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Decay rates of scattering states for Schrödinger operators

Dedicated to Professor Sigeru Mizohata on his 60 th birthday

By

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Introduction.

In this paper, we derive a best possible decay rate of scattering states for the Schrödinger operator $H=-\Delta+V(x)$ in $L^2(\boldsymbol{R}^n)$ $(n\geqq 2)$ with a long-range potential. We impose the following assumption on $V(x)$:

$$\text{(A)}\quad \begin{cases} V(x) \text{ is a real-valued } C^\infty\text{-function on } \boldsymbol{R}^n \text{ and for some constant } \varepsilon_0>0 \\ D_x^\alpha V(x)=O(|x|^{-|\alpha|-\varepsilon_0}) \quad \text{as} \quad |x|\to\infty \\ \text{for all multi-index } \alpha. \end{cases}$$

Here for $\alpha=(\alpha_1, \cdots, \alpha_n)$, $D_x^\alpha=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$ and $|\alpha|=\alpha_1+\cdots+\alpha_n$.

As is well-known, if f belongs to the absolutely continuous subspace for H, the local position probability of e^{-itH} decays in the sense that for any $R>0$

$$\int_{|x|<R}|e^{-itH}f(x)|^2dx \longrightarrow 0 \qquad \text{as} \quad |x|\to\infty.$$

It is a rather difficult problem to obtain the rate of decay. In order to study it, one usually considers the operator norm of e^{-itH} in various function spaces different from $L^2=L^2(\boldsymbol{R}^n)$. A convenient choice is the so-called weighted L^2-spaces, and one studies the operator norm in L^2 of $\langle x\rangle^{-\sigma}e^{-itH}\langle x\rangle^{-\rho}$ $(\sigma, \rho>0)$, where $\langle x\rangle=(1+|x|^2)^{1/2}$. In our previous work [2], we have already proved some decay rates for e^{-itH}. Combining the result of [2] with the estimates for the parametrix of e^{-itH} introduced in [5] enables us to prove the following

THEOREM 1. *Let $\chi(\lambda)\in C^\infty(\boldsymbol{R}^1)$ be such that for some $d>0$, $\chi(\lambda)=1$ if $\lambda>2d$, $\chi(\lambda)=0$ if $\lambda<d$. Then for any $s\geqq 0$, there exists a constant $C_s>0$ such that*

$$\|\langle x\rangle^{-s}e^{-itH}\chi(H)\langle x\rangle^{-s}\|\leqq C_s(1+|t|)^{-s},$$

for any $t\in\boldsymbol{R}^1$, where $\|\cdot\|$ is the operator norm in L^2.

This estimate is seen to be best possible if one examines the case of $H_0=-\Delta$. One can also allow some local singularities for V. Suppose V is split into two

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parts: $V=V_L+V_S$, where V_L satisfies the assumption (A) and V_S is a real function of compact support belonging to the Stummel class. Let $\chi(\lambda)$ be as above. Then one can show for any $s\geqq 0$

$$\|\langle x\rangle^{-s}e^{-itH}\chi(H)(H+i)^{-1}\langle x\rangle^{-s}\|\leqq C_s(1+|t|)^{-s},$$

which we do not prove here, however.

Decay rates for scattering states have been studied by many authors. Assuming sufficiently rapid decay on the potential V, Rauch [12] and Jensen-Kato [7] studied the operator $e^{-itH}P_{ac}$, where P_{ac} denotes the projection onto the absolutely continuous subspace for H. More general elliptic operators were studied by Murata [10], [11]. In these cases, a delicate problem (that of resonance) arises from the low energy part of $e^{-itH}P_{ac}$. As for the long-range potential, Kitada [9] studied the high-energy part of e^{-itH}. In the recent work of Jensen-Mourre-Perry [8], they obtained a weaker estimate than Theorem 1 using the simpler commutator method. Cycon-Perry [1] also discussed the decay property of the high-energy part. Combining the results of [1] and [8], one can derive almost the same results as ours. Hewever, we develope here our own method.

The notation used in this paper are as follows: For $x\in \boldsymbol{R}^n$, $\langle x\rangle=(1+|x|^2)^{1/2}$, $\hat{x}=x/|x|$. For a Banach space X, $\boldsymbol{B}(X;X)$ denotes the totality of bounded linear operators on X. $C_0^\infty(\boldsymbol{R}^n)$ is the space of smooth functions on $\boldsymbol{R}^n$ with compact support. $\hat{f}(\xi)$ means the Fourier transform:

$$\hat{f}(\xi)=\int e^{-ix\xi}f(x)\bar{d}x\,, \qquad \bar{d}x=(2\pi)^{-n/2}dx\,.$$

By a F. I. Op. and a Ps. D. Op. we mean a Fourier integral operator and a pseudodifferential operator, respectively.

§1. Proof of Theorem 1.

1.1. First we recall the parametrix for e^{-itH} introduced in [5]. Let $\varepsilon>0$ be a sufficiently small constant. Then there exists a real C^∞-function $\phi(x, \xi)$ on $\boldsymbol{R}^n\times\boldsymbol{R}^n$ satisfying

$$|\nabla_x\phi(x, \xi)|^2+V(x)=|\xi|^2 \tag{1.1}$$

for $|x|>R$ for some $R>0$, $|\xi|>\varepsilon$, $\hat{x}\cdot\hat{\xi}>-1+\varepsilon/2$, and

$$|D_x^\alpha D_\xi^\beta(\phi(x, \xi)-x\cdot\xi)|\leqq C_{\alpha\beta}\langle x\rangle^{1-|\alpha|-\varepsilon_0}\langle\xi\rangle^{-1} \tag{1.2}$$

for $x, \xi\in\boldsymbol{R}^n$ ([5], Theorem 2.2). We construct $a(x, \xi)$ and $G(x, \xi)=e^{-i\phi(x,\xi)}(-\Delta+V-|\xi|^2)e^{i\phi(x,\xi)}a(x, \xi)$ in such a way that

$$|D_x^\alpha D_\xi^\beta(a(x, \xi)-1)|\leqq C_{\alpha\beta}\langle x\rangle^{-|\alpha|-\varepsilon_0}\langle\xi\rangle^{-1} \tag{1.3}$$

for $|\xi|>\varepsilon$, $\hat{x}\cdot\hat{\xi}>-1+\varepsilon$, $|x|>2R$ and

$$|D_x^\alpha D_\xi^\beta G(x, \xi)|\leqq C_{\alpha\beta N}\langle x\rangle^{-N}\langle\xi\rangle \tag{1.4}$$

for any $N>0$, $|\xi|>\varepsilon$, $\hat{x}\cdot\hat{\xi}>-1+\varepsilon$ ([5], Theorem 2.3). Choose a constant μ_+ such that $-1+\varepsilon<\mu_+<1$ and $b_+(x, \xi)\in C^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ satisfying

$$b_+(x, \xi)=0 \quad \text{if} \quad |\xi|<\varepsilon/2 \text{ or } \hat{x}\cdot\hat{\xi}<\mu_+ \text{ or } |x|<1, \tag{1.5}$$

$$|D_x^\alpha D_\xi^\beta b_+(x, \xi)|\leqq C_{\alpha\beta}\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}. \tag{1.6}$$

We define F. I. Op.'s A_+, B_+ and G_+ by

$$A_+f(x)=\int e^{i\phi(x,\xi)}a(x, \xi)\hat{f}(\xi)d\xi, \tag{1.7}$$

$$B_+f(x)=\int e^{i\phi(x,\xi)}b_+(x, \xi)\hat{f}(\xi)d\xi, \tag{1.8}$$

$$G_+f(x)=\int e^{i\phi(x,\xi)}G(x, \xi)\hat{f}(\xi)d\xi. \tag{1.9}$$

Our parametrix is then defined by

$$U_+(t)=A_+e^{-itH_0}B_+^*, \tag{1.10}$$

([5], Definition 2.4). A simple calculation shows that

$$e^{-itH}A_+B_+^*=U_+(t)-i\int_0^t e^{-i(t-s)H}G_+(s)ds, \tag{1.11}$$

where

$$G_+(t)=G_+e^{-itH_0}B_+^*. \tag{1.12}$$

The estimates for $U_+(t)$ and $G_+(t)$ are summarized in

Lemma 1.1. *For any ρ, $\sigma\geqq 0$ and $t>0$*

$$\|\langle x\rangle^{-(\rho+\sigma)}U_+(t)\langle x\rangle^\sigma\|\leqq C_{\rho\sigma}(1+t)^{-\rho}, \tag{1.13}$$

$$\|\langle x\rangle^\sigma\langle D_x\rangle^{-1}G_+(t)\langle x\rangle^\sigma\|\leqq C_{\rho\sigma}(1+t)^{-\rho}, \tag{1.14}$$

where $\langle D_x\rangle^{-1}$ is the Ps. D. Op. with symbol $\langle\xi\rangle^{-1}$,

$$\|\langle x\rangle^\sigma G_+(t)\langle x\rangle^\sigma\|\leqq C_{\rho\sigma}t^{-1}(1+t)^{-\rho}. \tag{1.15}$$

Proof. (1.13) and (1.14) have been proved in [5], Lemma 2.5. We prove (1.15), which must be treated carefully, since G_+ is not L^2-bounded. Choose a constant $\tilde{\mu}$ such that $-1+\varepsilon<\tilde{\mu}<\mu_+$ and C^∞-functions $\rho_1(t)$, $\rho_2(t)$ such that $\rho_1(t)+\rho_2(t)=1$, $\rho_1(t)=1$ for $t>\tilde{\mu}+3(\mu_+-\tilde{\mu})/4$, $\rho_2(t)=1$ for $t<\tilde{\mu}-(\tilde{\mu}+1-\varepsilon)/4$. Split $G_+=G_+^{(1)}+G_+^{(2)}$, where

$$G_+^{(j)}f(x)=\int e^{i\phi(x,\xi)}G(x, \xi)\rho_j(\hat{x}\cdot\hat{\xi})\hat{f}(\xi)d\xi.$$

The idea of the estimation for $G_+^{(1)}e^{-itH_0}B_+^*$ has been given in [5], § 1. Looking at the proof of [5], Lemma 1.1 carefully and noting (1.4), we see that

$$\langle x\rangle^\sigma G_+^{(1)}e^{-itH_0}B_+^*\langle x\rangle^\sigma=\sum_m^{\text{finite}} A_m e^{-itH_0}B_m(t)^*$$

where A_m is an L^2-bounded F. I. Op. and $B_m(t)$ is a F. I. Op. similar to B_+ with symbol $b_m(x, \xi, t)$ satisfying

$$|D_x^\alpha D_\xi^\beta b_m(x, \xi, t)| \leqq C_{m\alpha\beta} t^{-1}(1+t)^{-\rho}.$$

This shows that $\langle x\rangle^\sigma G_+^{(1)} e^{-itH_0} B_+^* \langle x\rangle^\sigma$ has the decay rate of $t^{-1}(1+t)^{-\rho}$. We can treat $\langle x\rangle^\sigma G_+^{(2)} e^{-itH_0} B_+^* \langle x\rangle^\sigma$ similarly, if we look at the proof of [5], Lemma 1.3 carefully. □

1.2. Let $\chi(\lambda)$ be as in Theorem 1. In [2], Theorem 3.1, we eave already shown the following decay rates for e^{-itH}:

$$\|\langle x\rangle^{-\rho} e^{-itH} \chi(H) \langle x\rangle^{-\rho-2}\| \leqq C_\rho (1+|t|)^{-\rho}, \quad \rho \geqq 3, \tag{1.16}$$

$$\|\langle x\rangle^{-\rho} e^{-itH} (H+i) \chi(H) \langle x\rangle^{-\rho-2}\| \leqq C_\rho |t|^{-\rho}, \quad \rho \geqq 3. \tag{1.17}$$

We also note the following proposition whose proof will be given in § 2.

Proposition 1.2. *For any* $\phi \in C_0^\infty(\boldsymbol{R}^1)$ *and any* $N>0$,

$$\langle x\rangle^N \phi(H) \langle x\rangle^{-N} \in \boldsymbol{B}(L^2; L^2).$$

1.3. With the above preparations one can show the following lemma which is a generalization of the propagation properties for e^{-itH_0} (see [5], § 1 and also [6]).

Lemma 1.3. *Let* $P_\pm$ *be the Ps. D. Op.'s with symbols* $p_\pm(x, \xi)$ *such that for some constants* $0<\varepsilon<1$ *and* $-1<\mu_\pm<1$,

$$p_+(x, \xi)=0 \quad \textit{if} \ |\xi|<\varepsilon \ \textit{or} \ \hat{x}\cdot\hat{\varepsilon}<\mu_+ \ \textit{or} \ |x|<1,$$

$$p_-(x, \xi)=0 \quad \textit{if} \ |\xi|<\varepsilon \ \textit{or} \ \hat{x}\cdot\hat{\xi}>\mu_- \ \textit{or} \ |x|<1,$$

$$|D_x^\alpha D_\xi^\beta p_\pm(x, \xi)| \leqq C_{\alpha\beta} \langle x\rangle^{-|\alpha|} \langle \xi\rangle^{-|\beta|}.$$

Let $\chi(\lambda)$ *be as in Theorem* 1. *Then we have*

$$\|\langle x\rangle^{-(\rho+\sigma)} e^{-itH} \chi(H) P_\pm \langle x\rangle^\sigma\| \leqq C(1+|t|)^{-\rho} \tag{1.18}$$

for $\sigma \geqq 0$, $\rho \geqq 3$ *and* $\pm t>2$, *respectively.*

Proof. We prove the lemma for P_+ and $t \geqq 2$. By (1.11) we have for $t \geqq 2$

$$\begin{aligned}
&\|\langle x\rangle^{-(\rho+\sigma)} e^{-itH} \chi(H) A_+ B_+^* \langle x\rangle^\sigma\| \\
&\leqq \|\langle x\rangle^{-(\rho+\sigma)} \chi(H) \langle x\rangle^{\rho+\sigma}\| \times \|\langle x\rangle^{-(\rho+\sigma)} U_+(t) \langle x\rangle^\sigma\| \\
&\quad + \int_0^{t/2} \|\langle x\rangle^{-(\rho+\sigma)} \chi(H)(H+i) e^{-i(t-s)H} \langle x\rangle^{-(\rho+\sigma+2)}\| \\
&\quad \times \|\langle x\rangle^{\rho+\sigma+2} (H+i)^{-1} G_+(s) \langle x\rangle^\sigma\| ds \\
&\quad + \int_{t/2}^t \|\langle x\rangle^{-(\rho+\sigma)} \chi(H) e^{-i(t-s)H} \langle x\rangle^{-(\rho+\sigma+2)}\| \\
&\quad \times \|\langle x\rangle^{\rho+\sigma+2} G_+(s) \langle x\rangle^\sigma\| ds
\end{aligned}$$

$$\leqq Ct^{-\rho}+C\int_0^t(1+t-s)^{-\rho}(1+s)^{-\rho}ds,$$

where we have used Proposition 1.2 and (1.13) for the first term (note that $\chi(H)=1-\phi(H)$ for some $\phi\in C_0^\infty(\boldsymbol{R})$), (1.14) and (1.17) for the second term, (1.15) and (1.16) for the third term. Therefore

$$\|\langle x\rangle^{-(\rho+\sigma)}e^{-itH}A_+B_+^*\langle x\rangle^\sigma\|\leqq C(1+t)^{-\rho}. \tag{1.19}$$

As has been proved in [5], Theorem 7.4, for any $N>0$ there exists a Ps. D. Op. P_N such that

$$P_+=A_+B_+^{(N)*}+P_N. \tag{1.20}$$

where $B_+^{(N)}$ is a F. I. Op. with symbol $b_N(x, \xi)$ satisfying (1.5) and (1.6) so that (1.19) holds with B_+ replaced by $B_+^{(N)}$, and the symbol $p_N(x, \xi)$ verifies

$$|D_x^\alpha D_\xi^\beta p_N(x, \xi)|\leqq C\langle x\rangle^{-N-|\alpha|}.$$

Choosing N large enough, we have in view of (1.16), (1.19), (1.20),

$$\begin{aligned}&\|\langle x\rangle^{-(\rho+\sigma)}e^{-itH}\chi(H)P_+\langle x\rangle^\sigma\|\\ &\leqq\|\langle x\rangle^{-(\rho+\sigma)}e^{-itH}\chi(H)A_+B_+^{(N)*}\langle x\rangle^\sigma\|+\|\langle x\rangle^{-(\rho+\sigma)}e^{-itH}\chi(H)P_N\langle x\rangle^\sigma\|\\ &\leqq C(1+t)^{-\rho},\end{aligned}$$

which proves the lemma. □

Taking the adjoint in Lemma 1.3, one can easily see that

$$\|\langle x\rangle^\sigma P_+\chi(H)e^{itH}\langle x\rangle^{-(\rho+\sigma)}\|\leqq C(1+|t|)^{-\rho} \tag{1.21}$$

for $\sigma\geqq 0$, $\rho\geqq 3$, $\pm t>2$, where we have used the asymptotic expansion of the symbol of $P_\pm^*$ ([4], Theorem 2.4).

1.4. We turn to the proof of Theorem 1. Let $\phi_0(\xi)$, $\phi_\infty(\xi)\in C^\infty(\boldsymbol{R}^n)$ be such that $\phi_0(\xi)+\phi_\infty(\xi)=1$, $\phi_0(\xi)=1$ for $|\xi|^2<d/2$, $\phi_0(\xi)=0$ if $|\xi|^2>3d/4$. Choose $\rho_\pm(t)\in C^\infty(\boldsymbol{R}^1)$ such that $\rho_+(t)+\rho_-(t)=1$, $\rho_+(t)=0$ if $t<-1/2$, $\rho_-(t)=0$ if $t>1/2$. Let A, B, $P_\pm$ be Ps. D. Op.'s with symbols $\phi_0(\xi)$, $\phi_0(x)\phi_\infty(\xi)$, $\phi_\infty(x)\rho_\pm(\hat{x}\cdot\hat{\xi})\phi_\infty(\xi)$, respectively. Since $A+B+P_++P_-=1$, we have

$$\begin{aligned}&\langle x\rangle^{-s}\chi(H)^2e^{-2itH}\langle x\rangle^{-s}\\ &=\langle x\rangle^{-s}\chi(H)e^{-itH}Ae^{-itH}\chi(H)\langle x\rangle^{-s}+\langle x\rangle^{-s}\chi(H)e^{-itH}Be^{-itH}\chi(H)\langle x\rangle^{-s}\\ &\quad+\langle x\rangle^{-s}\chi(H)e^{-itH}P_+e^{-itH}\chi(H)\langle x\rangle^{-s}+\langle x\rangle^{-s}\chi(H)e^{-itH}P_-e^{itH}\chi(H)\langle x\rangle^{-s}.\end{aligned} \tag{1.22}$$

Here we quote the following proposition whose proof will be given in § 2.

Proposition 1.4. *For any* $N>0$, $\langle x\rangle^N\chi(H)A\langle x\rangle^N\in\boldsymbol{B}(L^2; L^2)$.

Using this proposition (1.16) and the fact that the symbol of B is compactly supported in x, we see that the norms of the first and the second terms of (1.22) are bounded from above by

$$\|\langle x\rangle^{-s}\chi(H)e^{-itH}\langle x\rangle^{-s-2}\|\leqq C(1+|t|)^{-s}.$$

In view of (1.18) and (1.21), we see that the norms of the third and the fourth terms are majorized by

$$\|\langle x\rangle^{-s}\chi(H)e^{-itH}P_{+}\|+\|P_{-}e^{-itH}\chi(H)\langle x\rangle^{-s}\|\leqq C(1+t)^{-s}$$

for $s\geqq 3$ and sufficiently large $t>0$ (take $\sigma=0$, $\rho=s$), which proves the theorem for $t\geqq 0$, $s\geqq 3$. Since the case $s=0$ is evident, the case $s\leqq 3$ follows from this by an interpolation.

§ 2. Asymptotic expansion of functions of *H*.

2.1. First we prove Proposition 1.2. Since

$$\langle x\rangle^{N}e^{-itH}\langle x\rangle^{-N}(H+i)^{-N}=e^{-itH}(H+i)^{-N}$$
$$-i\int_{0}^{t}e^{-i(t-s)H}[\langle x\rangle^{N}, H]e^{-isH}\langle x\rangle^{-N}(H+i)^{-N}ds,$$

one can show by induction on N,

$$\|\langle x\rangle^{N}e^{-itH}\langle x\rangle^{-N}(H+i)^{-N}\|\leqq C_{N}(1+|t|)^{N}.$$

Using the relation

$$\phi(H)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}\check{\phi}(t)e^{-itH}dt,$$

where $\check{\phi}(t)$ is the inverse Fourier transform of ϕ, we see that $\langle x\rangle^{N}\phi(H)\langle x\rangle^{-N}(H+i)^{-N}$ is bounded, from which follows Proposition 1.2.

Proposition 1.4 can be derived from the following theorem.

Theorem 2.1. *Let $\phi(\lambda)\in C_0^{\infty}(\boldsymbol{R}^1)$. Then for any $N\geqq 2$,*

$$\phi(H)=\phi(H_0)+\sum_{m=1}^{N-1}P_m\phi^{(m)}(H_0)+R_N, \tag{2.1}$$

where $\phi^{(m)}(\lambda)=(d/d\lambda)^m\phi(\lambda)$, P_m is a Ps. D. Op. with symbol $p_m(x, \xi)$ such that

$$|D_x^{\alpha}D_{\xi}^{\beta}p_m(x, \xi)|\leqq C_{\alpha\beta m}\langle x\rangle^{-|\alpha|-m\varepsilon_0},$$

$$\langle x\rangle^{N\varepsilon_0/2}R_N\langle x\rangle^{N\varepsilon_0/2}\in\boldsymbol{B}(L^2; L^2). \tag{2.2}$$

In particular, $p_1(x, \xi)=V(x)$.

Remark 2.2. As ca be seen from the proof given later, P_m is a polynomial of V and the multiple commutators of H_0 and V.

In order to prove Proposition 1.4, we choose $\psi(\lambda)\in C_0^{\infty}(\boldsymbol{R}^1)$ such that $\psi(\lambda)=1-\chi(\lambda)$ for $\lambda>0$ and $\psi(H)+\chi(H)=1$. From Theorem 2.1 it follows that

$$\psi(H)=\psi(H_0)+\sum_{m=1}^{k-1}P_m\psi^{(m)}(H_0)+R_k$$

with P_m and R_k having the properties stated above. Then

$$\langle x\rangle^N \chi(H) A\langle x\rangle^N$$
$$=\langle x\rangle^N(1-\phi(H))A\langle x\rangle^N$$
$$=\langle x\rangle^N \chi(H_0) A\langle x\rangle^N-\sum_{m=1}^{k-1}\langle x\rangle^N P_m \phi^{(m)}(H_0) A\langle x\rangle^N-\langle x\rangle^N R_k A\langle x\rangle^N .$$

Recall that A is a Ps. D. Op. with symbol $\phi_0(\xi)$ and the supports of $\phi_0(\xi)$ and $\chi(|\xi|^2)$, $\phi^{(m)}(|\xi|^2)$ $(m\geqq 1)$ are disjoint. Then we have

$$\langle x\rangle^N \chi(H) A\langle x\rangle^N=-\langle x\rangle^N R_k A\langle x\rangle^N .$$

Choosing k large enough and using (2.2), we conclude Proposition 1.4.

2.2. We turn to the proof of Theorem 2.1. It is convenient to introduce a class of Ps. D. Op.'s. We define: a Ps. D. Op. P belongs to $S(\sigma, m)$ if its symbol $p(x, \xi)$ verifies

$$|D_x^\alpha D_\xi^\beta p(x, \xi)| \leqq C_{\alpha\beta}\langle x\rangle^{-\sigma-|\alpha|}\langle\xi\rangle^{m-|\beta|} .$$

Note that if $P\in S(\sigma, m)$, $[H_0, P]=H_0P-PH_0\in S(\sigma+1, m+1)$.

Now, we construct Q_m $(m=1, 2, \cdots)$ in such a way that

$$Q_1=-iV , \quad Q_m=-i[H_0, Q_{m-1}] \quad (m\geqq 2). \tag{2.3}$$

One can see by induction that

$$Q_m\in S(m-1+\varepsilon_0, m-1), \qquad m=1, 2, \cdots . \tag{2.4}$$

Let us define

$$U_t=e^{-itH} \quad (t\geqq 0), \qquad U_t=0 \quad (t<0),$$
$$U_t^0=e^{-itH_0} \quad (t\geqq 0), \qquad U_t^0=0 \quad (t<0).$$

Then we have

$$[U_t^0, Q_m]=U_t^0 Q_{m+1}*U_t=\int_0^t e^{-i(t-s)H_0}Q_{m+1}e^{-isH_0}ds ,$$

where $*$ denotes the convolution. We consider the perturbation expansion for U_t:

$$\begin{aligned} U_t=U_t^0+(-i)(U_t^0V)*U_t^0+(-i)^2(U_t^0V)*(U_t^0V)*U_t^0 \\ +\cdots+(-i)^N(U_t^0V)*(U_t^0V)*\cdots *U_t . \end{aligned} \tag{2.5}$$

The idea of the proof consists in calculating the multiple commutator of V and U_t^0. Since $(t^nU_t^0)*U_t^0=(t^{n+1}/n+1)U_t^0$, we have

$$\begin{aligned} -iU_t^0V*U_t^0 &=(U_t^0Q_1)*U_t^0 \\ &=(Q_1U_t^0)*U_t^0+(U_t^0Q_2*U_t^0)*U_t^0 \\ &=Q_1(U_t^0*U_t^0)+Q_2(U_t^0*U_t^0)*U_t^0+((U_t^0Q_3*U_t^0)*U_t^0)*U_t^0 \\ &=Q_1tU_t^0+Q_2\frac{t^2}{2!}U_t^0+Q_3\frac{t^3}{3!}U_t^0+\cdots+R_N^{(1)}(t), \end{aligned}$$

where $\|\langle x\rangle^{N\varepsilon_0/2}R_N^{(1)}(t)\langle x\rangle^{N\varepsilon_0/2}(H_0+i)^{-N}\|$ is of polynomial growth in $t\geqq 0$, here we use the estimate of $\langle x\rangle^m e^{-itH_0}\langle x\rangle^{-m}(H_0+i)^{-m}$ proved at the beginning of this section. Repeating this procedure for each term of the right-hand side of (2.5), we have for $t\geqq 0$

$$U_t=U_t^0+P_1tU_t^0+P_2\frac{t^2}{2!}U_t^0+\cdots+R_N(t), \tag{2.6}$$

where $P_m\in S(m\varepsilon_0, l(m))$ with an integer $l(m)$ depending on m, $\|\langle x\rangle^{N\varepsilon_0/2}R_N(t)\langle x\rangle^{N\varepsilon_0/2}(H+i)^{-N}\|$ is of polynomial growth in $t\geqq 0$, and $P_1=-iV$. (2.6) also holds for $t\leqq 0$. We multiply the inverse Fourier transform of $\phi(\lambda)$ to (2.6) and integrate with respect to t to obtain

$$\phi(H)=\phi(H_0)+\sum_{m=1}^{N-1}\tilde{P}_m\phi^{(m)}(H_0)+R_N. \tag{2.7}$$

where $\tilde{P}_m\in S(m\varepsilon_0, 0)$, $P_1=V$ and

$$\langle x\rangle^{N\varepsilon_0/2}R_N\langle x\rangle^{N\varepsilon_0/2}(H+i)^{-N}\in \boldsymbol{B}(L^2; L^2). \tag{2.8}$$

In order to complete the proof, choose $\psi(\lambda)\in C_0^\infty(\boldsymbol{R}^1)$ such that $\psi(\lambda)=1$ on supp ϕ. Then by (2.7)

$$\begin{aligned}\phi(H)&=\psi(H)\phi(H)\\ &=\psi(H)\phi(H_0)+\sum_{m=1}^{N-1}\psi(H)\tilde{P}\psi^{(m)}(H_0)+\psi(H)R_N.\end{aligned}$$

From (2.8) it follows that $\langle x\rangle^{N\varepsilon_0/2}\psi(H)R_N\langle x\rangle^{N\varepsilon_0/2}\in \boldsymbol{B}(L^2; L^2)$. Since $\psi(H)$ admits an asymptotic expansion similar to (2.7), we have

$$\begin{aligned}&\psi(H)\phi(H_0)+\sum_{m=1}^{N-1}\psi(H)\tilde{P}_m\phi^{(m)}(H_0)\\ &=\phi(H_0)+\sum_{m=1}^{N-1}\tilde{P}_m\phi^{(m)}(H_0)+\tilde{R}_N,\end{aligned}$$

where $\tilde{R}_N$ satisfies (2.2). This completes the proof.

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References

[1] H.L. Cycon and P. Perry, Local time-decay of high energy scattering states for the Schrödinger equation, (preprint).
[2] H. Isozaki, Differentiability of generalized Fourier transforms associated with Schrödinger operators, J. Math. Kyoto Univ., **25** (1985), 789-806.
[3] H. Isizaki and H. Kiɿada, Modified wave operators with time-independent modifiers, J. Fac. Sci. Univ. Tokyo Sec. IA, **32** (1985), 77-104.
[4] H. Isozaki and H. Kitada, Micro-local resolvent estimates for two-body Schrödinger operators, J. Functional Anal., **57** (1984), 270-300.
[5] H. Isozaki and H. Kitada, A remark on the micro-local resolvent estimates for two-body Schrödinger operators, Publ. R.I.M.S. Kyoto Univ., **21** (1985), 889-910.
[6] H. Isozaki and H. Kitada, Asymptotic behavior of the scattering amplitude at high

energies, Differential Equations, I. W. Knowels and R. T. Lewis (Editors), Elsevier Science Publishers B. V., 1984, 327-334.

[7] A. Jensen and T. Kato, Spectral properties of Schrödinger oprators and time-decay of the wave functions, Duke Mrth. J., **46** (1979), 583-611.

[8] A. Jensen, E. Mourre and P. Perry, Multiple commutator estimates and resolvent smoothness in quantum scattering theory, Ann. Inst. H. Poincaré, Sect. A, **41** (1984), 207-225.

[9] H. Kitada, Time-decay of the high-energy part of the solution for a Schrödinger equation, (preprint).

[10] M. Murata, Rate of decay of local energy and spectral properties of elliptic operators, Japan. J. Math., **6** (1980), 77-127.

[11] M. Murata, Asymptotic expansions in time for solutions of Schrödinger type equations J. Funct. Anal., **49** (1982), 10-56.

[12] J. Rauch, Local decay of scattering solutions to Schrödinger equations, Comm. Math. Phys., **61** (1978), 149-168.