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On well-posedness of the Cauchy problem for p-parabolic systems

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§ 1. Introduction.

We are concerned with the Cauchy problem for the following p-parabolic systems

$$\frac{d}{dt}U(x,t)=\mathcal{A}(x,t;D)U(x,t)+F(x,t), \qquad (x,t)\in R^n\times[0,T] \tag{1.1}$$

$$U(x,0)=U_0(x)\in H^p(R^n), \tag{1.2}$$

where $U(x, t)$ and $U_0(x)$ are m-vectors, and

$$\mathcal{A}(x,t;D)=\mathcal{H}(x,t;D)\Lambda^p+\mathcal{B}(x,t;D). \tag{1.3}$$

Here $\widehat{(\Lambda u)}(\xi)=|\xi|\hat{u}(\xi)$ and p is a positive number. $\mathcal{H}(x, t; \xi)$ is homogenuous of degree 0 in ξ and all its derivatives $\partial_x^\beta\partial_\xi^\alpha\mathcal{H}(x, t; \xi)$ are assumed to be bounded for $(x, \xi)\in R^n\times\{\xi:|\xi|\geq 1\}$. $\mathcal{B}(x, t; \xi)$ belongs to the class $S^{p_0}_{1,0}$, $0\leq p_0<p$, modulo smoothing operators. $\mathcal{H}(x, t; \xi)$ and $\mathcal{B}(x, t; \xi)$ are Hölder continuous in t, (see section 3).

Historically, p-parabolic systems were defined by I.G. Petrowsky [2] for systems of differential operators. However, we can start our considerations from systems of pseudo-differential operators. We believe that this will have good applications in the future. Here we assumed only $p>0$. Assume also that $F(x, t)$ satisfies, for some $\sigma\in(0, 1]$,

$$\|F(x,t)-F(x,\tau)\|\leq C|t-\tau|^\sigma, \qquad \text{for any } t, \tau\in[0,T]. \tag{1.4}$$

We suppose there exists a positive constant δ, such that it holds

$$\operatorname{Re}\lambda_j(x,t;\xi)\leq-\delta, \qquad \xi\in S^{n-1}_\xi, \tag{1.5}$$

where $\lambda_j(x, t, \xi)$, $(j=1, 2, \cdots, m)$ are the roots of the equation

$$\det(\lambda I-\mathcal{H}(x,t;\xi))=0.$$

I.G. Petrowsky [2] treated this problem with constant coefficients. Note that S.O. Eidel'man [9] has studied this problem but his point of view is different from

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ours. Also S. Mizohata [8] treated this problem when the right-hand side $F(x, t)$ is continuous in t with values in H^p. Here we apply a theory of parabolic semi-group in order to consider the Cauchy problem (1.1)–(1.2) under the condition (1.4). P.E. Sobolevskii [3] and H. Tanabe [10] have has treated the following evolution equation

$$\text{(P)} \qquad \frac{dv}{dt}+\mathcal{A}(t)v=f(t)$$

$$v(0)=v_0$$

under the following assumptions:

1) $\mathcal{A}(t)$ is a linear closed operator acting on a Banach space E and the domain of the definition D is dense and independent of t.

2) The operator $(\lambda I+\mathcal{A}(t))$ has a bounded inverse satisfying

$$\|(\lambda I+\mathcal{A}(t))^{-1}\|\leq\frac{C}{|\lambda|+1},$$

for any λ with Re $\lambda\geq\beta>0$, where C and β are positive constants.

3) There exists a positive constant C such that, for some $\sigma\in(0, 1]$,

$$\|(\mathcal{A}(t)-\mathcal{A}(\tau))\mathcal{A}_\beta^{-1}(s)\|\leq C|t-\tau|^\sigma,$$

holds for any t, τ, $s\in[0, T]$, where $\mathcal{A}_\beta(s)=\mathcal{A}(s)+\beta I$.

4) The function $f(t)$ satisfies the following Hölder condition

$$\|f(t)-f(\tau)\|\leq C|t-\tau|^\sigma, \qquad \text{for any } t,\ \tau\in[0, T].$$

He proved that for any $v_0\in E$ there exists a unique solution $v(x, t)$ for (P) which is continuous for all $t\in[0, T]$ and continuously differentiable for $t>0$. In case of $v_0\in D$, the solution is continuously differentiable for $t=0$ too.

In this article we shall apply the results of Soboleveskii and Tanabe on the Cauchy problem (1.1)—(1.2). Our purpose is to show that the operator $\mathcal{A}(x, t; D)$ satisfies the conditions 1), 2) and 3) mentioned above. These properties of $\mathcal{A}(x, t; D)$ are derived from the following a priori estimate (1.6) below. The statment of our theorem is given in detail at the end of § 3.

Fundamental Proposition. *If we take $\beta(>0)$ sufficiently large, then for any $t\in[0, T]$ and any $U\in H^p$ we have the following estimate*

$$\text{(1.6)} \qquad \|(\lambda I-\mathcal{A}(x, t; D)U\|^2\geq C\{\|U\|_p^2+(|\lambda|^2-\beta^2)\|U\|^2\}, \qquad \text{Re}\,\lambda\geq\beta>0,$$

where $\|\cdot\|$, $\|\cdot\|_p$ denote L^2 and H^p—norm respectively and C is a positive constant independent of t.

The proof of the fundamental proposition is not derived from Garding's inequality differently from the case $m=1$. In fact, consider the case when λ is real positive, we get

$$\|(\lambda I-\mathcal{A})U\|^2=\lambda^2\|U\|^2-2\lambda\,\text{Re}\,(\mathcal{A}U, U)+\|\mathcal{A}U\|^2.$$

First, since $\mathcal{A}$ is elliptic operator of order p, we obtain

$$||\mathcal{A}U||^2 \geq r||U||_p^2 - c||U||^2, \qquad (r \text{ is a positive constant}).$$

Hence, if we obtain a estimate of the form

$$(*) \qquad -\mathrm{Re}(AU, U) \geq -\text{const. } ||U||^2,$$

we arrive at the desired estimate. But this last estimate is not true in general in our case. We explain it by taking a simple example. Let $\mathcal{H} = \begin{bmatrix} -1 & 0 \\ a & -1 \end{bmatrix}$, $m=2$ and a is real. H satisfies (1.5) since its eigen-values are double of -1. Now consider, taking $\mathcal{A} = \mathcal{H}\Lambda^p$

$$-2\,\mathrm{Re}(\mathcal{A}U, U) = (S\Lambda^p U, U),$$

where $S = \begin{bmatrix} 2 & -a \\ -a & 2 \end{bmatrix}$. Using a unitary matrix N_0, we have $S_1 = N_0 S N_0^{-1} = \begin{bmatrix} 2-a & 0 \\ 0 & 2-a \end{bmatrix}$. Put $N_0 U = V = {}^t(v_1(x), v_2(x))$. Then taking account of $N_0^* = N_0^{-1}$, we get $(S\Lambda^p U, U) = (S_1 \Lambda^p V, V)$. By choosing as V, the function of the form $V_0 = {}^t(v(x), 0)$, we obtain

$$(S_1 \Lambda^p V_0, V_0) = (2-a)||\Lambda^{p/2} v_0||^2.$$

Denoting $N_0 U_0 = V_0$, we get

$$-2\,\mathrm{Re}(\mathcal{H}U_0, U_0) = -(a-2)||\Lambda^{p/2} U_0||^2.$$

Now since $v_1(x)$ is arbitrary, we see that the inequality of the form (*) fails to hold if $a > 2$.

The above example suggests that a little detailed argument will be required in order to obtain (1.6). For this purpose we use a partition of unity of the unite sphere S_ξ^{n-1} and a partition of unity in R_x^n as in S. Mizohata [8]. In actual case the inequality (1.6) is sharper and of different character than those obtained in [8]. Our main aim is to show clearly how to derive the inequality (1.6). In § 4 a direct application of the Cauchy problem for a higher order single equation is given.

§ 2. Proof of the fundamental proposition.

We start from the basic lemma due to Petrowsky [2].

Lemma 2.1. *Let $\mathcal{A} = (a_{ij})$ be a constant $m \times m$ matrix with eigen-values $\lambda_1, \lambda_2, \cdots, \lambda_m$, then there exists a constant non-singular matrix $C = (c_{ij})$, such that*

i) $C\mathcal{A} = DC$, *where*

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ a_{ij}^* & & \ddots \\ & & & \lambda_m \end{bmatrix}.$$

ii) $|\det C| \equiv 1$, $|c_{ij}| \leq 1$.

iii) $|a_{ij}^*| \leq (m-1)!2^m |\mathcal{A}|$, *where* $|\mathcal{A}| = \max_{i,j} |a_{ij}|$. (See [4]).

By applying this lemma to the matrix $\mathcal{H}(x_0, t_0; \xi_0)$, for an arbitrary point $(x_0, t_0; \xi_0) \in R^n \times [0, T] \times S_\xi^{n-1}$, there exists a constant non singular matrix $N_0(x_0, t_0, \xi_0)$ satisfying the properties in Lemma 2.1. Namely

$$N_0(x_0, t_0; \xi_0)\mathcal{H}(x_0, t_0; \xi_0) = \begin{bmatrix} \lambda_1(x_0, t_0; \xi_0) & & 0 \\ & \ddots & \\ h_{ij}^* & & \lambda_m(x_0, t_0; \xi_0) \end{bmatrix} N_0\,, \tag{2.1}$$

where $|h_{ij}^*| \leq (m-1)!2^m M_{\mathcal{H}}$ and $M_{\mathcal{H}} = \sup_{x,t,\xi} |\mathcal{H}(x, t; \xi)|$. Put

$$I_{\varepsilon_0} = \begin{pmatrix} 1 & & & & 0 \\ & \varepsilon_0 & & & \\ & & \varepsilon_0^2 & & \\ & & & \ddots & \\ 0 & & & & \varepsilon_0^{n-1} \end{pmatrix}.$$ We fix ε_0 (small) such as

$$\varepsilon_0 = \min(1, \delta/(m-1)!2^m M_{\mathcal{H}} 4m)\,. \tag{2.2}$$

Putting $N(x_0, t_0; \xi_0) = I_{\varepsilon_0} N_0(x, t_0; \xi_0)$, then we have

$$N(x_0, t_0; \xi_0)\mathcal{H}(x_0, t_0; \xi_0) = D_0(x_0, t_0; \xi_0)N(x_0, t_0; \xi_0)\,,$$

where

$$D_0 = \begin{pmatrix} \lambda_1(x_0, t_0; \xi_0) & & 0 \\ & \ddots & \\ h_{ij}^{**} & & \lambda_m(x_0, t_0; \xi_0) \end{pmatrix}$$

and

$$h_{ij}^{**}(x_0, t_0; \xi_0) = \varepsilon_0^{i-j} h_{ij}^*(x_0, t_0; \xi_1)\,.$$

Hence,

$$|h_{ij}^{**}| \leq \varepsilon_0 |h_{ij}^*| \leq (m-1)!2^m M_{\mathcal{H}} \varepsilon_1 \leqq \delta/4m\,. \tag{2.3}$$

Since $N = I_{\varepsilon_0} N_0$, then $|\det N| = |\det I_{\varepsilon_0}| = \varepsilon_0^{m(m-1)/2}$ holds. Considering $N_0^{-1} = (m_{ij})$, then $m_{ij} = \Delta_{ji}/\det N_0$, where Δ_{ji} is the (j, i) co-factor of N_0. Since $|\text{entry of } N_0| \leq 1$, by virture of Hadamard's inequality, we get $|\Delta_{ji}| \leq (m-1)^{(m-1)/2}$. Taking into account that $|\det N_0| = 1$, we see $|m_{ij}| \leq (m-1)^{(m-1)/2}$. Since $N^{-1} = N_0^{-1} I_{\varepsilon_0}^{-1}$, so it holds

$$|\text{entry of } N^{-1}| \leq (m-1)^{(m-1)/2} \varepsilon_0^{-(m-1)}\,. \tag{2.4}$$

The above results lead to

Lemma 2.2. *The matrix* $N(x_0, t_0; \xi_0)$ *satisfies the following property*

$$|\det \boldsymbol{N}(x_0, t_0; \xi_0)| = \varepsilon_0^{m(m-1)/2},$$

$$|\mathit{entry\ of}\ \boldsymbol{N}^{-1}(x_0, t_0; \xi_0)| \leq (m-1)^{(m-1)/2}\varepsilon_0^{-(m-1)}.$$

For $(x, t_0; \xi) \in R^n \times [0, T] \times S_\xi^{n-1}$, we decompose $\boldsymbol{N}(x_0, t_0; \xi_0)H(x, t_0; \xi)\times \boldsymbol{N}^{-1}(x, t_0; \xi_0)$ as follows:

$$\begin{aligned}(2.5)\qquad & \boldsymbol{N}(x_0, t_0; \xi_0)\mathcal{H}(x, t_0; \xi)\boldsymbol{N}^{-1}(x_0, t_0; \xi_0) \\ &= \boldsymbol{N}(x_0, t_0; \xi_0)\mathcal{H}(x_0, t_0; \xi_0)\boldsymbol{N}^{-1}(x_0, t_0; \xi_0) \\ &+\boldsymbol{N}(x_0, t_0; \xi_0)\mathcal{H}((x, t_0; \xi)-\mathcal{H}(x_0, t_0; \xi_0))\boldsymbol{N}^{-1}(x_0, t_0; \xi_0) \\ &\equiv D_0(x_0, t_0; \xi_0)+\tilde{D}_0(x, x_0, t_0; \xi_0; \xi_0).\end{aligned}$$

Observed that it holds

$$(2.6)\qquad |h_{ij}(x, t_0; \xi)-h_{ij}(x_0, t_0; \xi_0)| \leq c_0|\xi-\xi_0|+c_0'|x-x_0|,$$

where

$$(2.7)\qquad \begin{cases} c_0 = \pi \sum\limits_k \sup\limits_{i,j,x,t,\xi} \left|\dfrac{\partial h_{ij}}{\partial \xi_k}(t, x; \xi)\right|, \\ c_0' = \sum\limits_{k=1}^n \sup\limits_{i,j,x,t,\xi} \left|\dfrac{\partial h_{ij}}{\partial x_k}(x, t; \xi)\right|, \qquad i, j = 1, 2, \cdots, m. \end{cases}$$

Denote $\tilde{D}(x_0, x_0\ t_0; \xi; \xi_0)=(d_{ij}(x, x_0, t_0; \xi; \xi_0))$. In view of Lemma 2.2 and |entry of $\boldsymbol{N}(x_0, t_0; \xi_0)| \leq 1$, we obtain

$$(2.8)\qquad |d_{ij}(x, x_0, t_0; \xi; \xi_0)| \leq m^2(m-1)^{(m-1)/2}\varepsilon_0^{-(m-1)}\tilde{c}_0(|\xi-\xi_0|+|x-x_0|),$$

where $\tilde{c}_0=\max(c_0, c_0')$. If

$$(2.9)\qquad |\xi-\xi_0|+|x-x_0| \leq \delta\,\varepsilon_0^{(m-1)}/\{8m^3(m-1)^{(m-1)/2}\tilde{c}_0\} = 2\varepsilon,$$

then

$$(2.10)\qquad |d_{ij}(x, x_0, t_0; \xi; \xi_0)| \leq \delta/8m.$$

In view of (2.2) we express ε in more explicit form

$$\begin{aligned}(2.10)\qquad \varepsilon &= \varepsilon(\delta, m, c_0, M_{\mathcal{H}}) \\ &= \delta/\{16m^3(m-1)^{(m-1)/2}c_0\} \min(1, \delta/\{(m-1)!2^m 4m M_{\mathcal{H}}\})^{m-1}.\end{aligned}$$

The condition (2.9) follows if (x, ζ) satisfies

$$(2.12)\qquad |x-x_0| \leq \varepsilon \quad \text{and} \quad |\xi-\xi_0| \leq \varepsilon.$$

Summing up the above results we state

Proposition 2.1. *Denoting*

(2.13) $$N(x_0, t_0; \xi_0)H(x, t_0; \xi)N^{-1}(x_0, t_0; \xi_0)$$
$$=\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_m \end{pmatrix} + \begin{pmatrix} 0 & & & 0 \\ & \ddots & & \\ & h^{**}_{ij} & \ddots & \\ & & & 0 \end{pmatrix} + (d_{ij}(x, x_0, t_0; \xi; \xi_0)),$$

we have the following properties

$$|h^{**}_{ij}| \leq \delta/4m\,,$$
$$|d_{ij}(x, x_0, t_0; \xi; \xi_0)| \leq \delta/8m \qquad \textit{if } |x-x_0| \leq \varepsilon \textit{ and } |\xi-\xi_0| \leq \varepsilon\,.$$

Remark 1. The quantity ε which is defined by (2.11) is independent of $(x_0, t_0; \xi_0)$.

Partition of unity. On S^{n-1}_ξ we choose finite points $\xi_1, \xi_3, \cdots, \xi_l$ satisfying the following property. For any point $\xi \in S^{n-1}_\xi$, there exists at least one point, say ξ_p, such that

$$|\xi-\xi_p| \leq \varepsilon/4\,.$$

Now, for each j we define a function $\tilde{\alpha}_i(\xi)=\tilde{\alpha}(\xi-\xi_j)$, where $\tilde{\alpha}_j(\xi) \in C^\infty_0$ satisfies $0 \leq \tilde{\alpha}_j(\xi) \leq 1$ and $=1$ for $|\xi-\xi_j| \leq \varepsilon/4$, $=0$ for $|\xi-\xi_j| \geq \varepsilon/2$. Since $\sum_j \tilde{\alpha}_j(\xi) \geq 1$ for any ξ, we define $\alpha_j(\xi)=\tilde{\alpha}_j(\xi)/\{(\sum_j \tilde{\alpha}_j(\xi)^2\}^{1/2}$. Then $\alpha_j(\xi)$ has the same support as $\tilde{\alpha}_j(\xi)$ and it holds

$$\sum_{j=1}^{l} \alpha_j(\xi)^2 = 1\,.$$

On the other hand we define a partition of unity in R^n_x. Let x_i be a η-lattice point $(m_1\eta, m_2\eta, \cdots, m_n\eta)$, where $m_i \in Z$, $(i=1, 2, \cdots, n)$ and $\eta=\varepsilon/4\sqrt{n}$. Now, we define for each i a function $\tilde{\beta}_i(x)=\tilde{\beta}(x-x_i)$, where $\tilde{\beta}(x) \in C^\infty_0$, $=1$ for $|x| \leq \varepsilon/4$, $=0$ for $|x| \geq \varepsilon/2$, $0 \leq \tilde{\beta}_i(x) \leq 1$. Since $\sum \tilde{\beta}_i(x)$ is bounded and larger than 1, we define

$$\beta_i(x) = \tilde{\beta}_i(x)/\{\sum_i \tilde{\beta}_i(x)^2\}^{1/2}\,.$$

Then $\beta_i(x)$ has the same support as $\tilde{\beta}_i(x)$ and it holds

$$\sum_{i=1}^{\infty} \beta_i(x)^2 = 1\,.$$

For $t \in [0, T]$, we can associate $\{N(x_i, t; \xi_j)\}$, $1 \leq i, j \leq m$, which was explained in Proposition 2.1, replacing $(x_0, t_0; \xi_0)$ by $(x_i, t; \xi_j)$. Since t is fixed, we write $N(x_i, t; \xi_j)$ simply by $N_{ij}(t)$. Applying Proposition 2.1 by taking $(x_0, t_0; \xi_0)=(x_i, t; \xi_j)$, we get

(2.14) $$N_{ij}(t)\mathcal{H}(x, t; \xi)N_{ij}(t)^{-1} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{bmatrix} + D'_{ij}(t) + D''_{ij}(x, t; \xi)\,,$$

where $D'_{ij} = \begin{bmatrix} 0 & & 0 \\ & \ddots & \\ h^{ij}_{\kappa l} & \ddots & \\ & & 0 \end{bmatrix}$, $D''_{ij}(x, t; \xi) = (d^{ij}_{\kappa l}(x, t; \xi))_{1\le\kappa, l\le m}$.

Then we have

$$(2.15)\qquad \begin{cases} |h^{ij}_{\kappa l}| \le \delta/4m\,, \\ |d^{ij}_{\kappa l}(x, t; \xi)| \le \delta/8m\,, \qquad \text{for } \{x;\ |x-x_i| \le \varepsilon\} \text{ and } \{\xi;\ |\xi-\xi_j| \le \varepsilon\}\,. \end{cases}$$

For the proof of the inequality (1.6), it is convenient to introduce

$$(2.16)\qquad \|U\|_K^2 = \sum_{i,j} \|N_{ij}\alpha_j(D)\beta_i(x)U\|^2\,.$$

From (2.4), we see easily that

$$(2.17)\qquad c_2\|U\|^2 \le \|U\|_K^2 \le c_1\|U\|^2\,,$$

where c_1 and c_2 are positive constants independent of i and j.

Let us consider

$$(2.18)\qquad \mathcal{H}(x, t; D) = \mathcal{H}(x_i, t; \xi_j) + (\mathcal{H}(x, t; D) - \mathcal{H}(x_i, t; \xi_j))\,.$$

Denoting the constant matrix $\mathcal{H}(x_i, t; \xi_j)$ by $\mathcal{H}_{ij}(t)$, we get

$$(x.19)\qquad \begin{aligned} &N_{ij}(t)\mathcal{H}_{ij}(t)\alpha_j(D)\beta_i(x)\Lambda^p U \\ &= D^0_{ij}(t)N_{ij}(t)\alpha_j(D)\beta_i(x)\Lambda^p U + D'_{ij}(t)N_{ij}(t)\alpha_j(D)\beta_i(x)\Lambda^p U\,, \end{aligned}$$

where $D^0_{ij} = \begin{pmatrix} \lambda_1(x_i, t; \xi_j) & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & \lambda_m(x_i, t; \xi_j) \end{pmatrix}$.

By commuting Λ^p with $\beta_i(x)$ in the right-hand side, we obtain

$$(2.19)'\qquad \begin{aligned} &N_{ij}(t)\mathcal{H}_{ij}(t)\alpha_j(D)\beta_i(x)\Lambda^p U \\ &= (D^0_{ij}(t) + D'_{ij}(t))\Lambda^p N_{ij}(t)\alpha_j(D)\beta_i(x)U \\ &\quad + (D^0_{ij}(t) + D'_{ij}(t))N_{ij}(t)\alpha_j(D)[\beta_i(x), \Lambda^p]U\,. \end{aligned}$$

Next, we consider

$$(2.20)\qquad N_{ij}(t)\alpha_j(D)\beta_i(x)(\mathcal{H}(x, t; D) - \mathcal{H}_{ij}(t))\Lambda^p U\,.$$

Now we microlocalize the symbol $\mathcal{H}(x, t; \xi)$. First we define a smooth function $X_i(x)$, $x \in R^n$ as follows:

$X(x) = x$ for $|x| \le \varepsilon/2$, $= x_i$ for $|x| \ge \varepsilon$. If $\varepsilon/2 \le |x| \le \varepsilon$, then $|X(x)| \le \varepsilon$ and define $X_i(x) = X(x_i - x) + x_i$. Similarly, we define $\tilde{\xi}_j(\xi)$ for $\xi \in S^{n-1}_\xi$ as follows:

$\tilde{\xi}_j(\xi)=\xi$ for $|\xi-\xi_j|\leq\varepsilon/2$, $=\xi_j$ for $|\xi-\xi_j|\geq\varepsilon$. If $\varepsilon/2\leq|\xi-\xi_j|\leq\varepsilon$, then $|\tilde{\xi}_j(\xi)-\xi_j|\leq\varepsilon$.

With these preparations we return to (2.20). Since $X_i(x)=x$ on the support of $\beta_i(x)$, we obtain

$$\beta_i(x)\mathcal{H}(x, t; D) = \beta_i(x)\mathcal{H}(X_i(x), t; D).$$

Hence, by commuting $\beta_i(x)$ with $(\mathcal{H}(X_i(x), t; D)-\mathcal{H}_{ij}(t))$, we get

$$\begin{aligned}(2.21)\qquad & \boldsymbol{N}_{ij}(t)\alpha_j(D)\beta_i(x)(\mathcal{H}(x, t; D)-\mathcal{H}_{ij}(t))\Lambda^p U \\ &= \boldsymbol{N}_{ij}(t)\alpha_j(D)(\mathcal{H}(X_i(x), t; D)-\mathcal{H}_{ij}(t))\beta_i(x)\Lambda^p U \\ &\quad +\boldsymbol{N}_{ij}(t)\alpha_j(D)[\beta_i(x), \mathcal{H}(X_i(x); D)]\Lambda^p U.\end{aligned}$$

By commuting α_j with $(\mathcal{H}(X_i(x), t; D)-\mathcal{H}_{ij}(t))$, the first part of the right-hand side of (2.21) becomes

$$\begin{aligned}(2.22)\qquad & \boldsymbol{N}_{ij}(t)\mathcal{H}((X_i(x), t; D)-\mathcal{H}_{ij}(t))\alpha_j\beta_j\Lambda^p U \\ &\quad +\boldsymbol{N}_{ij}(t)[\alpha_j, \mathcal{H}(X_i(x), t; D)]\beta_i\Lambda^p U.\end{aligned}$$

Since $\mathcal{H}(X_i(x), t; \xi)\alpha_j(\xi)=\mathcal{H}(X_i(x), t; \xi(\tilde{\xi}_j))\alpha_j(\xi)$, denoting

$$(2.23)\qquad \mathcal{H}(X_i(x), t; \tilde{\xi}_j(\xi)) = \mathcal{H}_{ij}(x, t; \xi),$$

we get
$$\begin{aligned}& \boldsymbol{N}_{ij}(t)(\mathcal{H}(X_i(x), t; D)-\mathcal{H}_{ij}(t))\alpha_j(D)\beta_i\Lambda^p U \\ &= \boldsymbol{N}_{ij}(t)(\mathcal{H}_{ij}(x, t; D)-\mathcal{H}_{ij}(t))\alpha_j(D)\beta_i\Lambda^p U.\end{aligned}$$

Now denote again $D''_{ij}(x, t; \xi)=\boldsymbol{N}_{ij}(t)(\mathcal{H}_{ij}(x, t; \xi)-\mathcal{H}_{ij}(t))\boldsymbol{N}_{ij}^{-1}(t)$, then

$$\begin{aligned}(2.24)\qquad & \boldsymbol{N}_{ij}(t)(\mathcal{H}_{ij}(x, t; D)-\mathcal{H}_{ij}(t))\alpha_j(D)\beta_i\Lambda^p U \\ &= D''_{ij}(x, t; D)\Lambda^p\boldsymbol{N}_{ij}(t)\alpha_j\beta_i U+D''_{ij}(x, t; D)\boldsymbol{N}_{ij}(t)\alpha_j[\beta_i, \Lambda^p]U.\end{aligned}$$

Summing up the above relations, we get:

$$\begin{aligned}(2.25)\qquad & \boldsymbol{N}_{ij}(t)\alpha_j\beta_i\mathcal{H}(x, t; D)\Lambda^p U \\ &= (D^0_{ij}(t)+D'_{ij}(t)+D''_{ij}(x, t; D))\Lambda^p\boldsymbol{N}_{ij}(t)\alpha_j\beta_i U \\ &\quad +(D^0_{ij}(t)+D'_{ij}(t)+D''_{ij}(x, t; D))\boldsymbol{N}_{ij}(t)\alpha_j[\beta_i, \Lambda^p]U \\ &\quad +\boldsymbol{N}_{ij}(t)[\alpha_j, \mathcal{H}(X_i(x), t; D)]\beta_i\Lambda^p U \\ &\quad +\boldsymbol{N}_{ij}(t)\alpha_j[\beta_i, \mathcal{H}(X_i(x), t; D)]\Lambda^p U.\end{aligned}$$

Proof of the Fundamental Proposition. For $U\in H^p$, we denote

$$(2.26)\qquad \boldsymbol{N}_{ij}(t)\alpha_j\beta_i U = W_{ij}.$$

Since
$$\begin{aligned}& \boldsymbol{N}_{ij}(t)\alpha_j\beta_i(\lambda I-\mathcal{A}(x, t; D))U \\ &= \boldsymbol{N}_{ij}(t)\alpha_j\beta_i(\lambda I-\mathcal{H}(x, t; D)\Lambda^p)U-\boldsymbol{N}_{ij}(t)\alpha_j\beta_i\mathcal{B}(x, t; D)U,\end{aligned}$$

using (2.25) and (2.26), we get

$$(2.27)\qquad ||N_{ij}(t)\alpha_j\beta_i(\lambda I-\mathcal{A}(x,t;D))U||^2$$
$$\geq(1-\theta_1)||(\lambda I-D^0_{ij}(t)\Lambda^p-D'_{ij}(t)\Lambda^p-D''_{ij}(x,t;D)\Lambda^p)W_{ij}||^2$$
$$-M_1(\theta_1)||(D^0_{ij}(t)+D'_{ij}(t)+D''_{ij}(x,t;D))N_{ij}(t)\alpha_j[\beta_i,\Lambda^p]U$$
$$+N_{ij}(t)[\alpha_j,\mathcal{H}(X_i(x),t;D)]\beta_i\Lambda^pU+N_{ij}(t)\alpha_j[\beta_i,\mathcal{H}(X_i(x),t;D)]\Lambda^pU$$
$$+N_{ij}(t)\alpha_j\beta_i\mathcal{B}(x,t;D)U||^2,$$

where θ_1 is an arbitrary number in (0, 1).
Now denote the first term of the right-hand side of (2.27) by I_{ij}, then

$$I_{ij}\geq\frac{1}{2}(1-\theta_1)||(\lambda I-D^0_{ij}(t)\Lambda^p)W_{ij}||^2-(1-\theta_1)||(D'_{ij}+D''_{ij})\Lambda^pW_{ij}||^2.$$

In order to estimate the first part of I_{ij} from below, we use the following Lemma.

Lemma 2.2. *For any $(x_0, t_0; \xi_0)\in R^n\times[0, T]\times S^{n-1}_\xi$, and any θ $(0<\theta<1)$, we have*

$$(2.28)\qquad |\lambda-\lambda_k(x_0,t_0;\xi_0)|\xi|^p|^2\geq(1-\theta)\delta^2|\xi|^2+M(\theta)|\lambda|^2,\ \mathrm{Re}\,\lambda>0,$$

where $M(\theta)$ is a positive number independent of λ, $(x_0, t_0; \xi_0)$ and $|\xi|$, $k=1, 2, \cdots, m$.

Proof of Lemma 2.2. Put

$$(2.29)\qquad c_3=\max_k\sup_{x,t,\xi}\mathrm{Im}\,\lambda_k(x,t;\xi)|.$$

Then we devide the proof into two cases according to the location of λ.

Case 1. Where $|\mathrm{Im}\,\lambda|\leq 2c_3|\xi|^p$ holds. Denoting

$$J=\max_k|\lambda-\lambda_k(x_0,t_0;\xi_0)|\xi|^p|^2,\text{ then from (1.5)},$$

we get

$$J\geq(\mathrm{Re}\,\lambda-\mathrm{Re}\,\lambda_k(x_0,t_0;\xi_0)|\xi|^p)^2$$
$$\geq\delta^2|\xi|^{2p}+(\mathrm{Re}\,\lambda)^2.$$

Since $\theta\delta^2|\xi|^{2p}\geq\theta\delta^2/4c_3^2|\mathrm{Im}\,\lambda|^2$ holds. So, we have

$$J\geq(1-\theta)\delta^2|\xi|^{2p}+(\theta\delta^2/4c_3^2)|\mathrm{Im}\,\lambda|^2+(\mathrm{Re}\,\lambda)^2$$
$$\geq(1-\theta)\delta^2|\xi|^{2p}+\min\{1,\theta\delta^2/4c_3^2\}|\lambda|^2.$$

Case 2. Where $|\mathrm{Im}\,\lambda|\geq 2c_3|\xi|^p$ holds. We get

$$J\geq(\mathrm{Re}\,\lambda+\delta|\xi|^p)^2+(\mathrm{Im}\,\lambda-\mathrm{Im}\,\lambda_k(x_0,t_0;\xi_0)|\xi|^p))^2$$
$$\geq(\mathrm{Re}\,\lambda+\delta|\xi|^p)^2+(1/2|\mathrm{Im}\,\lambda|)^2$$
$$\geq\delta^2|\xi|^{2p}+1/4|\lambda|^2.$$

Hence, by taking $M(\theta)=\min(1/4, \delta^2\theta/4c_3^2)$, we get **Lemma 2.2.**

Applying this Lemma we obtain

$$||(\lambda I-D_{ij}^0)\Lambda^p W_{ij}||^2\geq(1-\theta)\delta^2||\Lambda^p W_{ij}||^2+M(\theta)|\lambda|^2||W_{ij}||^2.$$

The second part of I_{ij} is estimated as follows:
Since (2.15) yields

$$|\text{entry of } (D'_{ij}+D''_{ij}(x, t; \xi))|\leq 3\delta/8m, \qquad \xi\in S_\xi^{n-1}$$

applying sharp form Gårding's inequality, we have

$$\begin{aligned}&||(D'_{ij}+D''_{ij}(x, t; D))\Lambda^p W_{ij}||^2\\ &\leq\{(3\delta/8)^2+\varepsilon'\}||\Lambda^p W_{ij}||^2+C(\varepsilon')||W_{ij}||^2,\end{aligned}$$

where ε' is an arbitrary.
Hence

$$\begin{aligned}(2.30)\qquad I_{ij}\geq&(1-\theta_1)\{(1-\theta)\delta^2/2-(3/8)^2\delta^2-\varepsilon'\}||\Lambda^p W_{ij}||^2\\ &+(1-\theta_1)\left\{\frac{1}{2}M(\theta)|\lambda|^2-C(\varepsilon')\right\}||W_{ij}||^2.\end{aligned}$$

In order to estimate the second term of (2.27), we use the following Lemma.

Lemma 2.4. *Let $a(x; D)$ be a pseudo-differential operator of class $S_{1,0}^p$, $(p>0)$. Then there exsits a positive constant C, such that*

$$\sum_i ||[\beta_i, a(x; D)]u||^2\leq C||u||_{p-i}^2, \qquad \textit{for } u\in H^p.$$

(*See appendix for the proof*).

The decomposition $\Lambda^p=\Lambda^p\alpha_0(D)+\Lambda^p(1-\alpha_0(D))$, $(\alpha_0(\xi)\in C_0^\infty, \alpha_0(\xi)=1, |\xi|\leq 1)$ and the application of Lemma 2.4 yield

$$(2.31)\qquad \sum_j ||[\Lambda^p, \beta_i]U||^2\leq\text{const.}\ ||U||_{p-1}^2.$$

Put $\mathcal{H}_i(x, t; \xi)=\mathcal{H}_i(X_i(x), t; \xi)$ and decompose U as $U=U_1+U_2$. Here $U_1=\alpha_0(D)U$ and $U_2=(1-\alpha_0(D))U$. Concerning (2.27) we consider the following estimate

$$(2.32)\qquad \sum_{i,j} ||[\alpha_j, \mathcal{H}_i(x, t; D)]\beta_i\Lambda^p U||^2\leqq\text{const.}\ ||U||_{p-1}^2,$$

only for $U=U_2$, since the estimate (2.32) for $U=U_1$ is simpler. Then it suffices to show

$$\begin{aligned}(2.32)'\qquad &\sum_{i,j}\{||[\alpha_j, \mathcal{H}_i]\Lambda\beta_i\Lambda^{p-1}U||^2+||[\alpha_i, \mathcal{H}_i][\beta_i, \Lambda]\Lambda^{p-1}U||^2\}\\ &\leq\text{const.}\ ||\Lambda^{p-1}U||^2.\end{aligned}$$

Here we apply Calderon-Zygmund theorem. In fact, $\alpha_j(\xi)$ and $\mathcal{H}_i(x, t; \xi)$ satisfies the conditions of this theorem. So, $[\alpha_j, \mathcal{H}_j]\Lambda$ is a bounded operator in L^2. Hence the first term of (2.32)′ is estimated as follows:

$$\sum_{i,j} \| [\alpha_j, \mathcal{H}_i] \Lambda\beta_i\Lambda^{p-1} U\|^2 \leq C \sum_j \|\beta_i\Lambda^{p-1} U\|^2 = C\|\Lambda^{p-1} U\|^2 ,$$

where C is a positive constant independent of i and j.
The similar argument is valid also for $\alpha_j[\beta_i, \mathcal{H}_i(x, t; D)]\, \Lambda^p U$ in (2.27). In view of Lemma 2.3, the second term of (2.32)′ is smaller than $C\|U\|_{p-2}^2$. Finally, from (2.17) and (2.26) we get

$$\text{(2.33)} \qquad \sum_{i,j} \|\Lambda^p W_{ij}\|^2 = \sum_{i,j} \|N_{ij}\, \alpha_j\, \Lambda^p \beta_i\, U\|^2$$

$$\geq c_2 \sum_i \|\Lambda^p \beta_i\, U\|^2 \geq \frac{1}{2}\, c_2\|\Lambda^p\, U\|^2 - c_2'\|U\|_{p-1}^2 .$$

Summing up (2.27) for i, j and use the inequalities (2.30), (2.32) and (2.33), we obtain:

$$\text{(2.34)} \quad \|(\lambda I - \mathcal{A}(x, t; D)\, U\|_k^2$$

$$\geq (1-\theta_1) \left\{\left(\frac{1-\theta_1}{2} - \left(\frac{3}{8}\right)^2\right) \delta^2 - \varepsilon'\right\} \left(\frac{1}{2}\, c_2\|\Lambda^p U\|^2 - c_2'\|U\|_{q-1}^2\right)$$

$$+ (1-\theta_1) \left(\frac{1}{2}\, M(\theta)\, |\lambda|^2 - c(\varepsilon')\right) c_2\|U\|^2 - c_4\|U\|_{p-1}^2 ,$$

Now, we fix θ, θ_1 and ε' in such a way that the coefficients of $\|\Lambda^p U\|^2$ becomes positive. For example, we choose $\theta = \frac{1}{16}$, $\theta_1 = \frac{1}{5}$ and $\varepsilon' = \frac{5}{64}\, \delta^2$. Then we obtain

$$\text{(2.34)}' \qquad \|\lambda I - \mathcal{A}(x, t; D)\, U\|_K^2$$

$$\geq \frac{1}{10}\, c_2 \delta^2\|\Lambda^p U\|^2 - \left(\frac{1}{5}\, \delta^2 c_2' + c_4\right) \|U\|_{p-1}^2$$

$$+ \frac{4}{5}\, c_2 \left(\frac{1}{2}\, \mathcal{M}(\theta)\, |\lambda|^2 - C(\varepsilon')\right) \|U\|^2 .$$

Since the following inequalities

$$\|\Lambda^p U\|^2 \geq (1-\varepsilon'')\|U\|_p^2 - \mathcal{M}''(\varepsilon'')\|U\|^2 ,$$

$$\|U\|_{p-1}^2 \geq \tilde{\varepsilon}''\|U\|_p^2 + \tilde{\mathcal{M}}''(\varepsilon'')\|U\|^2 ,$$

hold for any positive numbers ε'' and $\tilde{\varepsilon}''$, we get

$$\text{(2.35)} \qquad \| (\lambda I - \mathcal{A}(x, t; D)\, U\|^2 \geq \delta_0\|U\|_p^2 + C(|\lambda|^2 - \beta^2)\|U\|^2 ,$$

where δ_0, β and C can be taken as positive numbers satisfying the following relations:

$$\delta_0 = \frac{1}{c_1} \left\{\frac{c_2}{10}(1-\varepsilon'')\, \delta^2 - \left(\frac{1}{5}\, c_2'\, \delta^2 + c_4\right) \tilde{\varepsilon}''\right\},\ C = \frac{2c_2}{5c_1}\, \mathcal{M}(\theta) ,$$

$$\beta^2 = \frac{1}{c_1 C}\left\{\frac{4}{5}c_2 C(\varepsilon')+\frac{1}{10}c_2\mathcal{M}''(\varepsilon'')\delta^2+\left(\frac{\delta^2}{5}c_2'+c_4\right)\tilde{\mathcal{M}}''(\varepsilon'')\right\}.$$

Thus the proof of the Fundamental Proposition is completed.

§ 3. The conditions of Sobolevskii and Tanabe.

In this section we show that the operator $A(x, t; D)$ which is defined by (1.3) satisfies the conditions 1), 2) and 3) in §1. As we will see below, these properties are derived from the inequality (1.6).

Propositin 3.1. *Assume* (1.6). *Then* $(\lambda I-\mathcal{A}(x, t; D))$ *defines a one to one surjective mapping from* H^p *onto* L^2, *for* $\mathrm{Re}\,\lambda>\beta_0$, *where* β_0 *is a positive number larger than* β.

Proof of Proposition 3.1. From (1.6) it follows that $(\lambda I-\mathcal{A}(x, t; D))$ is one to one mapping from H^p into L^2. Now, we show that the image $(\lambda I-\mathcal{A}(x, t; D)) H^p$ is closed in L^2. Indeed, $(\lambda I-\mathcal{A})U_n\rightarrow V_0$ implies that $\{U_n\}$ is a Cauchy sequence in H^p. Since H^p is complete, we get $U_n\rightarrow U_0$ in H^p and $(\lambda I-A)\, U_0=V_0$. Therefore, we have to show only that the image $(\lambda I-A)H_0$ is dense in L^2. We will show this by a contradition. If not dense, then there exists $\Psi(\neq 0)\in L^2$, such that

$$((\lambda I-\mathcal{A})\, U, \Psi)=0, \qquad \text{for all} \quad U\in H^p.$$

Hence, we have

$$(\bar{\lambda} I-\mathcal{A}^*)\,\Psi=0 \quad \text{in} \quad \mathcal{D}', \tag{3.1}$$

where $\mathcal{A}^*$ is the formal adjoint of $\mathcal{A}$ denoted by $\mathcal{A}^*=\mathcal{H}^*\Lambda^p+\tilde{\mathcal{B}}$, where $\mathcal{B}=[H^*, \Lambda^p]+B^*$. Since $\Psi\in L^2$, (3.1) shows that $\mathcal{A}^*\Psi=\bar{\lambda}I\Psi\in L^2$. We can show that $\Psi\in H^p$ in view of the Lemma 3.1 below. Now, we show that $\mathcal{A}^*$ satisfies the same conditions as $\mathcal{A}$. It is sufficient to prove that the eigen-values of $\mathcal{H}^*$ satisfy (1.5). Namely, putting

$$\mathcal{P}(\lambda)=\det\,(\lambda I-\mathcal{H}(x, t; \xi))=0\,,$$

we get

$$\overline{\mathcal{P}(\lambda)}=\det\,(\bar{\lambda} I-\mathcal{H}^*(x, t; \xi))=0\,,$$

which implies that the eigen-values of $\mathcal{H}^*$ are equal to $\bar{\lambda}_j$, where $j=1, 2, \cdots, m$.

In order to show that $\Psi\in H^p$, we will use the following Lemma:

Lemma 3.1. *Let* $C(x; D)$ *be a matrix of pseudo-differential operators of class* $S^p_{1,0}$ *and assume the following estimate holds*:

$$\|C(x; D)\, V\|\geq c_0\|V\|_p, \quad \textit{for} \quad V\in H^p, \tag{3.2}$$

where c_0 *is a positive constant. Then the assumptions* $V\in L^2$ *and* $C(x; D)\, V\in L^2$ *imply* $V\in H^p$.

(a simple proof is given in the appendix).

Therefore, we can use the inequality (1.6) and have

$$0=\|(\lambda I-\mathcal{A}^*)\Psi\|^2\geq C'\{(|\lambda|^2-\beta^2|)|\Psi\|^2+\|\Psi\|_p^2\}\ .$$

This inequality requires that $\Psi=0$. This is contradictory to our assumption that $\Psi\neq 0$. Thus the proof of Proposition 3.1 is completed.

Proposition 3.2. *Assume all the coefficients in* (1.1) *are smooth in x and Hölder continuous in t. Then the following inequality holds*

$$\|\{\mathcal{A}(t)-\mathcal{A}(\tau)\}\mathcal{A}_{\beta_0}(s)^{-1}\|\leq c|t-\tau|^\sigma,\quad \textit{for some}\quad \sigma\in(0,1]\,,$$

for any t, τ and $s\in[0, T]$, where $\mathcal{A}_{\beta_0}(s)=\mathcal{A}(s)-\beta_0 I$, $\beta_0>\beta$.

Proof of Proposition 3.2. For any β_0 satisfying $\beta_0>\beta$, from above Proposition 3.1 $\mathcal{A}_{\beta_0}(x, t; D)$ is a one to one linear mapping from H^p onto L^2. Moreover, it satisfies

$$\|\mathcal{A}_{\beta_0}(x,s;D)\,U\|\geq c'\|U\|_p\,,\quad \text{for}\quad U\in H^p\,,$$

where c' is a positive constant independent of s and U. This implies

$$\|V\|\geq c'\|\mathcal{A}_{\beta_0}(x,s;D)^{-1}\,V\|_p\,,\quad \text{for all}\quad V\in L^2\,.$$

All the coefficients appearing in (1.1) are supposed to be smooth in x and Hölder continuous in t. Namely

$$\max_{\substack{|\beta|\leq l_0\\|\alpha|\leq l_0}}\ \sup_{x\in R^n,\xi\in S_\xi^{n-1}}|\{H_{(\beta)}^{(\alpha)}(x,t;\xi)-H_{(\beta)}^{(\alpha)}(x,\tau;\xi)\}\,|\xi|^p|\leq c|t-\tau|^\sigma\,,$$

$$\max_{\substack{|\beta|\leq l_0\\|\alpha|\leq l_0}}\ \sup_{x\in R^n,\xi\in S_\xi^{n-1}}|\{\mathcal{B}_{(\beta)}^{(\alpha)}(x,t;\xi)-\mathcal{B}_{(\beta)}^{(\alpha)}(x,\tau;\xi)\}|\leq c|t-\tau|^\sigma\,,$$

where $l_0=\left[\frac{n}{2}\right]+2$. Since $\mathcal{A}(x, t; D)$ is a matrix of pseudo-differential operators of class $S_{1,0}^p$, we get

$$\begin{aligned}&\|\{\mathcal{A}(x,t;D)-\mathcal{A}(x,\tau;D)\}\,\mathcal{A}_{\beta_0}(x,s;D)^{-1}V\|\\&\leq c|t-\tau|^\sigma\,\|\mathcal{A}_{\beta_0}(x,t;D)^{-1}V\|_p\leq c\,c'^{-1}|t-\tau|^\sigma\|V\|\,.\end{aligned}$$

Thus the proof of Proposition 3.2 is completed.

Theorem. *For any initial data $U_0\in H^p$ and for any right-hand side $F(t)$ satisfying the Hölder condition* (1.4), *then there exists a unique solution $U(x, t)$ for the Cauchy problem* (1.1)—(1.2) *belonging to $C_t^0([0, T], H^p)\cap C_t^1([0, T], L^2)$.*

Proof of Theorem. Since all condions of Sobolevskii and Tanabe 1), 2) and 3) are satisfied, so the solution $U(x, t)$ satisfies

$$U(x, t)\in C_t^0([0, T], L^2)\cap C_t^1([0, T], L^2)\,.$$

In order to prove the solution $U(x, t)\in C_t^0([0, T], H^p)$, we apply the following inequality,

$$\|\mathcal{A}(t)\, U\|\geq c_1\|U\|_p-c_2\|U\|\,,$$

where c_1 and c_2 are positive constants independent of t. Thus the proof is complete.

Definition 1. $u(x, t)\in C_t^k([0, T], H^p)$ means that $u(x, t)$ is continuous in t up to the k-th derivative with values in H^p.

Definition 2. $\mathcal{B}$ is said to be a smoothing operator if $\mathcal{B}$ is bounded from L^2 to H^∞.

Example. Let $\partial_x^\alpha a(x, \xi)$, $(|\alpha|\geq 0)$ be symbols of Calderon-Zygmund operator. Then for any $\alpha(\xi)\in\mathcal{D}$, $a(x; D)\,\alpha_0(D)$ is a smoothing operator since it holds for any $k>0$

$$\|\alpha(x; D)\,\alpha_0(D)\,u\|_k\leq C_k\sum_{|\beta_1+\beta_2|\leq k}\|\partial_x^{\beta_1} a(x; D)\, D^{\beta_2}\,\alpha_0(D)\,u\|$$
$$\leq C_k'\|\alpha_0\, u\|\leq C_k''\|u\|\,.$$

§ 4. Higher-order p-parabolic equation.

In this section, we consider the Cauchy problem for a single higher-order equation as a direct application. Let

$$\text{(4.1)}\qquad \partial_t^m u+\sum_{j=1}^{m} a_j(x, t; \partial_x)\,\partial_t^{m-j}\, u=f(x, t)\,,\quad (x, t)\in R^n\times[0, T]\,,$$

$$\text{(4.2)}\qquad \partial_t^j\, u(x, 0)=u_j(x)\in H^{p(m-j)}\,,\quad (j=1, 2, \cdots, m)\,,$$

where $a_j(x, t; \partial_x)=\sum_{|\alpha|\leq pj} a_{j,\alpha}(x, t)\,\partial_x^\alpha$.
Supposing that the coefficients $a_{j,\alpha}$ are smooth in x and Hölder continuous in t. Putting $u_j=(1+\Lambda)^{p(m-1-j)}\,\partial_t^j\, u$, $0\leq j\leq m-1$,

$$U={}^t(u_0, \cdots, u_{m-1})\,,\quad F(x, t)={}^t(0, \cdots, f(x, t))$$

and

$$H(x, t; \xi)=\begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & \text{\Large 0} & \\ & & 0 & 1 & & \\ & & & \cdot\ \cdot & & \\ & & & & \cdot\ \cdot & \\ & & & & \cdot\ \cdot & \\ & & & & & \cdot\ \cdot \\ & & & & & 0\ 1 \\ -\overset{0}{a}_m(x, t; \xi)/|\xi|^{mp} & \cdots & & & & \cdots \end{pmatrix},\qquad B(x, t; \xi)=\begin{pmatrix} 0 & 0 \\ -b_m & \cdots\cdots \end{pmatrix},$$

where $\overset{\circ}{a}_j(x, t; \xi)$ is the homogenuous part of degree pj of $a_j(x, t; \xi)$. Denote $a_j = \overset{\circ}{a}_j + a'_j$. Then b_j is given by

$$b_j = \overset{\circ}{a}_j((1+\Lambda)^{-p(j-1)} - \Lambda^{-p(j-1)}) + a'_j(1+\Lambda)^{-p(j-1)} .$$

So, we can see that (4.1) and (4.2) reduce to (1.1) and (1.2). In fact, put

$$\mathcal{B} = \mathcal{B}(1-\alpha_0(D)) + \mathcal{B}\,\alpha_0(D) \equiv \mathcal{B}_1 + \mathcal{B}_2 ,$$

where $\alpha_0 \in \mathcal{D}$ and $\alpha_0 \equiv 1$ for $\{\xi : |\xi| < 1\}$. Then $\mathcal{B}_1$ belongs to $S^{p-1}_{1,0}$, and we see, from Definition 2 and Example, that B_2 is smoothing operator. Hence, we have

Corollary. *Assume* (1.5). *Then for any initial data* $u_j(x) \in H^{p(m-j)}$ *and any right-hand side* $f(x, t)$ *satisfying* (1.5) *there exists a unique solution* $u(x, t)$ *for the Cauchy problem* (4.1) *and* (4.2) *belonging to* $\bigcap_{j=0}^{m} C_t^{m-j}([0, T], H^{pj})$.

Appendix

1. *Proof of Lemma* 3.1. We use the method of mollifier. Let $\Phi(\xi) \in C_0^\infty, =1$ for $|\xi| \leq 1$, $=0$ for $|\xi| \geq 2$, $0 \leq \Phi(\xi) \leq 1$. Also we use the operator $\Phi(\varepsilon D)$ defined by $\widehat{\Phi(\varepsilon D)\, u} = \Phi(\varepsilon\xi)\, \hat{u}(\xi)$. Now, let us apply $\Phi(\varepsilon D)$ to $C(x; D)\, U = F \in L^2$, then we get

$$\Phi(\varepsilon D)\, F = \Phi(\varepsilon D)\, C(x; D)\, U . \tag{a.1}$$

First, the right-hand side of (a.1) can be expressed as follows:

$$C(x; D)\,(\Phi(\varepsilon D)\, U) + \sum_{|\nu|=1}^{N} \varepsilon^{|\nu|}\, \nu!^{-1}\, C_{(\nu)}(x; D)\,(\Phi^{(\nu)}(\varepsilon D)\, U) + r_{N,0}\, U .$$

Put $N = p$. Then we have

$$\|r_{N,0}\, U\| \leq C(N)\, \varepsilon \|U\| .$$

Next, replacing $\Phi(\varepsilon D)$ in (a.1) by $\Phi^{(\mu)}(\varepsilon D)$, $(|\mu| \leq N)$, we have

$$\Phi^{(\mu)}(\varepsilon D)\, F = \Phi^{(\mu)}(\varepsilon D)\, C(x, D)\, U . \tag{a.2}$$

Denote by $I_{\mu,\varepsilon}(x, D)$ the right-hand side of (a.2), then we get

$$I_{\mu,\varepsilon} = C(x; D)\,(\Phi^{(\mu)}(\varepsilon D)\, U) + \sum_{1 \leq |\nu| \leq N - |\mu|} \varepsilon^{|\nu|}\, \nu!^{-1}\, C_{(\nu)}(x; D)\,(\phi^{(\mu+\nu)}\, U) + r_{N,\mu}\, U . \tag{a.3}$$

In the same way we see that

$$\|\varepsilon^{|\mu|}\, r_{N,\mu} U\| \leq C(N)\, \varepsilon \|U\| . \tag{a.4}$$

Hence,

(a.5)
$$\varepsilon^{|\mu|}\,||\Phi^{(\mu)}(\varepsilon D)\,F||\geq\varepsilon^{|\mu|}\,||C(x, D)\,(\Phi^{(\mu)}(\varepsilon D)\,U)|| -C(N)\sum_{1\leq|\nu|\leq N-|\mu|}\varepsilon^{|\mu+\nu|}\,||\Phi^{(\mu+\nu)}(\varepsilon D)\,U||_p-\varepsilon C(N)||U||\,,$$

for $|\mu|\leq N$. Adding these inequalities after the multiplication $M^{|\mu|}$, where M is a large constant, we obtain

(a.6)
$$\sum_{|\mu|\leq N}\varepsilon^{|\mu|}\,M^{|\mu|}\,||\Phi^{(\mu)}\,F||\geq\sum_{0\leq|\mu|\leq N}\varepsilon^{|\mu|}\,M^{|\mu|}\,||C(x, D)\,(\Phi^{(\mu)}\,U)|| -C(N)\sum_{0\leq|\mu|\leq N}\;\sum_{1\leq|\mu|\leq N-|\mu|}M^{|\mu|}\varepsilon^{|\mu+\nu|}\,||\Phi^{(\mu+\nu)}\,U||_p -C(N)\,\varepsilon\sum_{|\mu|\leq N}M^{|\mu|}\,||U||\,.$$

Applying the inequality (3.2) to the first term of the right-hand of (a.6), then we get

(a.7)
$$\sum_{0\leq|\mu|\leq N}\varepsilon^{|\mu|}M^{|\mu|}\,||C(x;\,D)\,(\Phi^{(\mu)}\,U)||\geq c_0\sum_{0\leq|\mu|\leq N}(\varepsilon M)^{|\mu|}\,||\Phi^{(\mu)}\,U||_p\,.$$

Next, the second term of the right-hand side of (a.6) is estimated by

(a.8)
$$C'(N)\sum_{|\mu|\leq N}\;\sum_{1\leq|\nu|\leq N-|\mu|}\frac{1}{M^{|\nu|}}\,(M\varepsilon)^{|\mu+\nu|}\,||\Phi^{(\mu+\nu)}\,U||_p\,.$$

If M is taken large (taking into account that $|\nu|\geq1$), then we get

(a.9)
$$\sum_{|\mu|\leq N}(M\varepsilon)^{|\mu|}\,||\Phi^{(\mu)}\,F||$$
$$\geq\frac{1}{2}\,c_0\sum_{0\leq|\mu|\leq N}(M\varepsilon)^{|\mu|}\,||\Phi^{(\mu)}\,U||_p-\varepsilon C(N)\sum_{|\mu|\leq N}M^{|\mu|}\,||U||\,.$$

From this inequality we see that, $||\Phi(\varepsilon D)\,U||_p$ remains bounded when $\varepsilon(>0)$ tends to 0, this implies $U\in H^p$. Thus the proof is completed.

2. *Proof of Lemma* 2.4. Let $\zeta_i(x)\in C_0^\infty$, $=1$ for $|x-x_i|\leq3\eta$, $=0$ for $|x-x_i|\geq4\eta$ and $0\leq\zeta_i(x)\leq1$. Denoting by C_i the commuteter $[\beta_i,\,a(x;\,D)]$, we get

$$C_i\,u=[\beta_i,\,a(x,\,D)]\;\zeta_i(x)\,u+\beta_i(x)\,a(x;\,D)\,(1-\zeta_i(x))\,u\,.$$

First, we consider

$$\beta_i(x)\,a(x;\,D)\,(1-\zeta_i(x))\,u\,.$$

Let ω_i be the ball of radius η and of center x_i in R^n which is the support of $\beta_i(x)$, then for any $x\in\omega_i$ and $y\in\complement3\omega_i$, we get

$$a(x;\,D)\,(1-\zeta_i(x))\,u=\lim_{\varepsilon\to0}\iint e^{-\varepsilon|\xi|^2}\,e^{i(x-y)\xi}\,a(x;\,\xi)\,(1-\zeta_i(y))\,u(y)\,dyd\xi$$
$$=\lim_{\varepsilon\to0}\iint e^{-\varepsilon|\xi|^2}\left(\frac{(-\Delta_\xi)^k\,e^{i(x-y)\xi}}{|x-y|^{2k}}\right)a(x;\,\xi)\,(1-\zeta_i(y))\,u(y)\,dyd\xi\,,$$

By integration by parts in ξ and taking the limt as $\varepsilon\to0$, we obtain

$$|a(x;D)(1-\zeta_i(x))u| \leq C\int \frac{|(1-\zeta_i(y))u(y)|}{|x-y|^{2k}}\,dy$$
$$\leq C\int_{C3\omega_i} \frac{|u(y)|}{|x-y|^{2k}}\,dy\,,$$

for fixed $k > \frac{1}{2}(p+n+1)$. By Shwartz's inequality, we get

$$|a(x;D)(1-\zeta_i(x))u|^2 \leq C^2\int_{C3\omega_i}\frac{|u(y)|^2}{|x-y|^{2k}}\,dy\int_{C3\omega_i}\frac{dy}{|x-y|^{2k}}$$
$$\leq C^2\,C(n)\int_{C3\omega_i}\frac{|u(y)|^2}{|x-y|^{2k}}\,dy$$
$$\leq C^2\,C(n)\sum_j\int_{\omega_j}\frac{|\beta_j(y)\,u(y)|^2}{|x-y|^{2k}}\,dy\,,$$

where the sum is taken over all ω_j, such that

$$\mathrm{dis}\,(\omega_i,\,\omega_j)\geq\eta\,.$$

Hence, we obtain

$$\|\beta_i(x)\,a(x;D)\,(1-\zeta_i(x))\,u\|^2$$
$$\leq C'|\omega_0|\sum_j\frac{\|\beta_j\,u\|^2}{\mathrm{dis}\,(\omega_i,\,\omega_j)^{2k}}\,.$$

Finally, summing up in i, we get

$$\sum_i\|\beta_i(x)\,a(x;D)\,(1-\zeta_i(x))\,u\|^2 \leq C''\sum_i\sum_j\frac{\|\beta_j\,u\|^2}{\mathrm{dis}\,(\omega_i,\,\omega_j)^{2k}}$$
$$\leq C''\sum_j\|\beta_j\,u\|^2\left\{\sum_i\frac{1}{\mathrm{dis}\,(\omega_i,\,\omega_j)^{2k}}\right\}$$
$$\leq C''\,K\sum_j\|\beta_j\,u\|^2 = C'\,K\|u\|^2\,,$$

where C'' and K are constants dependent on n.

Next, we consider

(b.1)
$$[\beta_i(x),\,a(x;D)]\,\zeta_i(x)\,u$$
$$= -\{\sum_{1\leq|\nu|\leq N}\nu!^{-1}\,\beta_{i(\nu)}(x)\,a^{(\nu)}(x;D)+r_{N,i}(x;D)\}\,\zeta_i(x)\,u\,.$$

The first part of the right-hand side of (b.1) is estimated as follows:

(b.2)
$$\sum_{1\leq|\nu|\leq N}\nu!^{-1}\|\beta_{i(\nu)}(x)\,a^{(\nu)}(x;D)\,\zeta_i(x)\,u\|^2$$
$$\leq C(N)\sum_{1\leq|\nu|\leq N}\sup_x|\beta_{i(\nu)}(x)|^2\|a^{(\nu)}(x;D)\,\zeta_i(x)\,u\|^2_{\omega_i}$$
$$\leq C(N)\,c'\|\langle\Lambda\rangle^{p-1}\,\zeta_i(x)\,u\|^2\,,$$

where c' is a constant independent of i.

Considering the second part of the right-hand side of (b.1), we fix N as the smallest integer satisfying $p-N-1\leq 0$. Since $r_{N,i}(x;\,D)\in S_{1,0}^{p-N-1}$, we obtain

(b.3) $$\sum_i \|r_{N,i}(x;\,D)\,\zeta_i(x)\,u\|^2 \leq \text{const.} \sum_i \|\zeta_i(x)\,u\|_{p-1}^2$$

$$\leq \text{const.}\ \|u\|_{p-1}^2\,,$$

where const. is independent of i. Now from (b.2) and (b.3), we have Lemma 2.3 for $p\in(0,\,1]$.
For general $p>1$, we decompose

$$\langle\Lambda\rangle^{p-1}\,\zeta_i(x)=\zeta_i(x)\,\langle\Lambda\rangle^{p-1}+[\langle\Lambda\rangle^{p-1},\,\zeta_i(x)]\,.$$

Assume that Lemma 2.3 is true for $P\in(k,\,k+1]$. Then we see that, Lemma 2.3 holds for $p\in(k+1,\,k+2]$. So, Lemma 2.3 holds for all $p>0$.

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