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Babylonian Tower Theorem on variety

By

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In this paper, we study a condition for a variety to be a weighted complete intersection in a weighted projective space.

We consider the following condition.

Let $\{X_n\}_{n\in N}$ and $\{Y_n\}_{n\in N}$ be two sequences of projective varieties where X_n and Y_n are divisors in X_{n+1} and Y_{n+1} with the closed embeddings $i_n: X_n\to X_{n+1}$ and $j_n: Y_n\to Y_{n+1}$ respectively.

Moreover let $\{f_n: X_n\to Y_n\}_{n\in N}$ be a sequence of finite and flat morphisms satisfying $f_n j_n = i_n f_{n+1}$ for any n.

Then we have

Main Theorem. (4.16) *Let two sequences of projective varieties $\{X_n\}$, $\{Y_n\}$ and a sequence of morphism $\{f_n: X_n\to Y_n\}$ be as above. Assume that for every positive integer n, X_n and Y_n are smooth and Y_n is a weighted complete intersection in the weak projective space $\boldsymbol{P}(\boldsymbol{e}_{n+r})$ (see 1.1 and 1.5). Moreover, assume that $\beta_n^{-1}(Y_{n+1})=Y_n$ with a canonical morphism $\beta_n: \boldsymbol{Q}(\boldsymbol{e}_{n+r})\to\boldsymbol{Q}(\boldsymbol{e}_{n+r}, 1)$ (see 4.3). Then,*

1) *If the characteristic of the ground field is zero, then X_n is a complete intersection in a vector bundle $\boldsymbol{V}(E_n)$ where E_n is a direct sum of line bundles on Y_n (see 1.6.4). More precisely, letting $E_n=\bigoplus_{i=1}^{s}\mathcal{O}_{Y_n}(b_i)$, $\boldsymbol{V}(E_n)$ is canonically embedded in the weak projective syace $\boldsymbol{P}(e_0, \cdots, e_{n+r}, b_1, \cdots, b_s)$ and X_n is a weighted complete intersection in it.*

2) *If the characteristic of the ground field is positive and Y_n is a projective space, then the same conclusion as in 1) holds well.*

The above theorem is an answer to the problem suggested by Fulton and posed by Lazarsfeld in [L2] in the more general form.

Moreover it provides us with the following results.

Corollary 5.6.1. (compare Conjecture 4.23 in [Fu]) *Let us consider a sequence $\{X_n, L_n\}$ of connected polarized schemes satisfying the following: for every n,*

(1) *X_n is an ample divisor in X_{n+1}.*

(2) *$L_{n+1|X_n}=L_n$ and $\mathcal{O}_{X_{n+1}}(X_n)=a_{n+1}L_{n+1}$ with some integer a_{n+1}.*

(3) *Letting $G(X_n, L_n)$ the graded algebra $\bigoplus_{t\geq 0}H^0(X_n, tL_n)$, the canonical homomorphism $G(X_{n+1}, L_{n+1})\to G(X_n, L_n)$ is surjective. Assume that X_n is smooth.*

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Then, for each n (X_n, L_n) is a weighted complete intersection. (*See* 1.5)

Theorem 5.8. *Let $\{X_n\}$ be a sequence of smooth projective varieties and X_n an ample divisor in X_{n+1} for any n. Assume that for each n, $\mathrm{Pic}\, X_n \cong \boldsymbol{Z} L_n$ with an ample line bundle L_n. Then in characteristic zero, (X_n, L_n) is a weighted complete intersection for a large n.*

Historically, there are several kinds of Babylonian Tower Theorem namely, about variety, vector bundle on the projective space and punctual spectrum. Babylonian Tower Theorem on smooth subvariety in the projective space was solved by Hartshorne [Ha], Barth and Van de Ven [BV], [B]. The one on vector bundle on the projective space was proved by Barth-Van de Ven [BV] for *rk* 2 bundles and Sato [S1, S2], Tjurin [T] in the general case. Moreover the one on vector bundle on punctual spectrum was solved by Flenner [Fl]. As the corollary, Flenner derived the theorem on a closed subscheme in the projective space which is locally complete intersection and on vector bundle on the projective space.

In this paper, we investigate Babylonian Tower Theorem for variety in more general form. In this case, our theorems depend heavily on the results [S1, S2] and [T].

In § 1 we review several results about the weighted projective space. In § 2 we study the formal duality in weighted projective space (Proposition 2.6). In § 3 we investigate a criterion for a subscheme to be a complete intersection and get Proposition 3.6 which is directly related with the proof of Main Theorem. In § 4 we prove Main Theorem. In § 5 we obtain applications of Main Theorem as stated above. We also get a Babylonian Tower Theorem (Theorem 5.3) on a closed, reduced subscheme in weighted projective space which is a locally complete intersection.

We work over an algebraically closed field k of any characteristic. Basically we use the customary notation in algebraic geometry. We use the terms vector bundle and locally free sheaf interchageably. For a vector bundle E on a scheme S, $\boldsymbol{V}(E)$ denotes $\mathrm{Spec}(S(E))$ where $S(E)$ is the $\mathcal{O}_S$-symmetric algebra of E and $E^{\vee}$ denotes the dual vector bundle of E.

§ 1. Several remarks about the weighted projective space

In this section, we shall review several results about the weighted projective space. We mainly quote the ones in [Mo] which are necessary for us.

First let us start with the definition of the weighted projective space.

(1.1) For positive integers $m, e_0, e_1, \cdots, e_m$, m-dimensional weighted projective space $\boldsymbol{Q}(e_0, \cdots, e_m)$ denotes $\mathrm{Proj}\, k[X_0, \cdots, X_m]$ (written as $\boldsymbol{Q}(\boldsymbol{e})$ simply) where the graduation of $k[X_0, \cdots, X_m]$ is given with $\deg X_i = e_i$ $(0 \leqq i \leqq n)$ and $\deg a = 0$ $(a \in k)$.

For an integer a, $\mathcal{O}_{\boldsymbol{Q}(\boldsymbol{e})}(a)$ is the coherent $\mathcal{O}_{\boldsymbol{Q}(\boldsymbol{e})}$-module corresponding to the homogeneous $k[X_0, \cdots, X_m]$-module $k[X_0, \cdots, X_m](a)$. Moreover, letting S_k the closed subset of $\boldsymbol{Q}(\boldsymbol{e})$ whose defining ideal is generated by $\{X_i \mid k \nmid e_i\}$, $\boldsymbol{P}(\boldsymbol{e})$ denotes $\boldsymbol{Q}(\boldsymbol{e}) - \bigcup_{k \geqq 2} S_k$, called the weak projective space.

Then, the following are well-known

Fact 1.2. I. There is a canonical morphism $\varphi: P^m \to \boldsymbol{Q}(\boldsymbol{e})$ by the corresponding $(Y_0, \cdots, Y_m) \to (X_0^{e_0}, \cdots, X_m^{e_m})$ where $Y_0, \cdots, Y_m$ are homogeneous coordinates of the projective space P^m.

II. $\varphi_* \mathcal{O}_{P^m} = \bigoplus \mathcal{O}_{Q(e)}\left(-\sum_{i=0}^{m} v_i\right)$ where $0 \leqq v_i \leqq e_i - 1$.

III. For every integer a, $\mathcal{O}_{Q(e)}(a)_{|P(e)}$ is the invertible sheaf $(=\mathcal{O}_{P(e)}(a))$ and $\mathcal{O}_{P(e)}(1)^{\otimes a} \cong \mathcal{O}_{P(e)}(a)$. Moreover, $\varphi^* \mathcal{O}_{P(e)}(1) \cong \mathcal{O}_{P^m}(1)_{|\varphi^{-1}(P(e))}$ and $\varphi: \varphi^{-1}(\boldsymbol{P}(\boldsymbol{e})) \to \boldsymbol{P}(\boldsymbol{e})$ is a flat morphism.

For the above properties, see § 1, § 2 and § 3, in [Mo].

Now we have

Proposition 1.3. *Let W be a closed subscheme in $\boldsymbol{Q}(\boldsymbol{e})$ and F a coherent sheaf on W. Then, we have*

(1) *$\varphi_* \mathcal{O}_{\varphi^{-1}(W)}$ has a trivial line bundle $\mathcal{O}_W$ as a direct summand.*

(2) *If $H^i(\varphi^{-1}(W), \varphi^* F)$ vanishes, so does $H^i(W, F)$.*

(3) *Assume $\varphi^* F$ is isomorphic to $\bigoplus \mathcal{O}_{P^m}(c_i)_{|\varphi^{-1}(W)}$ and $W \subset \boldsymbol{P}(\boldsymbol{e})$. Then F is isomorphic to $\bigoplus \mathcal{O}_{P(e)}(d_i)_{|W}$.*

Proof. (1) is trivial by II in 1.2 (see the proof of Theorem 3.7 in [Mo]). Since $H^i(\varphi^{-1}(W), \varphi^* F) \cong H^i(W, \varphi_* \varphi^* F)$, (2) is obtained. Moreover, noting that F is a direct summand of $\varphi_* \varphi^* F$ by (1), (3) is shown by II and III in Fact 1.2. q.e.d.

Now let Z be a closed subscheme in $\boldsymbol{P}(\boldsymbol{e})$ defined by homogeneous elements $f_1, \cdots, f_c$ in $k[X_0, \cdots, X_m]$ and E a locally free sheaf $\bigoplus_{i=1}^{r} \mathcal{O}_Z(-a_i)$ where all the a_i's are positive and $\mathcal{O}_Z(b) = \mathcal{O}_{P(e)}(1)^{\otimes b}|_Z$. Moreover, let $k[X_0, \cdots, X_m, \cdots, X_{m+r}]$ be the graded ring with $\deg X_i = e_i$ $(0 \leqq i \leqq m)$ and $\deg X_{j+m} = a_j$ $(1 \leqq j \leqq r)$ and let $\bar{E}$ be the closed subscheme of $\boldsymbol{Q}(\boldsymbol{e}, \boldsymbol{a})$ defined by the above elements $f_1, \cdots, f_c$ with $\boldsymbol{a} = (a_1, \cdots, a_r)$.

Then we can easily show

Proposition 1.4. *$\boldsymbol{V}(E)$ can be naturally considered as the open subset $\bigcup_{0 \leq i \leq m} \{x \in \bar{E} \mid X_i \neq 0\}$ and it is contained in $\boldsymbol{P}(\boldsymbol{e}, \boldsymbol{a})$. Letting $\varphi: \boldsymbol{Q}(\boldsymbol{e}, 1, \cdots, 1) \to \boldsymbol{Q}(\boldsymbol{e}, \boldsymbol{a})$ a canonical projection, $\varphi^{-1}(\boldsymbol{V}(E)) = \boldsymbol{V}(\mathcal{O}_Z(-1)^{\oplus r})$.*

Finally in this section let us recall

Definition 1.5. An algebraic k-scheme X is called a weighted complete intersection of $\boldsymbol{P}(\boldsymbol{e})$ if X is isomorphic to Proj of a graded ring R satisfying the following:

(#) R is isomorphic to $k[X_0, \cdots, X_m]/(f_1, \cdots, f_c)$ with homogeneous elements $f_1, \cdots, f_c$ in $k[X_0, \cdots, X_m]$ and have the following properties

(1) $(f_1, \cdots, f_c)$ is a regular sequence of $k[X_0, \cdots, X_m]$.

(2) $V_+(f_1, \cdots, f_c) \cap \bigcup_{k \geq 2} S_k = \varnothing$.

Moreover a polarised algebraic scheme (X, L) is called a weighted complete intersection if the graded ring $\bigoplus_t H^0(X, tL)$ is isomorphic to the one with the condition (#).

The following proposition is important for the proof of our main theorem.

Proposition 1.6. *Let X be a weighted complete intersection of dimension $\geqq 1$ in $\boldsymbol{P}(e_0, \cdots, e_m)$ defined as in* 1.5. *Then,*

(1) $k[X_0, \cdots, X_m]/(f_1, \cdots, f_c))_a \cong H^0(X, \mathcal{O}_X(a))$ $(a \in \boldsymbol{Z})$, *where R_a is the homogeneous part of degree a of R.*

(2) $H^j(X, \mathcal{O}_X(a))=0$ $(t \in \boldsymbol{Z},\ 0<j<\dim X)$,

(3) $\omega_X=\mathcal{O}_X(a_1+\cdots+a_c-(e_0+\cdots+e_m))$, *where ω_X is the dualizing sheaf of X and* $a_i=\deg f_i$.

(4) $\mathrm{Pic}(X)$ *is generated by $\mathcal{O}_{P(e)}(1)_{|X}$ if* $\dim X \geqq 3$.

See Proposition 3.3 and Theorem 3.7 in [Mo].

§2. Some remarks about the formal duality.

In this section, let us discuss about the Lefschetz condition.

First let us recall a definition due to Hironaka and Matsumura.

(2.1) Let X be a scheme, Y a closed subscheme in X. Then Y is called G3 in X if $K(X) \cong K(\hat{X})$ where $K(\hat{X})$ is the ring of formal-rational functions along Y.

Moreover let us recall

Theorem 2.2. (Hironaka-Matsumura [HM], Theorem 3.3) *Let $Y \subset P^n$ be a closed subscheme. Then, Y is* G3 *in P^n if and only if Y is connected and of dimension$\geqq 1$.*

Thus we have

Proposition 2.3. *Let X be a connected, closed set in P^n with* $\dim X \geqq 1$. *Then, $H^i(P^n - X, F)=0$ for every coherent sheaf F on P^n and $i=n-1, n$.*

See Proposition 3.2 (due to Speiser) in Chapter V [Ha].

Proposition 2.4. *Let W be a closed subscheme in an n-dimensional weighted projective space $\boldsymbol{O}(\boldsymbol{e})$ with a natural projection $\varphi: P^n \to \boldsymbol{Q}(\boldsymbol{e})$. Assume that $\varphi^{-1}(W)$ is connected and of dimension$\geqq 1$. Then, $H^i(\boldsymbol{Q}(\boldsymbol{e})-W, F)=0$ for every coherent sheaf F on $\boldsymbol{Q}(\boldsymbol{e})$ and $i=n-1, n$.*

Proof. By Proposition 2.3, $H^i(P^n-\varphi^{-1}(W), \varphi^* F)=0$ for $i=n-1, n$. By Proposition 1.3, $\varphi_* \mathcal{O}_{P^n-\varphi^{-1}(W)}$ has a trivial line bundle as a direct summand. Thus, since φ is an affine morphism, we have the desired result. q.e.d.

In the next place, we study about the formal duality. First we show

Proposition 2.5. *Let X be a projective, Cohen Macauley and equidimensional scheme of dimension n, ω_X° the dualizing sheaf of X and Y a closed subset in X. Then for a* *coherent sheaf F on X which is locally free on some neighbourhood of Y, letting $G=$ $\mathrm{Hom}_X(F, \omega_X^\circ)$, we have $H^0(\hat{X}, \hat{F})\cong(H_Y^n(X, G))^\vee$, where $^\vee$ denotes the dual vector space.*

Proof. We can show this proposition in the same way as in Theorem 3.3 in Chapter III [Ha 1]. Then we must notice the following: letting I_Y the defining ideal of Y in X,

$$
\begin{aligned}
&H^0(X, F\otimes\mathcal{O}_X/I_Y^m) \\
&=\mathrm{Ext}^n(F\otimes\mathcal{O}_X/I_Y^m, \omega_X^\circ)^\vee \qquad \text{(see Theorem 7.6 in Chapter III [Ha])} \\
&=\mathrm{Ext}^n(\mathcal{O}_X/I_Y^m, G)^\vee \qquad \text{(Since } F \text{ is locally free around } Y\text{)} \qquad \text{q.e.d.}
\end{aligned}
$$

The above yields an important proposition which will be used in the next section.

Proposition 2.6. *Let the notation and assumption be as in Proposition* 2.4. *Moreover assume F is a coherent sheaf on $\boldsymbol{Q}(\boldsymbol{e})$ which is locally free around some neighbourhood of W. Then, we have a canonical isomorphism:*

$$H_W^n(\boldsymbol{Q}(\boldsymbol{e}), F)\cong H^n(\boldsymbol{Q}(\boldsymbol{e}), F).$$

In other words, $H^0(\hat{\boldsymbol{Q}}(\boldsymbol{e}), \hat{F})\cong H^0(\boldsymbol{Q}(\boldsymbol{e}), F)$,

Proof. Note that $\boldsymbol{Q}(\boldsymbol{e})$ is a Cohen-Macauley variety. Then, there is the long exact sequence of local cohomology:

$$H^{n-1}(\boldsymbol{Q}(\boldsymbol{e})-W, F)\longrightarrow H_W^n(\boldsymbol{Q}(\boldsymbol{e}), F)\longrightarrow H^n(\boldsymbol{Q}(\boldsymbol{e}), F)\longrightarrow H^n(\boldsymbol{Q}(\boldsymbol{e})-W, F)$$

(See Corollary 1.9. in [G]). Thus Proposition 2.4 yields the former. The latter is obtained by Proposition 2.5. q.e.d.

§3. A criterion for a closed subscheme to be a complete intersection

In this section, we consider a sufficient condition for a subvariety to be a complete intersection in a ambient space.

(3.1) Let V be an n-dimensional complete variety and X a k-dimensional complete subscheme in V which is a locally complete intersection. Let L be an ample line bundle in V and $\mathcal{O}_X(m)$ denoted by $L^{\otimes m}|_X$.

We assume that

(3.2) 1. $H^1(X, \mathcal{O}_X(t))=0$ for every integer t.

2. The normal bundle $N_{X/V}$ is isomorphic to $\bigoplus_{i=1}^{n-k}\mathcal{O}_X(a_i)$ with positive integers $a_i(a_1\geqq a_2\geqq\cdots\geqq a_{n-k})$.

3. There is an open subscheme $\bar{U}$ $(\supset X)$ in V such that a canonical map $H^0(\bar{U}, \mathcal{O}_{\bar{U}}(a_i))\to H^0(\hat{V}, \mathcal{O}_{\hat{V}}(a_i))$ is surjective where $\hat{}$ is the completion along X.

Then we get

Proposition 3.3. *Under the above notation* 3.1, *assume* 3.2. *Then there is an open subscheme U in V satisfying: X is a complete intersection $H_1\cap H_2\cap\cdots\cap H_{n-k}$ in U where H_i is a divisor in $|\mathcal{O}_U(a_i)|$.*

Proof. Let I be the sheaf of ideals of X in V and V_m the closed subscheme of V defined by I^m. Then, we have an exact sequence

$$*_m \qquad 0 \longrightarrow I^m/I^{m+1} \longrightarrow \mathcal{O}_{V_{m+1}} \longrightarrow \mathcal{O}_{V_m} \longrightarrow 0.$$

Noting that I^m/I^{m+1} is the m-th symmetric power product of I/I^2 and tensoring the line bundle $L_i\,(=L^{\otimes a_i})$ to $*_m$, we obtain $*_{i,m}$ $0\to H^0(X, L_i\otimes I^m/I^{m+1})\to H^0(V_{m+1}, L_i)\to H^0(V_m, L_i)\to 0$ by assumption 1. Since $L_i\otimes I/I^2=\bigoplus_{j=1}^{n-k}\mathcal{O}_X(a_i-a_j)$, there is a nowhere vanishing and constant section s_i in $H^0(X, L_i\otimes I/I^2)$. Thus, we have a section $\hat{s}_i$ in $H^0(\hat{V}, \hat{L}_i)$ to which s_i is lifted. (If $a_i=a_{i+1}=\cdots=a_{i+h}\geqq a_{i+h+1}+1$, choose $(h+1)$-sections in $H^0(X, L_i\otimes I/I^2)$ which are linearly independent.)

Now, we have a section $\bar{s}_i$ in $H^0(U, \mathcal{O}_U(a_i))$ induced by $\hat{s}_i$ by means of 3. Since $s_{i|X}=0$, the divisor H_i defined by $\bar{s}_i$ contains X. Thus, applying Nakayama's Lemma to these elements $\bar{s}_1, \cdots, \bar{s}_{n-k}$, we see that X is a complete intersection of H_i's in a suitable open subset U in $\bar{U}$. q.e.d.

The above immediately yields

Corollary 3.4. *Under the above conditions and assumptions in* 3.3, *let us assume additionally that* $V=\boldsymbol{P}(\boldsymbol{e})$, $L=\mathcal{O}_{\boldsymbol{P}(\boldsymbol{e})}(1)$ *and* $\dim X\geqq\dim \boldsymbol{Q}(\boldsymbol{e})/2$. *Then X is a complete intersection of the closure's $\bar{H}_i$ in $\boldsymbol{P}(\boldsymbol{e})$.*

Proof. Let $\bigcup_{i\geqq 2} X_i$ be the irreducible decomposition of $\cap\bar{H}_i$ in $\boldsymbol{P}(\boldsymbol{e})$ with $X_1=X$. Since $\bar{H}_i$ is a Cartier divisor in $\boldsymbol{P}(\boldsymbol{e})$, we see that $\dim X_i\geqq\dim \boldsymbol{Q}(\boldsymbol{e})/2$ for every i, and therefore X_i intersects with X_j $(i\neq j)$. Thus we are done. q.e.d.

Moreover we have

Corollary 3.5. *Let X be a closed subscheme in P^n which is a locally complete intersection. Assume* 1) *and* 2) *in* (3.2) *where* $\mathcal{O}_X(*)=\mathcal{O}_{P^n}(*)_{|X}$. *Moreover, assume that* $\dim X\geqq n/2$. *Then X is a complete intersection in P^n.*

Proof. Since X is connected, we get this corollary by virtue of Proposition 2.6. q.e.d.

Under the above preparations in this section, we study a concrete case which is closely related with Main Theorem.

First, we consider the following

(3.6) Let Y be a complete subscheme in $\boldsymbol{P}(\boldsymbol{e})$, F a locally free sheaf on Y and X a complete subscheme in $\boldsymbol{V}(F)$ $(=\boldsymbol{F})$ which is a locally complete intersection where $\dim X=\dim Y=e$ and a natural morphism $X\to Y$ is a covering. Then we assume that

(1) Y is a weighted complete intersection $D_1\cap D_2\cap\cdots\cap D_c$ in $\boldsymbol{P}(\boldsymbol{e})$ with $D_i\in|\mathcal{O}_{\boldsymbol{P}(\boldsymbol{e})}(d_i)|$ and $d_i>0$ (see definition 1.5).

(2) F is isomorphic to $\bigoplus_{i=1}^{r}\mathcal{O}_Y(-a_i)$ with $a_i>0$ and $\mathcal{O}_Y(1)=\mathcal{O}_{\boldsymbol{P}(\boldsymbol{e})}(1)_{|Y}$. (Therefore, $\boldsymbol{F}$ is naturally a locally closed subscheme in $\boldsymbol{P}(\boldsymbol{e}, \boldsymbol{a})$ with $\boldsymbol{a}=(a_1, \cdots, a_r)$ by 1.4)

$N_{X/F} = \bigoplus_{i=1}^{r} \pi^* \mathcal{O}_Y(b_i)$ where b_i is a positive integer and $\bar{\pi}$ is a natural projection: $F \to Y$ and $\bar{\pi}_{|X} = \pi$.

(4) $\pi_* \mathcal{O}_X = \bigoplus_i \mathcal{O}_Y(c_i)$.

(5) $\dim X \geqq \dim \boldsymbol{P}(\boldsymbol{e}, \boldsymbol{a})/2$.

Now, we have

Proposition 3.6. *Under the above conditions and assumptions* (3.6), *we have*

(1) $N_{X/P(e,a)}$ *is isomorphic to* $N_{X/F} \oplus N_{F/P(e,a)|X} = \bigoplus_{i=1}^{r} \pi^* \mathcal{O}_Y(b_i) \bigoplus_{j=1}^{c} \pi^* \mathcal{O}_Y(d_j)$.

(2) *there is an open subset* U $(\supset X)$ *in* $\boldsymbol{P}(\boldsymbol{e}, \boldsymbol{a})$ *such that* X *is a complete intersection* $H_1 \cap H_2 \cap \cdots \cap H_{c+r}$ *in* U *and* H_i *is a divisor in* $|\mathcal{O}_U(m_i)|$ *with* $m_i = b_i$ $(1 \leqq i \leqq r)$ *and* $m_{j+r} = d_j$ $(1 \leqq j \leqq c)$. *In particular,* H_i $(1 \leqq i \leqq r)$ *can be taken as* $\varphi^{-1}(D_i)_{|U}$. *Consequently*

(3) X *is a complete intersection in* $\boldsymbol{F}$.

Proof. Let W be $\boldsymbol{P}(\boldsymbol{e}, \boldsymbol{a})$ and φ: $V(\oplus \mathcal{O}_{P(e)}(-a_i))$ $(=V) \dashrightarrow \boldsymbol{P}(\boldsymbol{e})$ a canonical projection. Noting $\mathcal{O}_W(1)_{|V} = \varphi^* \mathcal{O}_{P(e)}(1)$, we see that $\mathcal{O}_W(1)_{|X} = \pi^* \mathcal{O}_Y(1)_{|X}$ $(=L)$. Thus, we have a

Claim. $H^1(X, L^{\otimes a}) = 0$ for every integer a.

Proof. The morphism $X \to Y$ is a finite morphism. Thus we have only to show that $H^1(Y, \pi^*(L^{\otimes a})) = 0$. Since $\pi_* L = \mathcal{O}_Y(1) \otimes \pi_* \mathcal{O}_X$ by the projection formula, we get the desired fact by the assumption (1) (4) an Proposition 1.6.

Thus, since $N_{F/W|X} = \pi^* N_{Y/P(e)|X}$ and there is the following exact sequence on X:

$$0 \longrightarrow N_{X/F} \longrightarrow N_{X/W} \longrightarrow N_{F/W|X} \longrightarrow 0,$$

we infer that $N_{X/W}$ is isomorphic to a direct sum of line bundles $(= \oplus \mathcal{O}_X(b_i) \oplus \mathcal{O}_X(d_j))$ by virtue of the assumption (3) and the above claim, which gives (1). Moreover, by the assumption 5), $\sigma^{-1}(X)$ is connected in P^{c+e+r} with a canonical projection $\sigma: P^{c+e+r} \to \boldsymbol{Q}(\boldsymbol{e}, \boldsymbol{a})$ $(=\boldsymbol{Q})$, which implies that $H^0(\boldsymbol{Q}, \mathcal{O}_{\boldsymbol{Q}}(a)) \cong H^0(\hat{\boldsymbol{Q}}, \mathcal{O}_{\hat{\boldsymbol{Q}}}(a))$ for any a by Proposition 2.6. Thus, Proposition 3.3 yields (2), which provides us with (3) by Corollary 3.4.

q.e.d.

§ 4. Proof of Main Theorem

In this section let us consider an infinite sequence of algebraic k-schemes: $\{X_n\}_{n \in N}$ where X_n is a Cartier divisor in X_{n+1} with the closed embedding $i_n: X_n \to X_{n+1}$. This sequence with the above property is simply called an infinite sequence of schemes and is written as ISS $\{X_n, i_n\}$ often. Next, for each integer n, let E_n be a vector bundle on X_n. Then, an infinite sequence of vector bundles $\{E_n, X_n\}_{n \in N}$ is called infinitely extendable with respect to ISS $\{X_n, i_n\}$, if for each positive integer n, $i_n^* E_{n+1} \cong E_n$. For simplicity, such $\{E_n, X_n\}$ is written as an ISB w.r.t. $\{X_n, i_n\}$.

(4.1) let us consider two ISS's: $\{X_n, i_n\}$, $\{Y_n, j_n\}$ with a sequence of finite, flat morphisms: $\{f_n: X_n \to Y_n\}$ enjoying $i_n f_{n+1} = f_n j_n$.

Then we have

Proposition 4.2. *Under the above notations* 4.1, *let* $\{E_n, X_n\}$ *and* $\{E'_n, X_n\}$ *be two ISB w.r.t* $\{X_n, i_n\}$. *Then, we have*

(0) $\{E_n^{\vee}, X_n\}$ *and* $\{E_n \otimes E'_n, X_n\}$ *are ISB w.r.t* $\{X_n, i_n\}$.

(1) $f_{n*}E_n$ *is locally free for each positive integer n.*

(2) $f_{n*}i_n^* E_{n+1} \cong j_n^* f_{n+1*} E_{n+1}$, *namely* $\{f_{n*}E_n, Y_n\}$ *is a ISB w.r.t* $\{Y_n, j_n\}$.

Proof. (0) and (1) are trivial. (2) is shown by the base change theorem of Grothendieck. q.e.d.

(4.3) We fix an integer r and $(r+1)$ positive integers $e_0, \cdots, e_r$. For a positive integer m $(\geqq r)$, let $k[X_0, \cdots, X_m]$ $(=R_m)$ be the graded polynomial ring with $\deg X_i = e_i$ $(0 \leqq i \leqq r)$ and $\deg X_j = 1$ $(j \geqq r+1)$. Then we have a canonical surjective ring-homomorphism $\bar{\beta}: R_{n+r+1} \to R_{n+r}$ where $\bar{\beta}(X_i) = X_i$ $(0 \leqq i \leqq n+r)$ and $\bar{\beta}(X_{n+r+1}) = 0$.

The homomorphism yields a natural closed embedding $\beta_n: \boldsymbol{Q}(\boldsymbol{e}_{n+r}) \to \boldsymbol{Q}(\boldsymbol{e}_{n+r+1})$ with $\boldsymbol{e}_{m+r} = (e_0, \cdots, e_r, \underbrace{1, \cdots, 1}_{m})$. Such a sequence $\{\boldsymbol{Q}(\boldsymbol{e}_{n+r}), \beta_n\}$ is called an ISS of weighted projective space.

Now, let us recall

Theorem 4.4. (Theorem 2 and Theorem 3 in [T], Main Theorem in [S1])

Let $\{W_n, k_n\}$ *be an infinite sequence of schemes and* $\{E_n, W_n\}$ *is an ISB with respect to* $\{W_n, k_n\}$. *Assume that for each positive integer n,* W_n *is "a normal complete intersection in* $\boldsymbol{P}(\boldsymbol{e}_{n+r})$" *and* $\beta_n^{-1}(W_{n+1}) = W_n$ *under the notation* 4.3. *Then, if the characteristic of the ground field is zero, for any n* E_n *is a direct sum of line bundles on* W_n $(\bigoplus \mathcal{O}_{W_n}(c_i))$ *with* $\mathcal{O}_{W_n}(c) = \mathcal{O}_{P(\boldsymbol{e}_{n+r})}(c)_{|W_n}$. *Moreover,* $(c_1, \cdots)$ *is independent of a choice of n.*

Remark 4.5.1. In Theorem 3 in [T] (which is the more general one than Theorem 2 [T] and Main Theorem [S1]), Tjurin assumed not (*) "a normal complete intersection in $\boldsymbol{P}(\boldsymbol{e}_{n+r})$" but (**) "a smooth projective subvariety in $\boldsymbol{Q}(\boldsymbol{e}_{n+r})$". But it seems to the author that his proof is not complete under the condition (**). Therefore the author states Theorem 4.4 under another condition (*) which is sufficient for the study of our problem. See the Appendix the proof of Theorem 4.4.

Remark 4.5.2. In Theorem 5.3, we show that if an ISS $\{W_n, k_n\}$ of smooth projective subvarieties satisfies a condition: (#) $W_n \subset \boldsymbol{P}(\boldsymbol{e}_{n+r})$ and $\beta_n^{-1}(W_{n+1}) = W_n$ under the notation 4.3, then W_n is a weighted complete intersection. This result can be proved only by using Theorem 2 [T] or Main Theorem [S1] (see Remark 4.17.1).

Thus, Theorem 4.4 and Theorem 5.3 give us almost same conclusion as in Theorem 3 [T] under the condition (#) which is slightly more restrictive than the one in Theorem 3 [T].

Moreover as a result corresponding to Theorem 4.4 in any characteristic, let us recall

Theorem 4.6. ([S2]) *Let* $\{E_n, P^{n+r}\}$ *be an ISB w.r.t.* $\{P^{n+r}, \beta_n\}$. *Then the same conclsion as in Theorem* 4.4 *holds in any characteristic.*

(4.7) Hereafter, in characteristic zero, let us assume that for any n, Y_n in 4.1 is a normal weighted complete intersection in the weak projective space $\boldsymbol{P}(\boldsymbol{e}_{n+r})$ and the closed embedding j_n is the restriction of $\beta_n: \boldsymbol{Q}(\boldsymbol{e}_{n+r}) \to \boldsymbol{Q}(\boldsymbol{e}_{n+r+1})$ to Y_n (4.3).

In positive characteristic let Y_n be a projective space.

Hence note that X_n is also a projective (not necessarily irreducible) scheme because of the finiteness of $f_n: X_n \to Y_n$.

Now we state an easy

Proposition 4.8. *Let* $\{X_n\}$ *and* $\{Y_n\}$ *be as* 4.7. *Then under the notations and conditions in Proposition* 4.2, *we have*

(1) *There is a canonical surjective homomorphism* $f_n^* f_{n*} E_n \xrightarrow{g} E_n \to 0$ *and* $f_{n*} E_n$ *is a direct sum of line bundles* $(\cong \bigoplus_i \mathcal{O}_{Y_n}(b_i))$ *on* Y_n.

(2) *Letting* E'_n *the kernel of* g, $\{X_n, E'_n\}$ *and* $\{X_n, E'_n \otimes E_n^\vee\}$ *are sequences of ISB w.r.t.* $\{X_n, i_n\}$. *Consequently,* $f_n^* f_{n*} E_n = E_n \oplus E'_n$.

(2′) *Assume additionally that* $H^0(X_n, \mathcal{O}_{X_n}) \cong k$ *for a large* n (*e.g. connected and reduced or weighted complete intersection*). *Then for any* n, E_n *and* E'_n *are a direct summand of line bundles* $f_n^* \mathcal{O}_{Y_n}(b)$ *respectively.*

Proof. Since f_n is a finite and flat morphism, the first part of (1) is trivial. The latter part of (1) is obtained by (2) in Proposition 4.2, Theorem 4.4 and Theorem 4.6. The former part of (2) is trivial by Proposition 4.2. Hence, $E'_n \otimes E_n^\vee$ is a direct sum of line bundles similar to above (1). Moreover, since f_n is an affine morphism, $H^1(X_n, F) = H^1(Y_n, f_{n*}F)$ for a coherent sheaf F on X_n. Thus, the latter of (2) is shown by Proposition 1.6. (2′) is obtained by Krull-Schmit Theorem. Note that this theorem is applicable to an algebraic k-scheme X proper over k with $H^0(X, \mathcal{O}_X) = k$. q.e.d.

(4.9) Let $\{Z_n, k_n\}$ be an ISS and for each n let X_n (4.7) a closed subscheme in Z_n with a natural closed embedding $h_n: X_n \to Z_n$ which is a locally complete intersection. Assume that $h_n k_n = i_n h_{n+1}$ for every n.

Then we study the structure of the normal bundle N_{X_n/Z_n} $(=N_n)$ of X_n in Z_n.

Since $X_n = X_{n+1} \cap Z_n$, the following is the immediate consequence of Proposition 4.8.

Proposition 4.10. *Let* $X_n, Y_n, f_n: X_n \to Y_n, Z_n$ *and* N_n *be* (4.1), (4.7) *and* (4.9). *Then we have*

(1) $\{X_n, N_n\}$ *is an ISB w.r.t.* $\{X_n, i_n\}$.

(2) *There is a canonical surjective homomorphism* $f_n^* f_{n*} N_n \xrightarrow{g} N_n \to 0$. $f_n^* f_{n*} N_n$ *is a direct sum of line bundles.*

(3) *Letting* M_n *the kernel of* g, $f_n^* f_{n*} N_n \cong M_n \oplus N_n$.

(3′) *Assume additionally that* $H^0(X_n, \mathcal{O}_{X_n}) = k$ *for a large* n (4.8.2′). *Then* N_n *is a direct sum of line bundles* ($=\oplus f_n^* \mathcal{O}_{Y_n}(b_i)$) *for any* n.

The above provides us with

Theorem 4.11. (Corollary 3 in [F1]) *Let* $\{X_n\}$ *be an infinite sequence of schemes with* $\dim X_n = n+r$. *Assume that for every integer* n, X_n *is a closed reduced subscheme in* P^{n+m} *which is locally complete intersection and* $i_n^{-1}(X_{n+1}) = X_n$ *with a linear embedding* $i_n: P^{n+m} \to P^{n+m+1}$. *Then* X_n *is a complete intersection.*

Proof. Take an $(m-r-1)$-dimensional linear subspace V in P^{m+n} with $V \cap X_n = \emptyset$ and consider two projections $f_n: X_n \to P^{n+r}$ ($=Y_n$) and $f_{n+1}: X_{n+1} \to P^{n+r+1}$ ($=Y_{n+1}$) via the vertex V. Remark that f_n is finite and flat. Then, these projections induce a canonical linear embedding $j_n: Y_n \to Y_{n+1}$ with $i_n f_{n+1} = f_n j_n$. Thus, the sequence $\{f_n: X_n \to Y_n\}$ enjoys the conditions (4.1). In order to show this we have only to check that the conditions in Corollary 3.5 hold. Since $\{\mathcal{O}_{P^{n+m}}(\alpha)_{|X_n} (=L_n), X_n\}$ is an ISV w.r.t. $\{X_n, i_n\}$, $f_{n*} L_n$ is a direct sum of line bundles on Y_n by Proposition 4.8.1 which implies that $H^1(X_n, L_n)$ vanishes for every α. Moreover we see $N_{X_n/P^{n+m}} = \oplus f_n^* \mathcal{O}_{Y_n}(b_i) = \oplus \mathcal{O}_{P^{n+m+1}}(b_i)_{|X_n}$ by Proposition 4.10.3′. For the positivity of b_i, we need

SUBLEMMA 4.11.1. Let M be a complete subscheme in a weak projective space $\boldsymbol{P}$ which is a locally complete intersection. Assume that $N_{M/\boldsymbol{P}} \cong \oplus \mathcal{O}_{\boldsymbol{P}}(a_i)_{|M}$ with some integers a_i. Then all the a_i's are positive.

Proof. We have an exact sequence of vector bundles on $\boldsymbol{P}$:

$$0 \longrightarrow \mathcal{O} \longrightarrow \bigoplus_i \mathcal{O}_{\boldsymbol{P}}(e_i) \longrightarrow T_{\boldsymbol{P}} \longrightarrow 0.$$

with $\boldsymbol{e} = (e_1, \cdots)$ (see Remark 2.4 in [Mo]). Then the inclusion $M \subset \boldsymbol{P}$ induces a generally surjective homomorphism: $T_{\boldsymbol{P}|M} \to N_{M/\boldsymbol{P}}$ ($=N$). Therefore since N is a direct sum of line bundles and $T_{\boldsymbol{P}|M}$ is ample, we complete the proof. q.e.d.

Thus, we complete our proof of Theorem 4.11 by Corollary 3.5. q.e.d.

Remark 4.11.2. To prove the above, we use only the result of Theorem 4.4 in case that W_n is a projective space (and Theorem 4.6).

Now for the proof of Main Theorem let us consider the restrictive ISS $\{Z_n, k_n\}$ in (4.9).

(4.12) Let Z_n in 4.9 be a scheme $\boldsymbol{V}(E_n)$ where E_n is a vector bundle $\bigoplus_{i=1}^{d} \mathcal{O}_{Y_n}(-c_i)$ on Y_n ($c_1 \geqq c_2 \geqq \cdots \geqq c_d \geqq 1$) and $d, c_1, \cdots, c_d$ are independent of a choice of n. Moreover let $k_n: \boldsymbol{V}(E_n) \to \boldsymbol{V}(E_{n+1})$ be the closed embedding induced by a surjective homomorphism on $Y_{n+1}: E_{n+1} \to E_{n+1|Y_n} (=E_n) \to 0$ (Note $\{E_n, Y_n\}$ is an ISB w.r.t. $\{Y_n, j_n\}$). Moreover let us assume that $h_n p_n = f_n$ and $h_n k_n = i_n h_{n+1}$ with a canonical projection $p_n: Z_n \to Y_n$.

Then, by Proposition 1.4, we have the following diagram:

$$\begin{tikzcd}
\bar{X}_n \arrow[d, "h_n"] \arrow[dd, bend right=60, "\bar{f}_n"'] & \subset & \bar{Z}_n \arrow[d] & \underset{\text{open immersion}}{\subset} & P^{n+d+r} \arrow[d, "\varphi_n"] \\
X_n \arrow[d, "f_n"] & \subset & Z_n \arrow[dll, "p_n"] & \underset{\text{open immersion}}{\subset} & \boldsymbol{Q}(\boldsymbol{e}_{n+r}, \boldsymbol{c}) \\
Y_n & (\subset \boldsymbol{P}(\boldsymbol{e}_{n+r})) & & &
\end{tikzcd}$$

where $\boldsymbol{c}=(c_1, \cdots, c_d)$ in 4.12, $\bar{X}_i=\varphi_n^{-1}(X_i)$ and $\bar{f}_n$ is the composition of a canonical projection $\bar{X}_n \to X_n$ and f_n.

Hereafter till the end of this section, we assume for a large n, X_n is connected and reduced. Then by Proposition 4.10 (3′), $N_n=\bigoplus f_n^* \mathcal{O}_{Y_n}(b_i)$. Moreover since Y_n is a complete intersection in $\boldsymbol{P}(\boldsymbol{e}_{n+r})$, $N_{Y_n/\boldsymbol{P}(\boldsymbol{e}_{n+r})}=\bigoplus_i \mathcal{O}_{Y_n}(b_i')$. Therefore by 1) in Proposition 3.6, we see that

$$N_{X_n/\boldsymbol{Q}(\boldsymbol{e}_{n+r}, \boldsymbol{c})} \; (=N_n'')=N_{X_n/Z_n} \; (=N_n) \oplus N_{Z_n/\boldsymbol{Q}(\boldsymbol{e}_{n+r}, \boldsymbol{c}) \mid X_n} \; (=N_n')$$

where $N_n'=\bigoplus_i f_n^* \mathcal{O}_{Y_n}(b_i')$. Moreover all the b_i's and b_i''s above are positive by sublemma 4.11.1.

Under the above preliminaries, we get

Theorem 4.13. *Let* $\{X_n, i_n\}$, $\{Y_n, j_n\}$ *and* $\{Z_n, k_n\}$ *be ISS's and* $\{f_n: X_n \to Y_n\}$ *as in* (4.1) *and* (4.9). *We assume that*

0) *for a large* n, X_n *is connected and reduced.*
1) *for any* n, Y_n *has the property* (4.7).
2) *for any* n, (Z_n, k_n) *has the property* (4.12).

Then X_n *is a complete intersection in* Z_n.

Proof. It suffices to check the conditions in Proposition 3.6. The assumption 1 gives rise to 1 in (3.6). (2) in (3.6) is the definition of Z_n itself. (3) in (3.6) follows from the assumption 0), 2), Proposition 4.10 and sublemma 4.11.1. Since $\{\mathcal{O}_{X_n}, X_n\}$ is an ISB w.r.t. $\{X_n, i_n\}$, the condition 4 in (3.6) holds well by Proposition 4.8 (1). Finally, since $\operatorname{cod}_{\boldsymbol{P}(\boldsymbol{e}_{n+r}, \boldsymbol{c})} X_n=\operatorname{cod}_{\boldsymbol{P}(\boldsymbol{e}_{n+r})} Y_n+d$ (it is constant), $\dim X_n \geqq \dim \boldsymbol{P}(\boldsymbol{e}_{n+r}, \boldsymbol{c})/2$ for a large n. Thus we complete our proof. q.e.d.

From now on let us begin with the proof of Main Theorem. We maintain the conditions and assumptions of Main Theorem in Introduction.

The morphism $f_n: X_n \to Y_n$ induces an exact sequence of vector bundles on Y_n:

$$(4.14) \qquad 0 \longrightarrow \mathcal{O}_{Y_n} \xrightarrow{\alpha} f_{n*}\mathcal{O}_{X_n} \longrightarrow F_n \; (=\text{Cokernel of } \alpha) \longrightarrow 0$$

Hence it follows from Proposition 4.2 that

(4.14.1) $\{f_{n*}\mathcal{O}_{X_n}, Y_n\}$ and $\{F_n, Y_n\}$ are ISB's w.r.t. $\{Y_n, j_n\}$.

Moreover we have

Proposition 4.15. *$f_{n*}\mathcal{O}_{X_n}$ is isomorphic to $\bigoplus \mathcal{O}_{Y_n}(a_i)$ with $a_1=0$ and negative integers a_i $(i\geqq 2)$. F_n (4.14) is isomorphic to $\bigoplus_{i\geqq 2}\mathcal{O}_{Y_n}(a_i)$.*

Moreover, $(a_1, \cdots)$ is independent of the choice of n up to the order.

Proof. The decomposability of the first part is obtained by Proposition 4.8.1. As for the negativity, since f_n is finite, $f_n^*\mathcal{O}_{Y_n}(b)$ is a negative line bundle on X_n for each negative integer b. Therefore, we see that $H^0(X_n, f_n^*\mathcal{O}_{Y_n}(b))$ $(=H^0(X_n, f_*\mathcal{O}_{X_n}\otimes \mathcal{O}_{Y_n}(b))$ vanishes. This implies the negativity of a_i for $i\geqq 2$. The latter is obvious. q.e.d.

(4.16) By the above observation, a natural injective homomorphism $F_n \to f_{n*}\mathcal{O}_{X_n}$ as $\mathcal{O}_{Y_n}$-module induces a surjective $\mathcal{O}_{Y_n}$-algebra $\sum_n S_n(F_n) \to f_{n*}\mathcal{O}_{X_n}$, which yields a closed immersion $k_n: X_n \to \boldsymbol{V}(F_n)$ and $k_n\pi_n = f_n$ where $\pi_n: \boldsymbol{V}(F_n) \to Y_n$ is a natural projection.

Thus we have come to the final stage of the proof of Main Theorem.

First set Z_n as $\boldsymbol{V}(F_n)$ with the vector bundle F_n in 4.14.

Since X_n and Y_n are smooth for any n, X_n is a locally complete intersection in Z_n. Thus our condition enjoys the one in Theorem 4.13. Moreover (4.14.1), Proposition 4.15 and 4.16 yield the condition 2 in Theorem 4.13. Consequently we get Main Theorem. q.e.d.

§ 5. Applications

In this section we study the application of Theorem 4.11.

Let us consider

(5.1) a sequence of pairs consisting of graded ring $k[T_0, \cdots, T_{m+n}]$ $(=k[T]_{m+n})$ and its homogeneous ideal I_n satisfying the following:

(1) $\deg T_i = e_i$.

(2) $\operatorname{Proj} k[T]_{m+n}/I_n$ $(=X_n)$ is locally complete intersection in $\boldsymbol{P}(\boldsymbol{e}_{m+n})$ and of $(r+n)$-dimension.

(3) X_n is an element in $|L_{m+n+1}{}^{\otimes a_n}|$ where $\mathcal{O}_{P(e_{m+n})}(1)_{|X_n} = L_{m+n}$ and an integer a_n.

Let d_n be the positive integer $(\underbrace{L_{m+n}, \cdots, L_{m+n}}_{r+n \text{ times}})_{X_n}$.

Then, we immediately get

Proposition 5.2. *Let the condition and assumption be as in* 5.1. *Then we have an equality $a_{n+1}d_{n+1}=d_n$. Consequently, there is an integer n_0 such that for every integer $n\geqq n_0$, $a_n=1$.*

Moreover, L_{m+n} is not divisible in Pic X_n *for any n $(\geqq n_1)$, namely, $L_{m+n} \doteqdot cM$ for any $M\in \operatorname{Pic} X_n$ and any integer c $(\neq \pm 1)$.*

Proof. Since $d_n = a_{n+1}d_{n+1} = \cdots = \prod_{j=1}^{m} a_{n+j}\cdot d_{n+m}$, the set $\{i \mid a_i \geqq 2\}$ is at finite set, as required. As was shown above, the set $\{d_n : n \in \boldsymbol{N}\}$ is bounded. On the other hand, when $L_{m+n}=cM$, $d_n=c^{n+r}$ $(M, \cdots, M)$, which implies the latter part. q.e.d.

Under the condition 5.1 and the diagram 4.11, let us lift each X_n in the above sequence to the one $\varphi_n^{-1}(X_n)$ $(=\bar{X}_n)$ in P^{m+n} via φ_n. Then, the condition (3) implies that $\bar{X}_n$ is in $|\mathcal{O}_{P^{m+n+1}}(a_n)_{|\bar{X}_{m+n+1}}|$. Thus, we see that for every integer m bigger than a fixed integer n_0, the sequence of projective schemes $\{\bar{X}_n\}$ enjoys the assumptions in Theorem 4.11 except the reducedness of X_n by Proposition 5.1.

Therefore, we can show

Theorem. 5.3. *Let the condition and assumption be as in* 5.1. *Assume X_n is reduced for a large n.*

Then, X_n is a weighted complete intersection in $\boldsymbol{P}(\boldsymbol{e}_{m+n})$.

Proof. We have only to check that for a large $n \geqq n_0$, $\bar{X}_n \subset \boldsymbol{P}(\boldsymbol{e}_{n+m})$ satisfy the conditions 3.3 by Corollary 3.4. As for 1) of 3.3, since $H^1(\bar{X}_n, \varphi_n^*\mathcal{O}_{X_n}(a))=0$ for any a in the same way as in the proof of Theorem 4.11, we see that $H^1(X_n, \mathcal{O}_{X_n}(a))=0$ for any a by virtue of Proposition 1.3.2. Next letting $N_n=N_{X_n/P(e_{m+n})}$ and $\bar{N}_n=N_{\bar{X}_n/P^{m+n}}$, we have $\varphi_n^*N_n=\bar{N}_n$. Then Proposition 4.10 yields that $(\#)$ $f_n^*f_{n*}\varphi_n^*\bar{N}_n=\varphi_n^*\bar{N}_n \oplus M_n$ with a vector bundle M_n on X_n under the notations in Theorem 4.11. Note that $f_n^*\mathcal{O}_{Y_n}(1)=\varphi_n^*\mathcal{O}_P(1)_{|X_n}$ with $\boldsymbol{P}=\boldsymbol{P}(\boldsymbol{e}_{n+m})$. Now take the direct image φ_{n*} of $\#$. Then we infer that $N_n=\bigoplus_j \mathcal{O}(d_j)$ with some integers d_j by Proposition 1.3.3 and Krull-Schmidt Theorem (X_n is reduced). Moreover all the d_j's are positive by Sublemma 4.11.1. Therefore, by Proposition 2.6 and Corollary 3.4, we get desired fact.

q.e.d.

Finally we consider

(5.4) a sequence $\{X_n, L_n\}$ of connected polarised schemes satisfying the following: for each n,

(1) X_n is an ample divisor in X_{n+1}.

(2) $L_{n+1|X_n}=L_n$ and $\mathcal{O}_{X_{n+1}}(X_n)=a_{n+1}L_{n+1}$ with some integer a_{n+1}.

(3) Letting $G(X_n, L_n)$ the graded algebra $\bigoplus_{t \geqq 0} H^0(X_n, tL_n)$, the canonical homomorphism $G(X_{n+1}, L_{n+1}) \to G(X_n, L_n)$ is surjective.

Then, we get

Proposition 5.5. *Let $\{X_n, L_n\}$ be a sequence of polarised connected schemes with the above conditions* 5.4. *Then, there are an infinite of indeterminants $T_0, T_1, \cdots, T_m, \cdots$ and a sequence of homogeneous ideals I_n in the weighted polynomial ring $k[T_0, \cdots, T_{m+n}]$ such that X_n is isomorphic to* $\operatorname{Proj} k[T_0, \cdots, T_{m+n}]/I_n$ *with* $\deg T_i=e_i$.

In particularly there is an integer w such that $a_i=e_{i+w}$ for any positive integer i.

Proof. It is well-known that $G(X_n, T_n)$ is finitely generated. Thus, for X_1 there are indeterminants: $T_0, \cdots, T_w$ and a graded surjective homomorphism $f_1 : k[T_0, \cdots, T_w] \to G(X_1, L_1)$ with $\deg T_i=e_i$ and the homogeneous ideal I_1 (=Kernel of f_1). In the second place, by Theorem 3.6 in [Mo] or Theorem 3.1 in [F] we get a graded surjective homomorphism $f_2: k[T_0, \cdots, T_w, T_{w+1}] \to G(X_2, L_2)$ with $\deg T_{w+1} = a_1$. Thus, we can take an infinite indeterminants and homogeneous ideals inductively. At the same time the final is proved. q.e.d.

Corollary 5.5.1. *Under the condition in* 5.4, *assume that* $a_n = 1$ *for each* n. *Then any* X_n *is contained in* $\boldsymbol{P}(\boldsymbol{e}_{m+n})$.

Proof. By the argument of Theorem 3.6 [Mo], T_{n+1} induces the section defining X_n, which means that the coherent sheaf $\mathcal{O}_{Q(e_{m+n+1})}(1)$ $(=M)$ yields the invertible sheaf on X_n. Thus we see that M is an invertible sheaf on the neighbourhood of X_{n+1}. Therefore, $\boldsymbol{P}(\boldsymbol{e}_{m+n+1})$ contains X_{n+1} thanks to Theorem 1.7 [Mo]. q.e.d.

Therefore Proposition 5.2, Theorem 5.3 and Proposition 5.5 yield

Theorem 5.6. *Let a sequence* $\{X_n, L_n\}$ *be as in* 5.4. *Assume that* X_n *is a locally complete intersection in* $\boldsymbol{P}(\boldsymbol{e}_{m+n})$ *for any* n *(see Proposition* 5.5). *Moreover assume additionally that* X_n *is reduced for a large* n. *Then* (X_n, L_n) *is a weighted complete intersection. (Remark that* L_n *is not divisible for any* n $(\geqq n_0)$ *by Proposition* 5.2)

Corollary 5.6.1. (compare Conjecture 4.23 in [Fu]) *Let a sequence* $\{X_n, L_n\}$ *be as* 5.4. *Assume that* X_n *is smooth for each* n. *Then,* (X_n, L_n) *is a weighted complete intersection.*

Proof. In the same manner as in Proposition 5.2 we infer that $a_n = 1$ for any n $(\geqq n_0)$, and therefore X_n is contained in $\boldsymbol{P}(\boldsymbol{e}_{m+n})$ by Corollary 5.5.1. Thus, Theorem 5.6 yields this Corollary. q.e.d.

Moreover, we can prove

Corollary 5.7. *Let* $\{X_n, L_n\}$ *be a sequence of polarized smooth varieties. Assume that* X_n *is an ample divisor in* X_{n+1} *and the characteristic of the base field is zero. Furthermore, we suppose the following: for every* n, $-K_{X_n}$ *is ample,* $L_{n+1|X_n} = L_n$ *and* $\mathcal{O}_{X_{n+1}}(X_n) \cong a_{n+1} L_{n+1}$ *with some integer* a_{n+1}.

Then, (X_n, L_n) *is a weighted complete intersection. Moreover* L_n *is not divisible for any* n $(\geqq 3)$.

Proof. By virtue of Corollary 5.6.1, it suffices to check that the above assumption induces the condition 3 in 5.4, in other words, $H^1(X_n, tL_n) = 0$ for any t. But it follows from Kodaira's vanishing Theorem. q.e.d.

Proof of Theorem 5.8. By virtue of Corollary 5.7, it is sufficient to show that $-K_{X_n}$ is ample for a sufficiently big n. Now put $K_{X_n} = k_n L_n$ and $\mathcal{O}_{X_{n+1}}(X_n) = a_n L_{n+1}$ with some integer k_n, a_n by the assumption. Then note that a_n is positive. On the other hand, $k_{n+1} + a_n = k_n$ thanks to the adjunction formula. Thus we see that k_n is negative for a big n. q.e.d.

Appendix. Proof of Theorem 4.4

Here we use notations in [T]: infinite variety X_∞, infinite projective space P_∞ and infinite weighted projective space P_{ω_∞}, which can be considered as the ones: ISS $\{X_n, i_n\}$, ISS of projective spaces $\{\boldsymbol{P}^{n+r}, \beta_n\}$ and ISS of weighted projective spaces $\{\boldsymbol{Q}(\boldsymbol{e}_{n+r}), \beta_n\}$ (4.3).

Now as for the decomposability of a vector bundle on a nonsingular inifinite variety $X_\infty \subset \boldsymbol{P}_{\omega_\infty}$, it seems to the author that the proof of Theorem 3 [T] is not so clear. In fact, under the notations in [T] since the inverse image $\varphi_\infty^{-1}(X_\infty)$ of X_∞ via the covering $\varphi_\infty: \boldsymbol{P}_\infty \to \boldsymbol{P}_{\omega_\infty}$ (1.5 in [T]) is generally neither irreducible nor reduced, one cannot apply the case in question to the result (=Theorem 2) about the decomposability of a vector bundle on a nonsingular infinite variety of $\boldsymbol{P}_\infty$, namely we cannot infer that the inverse image $\varphi_\infty^* E$ of a vector bundle E on P_{ω_∞} is a direct sum of line bundles on $\varphi_\infty^{-1}(X_\infty)$.

Neverthless one can prove Theorem 4.4 with the slight modification.

Now let us maintain the notations in Theorem 4.4.

Letting $\varphi_{n+r}: \boldsymbol{P}^{n+r} \to \boldsymbol{Q}(\boldsymbol{e}_{n+r})$ a canonical projection, we can easily choose a sequence of subschemes in P^{n+r}: $\{\bar{X}_n\}$ satisfying the following: $\bar{X}_n$ is an irreducible component of $\varphi_{n+r}^{-1}(X_n)$ and $\bar{X}_{n+1} \cap P^{n+r} = \bar{X}_n$ in P^{n+r+1}.

Then § 1.2 [T] tells us the result:

A1) there is an integer N such that for every $n \geqq N$,

1) $\bar{X}_n$ is swept by lines in $\bar{X}_n$.
2) for two points p_1 and p_2 in $\bar{X}_n$, there are two lines l_1, l_2 in $\bar{X}_n$ where $p_i \in l_i$ and $l_1 \cap l_2 \neq \emptyset$.

Moreover let $\bar{E}_n$ be $\varphi_{n+r}^* E_{n|\bar{X}_n}$. Then as stated in Lemm 3.2 in [T] we have

A2) for every two lines l_1, l_2 on $\bar{X}_n$, $\bar{E}_{n|l_1} \cong \bar{E}_{n|l_2}$ and the decomposability is independent of a choice of n.

Thus, in order to prove theorem 4.4, we have only to show that E_n decomposes to a direct sum of line bundles for a sufficiently big n.

Hence we use notations X, $\bar{X}$ and E instead of X_n, $\bar{X}_n$ and E_n hereafter.

Letting Y the set $\{\text{line } l \text{ in } P^n \mid l \subset \bar{X}\}$ ($\subset$the Grassmann variety $\boldsymbol{Gr}(n, 1)$), we have the following diagram:

$$\begin{tikzcd}
 & & Z=q^{-1}(Y) \arrow[dll, "\bar{p}"'] \arrow[drr, "\bar{q}"] & & \\
\bar{X} \arrow[d] & \boldsymbol{P}_n \arrow[d, "\varphi"'] & \boldsymbol{Fl}(n, 1, 0) \arrow[l, "p"'] \arrow[r, "q"] & \boldsymbol{Gr}(n, 1) \arrow[r, phantom, "\supset"] & Y \\
X \arrow[r, phantom, "\subset"] & \boldsymbol{P}(\boldsymbol{e}_n) & & &
\end{tikzcd}$$

where $\boldsymbol{Fl}(n, 1, 0)$ is the flag variety $\{(x, l) \in \boldsymbol{P}^n \times \boldsymbol{Gr}(n, 1) \mid x \in l\}$

Then we have to remark

(#) Let E be a vector bundle on X. Assume that X is normal and for each point

y in Y,

$$(\bar{p}\varphi)^*E_{|o^{-1}(y)} \cong \bigoplus_{i=1} \mathcal{O}_{P^1}(a_i)^{\oplus r_i} \ (a_1 > a_2 > \cdots)$$

where $a_1, \cdots$ and r_i are independent of a choice of y.

Then E has a subbundle E_1 of rank r_1.

Proof. The assumption induces the following exact sequence of vector bundles on Z by the base change theorem:

$$\mathcal{O} \longrightarrow \bar{q}^*F_1 \longrightarrow (\bar{p}\varphi)^*E \longrightarrow H \longrightarrow \mathcal{O}$$

where F_1 is a vector bundle on Y of rank r_1 and H a vector bundle on Z.

Then the above yields a morphism

$$\begin{tikzcd} f: Z \arrow[r] \arrow[dr] & \mathrm{Grass}_{r_1}E \\ & X \arrow[u, leftarrow, "t"'] \end{tikzcd}$$

as stated in the last part of the proof of Theorem 1 [T]. Then, we see $f(Z)$ induces a section of a canonical projection t by the Rigidity Principal and Zariski main theorem, which give us this claim.

In the next place, we prove the following: if a vector bundle E on X has a property: for each point y in Y

$$(\bar{p}\varphi)^*E_{|q^{-1}(y)} \cong \mathcal{O}_{P^1}{}^{\oplus r}$$

then E is trivial.

This can be shown by the property A1) and in the same way as in Lemma 3.5 [T].

Thus, we could prove that E is a vector bundle with the extension of line bundles.

In order to complete the proof of Theorem 4.4, the author assumes an additional condition: X is a weighted complete intersection, by which Proposition 1.6 (2) provides us with the fact that the vector bundle E decomposes to a direct sum of line bundles.

In fact, the auther do not know the theorem of Barth and Larsen type in weighted projective space $\boldsymbol{Q}(\boldsymbol{e}_n)$: for a line bundle L on a smooth subvariety X in weighted projective space, $H^1(X, L)=0$ under some conditions about n and $\dim X$.

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