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On the propagation of a certain polarization set for semilinear systems of real principal type

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§0. Introduction

N. Dencker [3] defined the polarization set for vector-valued distributions (see definition 1.8). It is a refinement of the wave front set in the sense that it indicates the "most singular" components of a distribution. And he showed the propagation of the polarization sets of solutions for linear systems of real principal type. C. Gérard [4] pointed out that the above results also hold in the framework of H^s. Further he studied the reflection of ones for some linear systems under different boundary conditions.

On the other hand, there are a lot of studies on propagation and interaction of singularities of solutions for nonlinear partial differential equations (see [1]).

The aim of this paper is to extend Dencker's propagation results to nonlinear systems. In this process, the appearance of nonsmooth symbols seems to require the modified definition of the polarization set. So we try to introduce a new H^s polarization set, named a H^s energy polarization set, which is efficient in nonlinear problems and satisfies some basic properties. Namely it must be a refinement of the H^s wave front set and transform in the same way as the previous one when a vector-valued distribution is multiplied by systems of pseudo-differential operators. Such a definition is obtained by means of microlocal energy estimates in virtue of lemma 1.1 (see definition 1.3, 1.4). Since the definition is given in the form of estimates, it proves to be useful in nonlinear problems. Moreover it is contained in the previous H^s polarization set and there is an example which shows that the two polarization sets are not always equal.

Now we state our main results. One is that the H^s energy polarization set of a vector-valued distribution transforms similarly as above when it is acted by systems of nonlinear partial differential operators if s satisfies some inequalities depending on the regularity of the distribution. (See theorem 2.1). The other is that the H^s energy polarization set of a solution for a semilinear system of real principal type propagates in the same way as in the linear case, that is, it propagates along a uniquely determined line bundle, called a Hamilton orbit in [3], if s belongs to the interval depending on the regularity of the solution. (See theorem 2.6). In both cases the ranges of s are similar as in the scalar case [2].

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This paper consists of four sections. In section 1, we define H^s energy polarization sets and prove some basic properties of them. In section 2, we state our main results. Two examples are given to show that the former polarization set is rather unstable in nonlinear problems. In section 3, we review the paradifferential calculus which is used in the next section. In section 4, the proofs of the theorems in section 2 are given.

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§ 1. Energy polarization set

First we specify the microlocalizers that are necessary to define the new polarization set, and give two related lemmas.

Let $z = (x, \xi)$ be independent variables in $\mathbf{R}^d \times \mathbf{R}^d$. For $z_0 = (x_0, \xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$ and $r > 0$, we set:

$$M(z_0, r) = \{a(z) \in \mathscr{C}_0^\infty(\mathbf{R}^{2d});\ 0 \le a(z) \le 1,\ \operatorname{supp} a \subset \{z;\ |z - z_0| < r\},$$
$$a(z) = 1 \text{ in a neighborhood of } z_0\},$$
$$M(z_0) = \bigcup_{r>0} M(z_0, r).$$

Write $a \Subset b$ if and only if $b = 1$ in a neighborhood of $\operatorname{supp} a$. To each $a \in M(z_0)$, assign the bounded sequence $\{a_n(x, \xi)\}_1^\infty$ in $S^0_{1,0}(\mathbf{R}^d \times \mathbf{R}^d)$ where $a_n(x, \xi) = a(x, n^{-1}\xi)$. We call each sequence of pseudo-differential operators, $\{a_n(x, D)\}_1^\infty$, a microlocalizer at z_0. The idea of considering microlocalizers goes back to S. Mizohata (see Mizohata [8], [9], Takei [10] for the microloacl energy method).

By means of microlocalizers, we characterize the wave front set in the sense of H^s, which we denote by WF^s, for $s \in \mathbf{R} \cup \{\infty\}$. Here $WF^\infty = WF$.

Lemma 1.1 (see Takei [10]). *For $u \in H^{-\infty}(\mathbf{R}^d)$ and $z_0 = (x_0, \xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, the following conditions are equivalent:*

(1) $z_0 \notin WF^s(u)$.

(2) $\{\|a_n(x, D)u\|\}_1^\infty \in h^s$ *for some* $a \in M(z_0)$.

(3) *There exists* $r > 0$ *such that* $\{\|a_n(x, D)u\|\}_1^\infty \in h^s$ *for any* $a \in M(z_0, r)$.

Here $h^s = \{\{e_n\}_1^\infty;\ e_n \in \mathbf{R},\ \sum_{n=1}^\infty n^{2s-1} e_n^2 < \infty\}$ *if* $s \in \mathbf{R}$, $h^\infty = \bigcap_{s\in\mathbf{R}} h^s$, *and* $\|\cdot\| = \|\cdot\|_{L^2(\mathbf{R}^d)}$.

Let χ be a diffeomorphism from an open set U to another V, and T an induced diffeomorphism from $U \times \mathbf{R}^d$ to $V \times \mathbf{R}^d$, that is, $T(x, \xi) = (\chi(x), {}^t(\chi^{-1})'(\chi(x))\xi)$. Choose $z_0 \in V \times (\mathbf{R}^d \setminus 0)$ and a sufficiently small $r > 0$.

Lemma 1.2. *There exists a constant $K > 0$ such that*

$$\|a_n(x, D)u\| \le K \|(a \circ T)_n(x, D)\chi^* u\|$$

$$+ Cn^{-1}\|(b\circ T)_n(x, D)\chi^* u\| + e_n \qquad (n\in \mathbf{N})$$

for all $u\in\mathscr{E}'(V)$ *and* $a, b\in M(z_0, r)$ *satisfying* $a\Subset b$. *Here* $C>0$ *and* $\{e_n\}\in h^\infty$ *which may depend on* u, a, b ($\chi^* u$ *is a pull back of* u *by* χ, *see* [6, vol I]).

Proof. For $a, b\in M(z_0, r)$ such that $a\Subset b$,

$$\begin{aligned}\|a_n(x, D)u\| &= \|(\chi^{-1})^*\chi^* a_n(x, D)(\chi^{-1})^*\chi^* u\|\\ &\le K\|\chi^* a_n(x, D)(\chi^{-1})^*\chi^* u\|\\ &\le K\|(a\circ T)_n(x, D)\chi^* u\|\\ &\quad + Cn^{-1}\|(b\circ T)_n(x, D)\chi^* u\| + e_n \qquad (n\in\mathbf{N})\end{aligned}$$

with some $K, C>0$ and $\{e_n\}\in h^\infty$, where K depends only on χ.

Next we define the new polarization set. Let X be an open set in $\mathbf{R}^d$ and let $\mathscr{D}'(X, \mathbf{C}^N)$ be the set of all $\mathbf{C}^N$-valued distributions in X, that is, $u\in\mathscr{D}'(X, \mathbf{C}^N)$ means $u=(u_1,\dots,u_N)$ where $u_j\in\mathscr{D}'(X)$. Similarly $\mathscr{D}'(X, \mathbf{R}^N)$, $\mathscr{E}'(X, \mathbf{C}^N)$, $H^s_{loc}(X, \mathbf{R}^N)$ and so on are defined.

Definition 1.3. Let $s\in\mathbf{R}\cup\{\infty\}$, $u=(u_j)\in H^{-\infty}(\mathbf{R}^d, \mathbf{C}^N)$, $z_0=(x_0, \xi_0)\in\mathbf{R}^d\times(\mathbf{R}^d\setminus 0)$ and $h_0\in\mathbf{C}^N$. We say that $u\in H^s$ at (z_0, h_0) if for any $\varepsilon>0$, there exists $r>0$ such that

$$\|a_n(x, D)h_0\cdot u\| \le \varepsilon\|a_n(x, D)Iu\| + Cn^{-1}\|b_n(x, D)Iu\| + e_n \qquad (n\in\mathbf{N})$$

for all $a, b\in M(z_0, r)$ satisfying $a\Subset b$ with some $C>0$ and $\{e_n\}\in h^s$. Here $h_0\cdot u=\sum_{j=1}^N h_{0j}u_j\in H^{-\infty}(\mathbf{R}^d)$. If $x_0\in X$ and $u\in\mathscr{D}'(X, \mathbf{C}^N)$, we say that $u\in H^s$ at (z_0, h_0) if $\varphi u\in H^s$ at (z_0, h_0) for some $\varphi\in\mathscr{C}_0^\infty(X)$ such that $\varphi=1$ in a neighborhood of x_0. Of course it is independent of φ. Further we set

$$H^s(u) = \{(z, h)\in X\times(\mathbf{R}^d\setminus 0)\times\mathbf{C}^N;\ u\in H^s \text{ at } (z, h)\},$$

$$H^s(u, z) = \{h\in\mathbf{C}^N;\ (z, h)\in H^s(u)\}, \qquad z\in X\times(\mathbf{R}^d\setminus 0).$$

Definition 1.4 (H^s **energy polarization set,** $E^s(u)$). Let $u\in\mathscr{D}'(X, \mathbf{C}^N)$ and $s\in\mathbf{R}\cup\{\infty\}$. The H^s energy polarization set of u, $E^s(u)$, is defined as

$$E^s(u) = \{(z, w)\in X\times(\mathbf{R}^d\setminus 0)\times\mathbf{C}^N;\ h\cdot w=0 \text{ for all } h\in H^s(u, z)\}.$$

Here $h\cdot w=\sum_{j=1}^N h_j w_j$. We denote by $E^s(u, z)$ the fiber of $E^s(u)$ over $z\in X\times(\mathbf{R}^d\setminus 0)$.

To clarify the conicness of $H^s(u)$ and $E^s(u)$ for ξ, we show the next lemma.

Lemma 1.5. *For* $u\in H^{-\infty}(\mathbf{R}^d, \mathbf{C}^N)$, $u\in H^s$ *at* $(z_0, h_0)\in\mathbf{R}^d\times(\mathbf{R}^d\setminus 0)\times\mathbf{C}^N$ *if and only if for any* $\varepsilon>0$, *there exists* $r>0$ *such that*

$$\|a_t(x, D)h_0\cdot u\| \le \varepsilon\|a_t(x, D)Iu\| + Ct^{-1}\|b_t(x, D)Iu\| + f(t) \qquad (t\ge 1)$$

for all $a, b\in M(z_0, r)$ *satisfying* $a\Subset b$ *with some* $C>0$ *and* $f(\cdot)\in\tilde{h}^s$. *Here* $\tilde{h}^s=\{f\colon[1,\infty)\to\mathbf{R}$ *continuous and* $\int_1^\infty t^{2s-1}|f(t)|^2\,dt<\infty\}$ *if* $s\in\mathbf{R}$, $\tilde{h}^\infty=\bigcap_{s\in\mathbf{R}}\tilde{h}^s$

and $a_t(x, \xi) = a(x, t^{-1}\xi)$ *etc.*

Proof. Let $a, b, c \in M(z_0, r)$ satisfying $a \Subset b \Subset c$. Since

$$a_t(x, \xi) - a_n(x, \xi) = \int_0^1 (\nabla_\xi a)\left(x, \frac{\xi}{n} + \theta\left(\frac{\xi}{t} - \frac{\xi}{n}\right)\right) \cdot \xi d\theta \cdot \frac{n-t}{nt},$$

it follows that for $v \in H^{-\infty}(\mathbf{R}^d)$

$$\|a_t(x, D)v - a_n(x, D)v\| \leq Cn^{-1}\|b_n(x, D)v\| + e_n$$

$$\|a_t(x, D)v - a_n(x, D)v\| \leq C't^{-1}\|b_t(x, D)v\| + f(t)$$

for all $n \in \mathbf{N}$, $t \geq 1$ such that $|n - t| \leq 1$. Here $\{e_n\} \in h^\infty$, $f \in \tilde{h}^\infty$.

Suppose $u \in H^s$ at (z_0, h_0). Then

$$\begin{aligned}
&\|a_t(x, D)h_0 \cdot u\| \\
&\leq \|a_n(x, D)h_0 \cdot u\| + C_1 n^{-1}\|b_n(x, D)h_0 \cdot u\| + e_n^1 \\
&\leq \varepsilon(r)\|a_n(x, D)Iu\| + C_2 n^{-1}\|b_n(x, D)Iu\| + e_n^2 \\
&\leq \varepsilon(r)\|a_t(x, D)Iu\| + C_3 t^{-1}\|c_t(x, D)Iu\| + f(t), \qquad t \geq 1,\ n = [t].
\end{aligned}$$

Here $C_j > 0$, $\{e_n^1\} \in h^\infty$, $\{e_n^2\} \in h^s$, $f \in \tilde{h}^s$, $\varepsilon(r) > 0$ and $\varepsilon(r) \downarrow 0$ when $r \downarrow 0$. This proves the necessary part. The sufficiency is shown in a similar fashion.

Remark 1.6. $E^s(u)$ is conic for ξ by lemma 1.5 and linear for w. In particular $0 \equiv X \times (\mathbf{R}^d \setminus 0) \times 0 \subset E^s(u)$. $E^s(u)$ is not closed in $X \times (\mathbf{R}^d \setminus 0) \times \mathbf{C}^N$ in general. To obtain the closed one we denote by $H_*^s(u)$ the set of all $(z_0, h_0) \in X \times (\mathbf{R}^d \setminus 0) \times \mathbf{C}^N$ for which there exists a continuous map h from $X \times (\mathbf{R}^d \setminus 0)$ to $\mathbf{C}^N$, positively homogeneous of degree 0 with respect to ξ, such that $h(z_0) = h_0$ and that $h(z) \in H^s(u, z)$ for all $z \in X \times (\mathbf{R}^d \setminus 0)$. And define $E_*^s(u)$ in the same manner as $E^s(u)$ by means of $H_*^s(u)$ replacing $H^s(u)$. Then it is closed and the following relations hold:

$$E^s(u) \subset E_*^s(u) \subset WF_{pol}^s(u)$$

(see proposition 1.9). Since the other statements hold for both $E^s(u)$ and $E_*^s(u)$, we will take up only the case of $E^s(u)$.

Remark 1.7. Let $A = (A_{ij}) \in \mathscr{C}^\infty(X, GL(N, \mathbf{C}))$, $u \in \mathscr{D}'(X, \mathbf{C}^N)$ and χ be a diffeomorphism from Y to X. Then by lemma 1.2

$$E^s(\chi^* u, y, \eta) = E^s(u, \chi(y), {}^t(\chi^{-1})'(\chi(y))\eta)$$

for $(y, \eta) \in Y \times (\mathbf{R}^d \setminus 0)$ and by proposition 1.11

$$E^s(Au, x, \xi) = A(x)E^s(u, x, \xi)$$

for $(x, \xi) \in X \times (\mathbf{R}^d \setminus 0)$. So $E^s(u)$ can be regarded as a subset of the induced bundle $\pi^*(X \times \mathbf{C}^N)$ over $T^*X \setminus 0$, where $\pi\colon T^*X \setminus 0 \to X$ is the natural projection.

In general, let M be a $\mathscr{C}^\infty$ manifold, E a $\mathscr{C}^\infty$ vector bundle over M, E^* its dual bundle, Ω the $\mathscr{C}^\infty$ density bundle of M and $\mathscr{D}'(M, E) = \mathscr{C}_0^\infty(M, E^* \otimes \Omega)'$ the space of distributional sections of E over M. Then for $u \in \mathscr{D}'(M, E)$ we can define $E^s(u)$ and $H^s(u)$ as subsets of the induced bundles $\pi^* E$ and $\pi^* E^*$ respectively, where $\pi\colon T^*M \setminus 0 \to M$ is the natural projection.

For the sake of comparison we quote the definition of H^s polarization set due to Dencker [3] ($s = \infty$) and Gérard [4] ($s \in \mathbf{R}$).

Definition 1.8. Let $u \in \mathscr{D}'(X, \mathbf{C}^N)$. The H^s polarization set of u, $WF^s_{pol}(u)$, is defined as

$$WF^s_{pol}(u) = \cap N_A,$$
$$N_A = \{(z, w) \in T^*X \setminus 0 \times \mathbf{C}^N \colon w \in \operatorname{Ker} A_0(z)\},$$

where the intersection is taken over all $1 \times N$ system A of classical pseudo-differential operators of order 0 with its pricipal symbol $A_0(z)$ such that $Au \in H^s$. $WF^s_{pol}(u, z)$ is the fiber of $WF^s_{pol}(u)$ over $z \in T^*X \setminus 0$.

Proposition 1.9. $E^s(u) \subset WF^s_{pol}(u)$ *for* $u \in \mathscr{D}'(X, \mathbf{C}^N)$.

Remark. The following proof shows that $E^s(u) \subset E^s_*(u) \subset WF^s_{pol}(u)$.

Proof. We may assume $u \in \mathscr{E}'(X, \mathbf{C}^N)$. If $(z_0, w_0) \notin WF^s_{pol}(u)$, then there exists a $1 \times N$ system of classical pseudo-differential operators A of order 0 with principal symbol A_0 such that $A_0(z_0)w_0 \neq 0$ and that $Au \in H^s$. Then for any $a, b \in M(z_0, r)$ such that $a \Subset b$,

$$\begin{aligned}
&\| a_n(x, D) A_0(z_0) u \| \\
&\le \| a_n(x, D)(A_0(z_0) - A)u \| + \| a_n(x, D) Au \| \\
&\le \varepsilon(r) \| a_n(x, D) Iu \| + cn^{-1} \| b_n(x, D) Iu \| + e_n \qquad (n \in \mathbf{N})
\end{aligned}$$

with $\varepsilon(r) > 0$, $c > 0$ and $\{e_n\} \in h^s$. Here $\varepsilon(r)$ depends only on A and r, and $\varepsilon(r) \downarrow 0$ when $r \downarrow 0$ by the sharp Gårding inequality (cf. lemma 3.9). This implies $(z_0, w_0) \notin E^s(u)$.

The following propositions show that the H^s energy polarization set enjoys the two basic properties as announced in the introduction.

Proposition 1.10. $\pi(E^s(u) \setminus 0) = WF^s(u)$ *for* $u \in \mathscr{D}'(X, \mathbf{C}^N)$ *where* π *is a projection from* $X \times \mathbf{R}^d \times \mathbf{C}^N$ *to* $X \times \mathbf{R}^d$.

Proof. Let $z_0 \in X \times (\mathbf{R}^d \setminus 0)$. We prove $z_0 \notin WF^s(u)$ if and only if $E^s(u, z_0) = 0$. We may assume $u \in \mathscr{E}'(X, \mathbf{C}^N)$.

If $E^s(u, z_0) = 0$, it follows from definition 1.3 with $h_0 = e_j$, $j = 1, \ldots, N$, and a small $r > 0$ that

$$\| a_n(x, D) Iu \| \le cn^{-1} \| b_n(x, D) Iu \| + e_n \qquad (n \in \mathbf{R})$$

for all $a, b \in M(z_0, r)$ satisfying $a \Subset b$, with some $c > 0$ and $\{e_n\} \in h^s$. Take a sequence $a^0 \Subset a^1 \Subset a^2 \Subset \cdots \in M(z_0, r)$. Then

$$\| a_n^0(x, D) I u \| \leq c_j n^{-j} \| a_n^j(x, D) I u \| + e_n^i \qquad (n \in \mathbf{R})$$

with some $c_j > 0$ and $\{e_n^j\} \in h^s$, $j = 0, 1, 2, \ldots$. This implies $z_0 \notin WF^s(u)$ in virtue of lemma 1.1.

The sufficiency follows easily from lemma 1.1.

Proposition 1.11. *Let P be a $M \times N$ system of classical pseudo-differential operators of order m with principal symbol $P_m(x, \xi)$ and let $u \in \mathscr{D}'(X, \mathbf{C}^N)$. Then for any $z \in X \times \mathbf{R}^d \setminus 0$*

$$P_m(z) E^s(u, z) \subset E^{s-m}(Pu, z).$$

Especially if $M = N$ and P is elliptic at z, then

$$P_m(z) E^s(u, z) = E^{s-m}(Pu, z).$$

Proof. Let $z_0 = (x_0, \xi_0) \in X \times (\mathbf{R}^d \setminus 0)$, $|\xi_0| = 1$, $w \notin E^{s-m}(Pu, z_0)$. We may assume $u \in \mathscr{E}'(X, \mathbf{C}^N)$. Then there exists $h \in H^{s-m}(Pu, z_0)$ such that $h \cdot w \neq 0$. Let $f(\xi) \in \mathscr{C}^\infty(\mathbf{R}^d)$ such that $f(\xi) = |\xi|^{-m}(|\xi| \geq 1)$. For any $a, b, c \in M(z_0, r)$ such that $a \Subset b \Subset c$, and $n \in \mathbf{N}$,

$$\begin{aligned}
&\| a_n(x, D) h \cdot P_m(z_0) u \| \\
&\quad \leq \| a_n(x, D) h \cdot (P_m(z_0) - f(D) P) u \| + \| a_n(x, D) h \cdot f(D) P u \| \\
&\quad = A_n + B_n, \\
&A_n \leq \varepsilon_1(r) \| a_n(x, D) I u \| + C_1 n^{-1} \| c_n(x, D) I u \| + e_n^1, \\
&B_n \leq K n^{-m} \| a_n(x, D) h \cdot P u \| + C_2 n^{-1} \| c_n(x, D) I u \| + e_n^2 \\
&\quad \leq K n^{-m} \{ \varepsilon_2(r) \| a_n(x, D) I P u \| + C_3 n^{-1} \| b_n(x, D) I P u \| + e_n^3 \} \\
&\qquad + C_2 n^{-1} \| c_n(x, D) I u \| + e_n^2 \\
&\quad \leq \varepsilon_3(r) \| a_n(x, D) I u \| + C_4 n^{-1} \| c_n(x, D) I u \| + e_n^4.
\end{aligned}$$

Here $\{e_n^1\}$, $\{e_n^2\} \in h^\infty$, $\{e_n^3\} \in h^{s-m}$, $\{e_n^4\} \in h^s$, $C_j > 0$, $K > 0$ and $\varepsilon_j(r) > 0$. $\varepsilon_j(r) > 0$ and K are independent of a, b, c and $\varepsilon_j(r) \downarrow 0$ when $r \downarrow 0$. This implies ${}^t P_m(z_0) h \in H^s(u, z_0)$. So $w \notin P_m(z_0) E^s(u, z_0)$ follows.

Example 1.12. Take $x_j \in \mathbf{R}^d \setminus 0$ such that $x_j \to 0$ when $j \to \infty$ and $x_j \neq x_k (j \neq k)$, and set $v = (v_1, v_2) = (f(D)\delta_0, \sum_{j=1}^\infty j^{-2} \delta_{x_j}) \in H^{-\infty}(\mathbf{R}^d, \mathbf{C}^2)$, where δ_x is the Dirac measure at x and $f(\xi) = \log(2 + |\xi|^2)$, $\xi \in \mathbf{R}^d$. If $s \geq -\dfrac{d}{2}$, then

(1) $$E^s(v) \setminus 0 = \{(0, \xi, (w_1, 0)); \xi \in \mathbf{R}^d \setminus 0, w_1 \in \mathbf{C} \setminus 0\} \cup \Gamma,$$

(2) $$E_*^s(v) \setminus 0 = WF_{pol}^s(v) \setminus 0 = \{(0, \xi, w); \xi \in \mathbf{R}^d \setminus 0, w \in \mathbf{C}^2 \setminus 0\} \cup \Gamma.$$

Here $\Gamma = \{(x_j, \xi, (0, w_2)); j \in \mathbf{N}, \xi \in \mathbf{R}^d \setminus 0, w_2 \in \mathbf{C} \setminus 0\}$. In this case $E^s(v) \subsetneq E_*^s(v)$

$= WF^s_{pol}(v)$.

Proof. We can easily show that

$$\begin{cases} WF^s(v) = \phi & \text{if } s < -\dfrac{d}{2} \\ WF^s(v_1) = \{0\} \times (\mathbf{R}^d \setminus 0) & \text{if } s \geq -\dfrac{d}{2} \\ WF^s(v_2) = \{0, x_j; j \in \mathbf{N}\} \times (\mathbf{R}^d \setminus 0) & \text{if } s \geq -\dfrac{d}{2}. \end{cases}$$

Hereafter we assume $s \geq -\frac{d}{2}$. Let $z_0 = (0, \xi_0)$, $\xi_0 \in \mathbf{R}^d \setminus 0$. For $a \in M(z_0, r)(r > 0$ is small), take $b, c \in M(z_0, 2r)$ such that $b \Subset a \Subset c$ with their forms $b(x, \xi) = b^1(x)b^2(\xi)$, $c(x, \xi) = c^1(x)c^2(\xi)$. Then

$$\begin{aligned} \| a_n(x, D)v_2 \| &\leq C_1 \| c_n^2(D)v_2 \| \leq C_2 n^{\frac{d}{2}}, \\ \| a_n(x, D)v_1 \| &\geq \| b_n(x, D)v_1 \| - e_n^1 \\ &\geq \| b_n^2(D)v_1 \| - e_n^2 \geq C_3 n^{\frac{d}{2}} \log n - e_n^2, \end{aligned}$$

where $\{e_n^1\}, \{e_n^2\} \in h^\infty$, $C_j > 0$, $b_n^2(\xi) = b^2(n^{-1}\xi)$, $c_n^2(\xi) = c^2(n^{-1}\xi)$. So

$$\| a_n(x, D)v_2 \| \leq (\log n)^{-\frac{1}{2}} \| a_n(x, D)v_1 \| + e_n, \qquad n \geq 2$$

with $\{e_n\} \in h^\infty$, which implies $E^s(v, z_0) = \mathbf{C} \times \{0\}$. Thus we obtain (1). Further the closedness of $E^s_*(v)$ and $WF^s_{pol}(v)$ gives (2).

Example 1.13. Set $u = (u_1, u_2) = (f(D)\delta_0, \delta_0) \in H^{-\infty}(\mathbf{R}^d, \mathbf{C}^2)$, where f is in example 1.12. If $s \geq -\frac{d}{2}$, then

(1) $$E^s(u) \setminus 0 = E^s_*(u) \setminus 0 = \{(0, \xi, (w_1, 0)); \xi \in \mathbf{R}^d \setminus 0, w_1 \in \mathbf{C} \setminus 0\},$$

(2) $$WF^s_{pol}(u) \setminus 0 = \{(0, \xi, w); \xi \in \mathbf{R}^d \setminus 0, w \in \mathbf{C}^2 \setminus 0\},$$

that is, $E^s(u) = E^s_*(u) \subsetneq WF^s_{pol}(u)$.

Proof. We can easily show that

$$\begin{cases} WF^s(u) = \phi & \text{if } s < -\dfrac{d}{2} \\ WF^s(u) = \{0\} \times (\mathbf{R}^d \setminus 0) & \text{if } s \geq -\dfrac{d}{2}. \end{cases}$$

Henceforth we assume $s \geq -\frac{d}{2}$. We obtain (1) in the same way as the proof in example 1.12.

By the symmetry and remark 1.7, the fiber of $WF^s_{pol}(u)$ over $z_0 = (0, \xi_0)$ is independent of ξ_0. Suppose that the fiber is equal to $\mathbf{C} \times \{0\}$. Then for each $z_0 = (0, \xi_0)$ there exists a classical pseudo-differential operator $p(x, D)$ of order 0 with its principal symbol $p_0(x, \xi)$ such that

$$p_0(z_0) = 0, \ z_0 \notin WF^s(u_2 - p(x, D)u_1).$$

Using a pseudo-differential partition of unity, we get

$$a(x)u_2 = a(x)q(x, D)u_1 + g$$

where $q(x, D)$ is a classical pseudo-differential operator of order 0 with its principal symbol $q_0(x, \xi)$, $a(x) \in \mathscr{C}_0^\infty(\mathbf{R}^d)$ such that $a(x) = 1$ in a neighborhood of 0 and $g \in H^s(\mathbf{R}^d)$. Since $WF^s_{pol}(u) = E^s(u)$, $q_0(0, \xi) = 0$ and so

$$\begin{aligned} a(x)q(x, D)u_1 &= \textstyle\sum_{j=1}^d q_j(x, D)a(x)x_j u_2 + r(x, D)u_2 \\ &= (-\textstyle\sum_{j=1}^d q_j(x, D)a(x)(D_{\xi_j} f)(D) + r(x, D)f(D))\delta_0 \\ &\in H^{1-\frac{d}{2}-\varepsilon}, \ \varepsilon > 0, \end{aligned}$$

with $q_j \in S^0_{1,0}$, $r \in S^{-1}_{1,0}$, which contradicts $a(x)u_2 \notin H^{-\frac{d}{2}}$. Thus (2) follows.

§2. Propagation of energy polarization sets for semilinear systems

In this section we state our main results. All the proofs are given in section 4.

Let X be an open set in $\mathbf{R}^d$ and consider the system of nonlinear partial differential operators on X:

$$P[u] = F(x, \partial^\alpha u(x))_{|\alpha| \le m}, \tag{2.1}$$

where $u = (u_1, \dots, u_N)$ and $F(x, u_\alpha)_{|\alpha| \le m} = (F_j(x, u_\alpha)_{|\alpha| \le m})_{1 \le j \le M}$ be a $\mathscr{C}^\infty$ map from $\{(x, u_\alpha)_{|\alpha| \le m}; x \in X, u_\alpha = (u_{1,\alpha}, \dots, u_{N,\alpha}) \in \mathbf{R}^N\}$ to $\mathbf{C}^M$. If P is semilinear, (2.1) takes the following form:

$$P[u] = P_m(x, D)u + G(x, \partial^\alpha u)_{|\alpha| \le m-1}, \tag{2.2}$$

where $P_m(x, D)u$ is the linear part of highest order and $G(x, \partial^\alpha u)_{|\alpha| \le m-1}$ is the rest.

Theorem 2.1. *Suppose* $u \in H^s_{\mathrm{loc}}(X, \mathbf{R}^N)$, $m + \frac{d}{2} < s \le s_1 \le 2s - m - \frac{d}{2}$, $s - m - \frac{d}{2} \notin \mathbf{Z}$ *and* (2.1). *Then*

$$P_m(z)E^{s_1}(u, z) \subset E^{s_1 - m}(P[u], z), \qquad z \in X \times (\mathbf{R}^d \setminus 0), \tag{2.3}$$

where

$$P_m(x, \xi) = \sum_{|\beta| = m} \left(\frac{\partial F_j}{\partial u_{k,\beta}}(x, \partial^\alpha u)_{|\alpha| \le m} \right)_{\substack{j=1,\dots,M \\ k=1,\dots,N}} \cdot (i\xi)^\beta.$$

Especially if $WF^{s_1-m}(P[u]) = \phi$, *then*

$$E^{s_1}(u, z) \subset \operatorname{Ker} P_m(z), \qquad z \in X \times (\mathbf{R}^d \setminus 0). \tag{2.4}$$

If $M = N$ *and* $\det P_m(z_0) \neq 0$, *then the equality holds for* $z = z_0$ *in* (2.3).

Corollary 2.2. *Suppose* $u \in H^s_{\mathrm{loc}}(X, \mathbf{R}^N)$, $m - 1 + \frac{d}{2} < s \leq s_1 \leq 2s + 2 - m - \frac{d}{2}$, $s + 1 - m - \frac{d}{2} \notin \mathbf{Z}$ *and* (2.2). *Then all the statements in theorem* 2.1 *hold.*

Now let us recall some definitions ([3]).

Definition 2.3 (see [3]). A $N \times N$ smooth symbol $Q(x, \xi)$ on $X \times (\mathbf{R}^d \setminus 0)$, positively homogeneous for ξ, is of real principal type at $(x_0, \xi_0) \in X \times (\mathbf{R}^d \setminus 0)$ if there exists a $N \times N$ smooth symbol $\tilde{P}(x, \xi)$ on $X \times (\mathbf{R}^d \setminus 0)$, positively homogeneous for ξ, such that

$$\tilde{P}(x, \xi) Q(x, \xi) = q(x, \xi) I$$

in a neighborhood of (x_0, ξ_0), where $q(x, \xi)$ is a scalar symbol of real principal type.

Hereafter we assume $M = N$ and (2.2). We denote by R_{P_m} the set of all $z \in X \times (\mathbf{R}^d \setminus 0)$ such that $\det P_m(z) = 0$ and that P_m is of real principal type at z. For P_m, take q_1 and $\tilde{P}_{1-m}$ in definition 2.3 near each $z \in R_{P_m}$, positively homogeneous of degree 1 and $1 - m$ respectively. Then H_{q_1} defines a 1-demensional ($\mathscr{C}^\infty$) differential system on R_{P_m}. Let γ be an ($\mathscr{C}^\infty$) integral curve for it.

Definition 2.4 (see [3]). Suppose $u \in \mathscr{C}^{m-1}(X, \mathbf{R}^N)$ and (2.2). A Hamilton orbit of (P, u) over γ is a $\mathscr{C}^1$ line subbundle L of $\gamma \times \mathbf{C}^N$ such that $L \subset N_{P_m} = \{(x, \xi, w) \in X \times (\mathbf{R}^d \setminus 0) \times \mathbf{C}^N;\ w \in \operatorname{Ker} P_m(x, \xi)\}$ and that L is spanned by a $\mathscr{C}^1$ local section w satisfying

$$(H_{q_1} + 2^{-1}\{\tilde{P}_{1-m}, P_m\} + i\tilde{P}_{1-m}P^s_{m-1})w = 0, \tag{2.5}$$

where

$$\{\tilde{P}_{1-m}, P_m\} = \textstyle\sum_{j=1}^d (\partial_{\xi_j}\tilde{P}_{1-m} \cdot \partial_{x_j}P_m - \partial_{x_j}\tilde{P}_{1-m} \cdot \partial_{\xi_j}P_m),$$

$$P^s_{m-1} = P_{m-1} - (2i)^{-1}\textstyle\sum_{j=1}^d \partial_{x_j}\partial_{\xi_j}P_m,$$

$$P_{m-1}(x, \xi) = \sum_{|\beta| = m-1} \left(\frac{\partial G_j}{\partial u_{k,\beta}}(x, \partial^\alpha u)_{|\alpha| \leq m-1}\right)_{\downarrow j = 1, \dots, N \atop \to k = 1, \dots, N} \cdot (i\xi)^\beta.$$

Remark 2.5. Near $(x_0, \xi_0, w_0) \in L \setminus 0$. L is spanned by the following solution $(x(t), \xi(t), w(t))$:

$$\begin{cases} \dot{x}(t) = \nabla_\xi q_1(x(t), \xi(t)), & x(0) = x_0 \\ \dot{\xi}(t) = -\nabla_x q_1(x(t), \xi(t)), & \xi(0) = \xi_0 \\ \dot{w}(t) = -(2^{-1}\{\tilde{P}_{1-m}, P_m\} + i\tilde{P}_{1-m}P^s_{m-1})(x(t), \xi(t))w(t), & w(0) = w_0. \end{cases}$$

Theorem 2.6. *Suppose* $u\in H^s_{\mathrm{loc}}(X, \mathbf{R}^N)$, $m+\frac{d}{2}-1<s\le s_1\le 2s+1-\frac{d}{2}-m$, $\rho\equiv s+1-m-\frac{d}{2}\notin \mathbf{Z}$ *and* (2.2). *Let* L *be a Hamilton orbit of* (P, u) *over* γ. *If* $WF^{s_1+1-m}(P[u])\cap\gamma=\phi$, *then*

$$E^{s_1}(u)\cap L=\gamma\times 0 \quad \text{or} \quad L. \tag{2.6}$$

Remark 2.7. This theorem is true if $P_m(x, D)$ is replaced by a system of classical pseudo-differential operators of order m.

Remark 2.8. In the above statementes the ranges of s, s_1 are similar as in the scalar case [2].

The following examples show that the previous polarization set is rather unstable in nonlinear problems in comparison to ours. The reason is that the former is defined by means of (smooth) classical pseudo-differential operators (cf. definition 1.8). If another class of pseudo-differential operators, for example, paradifferential operators, is used instead, the polarization set might satisfy theorem 2.1; even so it seemes difficult to obtain an analogue of theorem 2.6 in this direction.

Example 2.9. Let us consider the 3×3 system:

$$P[u]\equiv\begin{bmatrix}\partial_1 & 0 & 0\\ 0 & \partial_1 & 0\\ -u_2\partial_1 & 0 & \partial_1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}=\begin{bmatrix}a(x_{1+})^{a-1}\\ 0\\ 0\end{bmatrix} \quad \text{on } \mathbf{R}^2,$$

where $a>0$ and $t_+=\max\{t, 0\}$. One of its solution is given by

$$u_1=(x_{1+})^a,\ u_2=(x_{2+})^a,\ u_3=(x_{1+})^a(x_{2+})^a.$$

Then for $j=1, 2, 3$

$$u_j\in H^{a+\frac{1}{2}-\varepsilon}_{\mathrm{loc}}(\mathbf{R}^2)(\varepsilon>0) \text{ and } u_j\notin H^{a+\frac{1}{2}}_{\mathrm{loc}}(\mathbf{R}^2).$$

Let $z_0=(0, (\xi_1, 0))\in\mathbf{R}^2\times(\mathbf{R}^2\setminus 0)$. By theorem 2.1

$$E^{s_1}(u, z_0)=\mathbf{C}\times 0\times 0 \quad \text{if } a+\frac{1}{2}\le s_1<2a-1$$

(actually this holds true for more general s_1). On the other hand

$$WF^{s_1}_{pol}(u, z_0)=\mathbf{C}\times 0\times\mathbf{C} \quad \text{if } s_1\ge a+\frac{1}{2}.$$

Example 2.10. Let us consider the 3×3 system:

$$P[u] \equiv \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_1 & 0 \\ 0 & 0 & D_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} iu_2u_3 \\ 0 \\ 0 \end{bmatrix} = 0 \qquad \text{on } \mathbf{R}^2.$$

One of its solution is the following:

$$u_1 = \frac{1}{a+1}(x_{1+})^{a+1}(x_{2+})^a, \quad u_2 = (x_{2+})^a, \quad u_3 = (x_{1+})^a,$$

where $a > 0$. Then for $j = 1, 2, 3$

$$u_j \in H_{\mathrm{loc}}^{a+\frac{1}{2}-\varepsilon}(\mathbf{R}^2)(\varepsilon > 0) \quad \text{and} \quad u_j \notin H_{\mathrm{loc}}^{a+\frac{1}{2}}(\mathbf{R}^2).$$

In this case $R_{P_1} = \{(x, \xi) \in \mathbf{R}^2 \times (\mathbf{R}^2 \setminus 0);\ \xi_1\xi_2 = 0\}$. Let γ be the null bicharacteristics of ξ_1 through $(x_0, \xi_0) = (0, (0, \xi_2))$. Any Hamilton orbit L of (P, u) over γ is given by

$$\begin{aligned} L &= \{(z(t), \alpha w(t));\ t \in \mathbf{R},\ \alpha \in \mathbf{C}\}, \\ & z(t) = (x(t), \xi(t)) = (t, 0, 0, \xi_2), \\ & w(t) = \left(\frac{1}{a+1} w_2(t_+)^{a+1} + w_1,\ w_2,\ 0\right), \\ & w_0 = (w_1, w_2, 0) \in \mathrm{Ker}\, P_1(\xi_0). \end{aligned}$$

By thorem 2.6 dim $E^{s_1}(u, z(t))$ is constant, and actually

$$E^{s_1}(u, z(t)) = \mathbf{C}\left(\frac{1}{a+1}(t_+)^{a+1},\ 1,\ 0\right), \qquad t \in \mathbf{R}$$

if $a + \frac{1}{2} \le s_1 < 2a$ (in fact this holds true for more general s_1), while

$$WF_{pol}^{s_1}(u, z(t)) = \begin{cases} \mathbf{C}\left(\dfrac{1}{a+1}(t_+)^{a+1},\ 1,\ 0\right), & t \neq 0 \\ \mathbf{C}^2 \times 0, & t = 0 \end{cases}$$

if $s_1 \ge a + \frac{1}{2}$.

§ 3. Reviews of paradifferential calculus

For notational convenience we present the paradifferential calculus due to Bony [2], Meyer [7]. Since all the statements are essentially contained in [2], [5], [7], we omit most of the proofs. Let C_ρ, $\rho \ge 0$, be the set of all locally integrable function $f(x)$ on $\mathbf{R}^d$ such that

$$\begin{cases} |f|_\rho = \sum_{|\alpha| \le \rho} \sup_{x \in \mathbf{R}^d} |\partial^\alpha f(x)| < \infty & \text{if } \rho \in \mathbf{Z} \\ |f|_\rho = |f|_{[\rho]} + \sum_{|\alpha| = [\rho]} \sup_{x, h \in \mathbf{R}^d} \dfrac{|\partial^\alpha f(x+h) - \partial^\alpha f(x)|}{|h|^{\rho - [\rho]}} < \infty & \text{if } \rho \notin \mathbf{Z}. \end{cases}$$

First we give the symbol classes.

Definition 3.1. For $m \in \mathbf{R}$ and $\rho \ge 0$, C_ρ^m is the set of all locally integrable function $p(x, \xi)$ on $\mathbf{R}^d \times \mathbf{R}^d$ such that for all α

$$|p^{(\alpha)}(\cdot, \xi)|_\rho \le C_\alpha (1 + |\xi|)^{m - |\alpha|}, \qquad \xi \in \mathbf{R}^d. \tag{3.1}$$

Definition 3.2. For $m \in \mathbf{R}$ and $\rho \ge 0$, A_ρ^m is the set of all $p(x, \xi) \in S_{1,1}^m(\mathbf{R}^d \times \mathbf{R}^d)$ such that

$$|p^{(\alpha)}(\cdot, \xi)|_\rho \le C_\alpha (1 + |\xi|)^{m - |\alpha|}, \qquad \xi \in \mathbf{R}^d, \tag{3.1}$$

$$|p_{(\beta)}^{(\alpha)}(x, \xi)| \le C_{\alpha,\beta} (1 + |\xi|)^{m - \rho + |\beta| - |\alpha|}, \qquad x, \xi \in \mathbf{R}^d \tag{3.2}$$

for all α, β if $|\beta| > \rho$. Here $p_{(\beta)}^{(\alpha)}(x, \xi) = D_x^\beta \partial_\xi^\alpha p(x, \xi)$.

Remark. Since only the case $\rho \notin \mathbf{Z}$ fits in with our analysis and will become necessary later, we simplify the definition of A_ρ^m when $\rho \in \mathbf{Z}$ (see [7]).

We set for any $S \subset \mathscr{S}'(\mathbf{R}^d \times \mathbf{R}^d)$ and $\mu > 0$

$$S[\mu] = \{p(x, \xi) \in S;\ \operatorname{supp} \hat{p}_{x \to \eta}(\eta, \xi) \subset \{|\eta| \le \mu |\xi|\}\}. \tag{3.3}$$

Especially we denote $A_\rho^m[\frac{1}{10}]$ by B_ρ^m.

Let us construct a "cut-off" map from $\mathscr{S}'(\mathbf{R}^d \times \mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d \times \mathbf{R}^d)[\mu]$. Take $\varphi(x) \in \mathscr{C}_0^\infty(\mathbf{R}^d)$ such that $0 \le \varphi(x) \le 1$, $\varphi(x) = 0$ $(|x| \ge 1)$, $\varphi(x) = 1$ $(|x| \le \frac{1}{2})$. Set

$$\chi_\mu(\eta, \xi) = \varphi((\mu |\xi|)^{-1} \eta)(1 - \varphi(\xi)), \quad \mu > 0. \tag{3.4}$$

We put for $p \in \mathscr{S}'(\mathbf{R}^d \times \mathbf{R}^d)$

$$T_p^\mu(x, \xi) = \mathscr{F}^{-1}[\chi_\mu(\eta, \xi) \hat{p}(\eta, \xi)] = \int \tilde{\chi}_\mu(x - y, \xi) p(y, \xi)\, dy. \tag{3.5}$$

Here the Fourier transformation is performed with respect to the first variables. T^μ is the desired map. Especially we denote $T^{\frac{1}{10}}$ by T.

Now we will state a series of lemmas which will be used later.

Lemma 3.3 (cf. [7]). *Let $m \in \mathbf{R}$, $\rho \ge 0$, $\mu, \mu_1, \mu_2 > 0$.*

(1) $T^\mu: S \to S[\mu]$ *is continuous if* $S = A_\rho^m, C_\rho^m, S_{1,1}^m$.
(2) $T^{\mu_1} - T^{\mu_2}: S \to S_{1,1}^{m-\rho}[\max\{\mu_1, \mu_2\}]$ *is continuous if* $S = A_\rho^m, C_\rho^m$.
(3) $I - T^\mu: A_\rho^m \to A_0^{m-\rho}$, $C_\rho^m \to C_0^{m-\rho}$, $S_{1,0}^m \to S^{-\infty}$ *are continuous.*
(4) $A_\rho^m \subset A_{\rho+k}^{m+k}$, $k = 0, 1, 2, \ldots$.
(5) $T_{p_1 p_2}^\mu - T_{p_1}^\mu T_{p_2}^\mu \in S_{1,1}^{m_1 + m_2 - \rho}[2\mu]$ *for* $p_j \in C_\rho^{m_j}$, $j = 1, 2$.
(6) *If* $p \in C_\rho^m$, *then* $T_{p(\beta)}^{\mu(\alpha)} - T_{p_{(\beta)}^{(\alpha)}}^\mu \in S^{m - \rho - |\alpha|}[\mu]$ *for all* α, β *satisfying* $|\beta| \le \rho$.

Lemma 3.4 (cf. [5]). *If* $p \in S^m_{1,1}[\mu]$, $0 < \mu < 1$, *then*

$$\|p(x, D)u\|_s \le C_{s,\mu}(p)\,\|u\|_{s+m}, \qquad u \in \mathscr{S}(\mathbf{R}^d)$$

for all $s \in \mathbf{R}$. *Here* $C_{s,\mu}(\cdot)$ *is a semi-norm in* $S^m_{1,1}$ *and* $\|\cdot\|_s = \|\cdot\|_{H^s(\mathbf{R}^d)}$.

Lemma 3.5 (cf. [5]). *If* $p \in S^m_{1,1}$, *then*

$$\|p(x, D)u\|_s \le C_s(p)\,\|u\|_{s+m}, \qquad u \in \mathscr{S}(\mathbf{R}^d)$$

for all $s > 0$. *Here* $C_s(\cdot)$ *is a semi-norm in* $S^m_{1,1}$.

Lemma 3.6 (cf. [5], [7]). *If* $p_1 \in S^{m_1}_{1,1}$, $p_2 \in A^{m_2}_{\rho_2}[\mu_2]$, $0 < \mu_2 < 1$, *then*

$$p_1(x, D)p_2(x, D) = q(x, D),\ q - \sum_{|\alpha| \le \rho_2} (\alpha!)^{-1} p_1^{(\alpha)} p_{2(\alpha)} \in S^{m_1+m_2-\rho_2}_{1,1}.$$

Especially if $p_1 \in A^{m_1}_{\rho_1}[\mu_1]$, *then* $q \in A^{m_1+m_2}_{\min\{\rho_1,\rho_2\}}[\mu_1\mu_2 + \mu_1 + \mu_2]$ *and*

$$q - \sum_{|\alpha| \le \rho_2} (\alpha!)^{-1} p_1^{(\alpha)} p_{2(\alpha)} \in S^{m_1+m_2-\rho_2}_{1,1}[\mu_1\mu_2 + \mu_1 + \mu_2].$$

Lemma 3.7 (cf. [5]). *If* $p \in A^m_\rho[\mu]$, $0 < \mu < 1$, *then*

$$p(x, D)^* = q(x, D),\ q \in A^m_\rho\left[\frac{\mu}{1-\mu}\right],$$

$$q - \sum_{|\alpha| \le \rho} (\alpha!)^{-1} D^\alpha_x \partial^\alpha_\xi \bar{p} \in S^{m-\rho}_{1,1}.$$

Lemma 3.8 (the sharp Gårding inequality, cf. [2], [5]). *If* $p \in A^m_\rho[\mu]$, $0 < \mu < 1$ *and* $\mathrm{Re}\, p \ge 0$, *then*

$$\mathrm{Re}\,(p(x, D)u, u) \ge -C_\mu(p)\,\|u\|^2_{\frac{m-\nu}{2}}, \qquad u \in \mathscr{S}(\mathbf{R}^d)$$

for $\nu = \min\left\{1, \dfrac{2\rho}{\rho+2}\right\}$. *Here* $C_\mu(\cdot)$ *is a semi-norm in* A^m_ρ.

Remark. Bony [2] claims this lemma with $\nu < \min\left\{1, \dfrac{\rho}{2}\right\}$. Hörmander [5] proves the one in more general situation that contains the above one with $\nu = \min\left\{1, \dfrac{\rho}{2}\right\}$.

Proof. The proof is carried along the comment in [2, theorem 6.8] with const. $\times\, |\xi|^{\frac{n}{4}} e^{-|\xi|\cdot|x|^2}$ replaced by $\langle\xi\rangle^{\frac{d\delta}{2}} a(\langle\xi\rangle^\delta x)$. Here $\langle\xi\rangle = (1 + |\xi|^2)^{\frac{1}{2}}$, $\delta = \max\left\{\dfrac{1}{2}, \dfrac{2}{\rho+2}\right\}$ and $a(x) \in \mathscr{S}(\mathbf{R}^d)$ is a real-valued even function such that $\int a(x)^2\,dx = 1$. We use lemma 3.4 to claim the boundedness of the terms of lower order.

More precisely we set

$$p(x, y, \xi) = \langle\xi\rangle^{d\delta} \int a(\langle\xi\rangle^{\delta}(x - z))a(\langle\xi\rangle^{\delta}(y - z))p(z, \xi)\,dz$$

where $0 \le \delta < 1$ is a parameter which will be fixed later. Then

$$\begin{cases} p(x, y, \xi) \in S^m_{1,\delta}(\mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R}^d) \\ \mathrm{Re}(p(x, y, D)u, u) \ge 0 \qquad \text{for all } u \in \mathscr{S}(\mathbf{R}^d) \\ p(x, y, D) = p(x, x, D) + i\sum_{j=1}^d (D_{y_j} D_{\xi_j} p)(x, x, D) + q(x, D). \end{cases}$$

Here $q(x, \xi) \in S^{m-2(1-\delta)}_{1,\delta}(\mathbf{R}^d \times \mathbf{R}^d)$ and

$$(p(x, y, D)u)(x) = (2\pi)^{-d} \int e^{i(x-y)\cdot\xi} p(x, y, \xi)u(y)\,dy\,d\xi.$$

Since

$$\begin{aligned} (D_{y_j}p)(x, x, \xi) &= \langle\xi\rangle^{d\delta} \int D_{x_j} \frac{1}{2} a(\langle\xi\rangle^{\delta}(x - z))^2 \cdot p(z, \xi)\,dz \\ &= \langle\xi\rangle^{d\delta} \int \frac{1}{2} a(\langle\xi\rangle^{\delta}(x - z))^2 \cdot D_{z_j} p(z, \xi)\,dz \\ &\in S^{m+\max\{0, 1-\rho\}}_{1,\delta}(\mathbf{R}^d \times \mathbf{R}^d), \end{aligned}$$

it follows

$$\sum_{j=1}^d (D_{y_j} D_{\xi_j} p)(x, x, \xi) \in S^{m-\min\{1,\rho\}}_{1,\delta}(\mathbf{R}^d \times \mathbf{R}^d).$$

By the assumption for $a(x)$,

$$\begin{aligned} p(x, x, \xi) - p(x, \xi) &= \langle\xi\rangle^{d\delta} \int a(\langle\xi\rangle^{\delta} z)^2 (p(x - z, \xi) - p(x, \xi))\,dz \\ &= \langle\xi\rangle^{d\delta} \int a(\langle\xi\rangle^{\delta} z)^2 (p(x - z, \xi) - p(x, \xi) + \nabla_x p(x, \xi) \cdot z)\,dz \\ &\in S^{m-\delta\min\{2,\rho\}}_{1,1}(\mathbf{R}^d \times \mathbf{R}^d)[\mu]. \end{aligned}$$

Here we use

$$\begin{aligned} &|a(\langle\xi\rangle^{\delta} z)^2 (p(x - z, \xi) - p(x, \xi)| \\ &\qquad \le |a(\langle\xi\rangle^{\delta} z)^2 (\langle\xi\rangle^{\delta}|z|)^{\rho} |p(\cdot, \xi)|_{\rho} \langle\xi\rangle^{-\delta\rho} \qquad \text{if } 0 < \rho \le 1, \\ &|a(\langle\xi\rangle^{\delta} z)^2 (p(x - z, \xi) - p(x, \xi) + \nabla_x p(x, \xi) \cdot z)| \\ &\qquad \le |a(\langle\xi\rangle^{\delta} z)^2 (\langle\xi\rangle^{\delta}|z|)^{\rho} |p(\cdot, \xi)|_{\rho} \langle\xi\rangle^{-\delta\rho} \qquad \text{if } 1 < \rho \le 2 \end{aligned}$$

and the same kind of estimates. So we obtain by lemma 3.4

$$\begin{aligned} \mathrm{Re}(p(x, D)u, u) = \mathrm{Re}((p(x, D) - p(x, x, D))u, u) \\ - \mathrm{Re}((i\textstyle\sum_{j=1}^d (D_{y_j} D_{\xi_j} p)(x, x, D) + q(x, D))u, u) \end{aligned}$$

$$+ \operatorname{Re}(p(x, y, D)u, u)$$
$$\geq - C\|u\|^2_{\frac{m-\nu}{2}} \quad \text{for all } u \in \mathscr{S}(\mathbf{R}^d).$$

Here $\nu = \min\{1, \rho, 2(1-\delta), 2\delta, \delta\rho\}$. ν takes the maximum, $\min\left\{1, \frac{2\rho}{\rho+2}\right\}$, when $\delta = \max\left\{\frac{1}{2}, \frac{2}{\rho+2}\right\}$ which completes the proof.

Lemma 3.9. *Let $p \in A^m_\rho[\mu]$, $0 < \mu < 1$, $u \in H^s(\mathbf{R}^d)$ and $r > 0$ is small. Then for any $a, b \in M(z_0, r)$ such that $a \Subset b$*

$$\|a_n(x, D)p(x, D)u\| \leq (\sup_{x,\xi\in\mathbf{R}^d} |p(x, \xi)b_n(x, \xi)| + C_1 n^{m-\frac{\nu}{2}}) \cdot \|a_n(x, D)u\|$$
$$+ C_2 n^{m-1}\|b_n(x, D)u\| + e_n, \; n = 1, 2, \dots,$$

where $\nu = \min\left\{1, \frac{2\rho}{\rho+2}\right\}$, $C_1, C_2 > 0$, $\{e_n\} \in h^{s+\rho-m}$.

Proof. It is sufficient to prove this lemma when $\mu = \frac{1}{100}$ in view of lemma 3.3, 3.4. Let $a, b \in M(z_0, r)$ satisfying $a \Subset b$. Then

$$\|a_n(x, D)p(x, D)u\| \leq \|p(x, D)a_n(x, D)u\| + \|[a_n(x, D), p(x, D)]u\|$$
$$= A_n + B_n,$$
$$A_n^2 \leq \|p(x, D)b_n(x, D)a_n(x, D)u\|^2 + e_n^1$$
$$\leq \operatorname{Re}(T_{|pb_n|^2}(x, D)a_n(x, D)u, a_n(x, D)u)$$
$$+ C_1 n^{2m-\min\{1,\rho\}}\|a_n(x, D)u\|^2 + e_n^1$$
$$\leq (\sup_{x,\xi\in\mathbf{R}^d} |p(x, \xi)b_n(x, \xi)|^2 + C_2 n^{2m-\min\{\rho,1\}}) \cdot \|a_n(x, D)u\|^2$$
$$+ C_3\|a_n(x, D)u\|^2_{m-\frac{\nu}{2}} + e_n^1$$
$$\leq (\sup_{x,\xi\in\mathbf{R}^d} |p(x, \xi)b_n(x, \xi)|^2 + C_4 n^{2m-\nu}) \cdot \|a_n(x, D)u\|^2 + e_n^2,$$

where $C_j > 0$ and $\{e_n^1\}, \{e_n^2\} \in h^\infty$. Since

$$\sigma(a_n(x, D)p(x, D))(x, \xi) = (2\pi)^{-d}\langle e^{ix\cdot\eta}a_n(x, \xi+\eta), \hat{p}(\eta, \xi)\rangle_\eta,$$

it follows

$$\operatorname{supp} \sigma(a_n(x, D)p(x, D)) \subset \left\{(x, \xi) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0); |n^{-1}\xi - \xi_0| < \frac{r + \mu|\xi_0|}{1-\mu}\right\}.$$

We may assume $\frac{r+\mu|\xi_0|}{1-\mu} < \frac{|\xi_0|}{4}$. Take $\varphi(\xi) \in \mathscr{C}_0^\infty(\mathbf{R}^d)$ such that $\varphi = 1$ when $|\xi - \xi_0| < \frac{|\xi_0|}{4}$, $\varphi = 0$ when $|\xi - \xi_0| > \frac{|\xi_0|}{2}$ and set $\varphi_n(\xi) = \varphi(n^{-1}\xi)$. Then

$$
\begin{aligned}
B_n &= \|[a_n(x, D), p(x, D)]\varphi_n(D)u\| \\
&\leq \|[a_n(x, D), p(x, D)]\, b_n(x, D)\varphi_n(D)u\| + C_5 n^{m-\rho}\|\varphi_n(D)u\| \\
&\leq C_6 n^{m-1}\|b_n(x, D)u\| + C_7 e_n^3,
\end{aligned}
$$

where $C_j > 0$, $\{e_n^3\} \in h^{s+\rho-m}$. The proof is completed if we sum up the both estimates.

Lemma 3.10 (cf. [7]). *Let* $u = (u_j) \in H^s(\mathbf{R}^d, \mathbf{R}^N)$, $s > \frac{d}{2} + m$ *and let* $F(x, u_\alpha)_{|\alpha| \leq m}$ *be a* $\mathscr{C}^\infty$ *map from* $\{(x, u_\alpha)_{|\alpha| \leq m};\ x \in \mathbf{R}^d,\ u_\alpha = (u_{1,\alpha}, \dots, u_{N,\alpha}) \in \mathbf{R}^N\}$ *to* $\mathbf{C}^M$ *with its support compact for* x. *Then for* $\mu > 0$

$$F(x, \partial^\alpha u)_{|\alpha| \leq m} = T_P^\mu(x, D)u + f$$

where $f \in H^{2s-2m-\frac{d}{2}}(\mathbf{R}^d, \mathbf{C}^M)$ *and*

$$P(x, \xi) = \sum_{|\beta| \leq m} \left(\frac{\partial F_j}{\partial u_{k,\beta}}(x, \partial^\alpha u)_{|\alpha| \leq m}\right)_{\downarrow j = 1, \dots, M \atop \rightarrow k = 1, \dots, N} \cdot (i\xi)^\beta.$$

§4. Proof of theorems in §2

Proof of theorem 2.1. The proof proceeds similarly as that of proposition 1.11 in view of the series of lemmas in section 3, especially lemma 3.9 and lemma 3.10.

Proof of theorem 2.6. Since the statment is microlocal, it is no restriction to assume that γ is sufficiently small and that $\operatorname{supp} u$, $\operatorname{supp} P_m(\cdot, \xi)$ and $\operatorname{supp} G(\cdot, u_\alpha)_{|\alpha| \leq m-1}$ are all contained in a compact set in X. Then there exists a $N \times N$ matrix $\tilde{P}_{1-m}(x, \xi)$ with conponents in $S_{1,0}^{1-m}(\mathbf{R}^d \times \mathbf{R}^d)$, positively homogeneous of degree $1 - m$ with respect to ξ, $|\xi| \geq 1$, such that

$$
\begin{cases}
\tilde{P}_{1-m}(x, \xi)P_m(x, \xi) = q_1(x, \xi)I \quad \text{on} \quad \mathbf{R}^d \times \mathbf{R}^d \\
\tilde{P}_{1-m}(x, D)P[u] \in H^{s_1}(\mathbf{R}^d, \mathbf{C}^N)
\end{cases}
$$

where q_1 is a scalar real symbol, is of real principal type on γ. By lemma 3.10

$$\tilde{P}_{1-m}(x, D)(P_m(x, D) + \sum_{l=0}^{m-1} T_{P_l}(x, D))u \in H^{s_1}(\mathbf{R}^d, \mathbf{C}^N),$$

where for $0 \leq l \leq m - 1$

$$P_l(x, \xi) = \sum_{|\beta| = l} \left(\frac{\partial G_j}{\partial u_{k,\beta}}(x, \partial^\alpha u)_{|\alpha| \leq m-1}\right)_{\downarrow j = 1, \dots, N \atop \rightarrow k = 1, \dots, N} \cdot (i\xi)^\beta,$$

whose components belong to C_ρ^l. As in the first part of [3, section 4], we obtain

$$(q(x, D)I + T_Q(x, D))u = f \in H^{s_1}(\mathbf{R}^d, \mathbf{C}^N).$$

Here $q = q_1 + (2i)^{-1}\sum_{j=1}^{d} \partial_{x_j}\partial_{\xi_j} q_1$, $Q = Q_0 + Q_{-1}$ and Q_{-j} is a $N \times N$ matrix with components in $C^{-j}_{\max\{\rho - j, 0\}}$, $j = 0, 1$ (in fact $Q_{-1} = 0$ if $\rho < 1$) and

$$Q_0 = (2i)^{-1}\{\tilde{P}_{1-m}, P_m\} + \tilde{P}_{1-m}P^s_{m-1} + \tilde{P}^s_{-m}P_m,$$

$$\{\tilde{P}_{1-m}, P_m\} = \sum_{j=1}^{d} (\partial_{\xi_j}\tilde{P}_{1-m} \cdot \partial_{x_j}P_m - \partial_{x_j}\tilde{P}_{1-m} \cdot \partial_{\xi_j}P_m),$$

$$P^s_{m-1} = P_{m-1} - (2i)^{-1}\sum_{j=1}^{d} \partial_{x_j}\partial_{\xi_j}P_m,$$

$$\tilde{P}^s_{-m} = -(2i)^{-1}\sum_{j=1}^{d} \partial_{x_j}\partial_{\xi_j}\tilde{P}_{1-m}.$$

Let $(z_0, w_0) = (x_0, \xi_0, w_0) \in L \setminus 0$, $(z(t), w(t)) = (x(t), \xi(t), w(t)) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0) \times (\mathbf{C}^N \setminus 0)$, $t \in \mathbf{R}$ be the solution of

$$\begin{cases} \dot{x}(t) = \nabla_\xi q_1(x(t), \xi(t)), & x(0) = x_0 \\ \dot{\xi}(t) = -\nabla_x q_1(x(t), \xi(t)), & \xi(0) = \xi_0 \\ \dot{w}(t) = -iQ_0(x(t), \xi(t))w(t), & w(0) = w_0. \end{cases}$$

Precisely q_1, Q_0 must be modified to be positively homogeneous for $\xi \neq 0$. However we often omit this kind of remarks if it makes no trouble.

We shall prove that $(x(t), \xi(t), w(t)) \notin E^{s_1}(u)$ for all $t \in \mathbf{R}$ if $(x_0, \xi_0, w_0) \notin E^{s_1}(u)$. By definition 1.4 there exists $h_0 \in \mathbf{C}^N$ such that $h_0 \in H^{s_1}(u, z_0)$ and $h_0 \cdot w_0 \neq 0$. Let $h(t)$ be the solution of

$$\begin{cases} \dot{h}(t) = i\,{}^t Q_0(x(t), \xi(t))h(t) \\ h(0) = h_0. \end{cases}$$

Then $\dfrac{d}{dt}(h(t) \cdot w(t)) \equiv 0$, which means $h(t) \cdot w(t) \equiv h_0 \cdot w_0 \neq 0$. So it is sufficient to prove $h(t) \in H^{s_1}(u, z(t))$ for all $t \in \mathbf{R}$.

Let $T > 0$ be fixed (the case $T < 0$ is treated similarly). We assign to each $a \in M(z(T), r)$ the solution $a(t) = a(t, x, \xi)$ of

$$\text{(4.1)} \qquad \begin{cases} (\partial_t + H_{q_1})a(t, x, \xi) = 0, & x, \xi \in \mathbf{R}^d,\ 0 \leq t \leq T \\ a(T, x, \xi) = a(x, \xi), & x, \xi \in \mathbf{R}^d. \end{cases}$$

Then $a(t) \in M(z(t), r(t))$ for $r(t) > 0$ such that $\sup_{0 \leq t \leq T} r(t) \downarrow 0$ if $r \downarrow 0$. Further $a_n(t, x, \xi) = a(t, x, n^{-1}\xi)$ is the solution of (4.1) with $a(x, \xi)$ replaced by $a_n(x, \xi) = a(x, n^{-1}\xi)$, $n = 1, 2, \ldots$.

Let $a \Subset b \Subset c \in M(z(T), r)$ ($r > 0$ is small) and let $a(t), b(t), c(t)$ be the solutions of (4.1) assigned to a, b, c respectively. Then

$$\begin{aligned} &(D_t + q(x, D))a_n(t, x, D)h(t) \cdot u \\ &\quad = [D_t + q(x, D), a_n(t, x, D)]h(t) \cdot u + a_n(t, x, D)D_t h(t) \cdot u \\ &\qquad + a_n(t, x, D)h(t) \cdot (f - T_Q(x, D)u) \end{aligned}$$

$$
\begin{aligned}
&= [D_t + q(x, D), a_n(t, x, D)]h(t)\cdot u + a_n(t, x, D)h(t)\cdot f \\
&\quad + a_n(t, x, D)h(t)\cdot(Q_0(z(t)) - T_Q(x, D))u \\
&\equiv f_n^1(t) + f_n^2(t) + f_n^3(t) \equiv f_n^0(t),
\end{aligned}
$$

$$\|f_n^1(t)\| \le C_0 n^{-1}\|b_n(t, x, D)Iu\| + e_n^1,$$

$$\|f_n^2(t)\| \le e_n^2,$$

$$
\begin{aligned}
\|f_n^3(t)\| &\le (\sup_{x,\xi\in\mathbf{R}^d} |h(t)\cdot(Q_0(z(t)) - T_Q(x, \xi))b_n(t, x, \xi)| + C_1 n^{-\frac{\nu}{2}}) \\
&\quad \times \|a_n(t, x, D)Iu\| + C_2 n^{-1}\|b_n(t, x, D)Iu\| + e_n^3, \ n\in\mathbf{N}, \qquad 0\le t\le T
\end{aligned}
$$

in virtue of lemma 3.9. Here $\{e_n^1\}\in h^\infty$, $\{e_n^2\}$, $\{e_n^3\}\in h^{s_1}$, $C_j > 0$, $\nu = \min\left\{1, \dfrac{2\rho}{\rho+2}\right\}$. So

$$\|f_n^0(t)\| \le \varepsilon_1(r)\|a_n(t, x, D)Iu\| + C_3 n^{-1}\|b_n(t, x, D)Iu\| + e_n^4 \tag{4.2}$$

for $\{e_n^4\}\in h^{s_1}$, $\varepsilon_1(r) > 0$, $C_3 > 0$ where $\varepsilon_1(r)$ depends only on Q, r and $\varepsilon_1(r)\downarrow 0$ if $r\downarrow 0$. Set $u_n(t) = a_n(t, x, D)h(t)\cdot u$. Since

$$
\begin{aligned}
\frac{d}{dt}\|u_n(t)\|^2 &= 2\,\mathrm{Im}\,(q(x, D)u_n(t) - f_n^0(t), u_n(t)) \\
&\le 2(K_1\|u_n(t)\| + \|f_n^0(t)\|)\cdot\|u_n(t)\|,
\end{aligned}
$$

we have

$$\frac{d}{dt}\|u_n(t)\| \le K_1\|u_n(t)\| + \|f_n^0(t)\|,$$

from which it follows that

$$
\begin{aligned}
\|u_n(t)\| &\le e^{K_1 T}\|u_n(0)\| + \int_0^T e^{K_1(T-s)}\|f_n^0(s)\|\,ds \\
&\le K_2(\|u_n(0)\| + \sup_{0\le t\le T}\|f_n^0(t)\|),
\end{aligned}
$$

where $K_1, K_2 > 0$ depend only on q, T. Since $h_0\in H^{s_1}(u, z_0)$,

$$
\begin{aligned}
\|u_n(0)\| &\le \varepsilon_2(r)\|a_n(0, x, D)Iu\| + C_4 n^{-1}\|b_n(0, x, D)Iu\| + e_n^4 \\
&\le \varepsilon_3(r)\|a_n(x, D)Iu\| + C_5 n^{-1}\|c_n(x, D)Iu\| + e_n^5
\end{aligned}
$$

for $\varepsilon_j(r) > 0$, $C_k > 0$, $\{e_n^4\}$, $\{e_n^5\}\in h^{s_1}$ where $\varepsilon_j(r)$ is independent of a, b, c and $\varepsilon_j(r)\downarrow 0$ if $r\downarrow 0$ in view of lemma 4.1 which will be given later. The similar estimate holds for $\sup_{0\le t\le T}\|f_n^0(t)\|$ by (4.2) and the same lemma. To sum up

$$\|a_n(x, D)h(T)\cdot u\| \le \varepsilon(r)\|a_n(x, D)Iu\| + Cn^{-1}\|c_n(x, D)Iu\| + e_n, \qquad n\in\mathbf{N}$$

for $\varepsilon(r) > 0$, $C > 0$, $\{e_n\}\in h^{s_1}$ where $\varepsilon(r)$ is independent of a, b, c and $\varepsilon(r)\downarrow 0$ if

$r \downarrow 0$. This implies $h(T) \in H^{s_1}(u, z(T))$.

Lemma 4.1. *For every* $a \Subset b$ *in* $M(z(T), r)$,

$$\sup_{0 \le t \le T} \| a_n(t, x, D) I u \|$$
$$\le K \| a_n(x, D) I u \| + C n^{-1} \| b_n(x, D) I u \| + e_n, \qquad n \in \mathbf{N},$$

where $\{e_n\} \in h^{s_1}$, $C, K > 0$ *and* K *depends only on* q, Q, T.

Proof. Let $a^j \in M(z(T), r)$ such that $a = a^0 \Subset a^1 \Subset \cdots \Subset a^k = b (k = [\rho] + 1)$, and let $a^j(t)$ be the solution of (4.1) assigned to a^j. Then

$$\begin{aligned}(D_t + q_1(x, D)) a_n^j(t, x, D) I u &= [D_t + q(x, D), a_n^j(t, x, D)] I u \\ &\quad + a_n^j(t, x, D) I f - a_n^j(t, x, D) I T_Q(x, D) u \\ &\equiv g_n^1(t) + g_n^2(t) + g_n^3(t) \equiv g_n^0(t),\end{aligned}$$

$$\| g_n^1(t) \| \le C_1 n^{-1} \| a_n^{j+1}(t, x, D) I u \| + e_n^1,$$
$$\| g_n^2(t) \| \le e_n^2,$$
$$\| g_n^3(t) \| \le K_1 \| a_n^j(t, x, D) I u \| + C_2 n^{-1} \| a_n^{j+1}(t, x, D) I u \| + e_n^3$$

for all $n \in \mathbf{N}$, $0 \le t \le T$. Here $\{e_n^1\} \in h^\infty$, $\{e_n^2\}, \{e_n^3\} \in h^{s_1}$, $C_1, C_2 > 0$. As before we obtain

$$-\frac{d}{dt} \| a_n^j(t, x, D) u \| \le K_2 \| a_n^j(t, x, D) u \| + C_3 n^{-1} \| a_n^{j+1}(t, x, D) u \| + e_n^4,$$

which implies

$$\sup_{0 \le t \le T} \| a_n^j(t, x, D) u \| \le K \| a_n^j(T, x, D) u \|$$
$$+ C n^{-1} \sup_{0 \le t \le T} \| a_n^{j+1}(t, x, D) u \| + e_n^4$$

for all $n \in \mathbf{N}$, $0 \le j \le k - 1$. Here $\{e_n^4\} \in h^{s_1}$, $C, K > 0$, and K depends only on q, Q, T. From this we have with $\{e_n^5\} \in h^{s_1}$

$$\sup_{0 \le t \le T} \| a_n^0(t, x, D) u \| \le K \sum_{j=0}^{k-1} (C n^{-1})^j \| a_n^j(T, x, D) u \|$$
$$+ (C n^{-1})^k \sup_{0 \le t \le T} \| a_n^k(t, x, D) u \| + e_n^5.$$

Since $k = [\rho] + 1$,

$$\sup_{0 \le t \le T} \| a_n(t, x, D) u \| \le K \| a_n(x, D) u \| + C_0 n^{-1} \| b_n(x, D) u \| + e_n$$

for $C_0 > 0$, $\{e_n\} \in h^{s_1}$ which completes the proof.

Remark. This lemma shows the propagation of the microlocal regularity of the solution u.

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