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Linear stochastic partial differential equations with constant coefficients

By

Yasuhiro FUJITA

1. Introduction

Stochastic partial differential equations (S. P. D. E.'s, in short) are considered as P. D. E.'s having random influence. They arise in several areas of applied mathematics [cf. 10]. In this paper we study linear S. P. D. E.'s with constant coefficients, namely

$$(1.1)\qquad \begin{cases} du(t, x)=\sum a_{\alpha_1\cdots\alpha_N}(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_N)^{\alpha_N}u(t, x)dt \\ \qquad +\sum b_{\beta_1\cdots\beta_N}(\partial/\partial x_1)^{\beta_1}\cdots(\partial/\partial x_N)^{\beta_N}u(t, x)dW(t) \\ \qquad\qquad\qquad\qquad 0<t\leqq T,\ x\in \boldsymbol{R}^N, \\ u(0, x)=u_0(x), \qquad\qquad x\in \boldsymbol{R}^N, \end{cases}$$

where $a_{\alpha_1\cdots\alpha_N}$ and $b_{\beta_1\cdots\beta_N}$ are complex numbers and W is a one-dimensional Wiener process. Similarly to P. D. E.'s, the Fourier transform furnishes a convenient method for our problems. Applying this method, we characterize the solvability and well posedness of (1.1) by the polynomial generated by $\sum a_{\alpha_1\cdots\alpha_N}(i\xi_1)^{\alpha_1}\cdots(i\xi_N)^{\alpha_N}$ and $\sum b_{\beta_1\cdots\beta_N}(i\xi_1)^{\beta_1}\cdots(i\xi_N)^{\beta_N}$ ($\xi=(\xi_1, \cdots, \xi_N)\in \boldsymbol{R}^N$) where $i=\sqrt{-1}$ (see Theorems 1 and 2 in § 2). In § 3, we approximate a Wiener process W by a piecewise linear one [cf. 6, 8] and prove the convergence of an approximate solution (see Theorem 3). We also prove the stability on the perturbation of coefficients.

2. The well-posedness of S. P. D. E.'s

In this paper, we treat complex-valued functions on $\boldsymbol{R}^N$, unless otherwise stated.

For $\phi\in L^2(\boldsymbol{R}^N)$, denote by $\mathcal{F}\phi$ and $\mathcal{F}^*\phi$ the Fourier transform and the inverse Fourier transform of ϕ defined by

$$(\mathcal{F}\phi)(\xi)=\underset{A\to\infty}{\mathrm{l.i.m.}}\,(2\pi)^{-N/2}\int_{\{|x|\leq A\}} e^{-i\xi\cdot x}\phi(x)dx\,,$$

$$(\mathcal{F}^*\phi)(\xi)=\underset{A\to\infty}{\mathrm{l.i.m.}}\,(2\pi)^{-N/2}\int_{\{|\xi|\leq A\}} e^{i\xi\cdot x}\phi(\xi)d\xi\,.$$

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Here $\xi\cdot x$ is the inner product in $\boldsymbol{R}^N$ and l.i.m. stands for the limit in $L^2(\boldsymbol{R}^N)$. Let $H^p=H^p(\boldsymbol{R}^N)$ $(p\in\boldsymbol{N}_0)$ be the Hilbert space with the norm

$$\|\phi\|_p=\left[\int_{\boldsymbol{R}^N}(1+|\xi|)^{2p}|\mathcal{F}\phi(\xi)|^2d\xi\right]^{1/2}$$

where $\boldsymbol{N}_0$ denotes the set of all nonnegative integers. Put $H^\infty=\bigcap_{p=0}^{\infty}H^p$. H^∞ is a Fre'chet space. Let $C([0, T]; H^\infty)$ be a Fre'chet space with seminorms $\sup_{0\leq t\leq T}\|f(t)\|_p$ $(p\in\boldsymbol{N}_0)$. We denote by $(\cdot,\cdot)$ the inner product in $L^2(\boldsymbol{R}^N)$. For a multiple index $\alpha=(\alpha_1,\cdots,\alpha_N)$, put $|\alpha|=\sum_{i=1}^{N}\alpha_i$ and $D^\alpha=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_N)^{\alpha_N}$ $(i\xi)^\alpha=(i\xi_1)^{\alpha_1}\cdots(i\xi_N)^{\alpha_N}$ $(\xi\in\boldsymbol{R}^N)$. Let $C_b^n(\boldsymbol{R}^N)$ be the set of all bounded continuous functions f on $\boldsymbol{R}^N$, which have bounded continuous derivatives up to the order n, with its norm

$$|f|_n=\sup_{x\in\boldsymbol{R}^N}\sum_{|\alpha|\leq n}|D^\alpha f(x)|\,.$$

By Sobolev's embedding theorem, every f in $H^{[N/2]+1+p}$ $(p\in\boldsymbol{N}_0)$ has a $C_b^p(\boldsymbol{R}^N)$-modification and satisfies

$$|f|_p\leqq C(N, p)\|f\|_{[N/2]+1+p}$$

where $C(N, p)$ is a constant depending on N and p, and $[\cdot]$ is Gauss' symbol. Put $C_b^\infty(\boldsymbol{R}^N)=\bigcap_{p=0}^{\infty}C_b^p(\boldsymbol{R}^N)$.

Define differential operators $A(D)$ and $B(D)$ by $A(D)=\sum_{|\alpha|\leq l}a_\alpha D^\alpha$ and $B(D)=\sum_{|\beta|\leq m}b_\beta D^\beta$ for $a_\alpha, b_\beta\in\boldsymbol{C}$ (the space of complex numbers) and $l, m\in\boldsymbol{N}_0$. Let $W(t)$ be a one-dimensional Wiener process starting at 0 on a complete probability space $(\Omega, \mathcal{F}, \boldsymbol{P}, \{\mathcal{F}_t\}_{t\geq 0})$.

Fix $T>0$ arbitrarily throughout this paper. We consider the following S.P.D.E.

$$\text{(2.1)}\qquad\begin{cases} du(t, x)=A(D)u(t, x)dt+B(D)u(t, x)dW(t) & 0<t\leqq T,\ x\in\boldsymbol{R}^N,\\ u(0, x)=u_0(x)\in H^\infty, & x\in\boldsymbol{R}^N.\end{cases}$$

Definition 2.1. A $\boldsymbol{C}$-valued function u on $[0, T]\times\boldsymbol{R}^N\times\Omega$ is called a *solution* of (2.1), if the following conditions are satisfied;

a) $u(t, x, \omega)$ is $\mathcal{F}_t$-measurable for each (t, x),
b) $u(t, x, \omega)$ is continuous in (t, x) for each ω,
c) $u(t, x, \omega)$ is infinitely differentiable with respect to x for every (t, ω), and all derivatives are continuous in (t, x) for each ω,
d) On a certain $\tilde{\Omega}$ of full probability, the equation

$$\text{(2.2)}\qquad u(t, x)=u_0(x)+\int_0^t A(D)u(s, x)ds+\int_0^t B(D)u(s, x)dW(s)$$

holds for all (t, x). Here, we choose a smooth version of u_0.

Definition 2.2. (2.1) is said to be *well-posed*, if:

(1) For any $u_0 \in H^\infty$, (2.1) has a unique solution in $L^2(\Omega; C([0, T]; H^\infty))$. Here we denote by $L^2(\Omega; C([0, T]; H^\infty))$ a Fre'chet space with seminorms $[E \sup_{0 \le t \le T} \|u(t)\|_p^2]^{1/2}$ ($p \in \boldsymbol{N}_0$).

(2) For any $\varepsilon > 0$ and $p \in \boldsymbol{N}_0$, there exist $\delta = \delta(\varepsilon, T) > 0$ and $q = q(p) \in \boldsymbol{N}_0$ such that if $\|u_0\|_q < \delta$, then

$$\boldsymbol{E} \sup_{0 \le t \le T} \|u(t)\|_p^2 < \varepsilon .$$

Now, we state the main results of this section. Put, for $\varepsilon > 0$

$$H_\varepsilon(\xi) \equiv 2 Re A(i\xi) - (1-\varepsilon)\{Re B(i\xi)\}^2 + \{Im B(i\xi)\}^2 \qquad \xi \in \boldsymbol{R}^N .$$

Theorem 1. *Suppose that, with some* $\varepsilon > 0$, $H_\varepsilon(\xi)$ *is bounded from above on* $\boldsymbol{R}^N$. *Then,* (2.1) *has a solution* u *which belongs to* $C([0, T]; H^\infty)$ *for every* ω. *Moreover the uniqueness holds in the following sense. If* v *is a solution in* $C([0, T]; H^\infty)$, *then*

$$\boldsymbol{P}\{u(t, x) = v(t, x), \ ^\forall(t, x)\} = 1 . \tag{2.3}$$

Theorem 2. (2.1) *is well-posed, if and only if* $H_2(\xi)$ *is bounded from above on* $\boldsymbol{R}^N$.

We give three examples to illustrate theorems.

Example 1. Let $A(D) = 0$ and $B(D) = \sum_{j=1}^N (\partial/\partial x_j)^2$ ($\equiv \Delta$). Then, $H_\varepsilon(\xi)$ is given by

$$H_\varepsilon(\xi) = -(1-\varepsilon)\left\{\sum_{j=1}^N \xi_j^2\right\}^2 .$$

Hence, (2.1) has a unique solution. But, (2.1) is not well-posed.

Example 2. Let $A(D)$ be a Schrödinger operator given by $A(D) = i\Delta$. Then (2.1) is well posed, if and only if $B(D)$ is a multiplicative operator i.e. $B(D) = a_0 I$ for some $a_0 \in \boldsymbol{C}$.

Example 3. It is easy to check that the initial value problem

$$\begin{cases} du/dt(t, x) = -\Delta u(t, x) & 0 < t \leqq T, \ x \in \boldsymbol{R}^1, \\ u(0, x) = 1/(1+x^2) \in H^\infty, & x \in \boldsymbol{R}^1 \end{cases}$$

has no solution in $C([0, T]; H^\infty)$. But, S.P.D.E.

$$\begin{cases} du(t, x) = -\Delta u(t, x)dt + \Delta u(t, x)dW(t) & 0 < t \leqq T, \ x \in \boldsymbol{R}^1. \\ u(0, x) = 1/(1+x^2) \in H^\infty, & x \in \boldsymbol{R}^1, \end{cases}$$

has a unique solution in $C([0, T]; H^\infty)$, since

$$H_\varepsilon(\xi) = \xi^2 - (1-\varepsilon)\xi^4 .$$

To prove Theorem 1, we need a lemma.

Lemma 1. *For* $\varepsilon > 0$, *let*

$$f_\varepsilon(t)=\sup_{x\in\boldsymbol{R}}\exp[-\varepsilon t x^2+2xW(t)].$$

Then, $t^a f_\varepsilon(t)$ $(a>0)$ is continuous on $[0, T]$ with probability 1.

Proof. It is easy to see that

$$f_\varepsilon(t)=\begin{cases} 1 & t=0 \\ \exp[|W(t)|^2/\varepsilon t] & t>0. \end{cases}$$

Therefore, it is sufficient to show that $t^a f_\varepsilon(t)$ is continuous at $t=0$. Let δ be a sufficiently small constant. By the law of the iterated logarithm, we have, for $0<t<\delta$,

$$|W(t)|\leqq C_\delta[t\log\log(1/t)]^{1/2}$$

with some $C_\delta>0$. So we have

$$f_\varepsilon(t)\leqq[\log(1/t)]^{C_\delta^2/\varepsilon}.$$

Hence, the assertion of lemma is clear.

Hereafter, we denote by C_j $(j\geqq 1)$ the positive constant depending on only T.

Proof of Theorem 1. Define $U(\xi, t)$ by

$$U(\xi, t)=\exp[\tilde{H}(\xi, t)](\mathcal{F}u_0)(\xi) \tag{2.4}$$

where $\tilde{H}(\xi, t)=t[A(i\xi)-(1/2)\{B(i\xi)\}^2]+B(i\xi)W(t)$.
Using the inequality

$$|e^z-e^{\tilde{z}}|\leqq e(e^{Re z}+e^{Re\tilde{z}})|z-\tilde{z}| \qquad {}^\forall z, \tilde{z}\in\boldsymbol{C}, \tag{2.5}$$

we obtain for any $0\leqq s,\ t\leqq T$

$$\begin{aligned}|U(\xi, t)-U(\xi, s)|^2 &\leqq 2e^2\{\exp[2Re\tilde{H}(\xi, t)]+\exp[2Re\tilde{H}(\xi, s)]\} \\ &\qquad \times|\tilde{H}(\xi, t)-\tilde{H}(\xi, s)|^2|\mathcal{F}u_0(\xi)|^2 \\ &\leqq C_1(1+|\xi|)^{2(l\vee 2m)}|\mathcal{F}u_0(\xi)|^2\{|t-s|^2+|W(t)-W(s)|^2\} \\ &\qquad \times\{\exp[H_\varepsilon(\xi)t]\exp[2ReB(i\xi)W(t)-\varepsilon t\{ReB(i\xi)\}^2] \\ &\qquad\quad +\exp[H_\varepsilon(\xi)s]\exp[2ReB(i\xi)W(s)-\varepsilon s\{ReB(i\xi)\}^2]\} \\ &\leqq C_2(1+|\xi|)^{2(l\vee 2m)}|\mathcal{F}u_0(\xi)|^2\{f_\varepsilon(t)+f_\varepsilon(s)\} \\ &\qquad \times\{|t-s|^2+|W(t)-W(s)|^2\}\end{aligned}$$

where $l\vee 2m=\max\{l, 2m\}$. So we have for any $p\in\boldsymbol{N}_0$

$$\begin{aligned}&\int_{\boldsymbol{R}^N}(1+|\xi|)^{2p}|U(\xi, t)-U(\xi, s)|^2 d\xi \\ &\leqq C_2\|u_0\|^2_{p+(l\vee 2m)}\{f_\varepsilon(t)+f_\varepsilon(s)\}\{|t-s|^2+|W(t)-W(s)|^2\}\end{aligned} \tag{2.6}$$

with probability 1. By Lemma 1, $u\equiv\mathcal{F}^*U$ belongs to $C([0, T]; H^\infty)$ on a certain Ω_0 of full probability. Extend u to Ω_0^c by putting 0. Then, by Sobolev's theorem, $u(t, \omega)(x)$ have $C_b^\infty(\boldsymbol{R}^N)$-modification for each (t, ω). We also denote it by $u(t, \omega)(x)$. For each fixed $\omega\in\Omega$, we get for any $0\leqq s,\ t\leqq T$ and $h\in\boldsymbol{R}^N$

$$|u(t, \omega)(\cdot)-u(s, \omega)(\cdot+h)|_p\leqq C(N, p)[\|u(t, \omega)-u(s, \omega)\|_{[N/2]+1+p}$$
$$+\|u(s, \omega)(\cdot)-u(s, \omega)(\cdot+h)\|_{[N/2]+1+p}],$$

which converges to 0 as $t-s\to 0$ and $h\to 0$, since $u(\omega)$ belongs to $C([0, T]; H^\infty)$ for every ω. Hence u possesses the properties a)~c).

Next we shall show that u satisfies (2.2). Applying Ito's formula to $U(\xi, t)$, we get with probability 1

$$U(\xi, t)=\mathcal{F}u_0(\xi)+\int_0^t A(i\xi)U(\xi, s)ds+\int_0^t B(i\xi)U(\xi, s)dW(s) \tag{2.7}$$

for any $t\geqq 0$. Hence we get for any $v\in L^2(\boldsymbol{R}^N)$

$$(U(t), \mathcal{F}v)=(\mathcal{F}u_0, \mathcal{F}v)+\int_0^t (A(i\cdot)U(s), \mathcal{F}v)ds+\int_0^t (B(i\cdot)U(s), \mathcal{F}v)dW(s).$$

By Parsevel's equality, the above equality implies

$$(u(t), v)=(u_0, v)+\int_0^t (A(D)u(s), v)ds+\int_0^t (B(D)u(s), v)dW(s) \quad \boldsymbol{P}\text{—a.s.} \tag{2.8}$$

for any $t\geqq 0$. We choose a smooth function $\chi\geqq 0$ with compact support on $\boldsymbol{R}^N$ such that

$$\int_{\boldsymbol{R}^N}\chi(x)dx=1.$$

Putting $\chi_\delta(\cdot)=\delta^{-N}\chi((x-\cdot)/\delta)$ $(\delta>0)$, we have for any $\rho,\ \sigma>0$

$$\boldsymbol{P}\left\{\left|\int_0^t B(D)u(s)*\chi_\delta(x)dW(s)-\int_0^t B(D)u(s)(x)dW(s)\right|>\rho\right\}$$
$$\leqq\sigma+\boldsymbol{P}\left\{C^2(N, 0)\int_0^t\|B(D)u(s)*\chi_\delta-B(D)u(s)\|^2_{[N/2]+1}ds>\rho^2\sigma\right\}.$$

Put $v=\chi_\delta$ in (2.8). As $\delta\downarrow 0$, we can see that, for any x, (2.2) holds for any t, with probability 1. This yields that u is a solution of (2.1), with a suitable modification of the stochastic integral [cf. 7].

Finally we consider the uniqueness. Let v be a solution in $C([0, T]; H^\infty)$. Then $u-v$ is a solution of (2.1) with $(u-v)(0)=0$. Using Parsevel's equality, the following equality holds in $L^2(\boldsymbol{R}^N)$:

$$\mathcal{F}(u-v)(t)=\int_0^t A(i\cdot)\mathcal{F}(u-v)(s)ds+\int_0^t B(i\cdot)\mathcal{F}(u-v)(s)dW(s) \qquad \boldsymbol{P}\text{—a.s.}$$

for any $t\geqq 0$. Hence we get, for a.s. $(t, \xi)\in[0, T]\times\boldsymbol{R}^N$,

$$\mathcal{F}(u-v)(t, \xi)=0, \qquad \boldsymbol{P}\text{—a.s.} \quad .$$

Since u and v are continuous in (t, x), we conclude (2.3). This completes the proof.

To prove Theorem 2, we recall the following Lemma 2, which is a martingale inequality.

Lemma 2. *For* $a\in \boldsymbol{R}$, *let*

$$X_a(t)=\exp[aW(t)-(1/2)a^2t].$$

Then

$$\boldsymbol{E}\sup_{0\leq t\leq T}X_a(t)\leqq C(1+a^2)$$

where $C>0$ *is a constant depending on only* T.

Proof of Theorem 2. Consider $U(\xi, t)$ defined by (2.4). Since

$$|U(\xi, t)|^2=\exp[H_2(\xi)t]\exp[2ReB(i\xi)W(t)-(1/2)\{2ReB(i\xi)\}^2t]|\mathcal{F}u_0(\xi)|^2,$$

we get by Lemma 2

$$\boldsymbol{E}\sup_{0\leq t\leq T}|U(\xi, t)|^2\leqq C_3[1+\{2ReB(i\xi)\}^2]|\mathcal{F}u_0(\xi)|^2.$$

So we obtain for any $p\in \boldsymbol{N}_0$

$$\boldsymbol{E}\sup_{0\leq t\leq T}\int_{R^N}(1+|\xi|)^{2p}|U(\xi, t)|^2d\xi\leqq C_4\|u_0\|_{p+2m}^2. \tag{2.9}$$

By Theorem 1, $u\equiv\mathcal{F}^*U$ is a solution of (2.1). Moreover, it is a unique solution in $L^2(\Omega;C([0, T];H^\infty))$ by (2.9). The well posedness is also an easy consequence of (2.9). This completes the proof of "*if part*".

Next, we show "*only if part*". According to [4], the following two conditions are equivalent:

1) $H_2(\xi)$ is bounded from above on $\boldsymbol{R}^N$,
2) There exist positive constants K_1 and K_2 such that

$$H_2(\xi)\leqq K_1\log(1+|\xi|)+K_2$$

for any $\xi\in \boldsymbol{R}^N$.

So, if $H_2(\xi)$ is not bounded from above on $\boldsymbol{R}^N$, there exist ξ_n and a neighborhood V_n of ξ_n such that $H_2(\xi)\geqq n\log(1+|\xi|)$ $(\xi\in V_n)$ for each $n\in \boldsymbol{N}_0$. Without loss of generality, we may assume that $\inf_{\xi\in V_n}|\xi|\geqq(1/2)|\xi_n|$, $\sup_{\xi\in V_n}|\xi|\leqq 2|\xi_n|$ and $\lim_{n\to\infty}|\xi_n|=\infty$. Choose $f_n\in H^\infty$ such that $\|\mathcal{F}f_n\|_0=1$ and the support of $\mathcal{F}f_n$ is contained in V_n. Define $u^n(x, t)$ by

$$u^n(x, t)=\mathcal{F}^*[\exp[\tilde{H}(\cdot, t)]\mathcal{F}u_0^n(\cdot)](x)$$

where $u_0^n(\cdot)=f_n(\cdot)/[1+(1/2)|\xi_n|]^{nT/2}$. There exists a modification $\tilde{u}^n$ of u^n such that $\tilde{u}^n$ is a solution of (2.1) in $L^2(\Omega;C([0, T];H^\infty))$ with $\tilde{u}^n(0)=u_0^n$, since $\sup_{\xi\in V_n}H_2(\xi)<\infty$ for each n. We have for any $q\in \boldsymbol{N}_0$

$$\|u_0^n\|_q^2=\int_{V_n}(1+|\xi|)^{2q}|\mathcal{F}u_0^n(\xi)|^2d\xi\leqq(1+2|\xi_n|)^{2q}/[1+(1/2)|\xi_n|]^{nT}.$$

So, $\lim_{n\to\infty} u_0^n=0$ in H^∞. But, we have

$$\boldsymbol{E}\|\tilde{u}^n(t)\|_0^2=\int_{V_n}\exp[H_2(\xi)t]|\mathcal{F}u_0(\xi)|^2d\xi\geqq[1+(1/2)|\xi_n|]^{nt}/[1+(1/2)|\xi_n|]^{nT}.$$

Hence, $\limsup_{n\to\infty}\boldsymbol{E}\sup_{0\leq t\leq T}\|\tilde{u}^n(t)\|_0^2\geqq 1$. This implies that (2.1) is not well-posed. This completes the proof of "*only if part*".

3. The approximation and the stability

First we consider the approximation of (2.1). Let $W^n(t)$ be a piecewise linear approximation of $W(t)$ defined by

$$W^n(t)=W(k/n)+n\Delta_k W[t-(k/n)],\ k/n\leqq t<(k+1)/n$$

for $k=0, 1, 2, \cdots$ where $\Delta_k W=W((k+1)/n)-W(k/n)$. We consider P.D.E.

$$\text{(3.1)}\quad \begin{cases} du^n/dt(t, x)=[A(D)-(1/2)B^2(D)]u^n(t, x)+B(D)u^n(t, x)dW^n/dt(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0<t\leqq T,\ x\in\boldsymbol{R}^N, \\ u^n(0, x)=u_0(x)\in H^\infty, \qquad\qquad\qquad\qquad x\in\boldsymbol{R}^N. \end{cases}$$

Definition 2.3. A $\boldsymbol{C}$-valued function u^n on $[0, T]\times\boldsymbol{R}^N\times\Omega$ is called a *solution* of (3.1), if the following conditions are satisfied:

a) $u^n(t, x, \omega)$ is a measurable function of (t, x, ω),
b) $u^n(t, x, \omega)$ is continuous in (t, x) for each ω,
c) $u^n(t, x, \omega)$ is infinitely differentiable with respect to x for every (t, ω), and all the derivatives are continuous in (t, x) for each ω,
d) On a certain Ω' of full probability, the equation

$$\text{(3.2)}\quad u^n(t, x)=u_0(x)+\int_0^t[A(D)-(1/2)B^2(D)]u^n(s, x)ds + \int_0^t B(D)u^n(s, x)dW^n(s),$$

holds for all t and x.

Theorem 3. *Let* $u_0\in H^\infty$. *Suppose that* $H_1(\xi)$ *and* $|ReB(i\xi)|$ *are bounded from above on* $\boldsymbol{R}^N$. *Then,*

(1) (3.1) *has a unique solution in* $L^2(\Omega; C([0, T]; H^\infty))$,
(2) $\boldsymbol{E}\sup_{0\leq t\leq T}|u^n(t)-u(t)|_p^2\to 0\ (n\to\infty)$ *for any* $p\in\boldsymbol{N}_0$.

Proof. (1) Consider $U^n(\xi, t)$ defined by

$$U^n(\xi, t)=\exp[A(i\xi)t-(1/2)\{B(i\xi)\}^2t+B(i\xi)W^n(t)]\mathcal{F}u_0(\xi).$$

Since $2Re[A(i\xi)-(1/2)\{B(i\xi)\}^2]$ is bounded from above on $\boldsymbol{R}^N$, we get

$$\boldsymbol{E}\sup_{0\leq t\leq T}|U^n(\xi, t)|^2\leqq C_5\boldsymbol{E}\sup_{0\leq t\leq T}\exp[2ReB(i\xi)W^n(t)]|\mathcal{F}u_0(\xi)|^2.$$

Using the inequality $\sup_{0\leq\theta<1} e^{a\theta}\leqq 1+e^{a}$ $(a\in\boldsymbol{R})$, we get by Lemma 2

$$\begin{aligned}
&\boldsymbol{E}\sup_{0\leq t\leq T}\exp[2ReB(i\xi)W^{n}(t)]\\
&\leqq\boldsymbol{E}\sup_{0\leq k\leq[nT]}\ \sup_{k/n\leq t<(k+1)/n}\exp[2ReB(i\xi)\{W(k/n)[1-n(t-k/n)]\\
&\qquad\qquad+n(t-k/n)W((k+1)/n)\}]\\
&\leqq\boldsymbol{E}\Big\{1+\sup_{0\leq k\leq[nT]+1}\exp[2ReB(i\xi)W(k/n)]\Big\}^{2}\\
&\leqq 2\Big\{1+\boldsymbol{E}\sup_{0\leq t\leq T+1}\exp[4ReB(i\xi)W(t)]\Big\}\\
&\leqq C_{6}.
\end{aligned}$$

Hence we obtain for any $p\in\boldsymbol{N}_{0}$

$$\boldsymbol{E}\sup_{0\leq t\leq T}\int_{R^{N}}(1+|\xi|)^{2p}|U^{n}(\xi,t)|^{2}d\xi\leqq C_{7}\|u_{0}\|_{p}^{2}.$$

Similarly to the proof of Theorem 1, it is easy to see that a suitable modification of $\mathcal{F}^{*}U^{n}$ is a unique solution of (3.1) in $L^{2}(\Omega;C([0,T];H^{\infty}))$. This completes the proof of (1).

(2) First we remark that

$$\boldsymbol{E}\sup_{0\leq t\leq T}|W^{n}(t)-W(t)|^{4}\leqq C_{8}n^{-3/2}.$$

Indeed, we get

$$\begin{aligned}
&\boldsymbol{E}\sup_{0\leq t\leq T}|W^{n}(t)-W(t)|^{4}\\
&\leqq\boldsymbol{E}\sup_{0\leq k\leq[nT]}\ \sup_{k/n\leq t<(k+1)/n}|\{W(k/n)-W(t)\}\{1-n(t-k/n)\}\\
&\qquad\qquad+n(t-k/n)\{W((k+1)/n)-W(t)\}|^{4}\\
&\leqq C_{9}\Big[\Big\{\sum_{k=0}^{[nT]}\boldsymbol{E}\sup_{0\leq\theta<1/n}|W(k/n)-W(k/n+\theta)|^{8}\Big\}^{1/2}\\
&\qquad\qquad+\Big\{\sum_{k=0}^{[nT]}\boldsymbol{E}|W((k+1)/n)-W(k/n)|^{8}\Big\}^{1/2}\Big]\\
&\leqq C_{10}n^{-3/2}.
\end{aligned}$$

Using the inequality (2.5), we get

$$\begin{aligned}
&\boldsymbol{E}\sup_{0\leq t\leq T}|U^{n}(\xi,t)-U(\xi,t)|^{2}\\
&=\boldsymbol{E}\sup_{0\leq t\leq T}\exp[2ReA(i\xi)t-Re\{B(i\xi)\}^{2}t]\\
&\qquad\times|\exp[B(i\xi)W^{n}(t)]-\exp[B(i\xi)W(t)]|^{2}|\mathcal{F}u_{0}(\xi)|^{2}\\
&\leqq C_{11}|B(i\xi)|^{2}|\mathcal{F}u_{0}(\xi)|^{2}\boldsymbol{E}\sup_{0\leq t\leq T}[\exp\{2ReB(i\xi)W^{n}(t)\}\\
&\qquad\qquad+\exp\{2ReB(i\xi)W(t)\}]|W^{n}(t)-W(t)|^{2}
\end{aligned}$$

$$\leqq C_{12}|B(i\xi)|^2|\mathcal{F}u_0(\xi)|^2\Big[\boldsymbol{E}\sup_{0\leq t\leq T}\exp\{4ReB(i\xi)W^n(t)\}$$
$$+\boldsymbol{E}\sup_{0\leq t\leq T}\exp\{4ReB(i\xi)W(t)\}\Big]^{1/2}$$
$$\times\Big[\boldsymbol{E}\sup_{0\leq t\leq T}|W^n(t)-W(t)|^4\Big]^{1/2}.$$

By the proof of (1) and Lemma 2, we get

$$\boldsymbol{E}\sup_{0\leq t\leq T}\exp\{4ReB(i\xi)W^n(t)\}+\boldsymbol{E}\sup_{0\leq t\leq T}\exp\{4ReB(i\xi)W(t)\}\leqq C_{13}.$$

So, we get

$$\boldsymbol{E}\sup_{0\leq t\leq T}|U^n(\xi, t)-U(\xi, t)|^2\leqq C_{14}n^{-3/4}|B(i\xi)|^2|\mathcal{F}u_0(\xi)|^2.$$

Hence we get for any $p\in\boldsymbol{N}_0$

$$\boldsymbol{E}\sup_{0\leq t\leq T}\|u^n(t)-u(t)\|_p^2\leqq C_{15}n^{-3/4}\|u_0\|_{p+2m}^2.$$

This completes the proof by Sobolev's theorem.

Next, we consider the stability of (2.1). Let $\{a_\alpha^{(\kappa)}\}$ and $\{b_\beta^{(\kappa)}\}$ be two $\boldsymbol{C}$-valued sequences ($|\alpha|\leqq l$, $|\beta|\leqq m$). Define $A^{(\kappa)}(D)$, $B^{(\kappa)}(D)$ and $H_\varepsilon^{(\kappa)}(\xi)$ by $A(D)$, $B(D)$ and $H_\varepsilon(\xi)$ with $a_\alpha^{(\kappa)}$ and $b_\beta^{(\kappa)}$ respectively.

Theorem 4. *Let $u_0\in H^\infty$. Suppose that there exists $\varepsilon>2$ such that $\sup_\kappa\sup_{\xi\in R^N}H_\varepsilon^{(\kappa)}(\xi)<\infty$ and that $\sum_{|\alpha|\leq l}|a_\alpha^{(\kappa)}-a_\alpha|+\sum_{|\beta|\leq m}|b_\beta^{(\kappa)}-b_\beta|$ converges to 0 as $\kappa\to\infty$. Then*

$$\boldsymbol{E}\sup_{0\leq t\leq T}|u^{(\kappa)}(t)-u(t)|_p^2\to 0\quad(\kappa\to\infty)$$

for any $p\in\boldsymbol{N}_0$, where $u^{(\kappa)}$ is a solution of (2.1) *with $A^{(\kappa)}(D)$ and $B^{(\kappa)}(D)$.*

Proof. The existence and the uniqueness of solutions $u^{(\kappa)}$ and u are clear by our assumptions. Define $U^{(\kappa)}(\xi, t)$ and $\tilde{H}^{(\kappa)}(\xi, t)$ by $U(\xi, t)$ and $\tilde{H}(\xi, t)$ with $a_\alpha^{(\kappa)}$ and $b_\beta^{(\kappa)}$ respectively. Put

$$\theta^{(\kappa)}=\sum_{|\alpha|\leq l}|a_\alpha^{(\kappa)}-a_\alpha|^2+\sum_{|\beta|\leq m}|b_\beta^{(\kappa)}-b_\beta|^2.$$

Then we get by the inequality (2.5)

$$|U^{(\kappa)}(\xi, t)-U(\xi, t)|^2\leqq 2e^2[\exp\{2Re\tilde{H}^{(\kappa)}(\xi, t)\}+\exp\{2Re\tilde{H}(\xi, t)\}]$$
$$\times|\tilde{H}^{(\kappa)}(\xi, t)-\tilde{H}(\xi, t)|^2|\mathcal{F}u_0(\xi)|^2.$$

We choose $\sigma>1$ such that $\sigma<\varepsilon/2$. Let τ be the conjugate number of σ. Then we get by Lemma 2

$$\boldsymbol{E}\sup_{0\leq t\leq T}|\tilde{H}^{(\kappa)}(\xi, t)-\tilde{H}(\xi, t)|^2\exp\{2Re\tilde{H}^{(\kappa)}(\xi, t)\}$$
$$\leqq C_{16}\theta^{(\kappa)}(1+|\xi|)^{2(l\vee 2m)}\Big[\boldsymbol{E}\sup_{0\leq t\leq T}\exp\{H_2^{(\kappa)}(\xi)t\}$$
$$\times\exp\{2ReB^{(\kappa)}(i\xi)W(t)-(1/2)(2ReB^{(\kappa)}(i\xi))^2t\}\Big]$$

$$+C_{17}\theta^{(\kappa)}(1+|\xi|)^{2m}\Big[\boldsymbol{E}\sup_{0\le t\le T}|W(t)|^{2\tau}\Big]^{1/\tau}$$
$$\times\Big[\boldsymbol{E}\sup_{0\le t\le T}\exp\{\sigma H_{2\sigma}^{(\kappa)}(\xi)t\}\exp\{2\sigma ReB^{(\kappa)}(i\xi)W(t)-(1/2)\{2\sigma ReB^{(\kappa)}(i\xi)\}^2t\}\Big]^{1/\sigma}$$
$$\leqq C_{18}\theta^{(\kappa)}(1+|\xi|)^{2[(l\vee 2m)+m]}.$$

We can show in the same way

$$\boldsymbol{E}\sup_{0\le t\le T}|\tilde{H}^{(\kappa)}(\xi,t)-\tilde{H}(\xi,t)|\exp\{2Re\tilde{H}(\xi,t)\}\leqq C_{19}\theta^{(\kappa)}(1+|\xi|)^{2[(l\vee 2m)+m]}.$$

Hence we obtain

$$\boldsymbol{E}\sup_{0\le t\le T}|U^{(\kappa)}(\xi,t)-U(\xi,t)|^2\leqq C_{20}\theta^{(\kappa)}(1+|\xi|)^{2[(l\vee 2m)+m]}|\mathcal{F}u_0(\xi)|^2.$$

So, we obtain

$$\boldsymbol{E}\sup_{0\le t\le T}\|u^{(\kappa)}(t)-u(t)\|_p^2\leqq C_{20}\theta^{(\kappa)}\|u_0\|_{(l\vee 2m)+m+p}^2.$$

This completes the proof.

DEPARTMENT OF MATHEMATICS
AND SYSTEM FUNDAMENTALS
DIVISION OF SYSTEM SCIENCE,
KOBE UNIVERSITY

References

[1] G. Da Prato, Some results on linear stochastic evolution equations in Hilbert spaces by the semi-group methods, Stochastic Anal. Appl., **1** (1983), 57-88.
[2] G. Da Prato, M. Iannelli and L. Tubaro, Some results on linear stochastic differential equations in Hilbert spaces, Stochastics, **6** (1982), 105-116.
[3] G. Da Prato and L. Tubaro, Some results on semilinear stochastic differential equations in Hilbert spaces, Stochastics, **15** (1985), 271-281.
[4] L. Gårding, Linear hyperbolic partial differential equations with constant coefficients, Acta Math., **85** (1950), 1-62.
[5] A. Ichikawa, Linear stochastic evolution equations in Hilbert spaces, J. Differential Equations, **28** (1978), 266-283.
[6] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, North-Holland/Kodansha (1981).
[7] N. V. Krylov and B. L. Rozovskii, Cauchy problem for linear stochastic partial differential equations, Izv. Acad. Nauk SSSR Ser. Mat., **41** (6) (1977), 1329-1347.
[8] H. Kunita, Limit theorems for stochastic differential equations and stochastic flows of diffeomorphisms, Stochastic differential systems, Proceedings of the Bohn Conference, to appear.
[9] S. Mizohata, The Theory of Partial Differential Equations, Cambridge University Press (1973).
[10] E. Pardoux, Stochastic partial differential equations and filtering of diffusion processes, Stochastics, **3** (1979), 127-167.