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Martin boundaries of Denjoy domains and quasiconformal mappings

Dedicated to Professor Tatuo Fuji'i'e on the occasion of his 60th birthday

By

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Consider a quasiconformal mapping f of an open Riemann surface R_1 onto another R_2. We denote by $R_i^*=R_i^*(X)$ the X-compactification of R_i for $X=M, R, W$, and K, where the M, R, W, and K-compactification mean the *Martin*, *Royden*, *Wiener* and *Kuramochi* compactification, respectively ($i=1, 2$). It seems to be natural to ask *whether f can be extended to a homeomorphism of R_1^* onto R_2^**. It is well-known that the answer is in the affirmative for $X=R$ (cf. e.g. Sario and Nakai [17]). Moreover the converse is also true in this case: a homeomorphism of R_1 onto R_2 that can be extended to a homeomorphism of R_1^* onto R_2^* is quasiconformal outside a compact set (Nakai [14]). Concerning $X=W$ and K the question seems to be entirely open (cf. e.g. Constantinescu and Cornea [9]). Hereafter, throughout this paper, we study only the Martin compactification. The above problem for the Martin compactification was first explicitly stated in an expository paper by Royden [16]. Since then the problem seemed to have been open till it was answered negatively in [19]. In the proof in [19], a plane domain with curious and complicated properties, which was considered by Ancona [2], plays a fundamental role. In view of this we wish to give a simple example showing the problem also in the negative, which is the main purpose of this paper. Namely,

Theorem. *There exists a quasiconformal mapping f of a Denjoy domain D_1 onto another D_2 such that f cannot be extended to a homeomorphism of D_1^* onto D_2^*.*

Here, a domain D in $\hat{\boldsymbol{C}}=\boldsymbol{C}\cup\{\infty\}$ is referred to as a *Denjoy domain* if the complement $\hat{\boldsymbol{C}}-D$ of D is contained in $\hat{\boldsymbol{R}}=\boldsymbol{R}\cup\{\infty\}$ (cf. Garnett and Jones [10]). Recently, Lyons [13] showed that there exist quasi-isometric Riemannian manifolds M_1 and M_2 such that M_1 has no nonconstant positive harmonic functions but M_2 has nonconstant bounded harmonic functions.

After preliminaries in § 1, we study fundamental properties of Martin boundaries of Denjoy domains in § 2. Let E_0 be a compact set of positive capacity in the interval $[0, 1]$ and $\{a_n\}$ and $\{b_n\}$ be increasing sequences of positive integers. Consider a domain $D=\boldsymbol{C}-\bigcup_{-\infty}^{\infty}E_n$ where $E_n=E_0+a_n=\{x+a_n : x\in E_0\}$ if $n>0$ and $E_n=E_0-b_{-n}$ if $n<0$. In § 3, we study the number of minimal boundary points 'over ∞' of the

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domain D. Next consider a Denjoy domain D which satisfies the following condition

$$\frac{|D\cap I_t|}{t}=O\left(\frac{1}{(\log(1/t))^{1/2}(\log\log(1/t))^{\mu}}\right) \quad (t\longrightarrow 0),$$

where $I_t=[-t, t]$ and $|\cdot|$ denotes the linear measure. In §4 we shall prove that if $\mu>1/2$, then there exist two minimal boundary points of D 'over 0'. In §§3 and 4, a criterion obtained by Benedicks [5] plays an important role. Based upon the results stated in §4, the construction of a triple (D_1, D_2, f) in Theorem is carried over in §5.

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§1. Martin boundaries of plane domains

1.1. Consider an open Riemann surface R with *positive boundary,* i.e. there exists the Green's function $g(\cdot, \cdot)$ on R. Fix a point a_0 in R. For $(a, b)\in R\times R$ we put

$$k_b(a)=k(a, b)=\frac{g(a, b)}{g(a_0, b)}.$$

In a word, the *Martin compactification* R^* of R is the 'smallest' compactification of R such that each function $k(a, \cdot)(a\in R)$ has the continuous extension to R^* in the extended sense. Continuous extensions of $k.(a)=k(a, \cdot)$ are also denoted by $k.(a)=k(a, \cdot)$. Thus $k_b(a)=k(a, b)$ is defined on $R\times R^*$. For each $b\in R^*$ the function $k_b(\cdot)=k(\cdot, b)$ is said to be the *Martin function with pole b.* The set $\Delta=\Delta(R)=R^*-R$ is said to be the *Martin boundary* of R. By definition, for each (p, q) in $\Delta\times\Delta$, p is different from q if and only if $k_p(\cdot)$ is different from $k_q(\cdot)$. It is easily seen that $k_p(\cdot)$ is a positive harmonic function on R with $k_p(a_0)=1$ for each $p\in\Delta$. A point p in Δ is said to be a *minimal point* of Δ if k_p is minimal, where we say that a positive harmonic function h on R is *minimal* if for every harmonic function u on R with $0\leqq u\leqq h$ there exists a constant c with $u=ch$. The set of minimal points is denoted by $\Delta_1=\Delta_1(R)$. It is well-known that R^* is metrizable, i.e. if $\{a_n\}$ is a countable dense sequence in R, then

$$d(p, q)=\sum_{n=1}^{\infty}2^{-n}\left|\frac{k(a_n, p)}{1+k(a_n, p)}-\frac{k(a_n, q)}{1+k(a_n, q)}\right| \tag{1}$$

is a metric on R^* compatible with the topology of R^*, where we make the convention that $\infty/(1+\infty)=1$. Thus R^* satisfies the first countability axiom (cf, e.g. [9] and Helms [11]).

Let R be an open Riemann surface with *null boundary,* i.e. there does not exist the Green's function on R. Choose a closed disk B and an open disk U in R such that $B\subset U$. Then the Martin compactification R^* of R is defined by $cl(R-U)\cup U$, where $cl(R-U)$ is the closure of $R-U$ in $(R-B)^*$.

1.2. We next consider the Kerékjártó-Stoïlow compactification R^*_S of R and denote by Γ the Kerékjártó-Stoïlow boundary R_S^*-R. For each $\zeta\in\Gamma$, we define Δ_ζ by the set of points p in the Martin boundary $\Delta=R^*-R$ such that there exists a sequence $\{a_n\}$ in R converging to ζ in R_S^* and also to p in R^*. It is well-known that R^* is 'larger' than R_S^*, that is, there exists a continuous mapping ψ of R^* onto R_S^* such that $\psi|R=id.$, $\psi(\Delta)=\Gamma$, and $\Delta_\zeta=\psi^{-1}(\zeta)$ for every $\zeta\in\Gamma$. We are especially interested in the following fact (cf. e.g. [9]):

Proposition A. *For every $\zeta\in\Gamma$, Δ_ζ contains at least one minimal point and if $p\in\Delta$, then there exists a unique positive measure μ on $\Delta_\zeta\cap\Delta_1$ such that $k_p(z)=\int k_q(z)d\mu(q)$.*

In no. 1.4, we shall establish another version of the above proposition for plane domains.

1.3. Hereafter we consider domains D in the sphere $\hat{C}=C\cup\{\infty\}$. For a set F in $\hat{C}$, we denote by $\bar{F}$ the closure of F in $\hat{C}$. Then, obviously, $\bar{D}$ is a compactification of D larger than the Kerékjártó-Stoïlow compactification D_S^* of D. We shall study relations between $\bar{D}$ and D^*, the Martin compactification of D. For the purpose, as Δ_ζ for $\zeta\in\Gamma=D_S^*-D$, we define $\Delta(\zeta)=\Delta(\zeta, D)$ for every $\zeta\in\partial D=\bar{D}-D$, i.e. $\Delta(\zeta)$ is the set of points p in $\Delta=\Delta(D)=D^*-D$ such that there exists a sequence $\{z_n\}$ in D converging to ζ in $\bar{D}$ and also to p in D^*, and set $\Delta_1(\zeta)=\Delta_1(\zeta, D)=\Delta_1(D)\cap\Delta(\zeta)$. Evidently $\bigcup_{\zeta\in\partial D}\Delta(\zeta)=\Delta$ and $\bigcup_{\zeta\in\partial D}\Delta_1(\zeta)=\Delta_1$. We are concerned with domains D which satisfies the following condition:

$$\Delta(\zeta)\cap\Delta(\eta)=\emptyset \tag{2}$$

for every pair (ζ, η) of distinct points of ∂D. If D is a domain with a totally disconnected boundary ∂D, then $\bar{D}$ is homeomorphic to D_S^*, and hence D satisfies (2). We also remark that there exists a domain not satisfying (2). Consider a square

$$Q=\{x+iy: 0<x<2, 0<y<2\}$$

and segments

$$I_n=\{x+i(1/n): 0\leqq x\leqq 1\} \qquad (n=1, 2, \cdots).$$

Set $D=Q-\bigcup_{n=1}^\infty I_n$. Since D is a simply connected domain, D is conformally equivalent to the unit disk $\{|z|<1\}$. Therefore the *boundary element* $\{x: 0\leqq x\leqq 1\}$ of D corresponds to a single point of the unit circle $\{|z|=1\}$. Thus $\Delta(\zeta)=\Delta(\eta)$ for every pair (ζ, η) in $\{x: 0\leqq x\leqq 1\}$, and hence D does not satisfy (2).

Our first observation is the following:

Proposition 1. *Suppose that D satisfies the condition* (2). *Then there exists a continuous mapping ϕ of D^* onto $\bar{D}$ such that $\phi|D=id.$, $\phi(\Delta)=\partial D$, and $\phi^{-1}(\zeta)=\Delta(\zeta)$ for every $\zeta\in\partial D$.*

Proof. We first define a mapping ϕ of D^* onto $\bar{D}$ as $\phi(p)=p$ if $p\in D$ and $\phi(p)=\zeta$ if $p\in\Delta(\zeta)$. By means of (2) and $\Delta=\bigcup_{\zeta\in\partial D}\Delta(\zeta)$, we see that ϕ is well-defined. It is easily seen that $\phi|D=id.$, $\phi(\Delta)=\partial D$, and $\phi^{-1}(\zeta)=\Delta(\zeta)$ for every $\zeta\in\partial D$. Let p belong to $\Delta(\zeta)$ $(\zeta\in\partial D)$ and $\{p_n\}$ be an arbitrary sequence in D^* converging to p. We must show that $\{\phi(p_n)\}$ converges to ζ in $\bar{D}$. Without loss of generality we may assume that $\phi(p_n)\neq\infty$ for every n. If $p_n\in\Delta$, by the definition of $\Delta(\phi(p_n))$, there exists a point $z_n\in D$ such that $d(z_n, p_n)<1/n$ and $|z_n-\phi(p_n)|<1/n$, where d is the metric on D^* defined by (1). Thus we can always find a sequence $\{z_n\}$ in D such that $d(z_n, p_n)<1/n$ and $|z_n-\phi(p_n)|<1/n$ for every n. Then $\{z_n\}$ converges to p in D^*, and hence also to ζ in $\bar{D}$ by (2). Therefore, from the fact $|z_n-\phi(p_n)|<1/n$ $(n=1, 2, \cdots)$, it follows that $\{\phi(p_n)\}$ converges to ζ. □

1.4. Let s be a positive superharmonic function on D and F be a closed set in D. We denote by $\Phi(s, F)$ the class of positive superharmonic functions h on D such that $h\geqq s$ on F *quasi-everywhere,* i. e. except for a set of capacity zero. Then the function s_F is defined by $s_F(z)=\inf_{h\in\Phi(s,F)}h(z)$. For each ζ in ∂D, let $HP_\zeta=HP_\zeta(D)$ be the class of positive harmonic functions on D which are bounded except for any neighborhood of ζ and vanishes at every regular boundary point of D except ζ. Then we obtain the following which is the main achievement of this section:

Proposition 2. *Suppose that*

$$\{k_p:\ p\in\Delta(\zeta)\}\subset HP_\zeta \tag{3}$$

for every $\zeta\in\partial D$. *Then* (i) D *satisfies the condition* (2), (ii) $\Delta(\zeta)$ *contains at least one minimal point for every* $\zeta\in\partial D$, *and* (iii) *if* $\zeta\in\partial D$ *and* $h\in HP_\zeta$, *then there exists a unique positive measure* μ *on* $\Delta_1(\zeta)=\Delta_1\cap\Delta(\zeta)$ *such that* $h(z)=\int k_p(z)d\mu(p)$.

Proof. Since $HP_\zeta\cap HP_\eta=\emptyset$ if $\zeta\neq\eta(\zeta, \eta\in\partial D)$, (i) follows from (3). Hence there exists a continuous mapping ϕ of D^* onto $\bar{D}$ stated in Proposition 1. Let $\zeta\in\partial D$ and $h\in HP_\zeta$. By the Martin representation theorem, there exists a unique positive measure μ on Δ_1 such that $h(z)=\int k_p(z)d\mu(p)$. For each $r>0$ we set $F_r=\{z\in D:\ |z-\zeta|\leqq r\}$. Since $\bigcap_{r>0}\phi^{-1}(\bar{F}_r)=\phi^{-1}(\zeta)=\Delta(\zeta)$, in order to show (iii), we only have to show that

$$\mu(\Delta_1-\phi^{-1}(\bar{F}_r))=0 \tag{4}$$

for every $r>0$. For an arbitrary $r>0$ set $\mu_1=\mu|\Delta_1\cap\phi^{-1}(\bar{F}_r)$ and $\mu_2=\mu-\mu_1$. Observe that $h_{F_r}=h$. By the fact that $\left(\int k_p d\mu\right)_{F_r}=\int(k_p)_{F_r}d\mu$,

$$h=h_{F_r}=\int k_p)_{F_r}d\mu_1+\int(k_p)_{F_r}d\mu_2. \tag{5}$$

Since $D^*-\phi^{-1}(\bar{F}_r)$ is a neighborhood of every p in $\Delta_1-\phi^{-1}(\bar{F}_r)$, $(k_p)_{F_r}$ is a potential and hence $\int(k_p)_{F_r}d\mu_2(p)$ is also a potential (cf. e. g. [9] and [11]). Therefore, by (5), we have (4) since h is harmonic on D. It is easily seen that (ii) follows from (iii). □

Although we do not have an explicit example, it seems to be impossible to replace (3) by (2) in Proposition 2.

1.5. Take a domain D with an irregular boundary point ζ. Then $\Delta(\zeta)$ consists of a single minimal point (Brelot [8] and also [9]). Thus, if D is of null boundary, i. e. $C-D$ is of capacity zero, D^* is homeomorphic to $\bar{D}=\hat{C}$. We shall give an example D with positive boundary such that D^* is homeomorphic to $\hat{C}$. For this purpose we maintain the following

Lemma 3. *Suppose that there exists a sequence of annuli A_n in D converging to a point ζ in ∂D such that every z in $\hat{C}-\{\zeta\}$ is separated from ζ by an A_n and*

(6) $$\inf_n \operatorname{mod} A_n>0,$$

where $\operatorname{mod} A_n$ *is the modulus of $A_n (n=1, 2, \cdots)$. Then $\Delta(\zeta)$ consists of a single point.*

Although it seems that the proof of Lemma 3 is standard, we include here the proof for the sake of completeness. For a fixed point a_0 in D and an arbitrary point a in D we put $k(a, \cdot)=g(a, \cdot)/g(a_0, \cdot)$. Let C_n be the 'middle' circle of A_n in the sense of modulus, i. e. if $A_n=\{\alpha<|z|<\beta\}$ (conformally), then $C_n=\{|z|=(\alpha\beta)^{1/2}\}$. Consider a defining sequence $\{\Omega_n\}$ of the boundary component $\{\zeta\}$ such that $\partial\Omega_n=C_n$. Since we consider only of sufficiently large n, we may assume that $\bar{\Omega}_1$ does not contain a_0 and a. Put $M_n=\max_{z\in C_n} k(a, z)$ and $m_n=\min_{z\in C_n} k(a, z) (n=1, 2, \cdots)$. Since $g(a, \cdot)$ and $g(a_0, \cdot)$ are bounded harmonic functions on Ω_n and have vanishing boundary values on $\partial\Omega_n-C_n$ except for irregular boundary points, we have

(7) $$m_n<k(a, z)<M_n \qquad (z\in\Omega_n).$$

Therefore $\{M_n\}$ is decreasing and $\{m_n\}$ is increasing. Put $M_\infty=\lim_{n\to\infty}M_n$ and $m_\infty=\lim_{m\to\infty}m_n$. Take points z_n on C_n such that $k(a, z_n)=M_n (n=1, 2, \cdots)$. For a positive harmonic function h on Ω_n, by means of (6), the Harnack inequality yields that there exists a universal constant K such that

(8) $$K^{-1}h(z_i)\leqq h(z)\leqq Kh(z_i) \qquad (z\in C_i,\ i>n).$$

Applying (8) to $M_n g(a_0, \cdot)-g(a, \cdot)$ and $g(a_0, \cdot)$, we have

$$M_n-k(a, z)\leqq K^2(M_n-k(a, z_i)) \qquad (z\in C_i,\ i>n),$$

and hence

$$M_n-m_i\leqq K^2(M_n-M_i) \qquad (i>n).$$

Letting $i\to\infty$ and then letting $n\to\infty$, we see that $M_\infty\leqq m_\infty$. Therefore, from (7) it follows that $\lim_{z\to\zeta} k(a, z)$ exists. This completes the proof.

Consider the Cantor ternary set E and a domain $D=\hat{C}-E$. Then D is of positive boundary since E is of capacity positive (cf. e. g. Tsuji [20]). It is easily seen that for every $\zeta\in\partial D=E$ there exists a sequence of annuli A_n in D stated in the Lemma 3. Then, by Lemma 3, $\Delta(\zeta)$ consists of a single point for every $\zeta\in\partial D$, i. e. D^* is homeomorphic to $\bar{D}=\hat{C}$.

§ 2. Martin boundaries of Denjoy domains

2.1. We recall the definition of Denjoy domains: A domain D in $\hat{C}$ is said to be a *Denjoy domain* if $\hat{C}-D$ is contained in $\hat{\boldsymbol{R}}=\boldsymbol{R}\cup\{\infty\}$. We first claim the following (cf. [2] and [5])

Proposition 4. *Let D be a Denjoy domain. Then D satisfies the condition* (2) *and $\Delta_1(\zeta)\neq\emptyset$ for every $\zeta\in\partial D$. Moreover if $h\in HP_\zeta$, then there exists a unique positive measure μ on $\Delta_1(\zeta)$ such that $h(z)=\int k_p(z)d\mu(p)$.*

By the Poisson integral formula, the following is easily verified.

Lemma 5. *Let h be a bounded positive harmonic function on $\{|z|<1\}$ with vanishing boundary values on $\{e^{i\theta}: |\theta|\leqq\alpha\}$ $(\alpha>0)$. Then there exists a constant K such that*

$$h(x)\leqq Kh(0) \qquad (0\leqq x<1),$$

where K does not depend on h and x.

Proof of Proposition 4. In view of Proposition 2, it is sufficient to show that $\{k_p: p\in\Delta(\zeta)\}\subset HP_\zeta$ for every $\zeta\in\partial D$. Let a_0 be a fixed point in D. Suppose that ζ is not contained in any interval in ∂D. Observe that there exists a sequence $\{B_n\}$ of disks such that $B_n\supset\bar{B}_{n+1}$, $a_0\notin B_1$, $\cap_n B_n=\{\zeta\}$, and $\partial B_n\subset D$. Then, by the Harnack inequality, there exists a constant K_n such that

$$k(a, z)=g(a, z)/g(a_0, z)\leqq K_n \qquad (a\in\partial B_n)$$

for every $z\in B_{n+1}\cap D$ since $k(a_0, z)=1$ and $k(\cdot, z)$ is positively harmonic on $D-\bar{B}_{n+1}$. This shows that if $p\in\Delta(\zeta)$, then k_p is bounded on $D-B_n$ and vanishes at every regular boundary point in $\partial D-B_n$ for each n, and hence $k_p\in HP_\zeta$. Next suppose that ζ is contained in an interval in ∂D. Then we can find a sequence $\{B_n\}$ of disks with center ζ such that $B_n\supset\bar{B}_{n+1}$, $a_0\notin B_1$, $\cap_n B_n=\{\zeta\}$, and if $\eta\in\partial B_n\cap\partial D$, then there exists an open interval in ∂D containing η. By Lemma 5 and the Harnack inequality, there exists a constant K_n such that

$$k(a, z)\leqq K_n \qquad (a\in\partial B_n\cap D)$$

for every $z\in B_{n+1}\cap D$. Therefore the preceding argument yields that $k_p\in HP_\zeta$ for every $p\in\Delta(\zeta)$. □

2.2. Ancona [3] and Benedicks [5], independently, showed the following, which is one of the most interesting results for Denjoy domains related to Martin boundaries.

Theorem B. *Let D be a Denjoy domain. Then $\Delta_1(\zeta)$ consists of at most two points for every $\zeta\in\partial D$.*

Applying the above result, we prove the following

Proposition 6. *Let D be a Denjoy domain. Then for every $\zeta\in\partial D$ one of the following alternatives must hold*: (i) *$\Delta(\zeta)$ consists of a single minimal point*; (ii) *$\Delta(\zeta)$ consists of two minimal points*; (iii) *$\Delta_1(\zeta)=\{p_1, p_2\}$ and $\Delta(\zeta)$ is homeomorphic to a closed interval with end points p_1 and p_2. Moreover,* (ii) *holds if and only if there exists an* *open interval in* ∂D *which contains* ζ.

Proof. By Theorem B and Proposition 4, $\varDelta_1(\zeta)$ consists of one or two points. First suppose that $D\cap\{|z-\zeta|<r\}$ is connected for every $r>0$. Then $cl(D\cap\{|z-\zeta|<r\})$, the closure of $D\cap\{|z-\zeta|<r\}$ in D^*, is connected in D^*. Therefore $\varDelta(\zeta)$ is connected since $\varDelta(\zeta)=\bigcap_{r>0} cl(D\cap\{|z-\zeta|<r\})$. If $\varDelta_1(\zeta)$ consists of a single point p_1, by means of Proposition 4, we see that for every $p\in\varDelta(\zeta)$ there exists a constant c with $k_{p_1}=ck_p$. Since $k_p(a_0)=k_{p_1}(a_0)=1$, $c=1$, i. e. $p=p_1$. Thus (i) holds. If $\varDelta_1(\zeta)$ consists of two points p_1 and p_2, by the same argument, we see that for every $p\in\varDelta(\zeta)$ there exists a constant $c\in[0, 1]$ with $k_p=ck_{p_1}+(1-c)k_{p_2}$. Hence $\{k_p: p\in\varDelta(\zeta)\}=\{ck_{p_1}+(1-c)k_{p_2}: 0\leqq c\leqq 1\}$ since $\varDelta(\zeta)$ is connected. Denoting the point p by $cp_1+(1-c)p_2$ if $k_p=ck_{p_1}+(1-c)k_{p_2}$, we can easily see that the mapping $c\mapsto cp_1+(1-c)p_2$ is a homeomorphism of $[0, 1]$ onto $\varDelta(\zeta)$. Thus (iii) holds. Next suppose that $D\cap\{|z-r|<r\}$ is disconnected for some $r>0$. Then the interval $[\zeta-r, \zeta+r]$ is contained in ∂D. Set $B_1=\{|z-\zeta|<r/2\}$ and $B_2=\{|z-\zeta|<r/4\}$. By the Harnack inequality and Lemma 5, there exists a constant K such that

$$k(a, z)=\frac{g(a, z)}{g(a_0, z)}\leqq K \qquad (a\in\partial B_1\cap D,\ z\in B_2\cap D).$$

This means that for every $p\in\varDelta(\zeta)$ if a sequence $\{z_n\}$ in $H^+=\{\operatorname{Im} z>0\}$ (resp. $H^-=\{\operatorname{Im} z<0\}$) converges to p, then k_p is bounded on H^-(resp. H^+). Hence, by Proposition 4, $\varDelta_1(\zeta)$ consists of two points p_1 and p_2 such that k_{p_1} (resp. k_{p_2}) is bounded on H^- (resp. H^+) and unbounded on H^+ (resp. H^-). If $p\in\varDelta(\zeta)-\{p_1, p_2\}$, then $k_p=ck_{p_1}+(1-c)k_{p_2}$ $(0<c<1)$. This contradicts that k_p is bounded on H^+ or H^-. Thus (ii) holds. □

2.3. Let D be a Denjoy domain with $\infty\in\partial D$. Denote by $Q(t, r)$, $t\in\boldsymbol{R}$, the square

$$\{x+iy: |x-t|<r/2,\ |y|<r/2\}.$$

For an arbitrary fixed α with $0<\alpha<1$ and every x in $\boldsymbol{R}-\{0\}$, let $\beta_x(\cdot)=\beta_x(\cdot\,;\partial D)=\beta_x(\cdot\,;\partial D, \alpha)$ be the solution of the Dirichlet problem on $Q(x, \alpha|x|)-\partial D$ with boundary values $\beta_x=1$ on $\partial Q(x, \alpha|x|)$ and $\beta_x=0$ on $\partial D\cap Q(x, \alpha x)$. Benedicks [5] also proved the following which is the other most interesting result for Denjoy domains.

Theorem C. *Let D be a Denjoy domain with $\infty\in\partial D$. Then, for every $\alpha\in(0, 1)$, $\varDelta_1(\infty)$ consists of a single point if $\int_{|x|\geqq 1}(\beta_x(x)/|x|)dx=\infty$ and $\varDelta_1(\infty)$ consists of two points if $\int_{|x|\geqq 1}(\beta_x(x)/|x|)dx<\infty$.*

The above theorem plays an essential role in § 3 and 4.

2.4. We are also interested in the following result which was originally proved by Maitani. For a proof we refer to [18].

Theorem D. *Suppose that D is a Denjoy domain such that $\bar{D}=\hat{\boldsymbol{C}}$ is homeomorphic to D^*, i.e. for every $\zeta\in\partial D$, $\varDelta_1(\zeta)$ consists of a single point. Then the linear measure*

$|\partial D|$ of ∂D is zero.

In other words, the above theorem means the following

Corollary E. *Let D be a Denjoy domain such that ∂D is of positive linear measure. Then there exists a point $\zeta \in \partial D$ such that $\Delta_1(\zeta)$ consists of two points.*

We here improve the above assertion. Namely

Proposition 7. *Let D be a Denjoy domain. Then for almost all $\zeta \in \partial D$, with respect to the linear measure, $\Delta_1(\zeta)$ consists of two points.*

Proof. We put

$$E^* = \{\zeta \in \partial D : \# \Delta_1(\zeta, D) = 1\},$$

where we denote by $\# \Delta_1(\zeta, D)$ the number of points in $\Delta_1(z, D)$. We first show that E^* is measurable. Let $\{a_n\}_{n=1}^{\infty}$ be a countable dense subset of D. Consider functions

$$\varepsilon_n(\zeta, y) = k(a_n, \zeta + iy) - k(a_n, \zeta - iy)$$

on $\partial D \times (0, 1)$ for every $n = 1, 2, \cdots$. It is shown by Ancona [3] that each of $\{\zeta + iy\}_{y>0}$ and $\{\zeta - iy\}_{y>0}$ converges, in D^*, to a point in $\Delta_1(\zeta, D)$ when y tends to 0 for every $\zeta \in \partial D$. Hence $\lim_{y \to 0} \varepsilon_n(\zeta, y) = \varepsilon_n(\zeta)$ exists for every n. Observe that

$$E^* = \bigcap_{n=1}^{\infty} \{\zeta \in \partial D : \varepsilon_n(\zeta) = 0\}.$$

Since each $\{\zeta \in \partial D : \varepsilon_n(\zeta) = 0\}$ is measurable, E^* is also measurable. Suppose that $|E^*| > 0$. Then we find a compact subset E_1 of E^* with $|E_1| > 0$. Consider a domain $D_1 = \boldsymbol{C} - E_1$. Then Theorem C implies that $\# \Delta_1(\zeta, D_1) = 1$ for every $\zeta \in \partial D_1 = E_1$ since $E_1 \subset \partial D$ and $E_1 \subset E^*$. However this contradicts Corollary E since $|E_1| > 0$. Thus we have that $|E^*| = 0$, which concludes the proof. □

§ 3. Denjoy domains with boundaries of positive capacity

3.1. Consider a Denjoy domain D with $\infty \in \partial D$. We denote by $B(c, r)$ the disk $\{|z - c| < r\}$ with center c and radius $r > 0$. For an arbitrary fixed α in $(0, 1/2)$ and every x in $\boldsymbol{R} - \{0\}$, let $\gamma_x(\cdot) = \gamma_x(\cdot\,; \partial D, \alpha)$ be the solution of the Dirichlet problem on $B(x, \alpha|x|) - \partial D$ with boundary values $\gamma_x = 1$ on $\partial B(x, \alpha|x|)$ and $\gamma_x = 0$ on $\partial D \cap B(x, \alpha|x|)$. Note that $Q(x, \sqrt{2}\alpha|x|) \subset B(x, \alpha|x|) \subset Q(x, 2\alpha|x|)$ or

$$\beta_x(\cdot\,; \partial D, 2\alpha) \leqq \gamma_x(\cdot\,; \partial D, \alpha) \leqq \beta_x(\cdot\,; \partial D, \sqrt{2}\alpha).$$

Therefore, by Theorem C, we obtain the following

Benedicks' criterion I. *Let D be a Denjoy domain with $\infty \in \partial D$. Then, for every $\alpha \in (0, 1/2)$, $\Delta_1(\infty) = \Delta_1(\infty, D)$ consists of a single point if $\int_{|x| \geq 1} (\gamma_x(x)/|x|)\,dx = \infty$ and $\Delta_1(\infty)$ consists of two points if $\int_{|x| \geq 1} (\gamma_x(x)/|x|)\,dx < \infty$.*

Let E_0 be a closed subset of the interval $[0, 1]$. For two increasing sequences $\{a_n\}$ and $\{b_n\}$ of positive integers, consider a Denjoy domain D with $\infty\in\partial D$ defined by

$$\text{(9)} \qquad \begin{cases} D=\boldsymbol{C}-\bigcup_{n=-\infty}^{\infty} E_n \\ E_n=E_0+a_n=\{x+a_n:\ x\in E_0\} & (n>0) \\ E_n=E_0-b_{-n} & (n<0). \end{cases}$$

The main purpose of this section is to prove the following

Theorem 8. *Let E_0 be a closed subset of the interval $[0, 1]$ of positive capacity. Suppose that for two sequences $\{a_n\}$ and $\{b_n\}$ of positive integers there exists an integer N such that*

$$\text{(10)} \qquad a_{n+1}-a_n\leqq N, \quad b_{n+1}-b_n\leqq N \quad (n=1, 2, \cdots).$$

Then, for a Denjoy domain D defined as in (9), $\Delta_1(\infty)=\Delta_1(\infty, D)$ consists of two points.

3.2. In order to prove Theorem 8, we prepare two lemmas. For simplicity we denote by B the unit disk $B(0, 1)=\{|z|<1\}$.

Lemma 9. *Let E be a union of finitely many closed intervals contained in $[-1, 1]$ and u be a bounded harmonic function on $B-E$ with boundary values $u=1$ on E and $u=0$ on ∂B. Then, for every $s\in(1/2, 1/2)$, $u(s+it)$ is a decreasing function of $t\in(0, 1/2)$.*

Proof. We denote by $g_B(\cdot, \cdot)$ the Green's function on B. We put $\hat{g}(z, w)=g_B(z, w)+g_B(z, \bar{w})$ for $w\in B^+=B\cap H^+$, where H^+ is the upper half plane. Observe that $(\partial/\partial y)\hat{g}(x, w)=0$ where $z=x+iy$. Hence, applying Green's formula to u and $\hat{g}$ on B^+, we have

$$\text{(11)} \qquad u(s+it)=-\frac{1}{2\pi}\int_{-1}^{1}\hat{g}(x, s+it)(\partial/\partial y)u(x)dx$$

for $s+it\in B^+$. It is easily seen that, for every $x\in(-1, 1)$ and every $s\in(-1/2, 1/2)$,

$$\hat{g}(x, s+it)=\log\frac{(1-xs)^2+(xt)^2}{(x-s)^2+t^2}$$

is a positive decreasing function of $t\in(0, 1/2)$ and that $(\partial/\partial y)u(x)\leqq 0$ for every $x\in(-1, 1)$. Therefore, by virtue of (11), we see that $u(s+it)$ is decreasing for every $s\in(-1/2, 1/2)$. □

Lemma 10. *Let E and u be as in Lemma 9. Then there exists a constant $c_0=c_0(|E|)(0<c_0<1)$ depending only on the length $|E|$ of E such that*

$$u(z)\geqq c_0 \qquad (z\in\overline{B(0, 1/2)}).$$

Proof. Let $g_+(z, w)$ be the Green's function on B^+. Then it is easily seen that $g_+(z, w)=g_B(z, w)-g_B(z, \bar{w})$. Applying Green's formula to u and g_+ on B^+, we have

$$u(s+i/2)=\frac{1}{2\pi}\int_{-1}^{1}u(x)\frac{\partial}{\partial y}g_{+}(x,\ s+i/2)dx\,, \tag{12}$$

where $z=x+iy$. Observe that

$$\begin{aligned}\frac{\partial}{\partial y}g_{+}(x,\ s+i/2)&=2\frac{\partial}{\partial y}g_{B}(x,\ s+i/2)\\ &=\frac{1}{(x-s)^2+1/4}-\frac{1}{(1-sx)^2+x^2/4}\\ &\geqq\frac{3(1-x^2)}{25}\end{aligned}$$

for every $s\in(-1/2,\ 1/2)$. Hence, by means of (12), we conclude that

$$\begin{aligned}u(s+i/2)&\geqq\frac{1}{2\pi}\int_{-1}^{1}u(x)\frac{3(1-x^2)}{25}dx\\ &\geqq\frac{3}{50\pi}\int_{E}(1-|x|)dx\\ &\geqq\frac{3}{200\pi}|E|^2\end{aligned}$$

for every $s\in(-1/2,\ 1/2)$. Therefore, from Lemma 9 and symmetry, it follows that $u(z)\geqq(3/200\pi)|E|^2$ for every z in $\overline{B(0,\ 1/2)}$. □

3.3. Proof of Theorem 8. An essential part of the proof is to show the following:

Lemma 11. *There exists a constant C with $0<C<1$ such that, for every integer n with $n>n_0$, for every integer k with $1\leqq k\leqq n-n_0$, and for every x with $|x|\in[2^n,\ 2^{n+1}]$,*

$$\gamma_x(z\,;\ \partial D,\ 1/2^k)\leqq C \qquad (z\in\overline{B(x,\ |x|/2^{k+1})})$$

where n_0 is an integer.

Before proving the above, we show that Theorem 8 follows from Lemma 11. By Lemma 11 and comparison of boundary values, we see that

$$\gamma_x(\cdot\,;\ \partial D,\ 1/2^k)\leqq C\gamma_x(\cdot\,;\ \partial D,\ 1/2^{k+1})$$

on $\overline{B(x,\ |x|/2^{k+1})}$ for every k with $1\leqq k\leqq n-n_0$. Therefore we conclude that

$$\gamma_x(\cdot\,;\ \partial D,\ 1/4)\leqq C^{n-n_0-1}\gamma_x(\cdot\,;\ \partial D,\ 1/2^{n-n_0+1})$$

on $\overline{B(x,\ |x|/2^{n-n_0+1})}$ if $n\geqq n_0+2$ and especially

$$\gamma_x(x)=\gamma_x(x\,;\ \partial D,\ 1/4)\leqq C^{n-n_0-1}$$

including the case $n=n_0,\ n_0+1$. Therefore

$$\begin{aligned}\int_{|x|\geqq 2^{n_0}}\frac{\gamma_x(x)}{|x|}dx&=\sum_{n=n_0}^{\infty}\int_{2^n\leqq|x|\leqq 2^{n+1}}\frac{\gamma_x(x)}{|x|}dx\\ &\leqq 2\sum_{n=n_0}^{\infty}\frac{C^{n-n_0-1}}{2^n}2^n<\infty\,.\end{aligned}$$

By Benedicks' criterion I, this completes the proof of Theorem 8.

We proceed to the proof of Lemma 11. Let n_0 be an integer with $2^{n_0}\geqq \mathrm{Max}(2a_1, 2b_1, 6N)$. We fix arbitrary x in $[2^n, 2^{n+1}]$ $(n>n_0)$ and k with $1\leqq k\leqq n-n_0$. Let $F(x, k)$ be the union of closed intervals $[a_i, a_i+1]$ which are contained in $[x-x/2^k+1, x+x/2^k-1]$. Then, by (10), we can verify that $|F(x, k)|\cdot N\geqq x/2^{k-1}-N-4$ and hence

$$|F(x, k)|\geqq\frac{1}{N}\frac{x}{2^k}. \tag{13}$$

Let $\delta(z)$ be the solution of the Dirichlet problem on $B(1/2, 3/2)-E_0$ with boundary values $\delta=1$ on $\partial B(1/2, 3/2)$ and $\delta=0$ on E_0. Set $c_1=\sup_{t\in[0,1]}\delta(t)$. Note that $0<c_1<1$. Then, by comparing $\delta(z-a_i)$ with $\gamma_x(z\,;\,\partial D, 1/2^k)$ on $B(a_i+1/2, 3/2)$, we see that

$$\gamma_x(t\,;\,\partial D, 1/2^k)\leqq c_1 \qquad (t\in F(x, k)). \tag{14}$$

Let $v(z)$ be a bounded harmonic function on $B(x, x/2^k)-F(x, k)$ with boundary values $v=1$ on $\partial B(x, x/2^k)$ and $v=0$ on $F(x, k)$. From (14) it follows that

$$\gamma_x(\cdot\,;\,\partial D, 1/2^k)\leqq c_1+(1-c_1)v(\cdot) \tag{15}$$

on $B(x, x/2^k)$. Consider the function $u(w)=1-v(xw/2^k+x)$ on $B-E(x, k)$ where $E(x, k)=\{2^k(t-x)/x : t\in F(x, k)\}$. Observe that u is a bounded harmonic function on $B-E(x, k)$ with boundary values $u=0$ on ∂B and $u=1$ on $E(x, k)$. In view of (13), we see that $|E(x, k)|\geqq 1/N$. Therefore, applying Lemma 10 to u, we obtain that there exists a constant c_0 with $0<c_0<1$ depending only on N such that

$$v(z)\leqq 1-c_0 \qquad (z\in\overline{B(x, x/2^{k+1})}),$$

Combining (15), this shows that

$$\gamma_x(z\,;\,\partial D, 1/2^k)\leqq C \qquad (z\in\overline{B(x, x/2^{k+1})})$$

where $C=c_1+(1-c_1)(1-c_0)<1$. Thus Lemma 11 is valid for $x>0$. Entirely the same argument also shows that Lemma 11 is valid for $x<0$. The proof is herewith complete. □

3.4. The condition (10) implies that there exists an integer N such that

$$a_n\leqq Nn, \qquad b_n\leqq Nn \qquad (n=1, 2, \cdots). \tag{16}$$

One might guess that the condition (10) can be replaced by the condition (16) in Theorem 8. We shall show that (10) cannot be replaced by (16) in Theorem 8.

Lemma 12. *For every $r\in(0, 1)$, we denote by v_r a bounded harmonic function on $B-([-1, -r]\cup[r, 1])$ with boundary values $v_r=1$ on ∂B and $v_r=0$ on $[-1, -r]\cup[r, 1]$. Then there exist positive constants C_1 and C_2 not depending on r such that $C_1r\leqq v_r(0)\leqq C_2r$.*

Proof. Consider two functions $f(z)$ and $g(z)$ such that

$$f(z)=\frac{\sqrt{(z+r)/(1+rz)}-\sqrt{r}}{1-\sqrt{r(z+r)/(1+rz)}}\frac{1+\sqrt{r(z+r)/(1+rz)}}{\sqrt{(z+r)/(1+rz)}+\sqrt{r}}$$

and

$$g(w)=\frac{\sqrt{(c-w)/(1-cw)}-\sqrt{c}}{1-\sqrt{c(c-w)/(1-cw)}}\frac{1+\sqrt{c(c-w)/(1-cw)}}{\sqrt{(c-w)/(1-cw)}+\sqrt{c}}$$

where $c=f(r)=\left(\frac{\sqrt{2r}+\sqrt{1+r^2}}{\sqrt{2}+\sqrt{1+r^2}}\right)^2$. Then it is verified that $g(f(z))$ is a conformal mapping of $B-([-1, -r]\cup[r, 1])$ onto B with $g(f(0))=0$. Hence $v_r(f^{-1}(g^{-1}(\zeta)))$ is a bounded harmonic function on B with boundary values 1 on $g(f(\partial B))$ and 0 on $\partial B-g(f(\partial B))$. Therefore

$$\begin{aligned} v_r(0)&=\frac{1}{2\pi}\ \text{(arc length of } g(f(\partial B)))\\ &=\frac{2}{\pi}\sin^{-1}\frac{6c-c^2-1}{(1+c)^2}\\ &\geqq\frac{1}{2\pi}(6c-c^2-1)\geqq\frac{\sqrt{2}+1}{\pi}\ (c-(3-2\sqrt{2}))\\ &\geqq\frac{\sqrt{2}+1}{\pi}(\sqrt{2}-1)r\geqq\frac{r}{\pi}\end{aligned}$$

and

$$\begin{aligned} v_r(0)&\leqq\frac{2}{\pi}\frac{\pi}{2}(6c-c^2-1)\\ &\leqq4\sqrt{2}(c-(3-2\sqrt{2}))\leqq4\sqrt{2}r\end{aligned}$$ □

We are in the stage to give an example showing that (10) cannot be replaced by (16) in Theorem 8. Consider open intervals

$$J_m=(2^m-[2^m/\sqrt{m}],\ 2^m+[2^m/\sqrt{m}])\qquad(m=10, 11, \cdots) \tag{17}$$

where $[t]$ is the greatest integer not exceeding $t\in\boldsymbol{R}$. Let $\{a_n\}$ be a sequence of positive integers such that

$$\bigcup_{n=1}^{\infty}[a_n, a_n+1]=[2, \infty)-\bigcup_{m=10}^{\infty}J_m. \tag{18}$$

From (17) if follows that $\lim_{n\to\infty}a_n/n=1$, and hence $\{a_n\}$ satisfies the condition (16). We consider a Denjoy domain $D=H^+\cup H^-\cup(\cup_{m=10}^{\infty}J_m)$, which is nothing but the Denjoy domain defined as in (9) for $E_0=[0, 1]$, $\{a_n\}$ satisfying (18), and $\{b_n\}$ with $b_n=n$. Set $J'_m=[(1-1/(3\sqrt{m}))2^m,\ (1+1/(3\sqrt{m}))2^m]$. Observe that

$$(x-2^m/(2\sqrt{m}),\ x+2^m/(2\sqrt{m}))\subset D\qquad(x\in J'_m).$$

By the definition of v_r in Lemma 12, this means that

$$\gamma_x(z)=\gamma_x(z\,;\,\partial D, 1/4)\geqq v_{r_m}(4(z-x)/x)\qquad(x\in J'_m)$$

on $B(x, x/4)$ where $r_m=1/\sqrt{m}$. Hence, by Lemma 12, we have

$$\gamma_x(x)\geqq v_{r_m}(0)\geqq C/\sqrt{m} \qquad (x\in J'_m)$$

for every $m=10, 11, \cdots$. Therefore

$$\begin{aligned}\int_{|x|\geq 1}\frac{\gamma_x(x)}{x}dx &\geqq \sum_{m=10}^{\infty}\int_{J'_m}\frac{\gamma_x(x)}{x}dx\\ &\geqq \sum_{m=10}^{\infty}\frac{C}{\sqrt{m}}\frac{1}{2^{m+1}}\frac{2^{m+1}}{3\sqrt{m}}\\ &=\sum_{m=10}^{\infty}\frac{C}{3m}=\infty,\end{aligned}$$

and hence Benedicks' criterion I yields that $\Delta_1(\infty, D)$ consists of a single point.

3.5. As a corollary of Theorem 8, we have the following, which was originally proved in [18].

Theorem F. *Suppose that E_0 is a closed subset of* [0, 1] *of positive capacity. Let $D=\boldsymbol{C}-\bigcup_{n=-\infty}^{\infty}E_n$ where $E_n=\{x+n: x\in E_0\}$. Then $\Delta_1(\infty)=\Delta_1(\infty, D)$ consists of two points.*

We here take the Cantor ternary set as E_0 in Theorem F. Then, for the above D, ∂D has zero linear measure but $\Delta_1(\infty)$ consists of two points p_1 and p_2. This shows that *the converse of Theorem D is not valid.* Moreover, by Proposition 6, we see that $\Delta(\infty)$ is homeomorphic to the 'interval' $\{cp_1+(1-c)p_2: c\in[0, 1]\}$ (cf. no. 2.2). We have shown in no. 1.5 that $\Delta_1(\zeta)$ consists of single point q_ζ for every $\zeta\in\partial D-(\infty)$. Therefore, by symmetry of the domain D, it is verified that if ζ tends to ∞ then q_ζ tends to the 'middle point' $(1/2)p_1+(1/2)p_2$ of p_1 and p_2. This means that the closure of $\Delta_1(D)$ in D^* coincides with $\Delta_1(D)\cup\{(1/2)p_1+(1/2)p_2\}$. Thus we see that *$\Delta_1(D)$ is not dense in $\Delta(D)$.* This fact was first shown by Ancona [3], using another Denjoy domain.

§4. Order criterion at a point of density

4.1. Consider a Denjoy domain D with $0\in\partial D$. Let $\gamma_x(\cdot)=\gamma_x(\cdot\,; \partial D, \alpha)$ be as in no. 3.1. For $F\subset\boldsymbol{C}$, we set $F^{-1}=\{1/z: z\in F\}$. Note that, for x in $\boldsymbol{R}$ with $0<|x|\leqq 1$ and α with $0<\alpha<1/3$,

$$B(1/x, \alpha_1/|x|)\subset B(x, \alpha|x|)^{-1}\subset B(1/x, \alpha_2/|x|)$$

where $\alpha_1=\alpha/(1+\alpha)$ and $\alpha_2=\alpha/(1-\alpha)$. Hence we have

$$\gamma_{1/x}(1/x\,; \partial D^{-1}, \alpha_2)\leqq\gamma_x(x\,; \partial D, \alpha)\leqq\gamma_{1/x}(1/x\,; \partial D^{-1}, \alpha_1).$$

Therefore, by Benedicks' criterion I, we have the following

Benedicks' criterion II. *Let D be a Denjoy domain with $0\in\partial D$. Then, for every $\alpha\in(0, 1/3)$, $\Delta_1(0)=\Delta_1(0, D)$ consists of a single point if $\int_{|x|\leq 1}\gamma_x(x)/|x|\,dx=\infty$ and $\Delta_1(0)$ consists of two points if $\int_{|x|\leq 1}\gamma_x(x)/|x|\,dx<\infty$.*

We denote by I_t the closed interval $[-t, t]$ for $t>0$. The main purpose of this section is to prove

Theorem 13. *Let D be a Denjoy domain with $0\in\partial D$. Suppose that there exists a constant $\mu>1/2$ such that*

$$\frac{|D\cap I_t|}{t}=O\Big(\frac{1}{(\log(1/t))^{1/2}(\log\log(1/t))^{\mu}}\Big)\qquad(t\longrightarrow 0). \tag{19}$$

Then $\Delta_1(0)=\Delta_1(0, D)$ consists of two points.

After some preparation the proof is carried over in no. 4.3.

4.2. The following is a modification of a theorem of Beurling [6] (also cf. e.g. Nevanlinna [15]).

Lemma 14. *Suppose that $0<a_1<b_1<\cdots<a_n<b_n\leqq 1$. Let $E=\cup_{i=1}^{n}([-b_i, -a_i]\cup[a_i, b_i])$ and $E_r=[-1, -r]\cup[r, 1]$ where $2r=2-|E|$, i.e. $|E_r|=|E|$. Let u (resp. u_r) be a harmonic function on $B-E$ (resp. $B-E_r$) with boundary values 0 on ∂B and 1 on E (resp. E_r). Then $u(0)\geqq u_r(0)$.*

Proof. We put $g^*(z, t)=g_B(z, t)+g_B(z, -t)$ for $t\in(0, 1)$. By Green's formula, we have

$$\begin{aligned}u(z)=&\frac{1}{\pi}\int_{a_1}^{b_1}g^*(z, t)(\partial/\partial u)u(t)dt\\&+\frac{1}{\pi}\sum_{i=2}^{n}\int_{a_i}^{b_i}g^*(z, t)(\partial/\partial n)u(t)dt\end{aligned}\tag{20}$$

where $\partial/\partial n$ is outer normal derivative. We denote by $u_1(z)$ (resp. $u_2(z)$) the first (resp. second) term of the right hand side of (20). Let $u^*(z)$ be a harmonic function on $B-E^*$ with boundary values $u^*=0$ on ∂B and $u^*=1$ on E^* where

$$\begin{aligned}E^*=&[-a_2, -a_1a_2/b_1]\cup[a_1a_2/b_1, a_2]\\&\cup\Big(\bigcup_{i=2}^{n}([-b_i, -a_i]\cup[a_i, b_i])\Big).\end{aligned}$$

Consider a function $\omega(z)=u_1(b_1z/a_2)+u_2(z)$. Note that $\omega(z)$ is a harmonic function on $B-E^*$. Observe that if $0<x\leqq a_2\leqq t<1$, then $g^*(x, t)\geqq g^*(b_1x/a_2, t)$. Hence $u_2(x)\geqq u_2(b_1x/a_2)$ and therefore

$$\omega(x)\geqq u_1(b_1x/a_2)+u_2(b_1x/a_2)=u(b_1x/a_2)=1$$

for $x\in[a_1a_2/b_1, a_2]$. On the other hand, observe that if $0<t\leqq b_1$ and $a_2\leqq x<1$, then $g^*(b_1x/a_2, t)>g^*(x, t)$. Hence $u_1(b_1x/a_2)\geqq u_1(x)$ and therefore

$$\omega(x)\geqq u_1(x)+u_2(x)=u(x)=1$$

for $x\in[a_i, b_i]$ $(i=2, \cdots, n)$. Moreover, by symmetry, we see that $\omega(x)\geqq 1$ for x in $[-a_2, -a_1a_2/b_1]\cup(\cup_{i=2}^{n}[-b_i, -a_i])$. Consequently we have that $\omega(z)\geqq u^*(z)$ on $B-E^*$ and especially

$$u(0)=\omega(0)\geqq u^*(0).$$

Since $|E^*|>|E|$, repeating the above argument, we have the conclusion. □

Lemma 15. *Let E be a closed subset of $[-1, 1]$ with $2-|E|\leqq 2r$ and v be the solution of Dirichlet problem on $B-E$ with boundary values $v=1$ on ∂B and $v=0$ on E. Then there exists a constant C not depending on r such that $v(0)\leqq Cr$.*

Proof. We may assume that E is a union of finitely many closed intervals and that $r<1/3$. Set $E_1=E\cap\{-x: x\in E\}$. Note that $E_1=\{-x: x\in E_1\}$ and $2-|E_1|\leqq 6r$. Let u be a harmonic function on $B-E_1$ with boundary values $u=0$ on ∂B and $u=1$ on E_1. Then, by Lemma 12 and 14, we have that

$$v(0)\leqq 1-u(0)\leqq 1-u_{3r}(0)=v_{3r}(0)\leqq Cr$$

where C does not depend on r. □

4.3. Proof of Theorem 13. By means of (19), there exists a constant C such that

$$|D\cap[-e^{-n+1}, e^{-n+1}]|\leqq Ce^{-n}/(\sqrt{n}(\log n)^{\mu}) \tag{21}$$

for all $n\geqq n_0$. Let x be in $[e^{-n-1}, e^{-n}]$ $(n\geqq n_0)$. Set $E_x=\{(t-x)/\alpha x: t\in\partial D\cap\overline{B(x, \alpha x)}\}$ where $\alpha\in(0, 1/3)$. Observe that E_x is a closed subset of $[-1, 1]$ and $v(z)=\gamma_x(x+\alpha xz)$ $=\gamma_x(x+\alpha xz; \partial D, \alpha)$ is the solution of the Dirichlet problem on $B-E_x$ with boundary values $v=1$ on ∂B and $v=0$ on E_x. Moreover, by virtue of (21), we see that

$$2-|E_x|=|D\cap[x-\alpha x, x+\alpha x]|/\alpha x\leqq C_1/(\sqrt{n}(\log n)^{\mu}).$$

Therefore, by Lemma 15, we have

$$\gamma_x(x)=v(0)\leqq C_2/(\sqrt{n}(\log n)^{\mu})$$

for every $x\in[e^{-n-1}, e^{-n}]$ $(n\geqq n_0)$. Consequently

$$\begin{aligned}\int_0^{e^{-n_0}}\frac{\gamma_x(x)}{x}dx &= \sum_{n=n_0}^{\infty}\int_{[e^{-n-1}, e^{-n}]\cap D}\frac{\gamma_x(x)}{x}dx \\ &\leqq \sum_{n=n_0}^{\infty}\frac{C_2e^{n+1}}{\sqrt{n}(\log n)^{\mu}}\frac{Ce^{-n}}{\sqrt{n}(\log n)^{\mu}} \\ &= \sum_{n=n_0}^{\infty}\frac{C_3}{n(\log n)^{2\mu}}<\infty\end{aligned}$$

since $2\mu>1$. Entirely the same argument yields that $\int_{-1}^{0}\gamma_x(x)/|x|dx<\infty$. Thus Benedicks' criterion II completes the proof. □

As a direct consequence of Theorem 13, we have the following, which was originally proved in [18] and plays an important role in § 5.

Corollary 16. *Let D be a Denjoy domain with $0\in\partial D$. Suppose that there exists a constant $\lambda>1/2$ such that*

$$\frac{|D\cap I_t|}{t}=O\Big(\frac{1}{(\log(1/t))^{\lambda}}\Big) \qquad (t\longrightarrow 0).$$

Then $\Delta_1(0, D)$ consists of two points.

4.4. We now give an example which shows that the condition $\mu>1/2$ in Theorem 13 is, in a sense, the best possible. Consider open intervals

$$J_n=((1-1/\sqrt{n}\,(\log n))e^{-n},\ \ (1+1/\sqrt{n}(\log n))e^{-n})$$

and a Denjoy domain D such that

$$D=H^+\cup H^-\cup\Big(\bigcup_{n=n_0}^{\infty}J_n\Big)$$

where n_0 is a sufficiently large integer. We put $r_n=1/(3\sqrt{n}\,(\log n))$ and $J'_n=[(1-r_n)e^{-n},\ (1+r_n)e^{-n}]\ (n\geqq n_0)$. Then, observe that

$$\partial D\cap B(x,\ x/4)\subset(3x/4,\ x-e^{-n}r_n)\cup(x+e^{-n}r_n,\ 5x/4)$$

for every x in J'_n. By the definition of γ_x and that of v_r in Lemma 12, this implies that

$$\gamma_x(z)=\gamma_x(z\,;\ \partial D,\ 1/4)\geqq v_{r_n}(4(z-x)/x)$$

for every z in $B(x,\ x/4)$ and x in J'_n. Hence, by Lemma 12, it is verified that

$$\gamma_x(x)\geqq v_{r_n}(0)\geqq C/\sqrt{n(\log n)}$$

for every x in $J'_n(n\geqq n_0)$. Therefore, we have

$$\begin{aligned}\int_{|x|\leq 1}\frac{\gamma_x(x)}{x}\,dx&\geqq\sum_{n=n_0}^{\infty}\int_{J'_n}\frac{\gamma_x(x)}{x}\,dx\\&\geqq\sum_{n=n_0}^{\infty}\frac{C}{e^{-n+1}\sqrt{n(\log n)}}\,\frac{e^{-n}}{3\sqrt{n(\log n)}}\\&=\sum_{n=n_0}^{\infty}\frac{C'}{n(\log n)}=\infty.\end{aligned}$$

Hence Benedicks' criterion II yields that $\Delta_1(0, D)$ consists of a single point. On the other hand, it is easily seen that

$$\frac{|D\cap I_t|}{t}=O\Big(\frac{1}{(\log(1/t)\cdot\log\log(1/t))^{1/2}}\Big) \qquad (t\longrightarrow 0).$$

Thus this shows that the condition $\mu>1/2$ cannot be replaced by $\mu\geqq 1/2$.

4.5. We put $L^1(t)=\log(1/t)$ and $L^n(t)=\log(L^{n-1}(t))\,(n\geqq 2)$. At the end of this section, we remark that the argument in this section yields the following

Theorem 17. *Let D be a Denjoy domain with $0\in\partial D$. Suppose that there exists a constant $\mu>1/2$ such that*

$$\frac{|D\cap I_t|}{t}=O\Big(\frac{1}{(L^1(t)\cdots L^{n-1}(t))^{1/2}(L_n(t))^{\mu}}\Big) \qquad (t\longrightarrow 0).$$

Then $\Delta_1(0, D)$ consists of two points. Moreover, the condition $\mu>1/2$ cannot be replaced by $\mu\geqq 1/2$.

§ 5. Martin boundaries of quasiconformally invariant Denjoy domains

5.1. For every positive real number λ, we consider open intervals

$$J_n(\lambda)=((1-n^{-\lambda})e^{-n},\ (1+n^{-\lambda})e^{-n})$$

and a Denjoy domain $D(\lambda)$ with $0\in\partial D(\lambda)$ such that

$$D(\lambda)=H^+\cup H^-\cup\Big(\bigcup_{n=n_0(\lambda)}^{\infty}J_n(\lambda)\Big) \tag{22}$$

where $n_0(\lambda)$ is a sufficiently large integer. Observe that

$$\frac{|D(\lambda)\cap I_t|}{t}=O\Big(\frac{1}{(\log(1/t))^{\lambda}}\Big) \qquad (t\longrightarrow 0).$$

Hence, in view of Corollary 16 and Proposition 6, we see that if $\lambda>1/2$ then $\Delta_1(0, D(\lambda))$ consists of two points p_1 and p_2 and hence $\Delta(0, D(\lambda))$ is homeomorphic to the interval $\{cp_1+(1-c)p_2: c\in[0, 1]\}$ (cf. no. 2.2). Suppose that $\lambda\leqq 1/2$. Let D be a Denjoy domain defined as in no. 4.4 for n_0 satisfying $n_0\geqq n_0(\lambda)$. Note that $D(\lambda)\supset D$. In no. 4.4, we have seen that $\Delta_1(0, D)$ consists of a single point. Therefore, by Benedicks' criterion II and Proposition 6, we see that $\Delta(0, D(\lambda))=\Delta_1(0, D(\lambda))$ consists of a single point. Consequently we obtain the following

Proposition 18. *Let $D(\lambda)$ be a Denjoy domain defined as in* (22). *Then, $\Delta(0, D(\lambda))$ is homeomorphic to a closed interval if $\lambda>1/2$ and $\Delta(0, D(\lambda))$ consists of a single point if $\lambda\leqq 1/2$.*

5.2. In order to prove the Theorem in Introduction, we need the following Beurling-Ahlfors theorem [7] (also cf. e. g. Ahlfors [1] and Lehto-Virtanen [12]).

Theorem G. *Let h be a continuous and increasing function on the real axis. Then there exists a quasiconformal mapping f of the upper half plane H^+ onto itself with boundary values $f(x)=h(x)$ if and only if there exists a constant ρ such that*

$$\frac{1}{\rho}\leqq\frac{h(x+t)-h(x)}{h(x)-h(x-t)}\leqq\rho \tag{23}$$

for every $x\in\boldsymbol{R}$ and $t>0$.

Moreover, for a continuous and increasing function h on the real axis satisfying (23), a quasiconformal mapping f of H^+ onto itself with boundary values $f(x)=h(x)$ is given by

$$f(x+iy)=\frac{1}{2}(\alpha(x, y)+\beta(x, y))+\frac{i}{2}(\alpha(x, y)-\beta(x, y))$$

where

$$\alpha(x, y)=\int_0^1 h(x+ty)dt, \quad \beta(x, y)=\int_0^1 h(x-ty)dt.$$

We take arbitrary positive real numbers λ and α. We consider a function h on the real axis such that

$$h(x)=\begin{cases} x & (x\notin\bigcup_{n=n_0(\lambda_0)}^{\infty}J_n(\lambda)) \\ n^{\lambda\alpha}e^{n\alpha}(x-e^{-n})^{\alpha+1}+e^{-n} & (x\in[e^{-n},\ (1+n^{-\lambda})e^{-n}]) \\ -n^{\lambda\alpha}e^{n\alpha}(e^{-n}-x)^{\alpha+1}+e^{-n} & (x\in[(1-n^{-\lambda})e^{-n},\ e^{-n}]) \end{cases} \tag{24}$$

where we take $n_0(\lambda)$ such that every pair of $\{J_n(\lambda)\}_{n\geq n_0(\lambda)}$ is mutually disjoint. Observe that h is a continuous and increasing function and $h(0)=0$. We show the following

Lemma 19. *For every positive λ and α, there exists a constant ρ such that h defined by* (24) *satisfies the condition* (23) *for every $x\in\boldsymbol{R}$ and $t>0$.*

Proof. Set $J_n'=J_n'(\lambda)=((1-n^{-\lambda}/3)e^{-n},\ (1+n^{-\lambda}/3)e^{-n})$. We assume that $x\in[(1+(n+1)^{-\lambda}/3)e^{-n-1},\ (1-n^{-\lambda}/3)e^{-n}]$ for some $n\geqq n_0(\lambda)$. Observe that if $x+t$ belongs to J_n' or J_{n+1}' then

$$2/3^{\alpha+1}\leqq(h(x+t)-h(x))/t\leqq\alpha+1.$$

This implies that if x does not belong to $\bigcup_{n=n_0(\lambda)}^{\infty}J_n$, $J_n=J_n(\lambda)$, then

$$1/\rho\leqq(h(x+t)-h(x))/(h(x)-h(x-t))\leqq\rho \tag{25}$$

where $\rho=3^{\alpha+1}(\alpha+1)/2$. Moreover we see that (25) is also valid if x belongs to J_n' for some $n\geqq n_0(\lambda)$ and neither of $x\pm t$ belongs to J_n'. We next assume that x and at least one of $x\pm t$ belong to J_n'. Then, note that x and $x\pm t$ belong to J_n. There is no loss of generality in assuming that $e^{-n}<x<x+t$. It is easily seen that $(h(x+t)-h(x))/(h(x)-h(x-t))\geqq1$. Observe that if $x-t\leqq0$, then $h(x+t)-h(x)\leqq n^{\lambda\alpha}e^{n\alpha}(2t)^{\alpha+1}$ and $h(x)-h(x-t)\geqq n^{\lambda\alpha}e^{n\alpha}(t/2)^{\alpha+1}$. Hence we see that

$$(h(x+t)-h(x))/(h(x)-h(x-t))\leqq4^{\alpha+1}$$

if $x-t\leqq0$. Finally, if $0<x-t<x<x+t$, then

$$\frac{h(x+t)-h(x)}{h(x)-h(x-t)}=\frac{(1+t/x)^{\alpha+1}-1}{1-(1-t/x)^{\alpha+1}}\leqq\frac{2^{\alpha+1}t/x}{t/x}=2^{\alpha+1}.$$

The proof is herewith complete. □

5.3. We give here the proof of Theorem in Introduction. We prove a bit more. Namely

Theorem 20. *For arbitrary positive real numbers λ_1 and λ_2 such that $\lambda_1\leqq1/2<\lambda_2$, let $D(\lambda_1)$ and $D(\lambda_2)$ be Denjoy domains defined as in* (22) *for sufficiently large $n_0(\lambda_1)=n_0(\lambda_2)$. Then there exists a quasiconformal mapping f of $\boldsymbol{C}$ onto itself such that $f(D(\lambda_1))=D(\lambda_2)$ but f cannot be extended to a homeomorphism of $D(\lambda_1)^*$ onto $D(\lambda_2)^*$.*

Proof. We take a λ with $0<\lambda<\lambda_1$. We may assume that $n_0=n_0(\lambda)=n_0(\lambda_1)=n_0(\lambda_2)$ and every pair of $\{J_n(\lambda)\}_{n\geq n_0}$ is mutually disjoint. Consider a continuous increasing function h on the real axis defined by (24) for λ and $\alpha=(\lambda_2-\lambda_1)/(\lambda_1-\lambda)$. Observe that

(26) $$h(J_n(\lambda_1))=J_n(\lambda_2) \qquad (n=n_0, n_0+1, \cdots).$$

In view of the Beurling-Ahlfors theorem and Lemma 19, there exists a quasiconformal mapping f of H^+ onto itself with boundary values $f(x)=h(x)$. Defining $f(z)=\overline{f(\bar{z})}$ for $z\in H^-$, f is extended to a quasiconfomal mapping of $\boldsymbol{C}$ onto itself (cf. e. g. [12]). By means of (26), we see that $f(D(\lambda_1))=D(\lambda_2)$. Note that $f(0)=0$. On the other hand, by Proposition 18, $\Delta(0, D(\lambda_1))$ consists of a single point but $\Delta(0, D(\lambda_2))$ is homeomorphic to an interval. Therefore f cannot be extended to a homeomorphism of $D(\lambda_1)^*$ onto $D(\lambda_2)^*$. □

5.4. Let $D(\lambda_1)$ and $D(\lambda_2)$ be as in Theorem 20. We remark that any quasiconformal mapping f of $\boldsymbol{C}$ onto itself with $f(D(\lambda_1))=D(\lambda_2)$ does not satisfy the condition $|f(z)|=f(|z|)$, which is a reason why we appeal to the Beurling-Ahlfors theorem to prove Theorem 20. To see this, suppsse that f satisfies the condition $|f(z)|=f(|z|)$. Consider ring domains

$$A_n(\lambda_i)=\{(1-n^{-\lambda_i})e^{-n}<|z|<(1+n^{-\lambda_i})e^{-n}\}$$

where $i=1, 2$ and $n=n_0, n_0+1, \cdots$. Then we have that

$$\begin{aligned}\operatorname{mod} A_n(\lambda_i)&=\log((1+n^{-\lambda_i})/(1-n^{-\lambda_i}))\\ &\sim 2/n^{\lambda_i} \qquad (n \longrightarrow \infty, i=1, 2)\end{aligned}$$

and hence

$$(\operatorname{mod} A_n(\lambda_1))/(\operatorname{mod} A_n(\lambda_2)) \longrightarrow \infty \qquad (n \longrightarrow \infty).$$

This contradicts the fact that $f(A_n(\lambda_1))=A_n(\lambda_2)$.

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