form $T_{ijk}=l_i p_{jk}-l_k p_{ji}$ for some $p_{jk}$, the left-hand side of (7.12) is written as $(p_{00}-p_{ij}g^{ij}+p_{0i}C^i+C^i_{|i})_{y=N}$. Further, if we put $p_{ij}=LC_{i|j}$, we have $p_{0j}=0$ and (8.2). This $C\Gamma(T)$ is nothing but Barthel's connection. But $A_{ijk}$ defined by (1.1) does not become simple in this case and it seems that this situation gives rise to a subject difficult of concrete solution.

Now we consider a $C\Gamma(T)$ whose $T$ is of a semi-symmetric form

$$T^i_{jk}=\delta^i_j p_k-\delta^i_k p_j, \tag{8.3}$$

for some $p_k$. Then the left-hand side of (7.12) is written as $\{(n-1)(p_0+p_iC^i)+C^i_{|i}\}_{y=N}$. If we further assume $p_i=pl_i$ for some scalar $p$, the above becomes $\{(n-1)p+C^i_{|i}\}_{y=N}$. Therefore, to reduce the left-hand side of (7.12) to zero, we notice the following form of the $T$-tensor:

$$T^i_{jk}=-C(\delta^i_j l_k-\delta^i_k l_j), \tag{8.4}$$

$$C=LC^i_{|i}/(n-1). \tag{8.4$_a$}$$

**Definition.** A Finsler connection is called a *Cartan C-connection* and denoted by $C\Gamma(T_c)$, if it satisfies the following five conditions:

(1) $h$-metrical, (2) deflection tensor$=0$,

(3) $(h)h$-torsion $T$ is given by (8.4) and (8.4$_a$),

(4) $v$-metrical, (5) $(v)v$-torsion tensor$=0$.

We consider $C\Gamma(T_c)$. From (8.4) it follows that $A_{ijk}$ defined by (1.1) is $A_{ijk}=C(l_i g_{jk}-l_j g_{ki})$. Thus (1.2) and (1.3) are written

$$F_{ijk}=\gamma_{ijk}-C_{ijr}N^r_k-C_{jkl}N^r_i+C_{kir}N^r_j+C(l_i g_{jk}-l_j g_{ki}), \tag{8.5}$$

$$F_{0jk}=\gamma_{0jk}-C_{jkr}N^r_0+CLh_{jk}, \qquad F_{0j0}=\gamma_{0j0}, \tag{8.6}$$

where $h_{jk}=g_{jk}-l_j l_k$ is the angular metric tensor. Thus the condition (2) and (8.6) lead to

$$N^i_j=G^i_j+CLh^i_j, \tag{8.7}$$

where $\{G^i_j(x, y)\}$ is the nonlinear connection of the Cartan connection $C\Gamma=$

$(\Gamma^{*i}_{jk}, G^i_j, C^i_{jk})$. Then (8.5) gives

$$F^i_{jk}=\Gamma^{*i}_{jk}+C(l_j\delta^i_k-l^ig_{jk}-LC^i_{jk}). \tag{8.8}$$

We now consider $(8.4_a)$. From (8.7) and (8.8) we get $C^i_{|i}=C^i_{\underset{c}{|}i}-CLC^i|_i$, where we use $l^j\dot{\partial}_jC^i=-C^i/L$, denote by $C^i_{\underset{c}{|}i}$ the $h$-covariant derivative of $C^i$ with respect to $C\Gamma$ and $C^i|_i=\dot{\partial}_jC^i+C^rC^i_{rj}$ is the $v$-covariant derivative of $C^i$ with respect to $C\Gamma$. Thus $(8.4_a)$ is written $C\{(n-1)+L^2C^i|_i\}=LC^i_{\underset{c}{|}i}$. Therefore, if the scalar

$$C^*=(n-1)+L^2C^i|_i \tag{8.9}$$

does not vanish, we have

$$C=LC^i_{\underset{c}{|}i}/C^*, \tag{8.10}$$

which gives $C$ by known quantities, and in consequence (8.8) determines $F^i_{jk}$. Consequently we have

**Theorem 8.1.** *The Cartan C-connection $C\Gamma(T_c)=(F^i_{jk}, N^i_j, C^i_{jk})$ is uniquely determined from the fundamental function L, provided that $C^*$ given by* (8.9) *does not vanish. $C^i_{jk}$ is Cartan's C-tensor and $F^i_{jk}$ and $N^i_j$ are given by* (8.8) *and* (8.7) *respectively, where C is written in the form* (8.10).

**Remark.** Putting $G=\log\sqrt{g}$, we have $C_i=\dot{\partial}_iG$ and the term $C^i|_i$ in (8.9) is written as $g^{ij}(\dot{\partial}_i\dot{\partial}_jG-\dot{\partial}_iG\dot{\partial}_jG)$. Is there a Finsler space of the vanishing $C^*$?

S. Watanabe communicated to the author: *If the indicatrices of $F^n$ is compact at every point of $F^n$, $C^*$ never vanish provided that $n\geqq 2$.*

*In case of $C\Gamma(T_c)$ we also have* (8.2) *as the characterizing equation of minimal hypersurface.* It is further observed that in case of $C\Gamma(T_c)$ and Barthel's connection as well as $C\Gamma$ we have $T^i_{jk}(x, N)B^j{}_\beta B^k_\gamma=0$, so that (6.6) and (6.7) show $T^\alpha_{\beta\gamma}=H_{\beta\rho}C^{\rho\alpha}_\gamma-H_{\gamma\rho}C^{\rho\alpha}_\beta$ and $H_{\alpha\beta}=H_{\beta\alpha}$.

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## References

[1] W. Barthel, Über die Minimalflächen in gefaserten Finslerräumen, Ann. di Mat., (4) **36** (1954), 159-190.
[2] E. Cartan, Les espaces de Finsler, Actualités 79, Paris, 1934, spécialement XI.
[3] E. T. Davies, Subspaces of a Finsler space, Proc. London Math. Soc., (2) **49** (1947), 19-39, specially 8 and 9.
[4] M. Hashiguchi, On Wagner's generalized Berwald space, J. Korean Math. Soc., **12** (1975), 51-61.
[5] M. Hashiguchi and Y. Ichijyō, On generalized Berwald spaces, Rep. Fac. Sci. Kagoshima Univ (Math. Phys. Chem.), **15** (1982), 19-32.

[6] H. Izumi and T. Sakaguchi, Identities in Finsler space, Memo. Nat. Defence Acad. Japan, **22** (1982), 7-15.

[7] M. Matsumoto, A Finsler connection with many torsions, Tensor, N. S., **17** (1966), 217-226.

[8] M. Matsumoto, The theory of Finsler connections, Publ. Study Group of Geometry 5, Deptt. Math. Okayama Univ., 1970.

[9] M. Matsumoto, The induced and intrinsic Finsler connections of a hypersurface and Finslerian projective geometry, J. Math. Kyoto Univ., **25** (1985), 107-144.

[10] B. Su, On the isomorphic transformations of minimal hypersurfaces in a Finsler space, Acta Math. Sinica, **5** (1955), 471-488, (Chinese).

[11] J. M. Wegener, Untersuchungen über Finslerschen Räume, Lotos Prag, **84** (1936), 4-7, besonders II.

[12] J. M. Wegener, Hyperflächen in Finslerschen Räumen als Transversalflächen einer Schar von Extremalen, Monatsh. für Math. und Phys., **44** (1936), 115-130.

Because the monograph [8] has long been out of print, the author would like to quote another monograph which was published quite recently:

[8′] M. Matsumoto, Foundations of Finsler geometry and special Finsler spaces, Kaiseisha Press, Saikawa 3-23-2, Otsu-shi, Shiga-ken 520, Japan, 1986.