J. Math. Kyoto Univ. (JMKYAZ)
29-1 (1989) 119-126

On uniqueness theorems for entire functions tending to zero along disjoint arcs

By

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1. Introduction

Let $f(z)$ be an entire function and let

$$M(r) = \max_{|z|=r} |f(z)|$$

be the maximum modulus of f. In [1, Theorem 2], A. Beurling proved the following uniqueness theorem:

If $\varepsilon(r)$ is a positive decreasing function of r with

$$\limsup_{r\to\infty} (\varepsilon(r) \log r) \geqslant \pi^2$$

and if for some $K > 0$

$$|f(r)| \leqslant KM(r)^{-(1+\varepsilon(r))} \qquad \text{for each} \quad r \geqslant 0\,,$$

then f is a constant function.

Beurling concerned the uniqueness theorem only by the behavior of a function on a ray. Instead of a ray, Hayman [2, Theorem V] proved that if Γ is a continuum strectching to infinity, then there exist a constant $A > 0$ such that the inequality $|f(z)| \leqslant M(r)^{-A}$ for each $z \in \Gamma$, implies that f is a constant. Hayman conjectured that the above uniqueness theorem should hold for any $A > 1$ [2]. This conjecture was recently solved by Hayman and Kjellberg [4] which mainly depends upon Beurling's argument (see Hayman [3, p. 218]). Naturally, we may ask instead of a single continuum can we extend the uniqueness theorem by considering a sequence of disjoint arcs tending to infinity? The purpose of this paper is to answer this question. We remark that in our case of "disjoint arcs" the reflexion principle used in the Beurling's argument is no longer applicable. What we need is the notion of harmonic measure, the Carleman-Milloux problem and two constants theorem of F. and R. Nevanlinna [8, p. 42]. In this connection, our method is similar to those of D. C. Rung [9] and the author [5-7].

To introduce our result, we let $\{\gamma_n\}$ be a sequence of Jordan arcs tending to infinity, and let

Communicated by Prof. Y. Kusunoki, August 17, 1987

$$l_n = \min_{z \in \gamma_n} |z| , \qquad L_n = \max_{z \in \gamma_n} |z| .$$

We say that $\{\gamma_n\}$ is an α-sequence if it satisfies

$$\liminf_{n \to \infty} l_n/L_n = \alpha \qquad (0 < \alpha < 1) .$$

We also need the usual notion of growth (ρ, τ) defined by

$$\limsup_{r \to \infty} \frac{\log \log M(r)}{\log r} = \rho , \qquad \limsup_{r \to \infty} r^{-\rho} \log M(r) = \tau .$$

With the notion of growth (ρ, τ), we first obtain the following reformulation of Hayman and Kjellberg's theorem.

Theorem 1. *Let $f(z)$ be an entire function of growth (ρ, τ) and let Γ be a continuum strectching to infinity such that*

$$|f(z)| \leqslant \exp(-\tau^* |z|^\rho) , \qquad \textit{for each} \quad z \in \Gamma ,$$

where τ^* is any number greater than τ. Then $f(z) = 0$ identically.

Note that the above theorem holds trivially if either ρ or $\tau = \infty$. Thus in the sequel we shall assume that both of ρ and τ are finite.

Also note that the above theorem is sharp due to the example $f(z) = \exp(\tau z^\rho)$, where ρ is a positive odd integer. Therefore, in order to extend the uniqueness theorem by replacing the whole continuum by disjoint arcs, we must pay the price by increasing the constant τ^*.

With the above definitions and remarks, we can now state the following main result.

Theorem 2. *Let $f(z)$ be an entire function of growth (ρ, τ) and let $\{\gamma_n\}$ be an α-sequence such that for each $n = 1, 2, \ldots$,*

$$(1) \qquad |f(z)| \leqslant \exp(-c|z|^\rho) \qquad \textit{for each} \quad z \in \gamma_n ,$$

where $c > c(\rho, \tau, \alpha) = \tau \alpha^{-\rho} \{(\pi/2)[\sin^{-1}((1-\alpha)/(1+\alpha))]^{-1} - 1\}$. *Then $f(z) = 0$ identically.*

2. Harmonic measure

To prove Theorem 2, we shall need a theorem of Nevanlinna [8, p. 102] about the lower estimate of a harmonic measure. Let $D = \{z : |z| < 1\}$ and let γ be a Jordan arc connecting the origin to a point on the boundary ∂D of D. Denote by $\omega(z; \gamma)$ the harmonic measure of the arc γ measured at the point z relative to the domain $D - \gamma$, that is the harmonic function which equals to 1 on γ and 0 on ∂D. According to Nevanlinna's theorem, we have

$$\omega(z; \gamma) \geqslant (2/\pi) \sin^{-1}((1 - |z|)/(1 + |z|)) .$$

If $\min_{z \in \gamma} |z| = a$ and $ae^{i\theta} \in \gamma$, then the conformal mapping $w(z) = (z - ae^{i\theta})/(1 - zae^{-i\theta})$ carries the point $ae^{i\theta}$ into the origin and hence by the invariance of harmonic measures under conformal mapping we obtain

$$\omega(z; \gamma) = \omega(w; w(\gamma)) \geqslant (2/\pi) \sin^{-1}((1 - |w|)/(1 + |w|)).$$

Since $w(0) = -ae^{i\theta}$, it follows that

$$\omega(0; \gamma) \geqslant (2/\pi) \sin^{-1}((1 - a)/(1 + a)).$$

This together with the continuity of $\omega(z; \gamma)$ at the origin yields the following result [6, Lemma].

Lemma 1. *Let γ be a Jordan arc in D tending to a point on ∂D. If $\min_{z \in \gamma} |z| = a > 0$ and $\varepsilon > 0$ is given, then there exists an δ $(0 < \delta < a)$ such that $|z| < \delta$ implies the harmonic measure*

$$\omega(z; \gamma) \geqslant (2/\pi) \sin^{-1}((1 - a)/(1 + a)) - \varepsilon.$$

3. Proof of Theorem 2

Let $0 < \varepsilon < \min(\alpha, 1 - \alpha)$ be given, $f(z)$ be an entire function of growth (ρ, τ), and let $\{\gamma_n\}$ be an α-sequence such that (1) holds. Then by the definitions (taking a subsequence if necessary) we have for each $n = 1, 2, \ldots,$

$$|f(z)| \leqslant \exp((\tau + \varepsilon)L_n^\rho), \qquad z \in \gamma_n \tag{2}$$

and

$$0 < \alpha - \varepsilon < l_n/L_n < \alpha + \varepsilon. \tag{3}$$

Furthermore, (1) gives

$$|f(z)| \leqslant \exp(-cl_n^\rho). \tag{4}$$

According to the two-constants theorem, inequalities (2), (3) and (4) allow us to write for each $z \in D_n - \gamma_n$, where $D_n = \{z: |z| < L_n\}$,

$$\begin{aligned} \log|f(z)| &\leqslant -cl_n^\rho \omega(z; \gamma_n) + (1 - \omega(z; \gamma_n))(\tau + \varepsilon)L_n^\rho \\ &\leqslant -l_n^\rho\{[c + (\tau + \varepsilon)/(\alpha - \varepsilon)^\rho]\omega(z; \gamma_n) - (\tau + \varepsilon)/(\alpha - \varepsilon)^\rho\}. \end{aligned} \tag{5}$$

We now estimate the harmonic measure. For this, we map D_n conformally onto the unit disk by

$$w(z) = z/L_n,$$

so that

$$\min_{w \in w(\gamma_n)} |w| = l_n/L_n > \alpha - \varepsilon.$$

Applying Lemma 1, there exists an $\delta(0 < \delta < a)$ such that $|w| < \delta$ implies

$$(6) \qquad \omega(w; w(\gamma_n)) > (2/\pi)\sin^{-1}((1-\alpha-\varepsilon)/(1+\alpha+\varepsilon)) - \varepsilon .$$

By the invariance of harmonic measures, we have

$$(7) \qquad \omega(z; \gamma_n) = \omega(w; w(\gamma_n)) .$$

Since $|z| < \delta$ implies $|w(z)| = |z|/L_n < \delta$, so by combining (5), (6), and (7), we find that

$$(8) \qquad \log|f(z)| < -l_n^\rho g(c, \varepsilon, \rho, \tau, \alpha), \qquad \text{for each } |z| < \delta ,$$

where

$$g(c, \varepsilon, \rho, \tau, \alpha) = [c + (\tau+\varepsilon)/(\alpha-\varepsilon)^\rho][(2/\pi)\sin^{-1}((1-\alpha-\varepsilon)/(1+\alpha+\varepsilon)) - \varepsilon] - (\tau+\varepsilon)/(\alpha-\varepsilon)^\rho .$$

The constant $c(\rho, \tau, \alpha)$ defined in (1) is to guarantee that when $\varepsilon = 0$ the function

$$g(c, 0, \rho, \tau, \alpha) > 0 .$$

Hence by the continuity, we conclude that the inequality (8) holds for each $|z| < \delta_0$ and for sufficiently large n, where the number δ_0 is fixed. Letting $n \to \infty$, we obtain $f(z) = 0$ for each $|z| < \delta_0$ and hence $f(z) = 0$ identically. This completes the proof.

4. Small arcs

In view of the definition, we see that the size of the arcs of α-sequence is fairly large. In this section, we shall consider arcs whose size is small. For this, we let $\{\gamma_n\}$ be a sequence of Jordan arcs tending to infinity associated with l_n and L_n defined before. We call it a β-sequence ($\beta \geqslant 1$) if it satisfies

$$\lim_{n\to\infty} l_n/L_n = 1 \quad \text{and} \quad \lim_{n\to\infty} l_n^\beta(L_n - l_n)/(L_n + l_n) > \pi .$$

Roughly speaking, the length $L_n - l_n$ of each arc in an β-sequence can be as small as $l_n^{-(\beta-1)} \to 0$ if $\beta > 1$. In this case, to extend Theorem 2, we need pay the price by increasing the order ρ in equation (1) as follows.

Theorem 3. *Let $f(z)$ be an entire function of growth (ρ, τ) and let $\{\gamma_n\}$ be a β-sequence such that for each $n = 1, 2, \ldots$,*

$$|f(z)| \leqslant \exp(-\tau|z|^{\rho+\beta}) \qquad \textit{for each} \quad z \in \gamma_n .$$

Then $f(z) = 0$ identically.

We remark that Lemma 1 cannot be used without omitting the ε there because $l_n/L_n \to 1$. Therefore, we shall need the following improved version of Lemma 1.

Lemma 2. *Let γ be a Jordan arc in D tending to a point on ∂D,* $\min_{z\in\gamma}|z| =$

a, $ae^{i\theta} \in \gamma$, $w(z) = (z - ae^{i\theta})/(1 - zae^{-i\theta})$, and let $E = z(\bar{D}_a)$, where $D_a = \{w\colon |w| < a\}$ and $z(w)$ is the inverse function of $w(z)$. Then the harmonic measure

$$\omega(z;\gamma) \geqslant (2/\pi)\sin^{-1}((1-a)/(1+a)), \qquad \textit{for} \quad z \in E.$$

Proof. According to the conformal invariance of harmonic measures and Nevanlinna's theorem, we obtain

$$\omega(z;\gamma) = \omega(w; w(\gamma)) \geqslant (2/\pi)\sin^{-1}((1-|w|)/(1+|w|)).$$

Since

$$(1-|w|)/(1+|w|) \geqslant (1-a)/(1+a), \qquad \text{for} \quad |w| \leqslant a,$$

the assertion now follows from the monotonicity of the function $\sin^{-1} x$.

5. Proof of Theorem 3

The proof here is the same as in Theorem 2 and therefore we shall only sketch the details. As in (6) and (7), applying Lemma 2 by substituting l_n/L_n for a, we obtain for $z \in E_n$, where $E_n = z(\bar{D}_a)$, $a = l_n/L_n$,

$$\begin{aligned}\omega(z;\gamma_n) &\geqslant (2/\pi)\sin^{-1}((1 - l_n/L_n)/(1 + l_n/L_n)) \\ &\geqslant (2/\pi)(L_n - l_n)/(L_n + l_n).\end{aligned}$$

In view of Lemma 2, we see that the set E_n is a disk of diameter greater than a and contained in the unit disk, and its boundary includes the origin. Since $l_n/L_n \to 1$, the sequence $\{E_n\} \to E$, contains a subsequence converging to a disk of diameter 1 whose boundary passing through the origin. We may, therefore, assume that $E_n \to E$, where $E = \{z\colon |z - \frac{1}{2}| < \frac{1}{2}\}$.

Replacing (1) by (9), then by the same argument as in (5), we have for each $z \in E$ and all sufficiently large n,

$$\begin{aligned}\log|f(z)| &\leqslant -\tau l_n^{\rho+\beta}\omega(z;\gamma_n) + (1 - \omega(z;\gamma_n))(\tau+\varepsilon)L_n^\rho \\ &\leqslant -l_n^\rho\{\tau l_n^\beta \omega(z;\gamma_n) - (\tau+\varepsilon)(1+\varepsilon)^\rho\} \\ &\leqslant -l_n^\rho\{\tau l_n^\beta(2/\pi)(L_n - l_n)/(L_n + l_n) - (\tau+\varepsilon)(1+\varepsilon)^\rho\}.\end{aligned} \tag{10}$$

The main condition in the β-sequence is to guarantee that the number inside the right brace of (10) is bounded away from zero for all sufficiently small ε. This yields the assertion.

6. Growth in sectors

Let $\{\gamma_n\}$ be a β-sequence associated with l_n and L_n, then the length $L_n - l_n$ tends to 0 if $\beta > 1$. But the diameter of γ_n can be very large, for instance

$$\gamma_n = \{z\colon |z| = n, |\arg z| \leqslant \theta\}.$$

Arcs of this kind have already been studied by the author [5]. The remaining case needs to be considered is the small arcs. For this, we call $\{\gamma_n\}$ a β_0-sequence if it is a β-sequence and the diameter of each arc tends to 0. For simplicity, we may assume that $\theta_n \to 0$, where $\theta_n = \min\{\arg z, z \in \gamma_n\}$. We then associate with the sector

$$S_n^\theta = \{z\colon 0 \leqslant |z| \leqslant L_n, |\arg z| \leqslant \theta/2\}\,, \qquad 0 < \theta < \pi\,.$$

For convenience, we write

$$M(f, S) = \max\left(\sup_{z \in S} \log|f(z)|, 1\right).$$

With these notions, we can now study the uniqueness theorem characterized by the growth of the function in the sectors.

Theorem 4. *Let $f(z)$ be an entire function and let $\{\gamma_n\}$ be a β_0-sequence such that*

$$\liminf_{n\to\infty} M(f, S_n^\theta)/A_n < M < \infty \tag{11}$$

and for each $n = 1, 2, \ldots,$

$$|f(z)| \leqslant \exp(-A_n|z|^{\beta+\pi/\theta})\,, \qquad \text{for each} \quad z \in \gamma_n\,, \tag{12}$$

where $0 < \theta < \pi$ and $0 < A_n \to \infty$. Then $f(z) = 0$ identically.

To prove this theorem, we shall need the following estimate of harmonic measure due to the author [7, Lemma 3].

Lemma 3. *Let $\{\gamma_n\}$ be a β_0-sequence associated with l_n, L_n, θ_n, and S_n^θ. Then for each $re^{i\phi} \in S_n^\theta - \gamma_n$, the harmonic measure*

$$\omega(re^{i\phi}; \gamma_n) \geqslant (1/8\pi)(rl_n/L_n^2)^{\pi/\theta}(1 - (r/L_n)^{2\pi/\theta})$$
$$\times (1 - (l_n/L_n)^{2\pi/\theta}) \cos(\pi(\phi - \theta_n)/\theta)\,,$$

where $\theta = \alpha_0$ and $\theta/2 = \delta$ in [7, Lemma 3], and $\theta_n \to 0$.

In view of the definition of β_0-sequence, we see that

$$1 - (l_n/L_n)^{2\pi/\theta} = ((L_n - l_n)/L_n)(2\pi/\theta)(l_n^*/L_n)^{2\pi/\theta - 1}$$
$$\sim \pi l_n^{-\beta}(2\pi/\theta)\,,$$

where $l_n \leqslant l_n^* \leqslant L_n$ and $l_n^*/L_n \sim 1$. Hence for all sufficiently large n and small $|\phi|$, we have

$$\omega(re^{i\theta}; \gamma_n) \geqslant (\pi/8\theta)r^{\pi/\theta}L_n^{-(\beta+\pi/\theta)}\,. \tag{13}$$

7. Proof of Theorem 4

Again, we sketch the details based upon the proof of Theorem 2. As in (5), the hypothesis (12) gives

$$\log|f(z)| \leqslant -A_n l_n^{\beta+\pi/\theta}\omega(z;\gamma_n) + M(f, S_n^\theta).$$

Applying (11) and (13), we obtain for $z = re^{i\phi}$, all sufficiently large n and small $|\phi|$,

$$\log|f(z)| \leqslant -A_n\{(\pi/8\theta)r^{\pi/\theta}(l_n/L_n)^{\beta+\pi/\theta} - M\}.$$

Since $A_n \to \infty$ and $l_n/L_n \to 1$, it follows that $f(z) = 0$ identically and the proof is complete.

8. Conjugate numbers

In harmonic analysis, the pair of conjugate numbers $p, q > 0$ and $1/p + 1/q = 1$ plays an important role. In [5], we introduced a uniqueness theorem by applying this "conjugate principle". We now study the same principle for β_0-sequences.

Theorem 5. *Let $f(z)$ be an entire function and let $\{\gamma_n\}$ be a β_0-sequence such that*

$$\liminf_{n\to\infty} A_n^{1/p}/L_n^\beta > a > 0$$

$$\liminf_{n\to\infty} M(f, S_n^\theta)/A_n^{1/q} < M < \infty$$

and for each $n = 1, 2, \ldots,$

$$|f(z)| \leqslant \exp(-A_n|z|^{\pi/\theta}), \qquad \textit{for each} \quad z \in \gamma_n,$$

where $p, q > 0$ are conjugate. Then $f(z) = 0$ identically.

Proof. As in the proof of Theorem 4, the hypotheses give

$$\begin{aligned}\log|f(z)| &\leqslant -A_n^{1/q}\{A_n^{1/p}l_n^{\pi/\theta}\omega(z;\gamma_n) - M(f, S_n^\theta)/A_n^{1/q}\} \\ &\leqslant -A_n^{1/q}\{(a\pi/8\theta)(rl_n/L_n)^{\pi/\theta} - M\}.\end{aligned}$$

This yields the assertion.

We remark that the above Theorems 4 and 5 can actually be described in terms of functions holomorphic in a half-plane instead of entire functions as were done in [5, 7].

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