J. Math. Kyoto Univ. (JMKYAZ)
34-3 (1994) 641-694

On super theta functions

By

Yoshifumi TSUCHIMOTO*

1. Introduction

The purpose of the present paper is to define super theta function of a super Riemann surface. It is identified with a section of a line bundle on a super Jacobian. In the non super case, it is known in conformal field theory that the tau function associated to a family of Virasoro uniformalized Riemann surfaces (Riemann surfaces with speciated points and coordinates around the points) is expressed by means of Riemann's theta function ([4]). Moreover, all the coefficients of Taylor expansion of Riemann's theta function is encoded in the tau function. We may thus use tau function to calculate theta function. In the present paper we use this observation and first we define a super tau function in an analogous way to the usual theory of tau function (which we review in section 2). It is defined as a Berezinian (super determinat of an infinite matrix representing the effect of multiplication by a function on a function space (Theorem 5.2). We use the space of formal power series as the function space. This enables us to develop our theory within a framework of (infinite dimensional) super algebraic geometry. A theory on super tau functions constructed by choosing L^2-space as the function space already appeared in [10]. Our approach seems to be easier to handle with when we use the tau function to study the moduli space of line bundles (super Jacobian in our terminology.).

Having defined the super tau function, our next task is to interpret it as a sort of super theta function. We need to define "super Jacobian" and identify the super tau function with a section of a line bundle on it. We will define in section 6 super Jacobian, which we call the n-super Jacobian, as the moduli space of line bundles (with trivialization at the distinguished point) on the original Riemann surfaces. We employ there analytic methods, and obtain a description of the n-super Jacobian using "periods". We finally show in section 7 that the super tau function may be interpreted as a section of a line bundle on an n-super Jacobian for some n. Our main theorems are the following.

Theorem. (*Theorem* 7.1) *The super tau function may be interpreted as a functional on the space of convergement power series.*

Received September 17, 1993, Reviced November 29, 1993
*Supported by JSPS Fellow/ships for Japanese Junior Scientists

Theorem. (*Theorem* 7.2) *The super tau function, interpreted as above, has a periodicity so that it may be identified as a section of a line bundle on n-super Jacobian for some n (outside an analytic subvariety).*

Let us describe here a background of the present paper.

In the process of development of quantum field theories, physicists noticed importance of so called super theories. It emerged from a consideration of a "classical" counterpart of Fermions. In mathematical words, this consideration corresponds to the fact that a limit of the Clifford algebra is an exterior algebra. The theory goes further and focuses on studies in nice representations of ring objects in derived categories. For example, we consider the cohomology ring $H^*(M)$ of a manifold M. It has a natural structure of a super commutative ring, the multiplication being defined by the cup product. We may also consider a ring object $\mathbf{R}\Gamma_M\mathbf{C}$ in the derived category $\mathrm{D}^+(\mathbf{C})$. This object is represented by the well-known de Rham complex. It goes without saying that the complex itself is interesting and important. Homological algebra gives many other interesting complexes and the corresponding objects in suitable derived categories. Importance of super theories is fairly increased by arguments of Witten. It suggests that geometrical invariants of a manifold is obtained by considering the distribution of super Riemann surfaces in the de Rham super space $(M, \wedge^*T^*M)$, the super space with the underlying topological space M and which has the de Rham complex as its structure sheaf. If we regard each of these super Riemann surfaces as a "particle" on the manifold, the number or the distribution of these particles seems to be a fundamental quantity of the manifold. We recall here three analogues of this kind of observation.

I. The number of rational points on a variety defined over a finite field $\mathbf{F}_q$. The number of rational points (defined over various extension fields $\mathbf{F}_{q^n}$ of $\mathbf{F}_q$) on the variety is an important invariant of the variety. The data is encoded in its generating function, the Weil zeta function of the variety.

II. The distribution of lengths of geodesics on the manifold. If we regard a manifold M as a space-time, classical particles appear as world lines, not points. We equip M with a Riemannian metric. We define the action of a world line to be its length, and follow the minimal principle. Then particles correspond to geodesics. Thus the distribution of lengths of geodesics on M is a fundamental invariant of M. It is again encoded in its generating function, which is also called a zeta function. We may also consider the distribution of length of all curves on M. This is equivalent to considering the length function on the moduli space of curves on M. Geodesics are clearly critical points of the length function on the moduli space.

III. Dimensions of the space of harmonic forms. In relativistic quantum mechanics, wave functions of particles obey the Klein-Gordon equation. When we suppose again that the space-time M is Riemannian with a positive definite metric, Klein-Gordon equation is interpreted as an eigen equation of the Laplacian of M;

$$\Delta f=m^2f\ .$$

Again we "count the number of particles" on the manifold, namely consider the dimension of harmonic functions of forms. When M is compact, the space of harmonic forms of degree d may be identified with the cohomology group $H^d(M)$. Thus in this case the number of particles gives the Betti numbers of M.

It is interesting to note that the data obtained in I and in III are related by the Weil conjecture, proved by Deligne. It is also true that II and III are related to each other in a similar way. (Physically, III may be regarded as a quantization of II.)

Let us put the data II in another way. The modulus of a compact 1-dimensional Riemannian manifold (circle) is its length. Thus the moduli space of compact Riemannian manifold of dimension 1 is given by $\mathbf{R}_{>0}$. The distribution of lengths of geodesics on the manifold is equivalently describedby a function N on the moduli space, where $N(l)$ is the number of geodesics of length l.

Similarly, we may describe the distribution of super Riemann surface on $(M, \wedge T^*M)$ a given manifold by considering a function on a moduli space of super Riemann surfaces. The description of the moduli space is thus of the fundamental importance. In the usual (non super) conformal field theory, the moduli space of Virasoro uniformized Riemann surfaces is embedded into the Sato's universal Grassmann manifold (UGM). Coefficients of tau function gives the Plücker coordinates of the point of UGM corresponding to a Riemann surface R [4]. We may thus use tau function or theta function to classify Virasoro uniformized Riemann surfaces. We want to use super theta function to classify Virasoro uniformized super Riemann surfaces in a similar way. When the base scheme S of the family is pure even and the odd dimension N of the super Riemann surface is equal to 1, then we will show in Proposition 5.3 that the super tau function of the family and the tau function associated to the reduction of the family together determines the original family.

The author is grateful to Professor Kenji Ueno for giving him good advice.

2. Review of non super case

To give a concrete picture of what is done in this paper, let us review the theory in [4] in precise. It is well known that the whole data of a Riemann surface X with a specified point Q is encoded in the affine coordinate ring $A=\Gamma(X; \mathcal{O}_X(*Q))$ of the afine variety $X\backslash Q$. In fact, we have $X\backslash Q=\mathrm{Spec}(A)(\mathbf{C})$, and the information of the neighbourhood of Q is determined uniquely by the requirement of non singularity of X. To extract the data of A we use the so-called Virasoro uniformization. We choose a formal coordinate z around

Q. We employ the word "Virasoro uniformized Riemann surface" to indicate a triple (X, Q, z) of a Riemann surface with a specified point and a coordinate. The Laurent power series expansion gives an inclusion

$$\iota_z: A \subset \mathbf{C}((z)) .$$

Thus we have a linear subspace $U = \iota_z(A)$ of a vector space $\mathbf{C}((z))$. The ambient space is independent of (X, Q, z), and U subjects to a remarkable constraint in size. Namely, it satisfies the following condition.

$$\text{(2.1)} \qquad \begin{cases} \dim(\mathbf{C}[[z]] \cap U)) < \infty , \\ \dim\left(\dfrac{\mathbf{C}((z))}{\mathbf{C}[[z]] + U}\right) < \infty . \end{cases}$$

in fact, arguments in formal Čech cohomology ([1]) suggests

$$\mathbf{C}[[z]] \cap \iota_z(A) \cong H^0(X; \mathcal{O}_X) ,$$

$$\frac{\mathbf{C}((z))}{\mathbf{C}((z)) + \iota_z(A)} \cong H^1(X; \mathcal{O}_X) ,$$

The set of all linear subspaces of $\mathbf{C}((z))$ which satisfies (2.1) has a natural structure of infinite dimensional scheme called the Sato universal Grassmann manifold (UGM). It is embedded in an infinite dimensional projective space by the Plücker embedding. Let us recall its definition here. The homogeneous coordinates (Plücker coordinates) of U is defined as follows. We first consider a "generic function" (formal power series with indeterminate coefficients)

$$F = F(a) = \sum_{-\infty < i < \infty} a_i z^i .$$

We put

$\iota: U \to \mathbf{C}((z))$	the inclusion,
$M_{\exp(F)}: \mathbf{C}((z)) \to \mathbf{C}((z))$	multiplication by $\exp(F)$,
$p: \mathbf{C}((z)) \to \mathbf{C}[[z]]$	the projection.

Then the "infinite determinant" of the composition of the above maps,

$$\text{(2.2)} \qquad \tau(\{a_i\}) = \tau(a; U) = \det(p \circ M_{\exp}(F) \circ \iota) ,$$

gives a generating function of the required Plücker coordinates. (The determinant is considered as a "formal function in $\{a_i\}$". See below.)

We need to explain the meaning of the above infinite determinant. The difficulties are that above determinant is a determinant of a map between infinite dimensional spaces, and that the domain and the target are not the same space. If we first are regardless of the infinite dimensionality, the determinant (2.2) is defined as a map between the top exterior powers (determinants) of vector spaces.

$$\det(p \circ M_{\exp}(F) \circ \iota) : \det U \to \det \mathbf{C}((z))/\mathbf{C}[[z]] \ .$$

In other words, τ is an element of a line

$$\ell = (\det U)^{\vee} \otimes (\det \mathbf{C}((z))/\mathbf{C}[[z]]) \ .$$

To explain this line in terms of determinants of finite dimensional vector spaces, we first note that the condition implies that the complex

$$U \overset{p \circ M_{\exp}(F) \circ \iota}{\longrightarrow} \mathbf{C}((z))/\mathbf{C}[[z]]$$

being quasi-isomorphic to a complex $C^{\cdot}$ of finite dimensional vector spaces. For example, it is quasi-isomorphic to the following complex.

$$K^0 \overset{0}{\longrightarrow} K^1$$

Where

$$K^0 = \ker(p \circ M_{\exp}(F) \circ \iota) \ ,$$

and

$K^1 =$ (complementary linear subspace to $I_1 = ((p \circ M_{\exp}(F) \circ \iota)(U) + \mathbf{C}[[z]]/\mathbf{C}[[z]]$ in $\mathbf{C}((z))/\mathbf{C}[[z]])$,

$$\mathbf{C}((z))/\mathbf{C}[[z]] = K^1 \oplus I^1 \ .$$

The fact guarantees us an existence of the determinant line,

$$\lambda = \det\left(U \overset{p \circ M_{\exp(F)} \circ \iota}{\longrightarrow} \mathbf{C}((z))\right) = \left(\bigotimes_{j:\mathrm{odd}} (\det C^j)\right)^{\vee} \otimes \left(\bigotimes_{j:\mathrm{even}} (\det C^j)\right) \ .$$

It is proved in [6] that this definition is independent of the choice of $C^{\cdot}$. If we assume U to be the one which corresponds to a Virasoro uniformized data (X, Q, z), then it is isomorphic to $\det \mathbf{R}\Gamma_X(\mathcal{O}_X)$ by virtue of the above cited cohomology argument in [1]. In any case, our line l is (by definition) equal to $\lambda^{\vee}$. To explain this intuitively, we take a complementary linear subspace I^0 to K^0 in U and employ the following decompositions.

$$U = K^0 \oplus I^0 \ ,$$

$$\mathbf{C}((z))/\mathbf{C}[[z]] = (A + \mathbf{C}[[z]]) \oplus K^1 \ .$$

Then we have

$$\det U = \det K^0 \otimes \det I^0 \ ,$$

$$\det \mathbf{C}((z))/\mathbf{C}[[z]] = \det I^1 \otimes \det K^1 \ .$$

So $\ell^{\vee}$ is isomorphic to

$$\det K^0 \otimes \det I^0 \otimes (\det I^1)^{\vee} \otimes (\det K^1)^{\vee} \ .$$

But by virtue of the homomorphism theorem, there is a canonical isomorphism between I^0 and I^1. We may thus substitute $\det I^0 \otimes (\det I^1)^\vee$ by a trivial line and obtain

$$\ell^\vee = \det K^0 \otimes (\det K^1)^\vee = \det(\mathbf{C}[[z]] \oplus U \to \mathbf{C}((z))) .$$

The above formal treatment of the determinant line including the use of determinants of infinite vector spaces, is justified by handling "semi-infinite forms" directly ([4], [12]). For example, we may represent an element of λ by assigning topological bases ("frames") $\{\xi_i\}$, $\{\eta_j\}$, $\{e_k\}$ of U, $\mathbf{C}[[z]]$, and $\mathbf{C}((z))$, respectively such that the triple $(\{\xi_i\}, \{\eta_j\}, \{e_k\})$ is consistent ([12]). We may denote it symbolically as

$$(\wedge_i \xi_i) \otimes (\wedge_j \eta_j) \otimes (\wedge_k e_k)^{-1} .$$

Now let us examine the geometric meaning of the determinant line λ. First of all, F determines, via formal Čech cohomology, an element of $H^1(X; \mathcal{O}_X)$. Let us denote by $[F]$ the cohomology class:

$$[F] \in \mathbf{C}((z))/\iota(A) + \mathbf{C}[[z]] \cong H^1(X; \mathcal{O}_X) .$$

$[F]$ in turn determines a line bundle $H^1(\exp)[F] = L_F$ of degree 0 via the map

$$H^1(X; \mathcal{O}_X) \cong H^1(X_{\mathrm{an}}; \mathcal{O}_{X_{\mathrm{an}}}) \xrightarrow{H^1(\exp)} H^1(X_{\mathrm{an}}; \mathcal{O}_{X_{\mathrm{an}}}{}^\times) \cong H^1(X_{\mathrm{an}}; \mathcal{O}_X{}^\times)$$

obtained by exponetial map and the GAGA isomorphisms. Intuitively speaking, it is a line bundle obtained by gluing a trivial line bundle on $\pi_* \mathcal{O}_{\dot{X}}$ and a trivial line bundle on $\mathrm{Spf}\mathbf{C}[[z]]$ by the transition function $\exp(F)$ on $\mathrm{Spf}\mathbf{C}((z))$. A simple diagram chasing shows that L_F corresponds to the following point of the Jacobian $\mathrm{Jac}(X) = H^0(X; \omega_{X/S})^\vee / H_1(X; \mathbf{Z})$.

$$\omega \mapsto \mathrm{Res}_Q(F_\omega) . \tag{2.3}$$

We may also be more explicit and rewrite the above. Choose a basis $\{\omega_i\}_{i=1}^g$ of $\omega_{X/S}$ and a symplectic homology basis $(\alpha_i, \beta_i)_{i=1}^g$ so that

$$\int_{\alpha_i} \omega_j = \delta_{ij} .$$

Then the Jacobian of X is given by a complex torus

$$\mathbf{C}^g/\Lambda = \mathbf{C}^g/\mathbf{Z}^g + \Omega\mathbf{Z}^g ,$$

where Ω is period matrix of X,

$$\Omega_{ij} = \int_{\beta_i} \omega_i .$$

Then L_F corresponds to $I(a)$ modulo Λ, where $I(a)$ is given by the following.

$$I(a) = (\mathrm{Res}_Q(F\omega_i))_{i=1}^g .$$

We may see that the determinant line λ is expressed as

$$\det\mathbf{R}\Gamma_X(L_F)=\det(H^0(X;L_F))\otimes\det(H^1(X;L_F))^\vee .$$

To obtain a number instead of an element of the determinant line, we have to trivialize the determinant line. To do this, we first clarify the meaning of "formal function of $\{a_i\}$". Our coefficient ring should be $\mathscr{B}=\mathbf{C}[[\{a_i\}]]$, the completion of the polynomial ring on infinite variables with respect to the gradation given by the following.

$$\deg(a_i)=\max(-i,1) .$$

We actually are considering things over a formal scheme $S=\mathrm{Spf}\mathscr{B}$. L_F should be regarded as a family of line bundles parametrized by S. Put

$$\mathscr{U}=U\widehat{\otimes}\mathscr{B} ,$$
$$\mathscr{B}[[z]]=\mathbf{C}[[z]]\widehat{\otimes}\mathscr{B} ,$$
$$\mathscr{B}\{\{z\}\}=\mathbf{C}((z))\widehat{\otimes}\mathscr{B}$$

To trivialize λ, we first twist L_F to make a degree $g-1$ line bundle.

$$\widetilde{L_F}=L_F\otimes\mathscr{O}_X((g-1)Q)$$

Then for a general F, we have

$$H^0(X;L_F)=H^1(X;L_F)=0$$

(Note that 2.3 implies that $\mathscr{B}\{\{z\}\}\ni F\to L_F\in\mathrm{Jac}$ gives a submersion (induces a surjective map on the tangent space).) We may thus trivialize the deptерminant line $\det\mathbf{R}\Gamma(X;\widetilde{L_F})$ of twisted line bundle L_F for general F. There is an isomorphism between $\det\mathbf{R}\Gamma(X;\widetilde{L_F})$ and $\det\mathbf{R}\Gamma(X;L_F)$. We may put this in concrete way. Instead of $\det(p\circ M_{\exp(F)}\circ\iota)$, we consider $\det(\widetilde{p}\circ M_{\exp(F)}\circ\iota)$, where we denote by $\widetilde{p}$ the following canonical projection

$$\widetilde{p};\ \mathbf{C}((z))\to\mathbf{C}((z))/z^{1-g}\mathbf{C}[[z]]$$

We fix an isomorphism

$$\Phi\colon\mathbf{C}((z))/z^{1-g}\mathbf{C}[[z]]\to A$$

so that there exist an integer N and we have the following Laurent expansions at Q.

$$\Phi(z^n)=f^n(z)\in z^{-n}+\mathbf{C}[[z]][\zeta]z^{-N}$$

(If we are given two such isomorphisms Φ_1, Φ_2, then the difference is given by a "compact operator", See the section 4)

We then obtain a number (or, as we have said, a formal function on $\{a_i\}$),

$$\tau=\det_{\mathbf{C}((z))/z^{1-g}\mathbf{C}[[z]]}(\widetilde{p}\circ M_{\exp(F)}\circ\iota\circ\Phi)$$

This function is called the tau function associated to the Virasoro uniformized

Riemann surface (X, Q, z). We may regard the tau function as a section of the determinant line on the Jacobian of the Riemann surface. It is well known that such sections are described in terms of theta functions. In our case, arguments in [4] give the following formula relating the tau function and theta functions.

Theorem 2.1. ([4]) *We have,*

$$\tau(a)=\exp(q(a))\vartheta(I(a)+\Delta) \ ,$$

where $q(a)$ is a quadratic function of a, Δ is the vector of Riemann constants associated to $(X, Q, (\alpha, \beta))$ and ϑ is the theta function on $\mathrm{Jac}X=\mathbf{C}^g\Lambda=\mathbf{C}^g/\mathbf{Z}^g+\Omega\mathbf{Z}^g$.

We may add a few comments to the above theorem. First we note that

$$\tau(a)=0 \Longrightarrow H^0(X; L_F(a))\neq 0 \Longrightarrow \vartheta(I(a)+\Delta; \Omega)=0 \ ,$$

which explains why we need the vector of Riemann constant in the above theorem.

Second we remark that the Jacobian comes into consideration because we consider an action $M_{\exp F}$ of a generic element of $\exp(\mathbf{C}((z)))$. We may of course consider

$$\mathrm{Ber}_{\mathbf{C}((z))/z^{1-g}\mathbf{C}[[z]]}(\tilde{p}\circ M_{\exp(F)}\circ\iota\circ\Phi)$$

for arbitrary continuous linear endomorphism M of $\mathbf{C}((z))$. But the following proposition (which is a corollary to the theorem of "Boson-Fermion correspondence") enables us to restrict ourselves to consider only $\exp(\mathbf{C}((z)))$.

Proposition 2.2. (*corollary to Boson-Fermion correspondence*) ([4]) *The map*

$$U\longmapsto\tau(a; U) \quad (\textit{modulo scalar multiples})$$

gives an embedding of UGM to a infinite dimensional projective space $\mathbf{P}(\mathscr{B})$. In other words, the tau function $\tau(a; U)$ determines completely the linear subspace U of $\mathbf{C}((z))$ with the property (2.1).

We may also consider a tau function of a Virasoro uniformized Riemann (X, Q, z) surface with a line bundle L and a trivialization t ("Virasoro uniformization") of L on $\mathrm{Spf}\mathbf{C}[[z]]$.

$$\tau(a; (X, Q, z, L, t))=\tau(a; \iota_{t,z}(H^0(X\backslash Q; L))) \ ,$$

where we denote by $\iota_{t,z}$ the Laurent expansion of sections of L in the coordinate z with the help of trivialization t of L. [4] [13] Then we have the following extension of the theorem.

Theorem 2.3. [4]

$$\tau(a; (X, Q, z, L, t)) = \exp(q(a))\vartheta(I(a)+\Delta+c(L)) ,$$

where $c(L)$ *denotes the point of the Jacobian corresponding to the line bundle* $L\otimes \mathcal{O}_X(-\deg(L)Q)$.

Our final remark is that the factor $\exp(q(a))$ in the theorem comes in the process of "renormalization", a method for dealing with determinants of matrices of infinite size. An explanation of this method is given in [4] in terms of representation theory of current algebras. In section 4 we give another explanation, defining a determinant of infinite matrix by using LU-decomposition.

3. Recollection of super geometry

In this section we record and give sketchy proofs for some results of super geometry. Most of them are well known to specialists, but they do not seems to be easy to access for non specialists. Many results go parallel to the usual (non super) case, and in that case we omit proofs. We refer [8] for the fundamental language of super geometry. Results of usual algebraic geometry are written in several texts, for example in [2].

3.1. Finiteness theorem. In this subsection we show existence of enough functions on families of super Riemann surfaces. (Proposition 3.9). To do this, we briefly recall super algebraic geometry.

A super commutative ring A is by definition a $\mathbf{Z}/2\mathbf{Z}$-graded ring

$$A = A_0 \oplus A_1$$

with the following commutation relation.

$$fg = (-1)^{\tilde{f}\tilde{g}} gf \qquad \text{for all} \quad f \in A_{\tilde{f}},\ g \in A_{\tilde{g}} . \tag{3.1}$$

As in here, for any homogeneous element f of A, we denote its parity by $\tilde{f}$: $f \in A_{\tilde{f}}$. Elements of A_0 are said to be even. Elements of A_1 is said to be odd.

We define a super commutative ringed space as a ringed space $(X, \mathcal{O}_X)$ with a $\mathbf{Z}/2\mathbf{Z}$-graded structure sheaf

$$\mathcal{O}_X = \mathcal{O}_{X,0} \otimes \mathcal{O}_{X,1} ,$$

with the analoguos commutation relation as 3.1 above. Each super commutative ringed spaces has its reduction, denoted by X_{rd}, defined as

$$X_{\mathrm{rd}} = (|X|, \mathcal{O}_X/\mathcal{J}_X) , \qquad \text{where} \quad \mathcal{J}_X = \mathcal{O}_{X,1} + \mathcal{O}_{X,1}^2 .$$

We define a locally super commutative ringed space to be a super commutative ringed space whose reduction is a locally ringed space in the usual sence.

Suppose we are given a super commutatuve ring A. Its even part A_0 forms a subalgebra of A, and is cotained in the center of A. A, as an

A_0-algebra, corresponds to a sheaf of super algebras $\widetilde{A}$ on the usual affine scheme SpecA_0. We thus obtain a locally super commutative ringed space

$$(\mathrm{Spec}A_0, \widetilde{A}) \ ,$$

which we will denote by Spec$A$. A locally super commutative ringed space which is isomorphic to Spec$A$ for some super algebra A is called an affine super scheme. A locally super commutative ringed space which is locally isomorphic to a an affine super scheme is called a super scheme. We explain here some basic topics about super schemes.

The first is a criterion of affineness for super schemes. It is an analog of the well known criterion of affineness for schemes due to Serre.

Lemma 3.1. *Let X be a super scheme. If $\mathcal{O}_{X,1}$ is a finitely generated module over $\mathcal{O}_{X,0}$, then the following statements are equivalent.*

(1) *X is an affine super scheme.*
(2) *X_{rd} is an affine scheme.*
(3) *$H^i(X_{\mathrm{rd}}; \mathcal{F})=0$ for all $i>0$ and for all $\mathcal{O}_{X_{\mathrm{rd}}}$-module $\mathcal{F}$.*
(4) *$H^i(X; \mathcal{F})=0$ for all $i>0$ and for all $\mathcal{O}_X$-module $\mathcal{F}$.*

Proof. (1) clearly implies (2). The equivalence of (2) and (3) is the result of the usual Serre's criterion. To derive (4) from (3), we introduce the following filtration on $\mathcal{F}$,

$$\mathcal{F}_n=\mathcal{J}_X^n\mathcal{F} \ ,$$

and note that $\mathrm{Gr}_n(\mathcal{F})=\mathcal{F}_n/\mathcal{F}_{n+1}$ is an $\mathcal{O}_{X_{\mathrm{rd}}}$-module. Finally, the same proof as in the proof on the usual Serre's criterion works, and we see that (4) implies (1).

As in usual non super case, we may consider a super projective space $\mathbf{P}^{(m|n)}$ ([8]). It represents the sheaf associated to the following presheaf of sets on the Zariski site of affine schemes.

$$\begin{aligned}\mathrm{Spec}B \mapsto F(B) &= (1|0)\text{-dimensional direct summand of} \bigoplus^{m+1}B\bigoplus^{n}\Pi B \\ &= \left\{\begin{matrix}(b_0, \dots, b_m, \beta_1, \dots, \beta_n);\ b_i\in B_0,\ \beta_i\in B_1, \\ \text{the ideal of } B \text{ generated by } b_0, \dots, b_m \text{ is } B \text{ itself}\end{matrix}\right\}/B_0^\times\end{aligned}$$

Let $(X_0, X_1, \dots, X_m, \Xi_1, \Xi_2, \dots, \Xi_n)$ be the homogeneous coordinate of $\mathbf{P}^{(m|n)}$. We denote by $H_0, \dots, H_m$ the divisors (hyperplanes) defined by $X_0, \dots, X_m$, respectively. We have the following result for the cohomology groups of the projective space.

Lemma 3.2. *For any super commutative algebra A, we have the following result.*

(1) *$H^k(\mathbf{P}_A^{(m|n)}; \mathcal{O}(pH_0))\neq 0$ only if $k=0$ or $k=m$.*
(2) *For $k=0$, we have,*

$H^0(\mathbf{P}^{(m|n)}; \mathcal{O}(pH_0)) \cong$ (*elements of* $A[X_0, \dots, X_m, \Xi_1, \dots, \Xi_n]$ of degree p).

(3) *For* $k=m$, *we have,*

$$H^m(\mathbf{P}_A^{(m|n)}; \mathcal{O}(pH_0))$$
$$\cong (\textit{elements of } A[X_0^{-1}, \dots, X_m^{-1}, \Xi_1, \dots, \Xi_n]\frac{1}{X_0 \dots X_m} \textit{ of degree } p).$$

The proof of the above lemma is the same as the usual case, by using the Čech cohomology. We note that the Lemma 3.1 guarantees us that cohomology groups may be calculated as Čech cohomology groups with respect to an affine open covering.

Lemma 3.3. *For any coherent sheaf* $\mathcal{F}$ *on* $\mathbf{P}^{(m|n)}$, *there exists an integer* M *such that* $\mathcal{F}(MH_0) = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}(MH_0)$ *is generated by its global sections.*

We call a super scheme X projective if it can be embedded in a projective space $\mathbf{P}^{(m|n)}$ for some m, n. We may define the notion of an ample line bundle on a projective super scheme in the obvious way. We have the following analog of Serre's vanishing theorem. The proof is again parallel to the usual case.

Lemma 3.4. (1) *For any coherent sheaf* $\mathcal{F}$ *on a super projective scheme* X *over a super commutative ring* A, *and for any ample invertible sheaf* $\mathcal{O}(1)$ *on* X, *there exists an integer* M *such that,*

$$H^i(X; \mathcal{F}(p)) = 0 \quad \textit{for all } i>0 \textit{ and for all } p>M \ .$$

(2) *Assume furthermore that the coefficient ring* A *is noetherian. Then cohomology groups* $H^i(X; \mathcal{F}(q))$ *are finitely generated over* A *for all* i *and for all* q.

Proposition 3.5. (*Super analogue of GAGA* [11]) *For any projective super scheme* X *over* $\mathbf{C}$, *denote by* X_{an} *its associated super analytic space. Then there exists a one to one correspondence between the set of coherent sheaves on* X *and the set of coherent sheaves on* X_{an}. *For any coherent sheaf* $\mathcal{F}$ *on* X, *we denote by* $\mathcal{F}_{\mathrm{an}}$ *the corresponding coherent sheaf on* X_{an}. *Then the cohomology groups of these two sheaves are canonically isomorphic.*

$$H^i(X; \mathcal{F}) \cong H^i(X_{\mathrm{an}}; \mathcal{F}_{\mathrm{an}}) \ .$$

Proof. Almost identical with the usual case.

We introduce a notion of "a family of Virasoro uniformized super Riemann surfaces". It is this object we will discuss extensively through this paper.

Definition 3.1. By "a family of Virsoro uniformized (compact) super Riemann surfaces (with the odd dimension N) $(X, \pi, (S, s_0), q, (z, \zeta_1, \dots, \zeta_N))$", we mean the following data.

(1) A proper smooth super scheme X of relative dimension $(1|N)$ over a smooth super scheme S (over Spec$\mathbf{C}$) with the structure morphism π.

(2) A section $q: S \to X$ of the structure morphism π. We will denote by Q the image of q.

(3) A local coordinate system $(z, \zeta_1, \ldots, \zeta_N)$ along Q relative to S, with the following condition.

$$z(Q)=\zeta_1(Q)=\zeta_2(Q)=\ldots=\zeta_N(Q)=0 .$$

(4) A distinguished closed point ($=\mathbf{C}$-valued point) s_0 of the base scheme S.

Furthermore, in the above situation, we employ the following notation.

$$\dot{X}=X \backslash Q .$$

Our arguments focus on properties which are local on S. So we sometimes shrink S, that is, replace S by an open neighbourhood U of s_0 in S.

Remark 3.1. Unlike the usual case, 'points' and 'divisors' of a super Riemann surface are not the same. But if D and E are divisors with the same reduction of their wupports, we see that there exists an integer M such that

$$MD > E \quad \text{and} \quad ME > D .$$

This is because odd dimensions give merely nilpotent functions.

Our first observation is the following.

Lemma 3.6. *X is projective over S.*

Proof. (We prove the lemma without assuming the existence of the section q of π and the coordinate system) We may shrink S and may assume that S is affine, and that there exist two disjoint sections p and q of π. Let us denote by P and Q their image. We shrink S again if necessary and find local coordinate systems $(z_P, \zeta_{1,P}, \ldots, \zeta_{NP})$ and $(z_Q, \zeta_{1,Q}, \ldots, \zeta_{NQ})$ around P and Q, respectively. Let D_P and D_Q be divisors on X defined by $z_Q=0$ and $z_P=0$ respectively.

By Lemma 3.1 we see that $X \backslash P$ and $X \backslash Q$ are both affine. So we have a positive integer M and finite collection of homogeneous sections $\{f_i\}$ of $\mathcal{O}(MD_P)$ and $\{g_i\}$ of $\mathcal{O}(MD_Q)$, such that they give an embedding of $X \backslash P$, $X \backslash Q$ to projective spaces. Since $X \backslash Q$ is affine, we have, for sufficiently large integer M', a section s of $\mathcal{O}(M'D_Q)$ with non zero reduction which vanishes at D_P. This gives an injection of sheaves,

$$\times s: \mathcal{O}(D_P) \to \mathcal{O}(M'D_Q) .$$

This implies that $\mathcal{O}(MM'(D_Q))$ has enough sections to embed X to a super projective space.

We next mention a powerful tool for the analysis of (super) Riemann surface, namely, the formal Čech cohomology. [1] [4]

Lemma 3.7. *Let $(\pi: X \rightarrow S,\ q,\ (z, \zeta_1, \ldots, \zeta_N))$ be a family of Virasoro uniformized super Riemann surfaces. Assume S is affine, $S = \mathrm{Spec} B$. Let $\mathcal{F}$ be a coherent sheaf on X which is locally free (of rank $(r|\rho)$) on a neighbourhood of Q. Then we have the following.*

$$H^0(X; \mathcal{F}) \cong \mathcal{F}(U) \cap \mathcal{F}(\dot{X}) \ , \tag{3.2}$$

$$H^1(X; \mathcal{F}) \cong \frac{\mathcal{F}(\dot{U})}{\mathcal{F}(U) + \mathcal{F}(\dot{X})} \ , \tag{3.3}$$

where we denote by $\mathcal{F}(U)$ the set of global sections of the completion $\widehat{\mathcal{F}}_Q$ of $\mathcal{F}$ along Q and $\mathcal{F}(\dot{U})$ the localization of $\mathcal{F}(U)$ at Q. If we choose a formal trivialization of Faround Q, then we may identify them as follows.

$$\mathcal{F}(U) \cong B[[z]][\zeta_1, \ldots, \zeta_N]^{(r|\rho)} \ , \tag{3.4}$$

$$\mathcal{F}(\dot{U}) \cong B((z))[\zeta_1, \ldots, \zeta_N]^{(r|\rho)} \tag{3.5}$$

Proof. Let D be a divisor on X defined by $z=0$. The lemma is an easy consequence of the cohomology exact sequence associated to the following short exact sequence of sheaves.

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{F}(+nD) \rightarrow \frac{\widehat{\mathcal{F}}_Q(+nD)}{\widehat{\mathcal{F}}_D} \rightarrow 0 \ .$$

Lemma 3.8. (*Special case of upper semi continuity theorem of cohomology.*)

Let $(\pi: X \rightarrow S,\ q,\ (z, \zeta_1, \ldots, \zeta_N))$ be a family of Virasoro uniformized super Riemann surfaces Assume S is affine, $S = \mathrm{Spec} B$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module flat over S. If $H^1(X; \mathcal{F}) = 0$, then $H^0(X; \mathcal{F})$ is a flat B-module.

Proof. We may assume that π admits two disjoint sections p_1, p_2. Then X is a union of two affine open sets, $U_i = X \backslash p_i(S)$, $i = 1, 2$. The cohomology of $\mathcal{F}$ is the cohomology of the following complex.

$$0 \rightarrow \mathcal{F}(U_1) \oplus \mathcal{F}(U_2) \rightarrow \mathcal{F}(U_1 \cap U_2) \rightarrow 0 \ .$$

Since U_1, U_2, $U_1 \cap U_2$ are all affine, $\mathcal{F}(U_1)$, $\mathcal{F}(U_2)$, $\mathcal{F}(U_1 \cap U_2)$ are flat B-modules. By the hypothesis on H^1, we see that $H^0(X; F)$ is a kernel of a surjection of flat modules, so it is flat.

Proposition 3.9. *Let $\mathcal{F}$ be a locally free super $\mathcal{O}_X$-module of rank $(r|\rho)$. Suppose $\mathcal{F}$ is formally trivialized around Q. That is, we are given an isomorphism*

$$\widehat{\mathcal{F}}_Q \cong \mathcal{O}_S((z))[\zeta_1, \ldots, \zeta_N]^{(r|\rho)} \ .$$

We denote the standard $\mathcal{O}_S((z))[\zeta_1, \ldots, \zeta_N]$-basis of the right hand side of the above formula as $\{e, \epsilon\}$, where $e = \{e_1, e_2, \ldots, e_r\}$ (even), and $\epsilon = \{\epsilon_1, \ldots, \epsilon_\rho\}$ (odd). Then, shrinking S if neccessary, there exists an integer M such that we may find *sets of sections*

$$\{f_{n,I,i};\ n\leq -M,\ I\subset [1, N],\ i=1, 2, \ldots, r\}\ ,$$

$$\{g_{n,I,j};\ n\leq -M,\ I\subset [1, N],\ j=1, 2, \ldots, \rho\}\ ,$$

of $\mathscr{F}$ *on* $\dot{X}$ *with the following Laurent expansion at* Q *(with respect to the formal trivialization).*

(1) $f_{n,I,i}(z)\equiv z^{-n}\zeta^{I}e_i \mathrm{mod}\,(\mathscr{O}_S(S)[[z]][\zeta_1, \ldots, \zeta_N]z^{-M})^{(r|\rho)}$,

(2) $g_{n,I,j}(z)\equiv z^{-n}\zeta^{I}\epsilon_j \mathrm{mod}\,(\mathscr{O}_S(S)[[z]][\zeta_1, \ldots, \zeta_N]z^{-M})^{(r|\rho)}$.

Proof. We combine Lemmas 3.4, 3.7 and 3.8.

Shrinking S if necessary, we may use the above proposition to obtain a $\mathscr{O}_S(S)$ -basis of $\mathscr{O}_X(\dot{X})$ (see Lemma 5.1), or, in other words, $\mathscr{O}_X(\dot{X})$ is a free $\mathscr{O}_S(S)$ -module (of infinite rank). This fact and Lemma 3.7 yields the following.

Lemma 3.10. *Let* $(\pi\colon X\to S,\ q,\ (z, \zeta_1, \ldots, \zeta_N))$ *be a family of Virasoro uniformized super Riemann surfaces. Assume* S *is affine,* $S=\mathrm{Spec}B$. *Let* $\mathscr{F}$ *be a locally free* $\mathscr{O}_X$*-module of finite rank. If* $H^0(X_s; \mathscr{F}_s)=0$ *for any closed point* s *of* S, *Then* $R^1\pi_*(\mathscr{F})$ *is a locally free sheaf on* S.

3.2. Relative de Rham complex and the residue theorem. Let us describe the most powerful result in this section, the super residue theorem. It plays a fundamental role for studying super Riemann surfaces. First we describe the relative de Rham complex of super manifolds. The absolute case is described in [8]). It is by definition a locally super commutative ringed spase $(M, \mathscr{O}_M)$ which is locally isomorphic to

$$(B_m(r),\ \mathscr{O}_{C^\infty}\otimes S^{\cdot}(\mathbf{C}^n\Pi))\ ,$$

where $B_m(r)$ is the ball of radius r in $\mathbf{R}^m$ and Π is the parity change [8]. We see from the definition that locally, the function space on a super manifold is isomorphic to

$$\mathrm{Func.}(B_m(r))\otimes S^{\cdot}(\mathbf{C}^n\Pi))\ ,$$

where Func. denotes some function space (depending on what kind of geometry we are going to focus on, for example C^∞, continuous, ...), With this identification we may equip the function space of a super manifolds several topologies. Among them we use C^∞-topology, C^1-topology, and uniform topology.

Let $\varpi\colon M\to B$ be a submersion between differentiable super manifolds. We denote by (m, μ) and (n, ν) the dimension of M and B, respectively. The relative tangent and cotangent sheaves are defined in an obvious way. (Actually these sheaves are locally free, so we may as well regard them as bundles. But in the sequel, we prefer the word sheaves rather than bundles.) We have,

$$\Omega^1_{M/B}=\Omega^1_M/\varpi^*\Omega^1_B\ , \tag{3.6}$$

(3.7) $\mathcal{T}_{M/B} = \ker\{\varpi^*: \mathcal{T}_M \to \varpi^*: \mathcal{T}_B\}$.

We state a few words about the parity of the cotangent sheaf. The even cotangent sheaf has the parity determined by

$$\Omega^1_{M/B,\mathrm{even}} = \mathrm{Hom}\,(\mathcal{T}_{M/B}, \mathcal{O}) ,$$

which is not a usual parity if it is regarded as the bundle of 1-forms. The cotangent sheaf with the odd (but usual) parity is denoted by $\Omega^1_{M/B,\mathrm{odd}}$. Namely,

$$\Omega^1_{M/B,\mathrm{odd}} = \Omega^1_{M/B,\mathrm{even}}\Pi .$$

We define the relative Berezinian line as the Berezinian of the cotangent sheaf.

$$\mathrm{Ber}_{M/B} = \mathrm{Ber}\,(\Omega^1_{M/B,\mathrm{even}}) .$$

We recall that each vector bundle on M has s Berezinian line, the transition function being the Berezinian of the original boundle. Each relative coordinate system $(x_1, \ldots, x_m, \xi_1, \ldots, \xi_n)$ on M over B determines a section Ber $(dx_1, \ldots, dx_m, d\xi_1, \ldots, d\xi_n)$. The parity of the Berezinian bundle differs in literature, but is not important for our argument. So we simply write $\widehat{\mathrm{Ber}}$ to denote the parity. We choose the following ([8]).
$\widehat{\mathrm{Ber}}\,(dx_1, \ldots, dx_{m-n}, d\xi_1, \ldots, d\xi_{\mu-\nu}) = m - n =$ even relative dimension of M over B. In the pure even case, the Berezinian line reduces to the bundle of forms of the top degree. We see from the relation 3.6 that the relative and absolute Berezinian lines are related in the following way.

(3.8) $\mathrm{Ber}_{M/B} \cong \mathrm{Ber}_M \otimes (\varpi^*\mathrm{Ber}_B)^\vee$.

As in the absolute case, each section of the relative Berezinian line (with proper support) can be integrated along the fibers. We may define this by the following identity.

(3.9) $\displaystyle\int_M (\alpha \otimes \varpi_* \eta) = \int_B \left(\int_{M/B} \alpha\right)\eta$ for all $\alpha \in C^\infty_{\mathrm{cpt}}(\mathrm{Ber}_{M/B}), \eta \in C^\infty_{\mathrm{cpt}}(\mathrm{Ber}_B)$.

In fact, the uniqueness of the $\int_{M/B}$ with the above property is clear. For any local coordinate system $(b_1, \ldots, b_n, \beta_1, \ldots, \beta_\nu)$ of B and $(x_1, \ldots, x_{m-n}, b_1, \ldots, b_n, \xi_1, \ldots, \xi_{\mu-\nu}, \beta_1, \ldots, \beta_\nu)$ of M, we have,

$$\int_{M/B} \mathrm{Ber}\,(dx, d\xi) f(x, \xi) = \int dx \ \text{(the coefficient of } \xi_1 \ldots \xi_{\mu-\nu} \text{ in } f(x, \xi)) ,$$

for any C^∞-function f on M which has a compact support contained in the coordinate patch. This provide the local existence of the integral and this, together with local uniqueness, provides the proof of the existence of the integral in general. We next introduce the sheaf of relative integral forms. It is defined as follows.

$$\Sigma_{M/B}=\mathrm{Ber}_{M/B}\otimes(\mathrm{S}^{\cdot}(T_{M/B}\Pi))$$

It is graded in a natural way. We define a derivation $\delta^{M/B}$ on this sheaf to make it a differential complex. To do this, we note that the isomorphism 3.8 gives an inclusion,

$$\Sigma_{M/B}\otimes\varpi^{*}\mathrm{Ber}_{B}\subset\Sigma_{M} .$$

Then the relative exterior derivative $\delta^{M/B}$ is defined as the unique differential satisfying the following relation.

$$\delta^{M/B}(\alpha)\otimes\varpi^{*}\eta=\delta^{M}(\alpha\otimes\varpi^{*}\eta) \quad \text{for all } \alpha\in\Sigma_{M/B},\ \eta\in\mathrm{Ber}_{B} . \tag{3.10}$$

The same reasoning as in the case of integral applies and we see that the relative exterior derivative exists. In local terms, it is expressed as follows.

$$\begin{aligned}
&\delta^{M/B}(\mathrm{Ber}(dx, d\xi)f(x, \xi)\otimes F)\\
&=\sum_{i}(\nabla_{r,\frac{\partial}{\partial x_i}})\otimes_{\mathcal{O}_B}\frac{\partial}{\partial\frac{\partial}{\partial x_i}\Pi}(\mathrm{Ber}(dx, d\xi)f(x, \xi)\otimes F)\\
&-\sum_{j}(\nabla_{r,\frac{\partial}{\partial \xi_j}})\otimes_{\mathcal{O}_B}\frac{\partial}{\partial\frac{\partial}{\partial \xi_i}\Pi}(\mathrm{Ber}(dx, d\xi)f(x, \xi)\otimes F) ,\\
&=(-1)^{(\widehat{\mathrm{Ber}}+\hat{f})+1}\sum_{i}\mathrm{Ber}(dx, d\xi)\frac{\partial f}{\partial x_i}\otimes_{\mathcal{O}_B}\frac{\partial F}{\partial\frac{\partial}{\partial x_i}\Pi} ,\\
&+(-1)^{(\widehat{\mathrm{Ber}})}\sum_{j}\mathrm{Ber}(dx, d\xi)\frac{\partial f}{\partial \xi_i}\otimes_{\mathcal{O}_B}\frac{\partial F}{\partial\frac{\partial}{\partial \xi_i}\Pi}
\end{aligned}$$

Where we put

f: a C^{∞} function on X

$f\in S^{\cdot}(T_{M/B}\Pi)$

One reason to introduce the super de Rham comlex is that we may integrate in parts. Namely, Following lemma holds.

Lemma 3.11.

$$\int_{M/B}\delta^{M/B}\alpha=0 \qquad \text{for all} \quad \alpha\in\Gamma_{\mathrm{cpt}}(\Sigma_{M/B}) .$$

Proof. This follows directly from the absolute case, using the equations 3.9 and 3.10.

We define the affine super line $\mathbf{A}^{(0|N)}$ of dimension $(0|N)$ to be the unique connected super manifold of dimension $(0|N)$. The ring of functions on $\mathbf{A}^{(0|N)}$ as $\tau_1, \ldots, \tau_N$.

Let O be an open subset of $\mathbf{C}$ which contains the origin. Let B be a complex super manifold.

Definition 3.2. Embedded smooth super paths on $O\times B$ is defined as a C^{∞}-map

$$\gamma\colon [0,1]\times \mathbf{A}^{(0|N)}\times B\to O\times \mathbf{A}^{(0|N)}\times B$$

which commutes with projections on B and is an embedding (in the sense of [8].)

Lemma 3.12. *Let*

$$\omega=\mathrm{Ber}\,(dz, d\zeta_1, \dots, d\zeta_N)f(z, \zeta_1, \dots, \zeta_N)$$

be a C^{∞}-relative integral form on O, whose support is contained in the image of $(0,1)\times \mathbf{A}^{(0|N)}\times B$. Then we have

$$\int_{\gamma}\omega=\int_I dt\left[\frac{dz}{dt}\frac{\partial\ldots\partial}{\partial\zeta_N\ldots\partial\zeta_1}f(z, \zeta_1, \dots, \zeta_N)\right]_{\tau_1=\dots=\tau_N=0}$$

Proof. We give here a computational proof of the lemma. We first assume that $\zeta_1, \dots, \zeta_N$ do not have constant terms if they are expanded in terms of $\tau_1, \dots, \tau_N$). In other words, we assume that

$$(3.11)\qquad (\zeta_1|_{\tau_1=0,\dots,\tau_N=0})=\dots=(\zeta_N|_{\tau_1=0,\dots,\tau_N=0})=0\ .$$

We expand f in terms of the odd coordinate $\zeta_1, \dots, \zeta_N$.

$$f=\sum_{I\subset[1,N]}f_I(z)\zeta_I$$

We know as in the absoulute case ([8]) that the integral form Ber $(dz, d\zeta)$ $f_I\zeta_I$ is δ-exact unless $I=[1,N]$. Let us denote the function $f_{[1,N]}$ simply as $\check{f}$. Then we have,

$$\int\mathrm{Ber}\,(dt, d\zeta)f(z,\zeta)=\int\mathrm{Ber}\,(dt, d\tau)\left[\mathrm{Ber}\left(\frac{dz,\, d\zeta}{dt,\, d\tau}\right)\check{f}(z)\,\zeta_1, \dots, \zeta_N\right]\ .$$

But the assumption 3.11, we see that $\zeta_1\ldots\zeta_N$ is already a multiple of $\tau_1, \dots, \tau_N$, so we see that the right hand side of the above formula equals to

$\int\mathrm{Ber}\ (dt,\ d\tau)\ \left[\mathrm{Ber}\ \left(\frac{dz,\, d\zeta}{dt,\, d\tau}\right)\check{f}\ (z)\right]_{\tau_1,\dots,\tau_N=0}\times$ (the coefficient of $(\tau_1\ldots\tau_N)$ in $(\zeta_1\ldots\zeta_N)$) .

$$=\int dt\check{f}(z(t), 0, \dots, 0)\left[\frac{\det\left(\frac{dz}{dt}-\frac{d\zeta}{dt}\left(\frac{d\zeta}{d\tau}\right)^{-1}\frac{dz}{d\tau}\right)}{\det\left(\frac{d\zeta}{d\tau}\right)}\right]_{\tau_1=\dots=\tau_N=0}\times\det\left[\frac{d\zeta}{d\tau}\right]_{\tau_1=\dots=\tau_N=0}$$

$$=\int dt\check{f}(z(t, 0, \dots, 0)\left[\frac{dz}{dt}\right]_{\tau_1=\dots=\tau_N=0}\quad(\text{since }\frac{d\zeta}{dt}|_{\tau_1=\dots=\tau_N=0}=0.)$$

This completes the proof of the lemma under the assumption 3.11.

Let us now consider the general case. We first notice that, using a partition of unity on $\mathbf{C}$, we may divide γ into pieces and assume γ sufficiently small. We also note that the set of paths which satisfy the lemma is a closed set in the C^1-topology. We may thus prove the lemma for paths of good nature which form a C^1-dense subset of the whole space of paths. In fact, replacing B by its relatively compact open submanifold, we deduce from the Weierstrass theorem that γ may be arbitrarily well approximated by paths which are analytic along each fibers. So we restrict ourselves to cases where γ is analytic along each fibers. In this case, γ extends to a map

$$\hat{\gamma}: U\times\mathbf{A}^{(0|N)}\times B\rightarrow O\times\mathbf{A}^{(0|N)}\times B\ ,$$

which is complex analytic along the fibers, where U is an open neighbourhood of $[0, 1]$ in $\mathbf{C}$. Shrinking O and U if necessary, we may assume that $\hat{\gamma}$ is an isomorphism of C^∞-super manifolds. Let us divide $\hat{\gamma}$ in components.

$$\hat{\gamma}(t, b, \beta)=\hat{\gamma}_0(t, \tau, b, \beta), \hat{\gamma}_1(t, \tau, b, \beta)), (b, \beta))$$

We see immediately that

$$\tilde{\gamma}(t, b, \beta)=\hat{\gamma}_0(t, 0, b, \beta): U\times B\rightarrow O\times B$$

is a diffeomorphism. From this we conclude the existence of a C^∞-map

$$\phi: U\times B\rightarrow\mathbf{A}^{(0|N)}\times B$$

which is defined on a neighbourhood $U\times B$ of the image of γ, commutes with the projection on B, and is complex anayltic on each fibers, such that,

$$\zeta(t, 0, b, \beta)=\phi(z(t, 0, b, \beta), (b, \beta))\ \text{for all}\ t\in[0, 1]\ \text{and for all}\ (b, \beta)\in B\ .$$

In fact, we define ϕ as follows.

$$\phi(z, b, \beta)=\gamma_1((\tilde{\gamma})^{-1}(z, b, \beta), 0, b, \beta)$$

Then we may change the coordinate of the target space $\mathbf{C}\times\mathbf{A}^{(0|N)}\times B$ of γ by

$$\bar{z}=z,\ \bar{\zeta}=\zeta-\phi(z)$$

and get,

$$\int_\gamma \mathrm{Ber}(dz, d\zeta)f(z, \zeta)$$

$$=\int_\gamma \mathrm{Ber}(dz, d\zeta)\check{f}(z)\,\zeta_1\ldots\zeta_N$$

$$=\int_{\bar{\gamma}} \mathrm{Ber}(d\bar{z}, d\zeta)\check{f}(\bar{z})\,(\bar{\zeta}_1+\phi_1(\bar{z}))\ldots(\bar{\zeta}_N+\phi_N(\bar{z}))\quad\text{(coordinate change)}$$

$$=\int_{\bar{\gamma}} \mathrm{Ber}(d\bar{z}, d\bar{\zeta})\check{f}(\bar{z})\,(\bar{\zeta}_1)\ldots(\bar{\zeta}_N)\quad\text{(other terms are 0)}$$

$$=\int \frac{d\bar{z}}{dt}(t, 0, \ldots, 0)\check{f}(\overline{z}(t, 0, \ldots, 0))dt \qquad \text{by the assumption 3.11 case}$$

$$=\int \frac{dz}{dt}(t, 0, \ldots, 0)\check{f}(\overline{z}(t, 0, \ldots, 0))dt$$

This completes the proof.

Proposition 3.13. (*Super residue theorem*) *Assume furthermore in the above lemma that ω is holomorphic and that the path is closed, that is, we have*

$$\gamma|_{\{0\}\times \mathbf{A}^{(0|N)}} \times S = \gamma|_{\{1\}\times \mathbf{A}^{(0|N)}} \times S \ ,$$

and winds exactly once around the origin 0 on each fiber. Then we have

$$\int_\gamma \mathrm{Ber}\,(dz, d\zeta)f(z, \zeta) = \mathrm{ress}_0(\mathrm{Ber}\,(dz, d\zeta)f) := (\textit{the coefficient of } z^{-1}\zeta_1 \ldots \zeta_N \textit{ in } f).$$

The formula shows in particular that super residue ress is invariantly defined.

It is very likely that the algebro-geomitric counterpart holds (for example in positive characteristics), but the author does not know a proof.

For any super scheme S which is smooth over $\mathbf{C}$, we may associate a C^∞-super manifold S_{C^∞} as in the usual way. That is, the topological space $S(\mathbf{C})$ equipped with the sheaf of ring $\mathcal{O}_{C^\infty}$, where $\mathcal{O}_{C^\infty}$ is defined as the completion of $\mathcal{O}_S$ with respect to the C^∞-topology on each compact subset of S. This procedure enables us to apply the above proposition to super schemes, and we obtain,

Corollary 3.14. *For all $f \in \pi_*\mathcal{O}_{\dot{X}}$, $\omega \in \pi_*\omega_{\dot{X}/S}$, we have*

$$\mathrm{ress}_Q \omega f = 0 \ .$$

Furthermore, $\pi_*\mathcal{O}_{\dot{X}}$ and $\pi_*\omega_{\dot{X}/S}$ are the annihilator of each other.

Proof. Same as the usual case.

4. Theory of Infinite determinants

4.1. motivations. In this section we give a way to deal with determinants of matrices of infinite size. The method we employ here uses a LU- and UL-decomposition of matrices. That is, we decompose a matrix to a product of an essentially lower-half matrix and an essentially upper-half matrix. Then the determinant of the original matrix is defined to be the product of the lower- and upper-half matrices. Unfortunately, the determinant defined in this way does not commute with multiplication. It is a feature of determinants of matrices of infinite size. We may as well define determinants of infinite matrices by using a representation theory of the current algebra. We will show in this section that the two definition of determinants actually coincide. But the determinants defined by representation theory does not seem to be convenient for application to super theory. The idea of defining infinite determinants using LU- decomposition already appears in Schwarz [10]. Our approach differs from his in the point that we use more algebraic machinery and formal topology, where his approach uses Hilbert spaces. The author admists that the latter approach is geometrically more interesting. But our approach has an advantage of being easy to handle with and may be extend to the positive characteristic case.

In this section we fix a family of Virasoro uniformized super Riemann surfaces (π: $X \to S$, q, $(z, \zeta_1, \ldots, \zeta_N)$) (definition 3.1). We consider the following algebras.

$$B=\Gamma(\mathcal{O}_S) ,$$
$$A=\Gamma(\dot{X}; \mathcal{O}_X) ,$$
$$\mathcal{B}^{(0)}=B[[\{T_{i,I}\}_{i\in\mathbf{Z},I\subset[1,N]}]] ,$$
$$\mathcal{A}^{(0)}=A[[\{T_{i,I}\}_{i\in\mathbf{Z},I\subset[1,N]}]] ,$$

The double brackets in the definition of the last two algebras mean the completion of polynomial algebras in infinite variables $\{T_{i,I}\}_{i\in\mathbf{Z},I\subset[1,N]}$. The completion is given with respect to the following grading on the polynomial algebras.

$$\deg T_{i,I}=\max(-i, 1)$$

We develop here a linear algebra over the tpoplogical vector space $\mathcal{B}^{(0)}$. There are two important features in this algebra.

(B1) The topology of $\mathcal{B}^{(0)}$ is determined by a conutable set of its topologically nilpotent open ideals J_n.

(B2) It is complete.

In fact, J_n is given by the following.

J_n = closed ideal of $\mathcal{B}^{(0)}$ generated by monomials of degree larger than n. The discussion of this section is based on the above two properties.

4.2. Various function spaces. In this section we define several function spaces. In contrast with arguments in [10] where one uses Hilbert space $(L^2(S^1))$ and linear operators on it, we use formal functions on $\mathrm{Spf}\mathbf{C}((z))$ and linear operators on it. As it was explained in section 2, for the later use we have to add formal variables to the base ring. So we fix, as a ring of coefficients, a topological ring $\mathcal{B}$ which satisfies the properties (B1) and (B2) above (with $\mathcal{B}^{(0)}$ replaced by $\mathcal{B}$). We define the completed ring of formal Laurent power series $\mathcal{B}\{\{z\}\}$ in the following way.

$$\mathcal{B}\{\{z\}\}=\left\{\sum_{i=-\infty}^{\infty} x_i z^i;\ x_i\in\mathcal{B},\ x_i\to 0 \text{ as } i\to-\infty\right\} .$$

The set of formal power series,

$$\mathcal{B}[[z]]=\left\{\sum_{i=0}^{\infty}x_iz^i;\ x_i\in\mathcal{B}\right\}\ .$$

forms a subalgebra of $\mathcal{B}\{\{z\}\}$. Our consideration is centered round properties of $\mathcal{B}\{\{z\}\}/\mathcal{B}[[z]]$ as a topological linear space. So we define an abstract topological vector space as follows.

$$\begin{aligned}\mathcal{U}&=C_0(\mathbf{Z}_{\leq 0};\mathcal{B})\\&=\left\{\sum_{i\leq 0}x_ie_i;\ x_i\in\mathcal{B},\ x_i\to 0\text{ as }i\to\infty\right\}\end{aligned}$$

We equip $\mathcal{U}$ the uniform topology: a net $\{f_\lambda\}$ converges to f if and only if for every open ideal J of $\mathcal{B}$ there exists an index λ_0 such that

$$f_\lambda-f\equiv 0\quad(\text{modulo}\sum_i Je_i)\qquad\text{for all}\quad\lambda>\lambda_0\ .$$

It is an easy exercise to show that $\mathcal{U}$ is complete with respect to this topology.

We denote the ring of continuous endomorphism of $\mathcal{U}$ by $E(\mathcal{U})$;

$$E(\mathcal{U})=\mathrm{Hom}_{\mathrm{cont}}(\mathcal{U},\mathcal{U})$$

We may represent each element M of $E(\mathcal{U})$ by a matrix $((M)_{ij})$:

$$M(\sum x_ie_i)=\sum(M)_{ij}x_je_i$$

(We use paretheses to express the martix elements of the corresponding linear operators.)

Next we define a subalgebra of $E(\mathcal{U})$, the algebra of "compact" operators.

$$\mathcal{K}(\mathcal{U})=\left\{M\in E(\mathcal{U});\ \begin{array}{l}\text{For all open ideal } J \text{ of } \mathcal{B},\ (M \bmod J) \text{ is of finite rank}\\ \text{i.e., there exists a positive integer } N \text{ such that}\\ (M)_{ij}\in J \text{ for all } i<-N \text{ and for all } j\end{array}\right\}$$

Note that $\mathcal{K}(\mathcal{U})$ is an ideal of $E(\mathcal{U})$. Its elements are "limits" of finite rank matrices. It plays a similar role as the set of compact operators on a Hilbert space in the theory of operator algebra. The definition of $\mathcal{K}$ is independent of a choice of a topological base $\{e_i\}$ of $\mathcal{U}$. In other words, it is stable under automorphisms of $\mathcal{U}$. This is an easy consequence of the fact that $\mathcal{K}$ is an ideal. We may use the fact to give the following definition.

Definition 4.1. Let $\mathcal{U}_I$, $\mathcal{U}_{II}$ be two topological vector space over $\mathcal{B}$ which is isomorphic to $\mathcal{U}$. Fix a topological linear isomorphism ϕ between the two vector space. Then we define,

$$\mathcal{K}(\mathcal{U}_I,\mathcal{U}_{II})=\{\phi\circ K;\ K\in\mathcal{K}(\mathcal{U})\}\ .$$

This actually does not depend on the choice of ϕ.

We further define the following subsets of $E(\mathcal{U})$.

$$U_{\langle 0\rangle}(\mathcal{U})=\left\{M\in E(\mathcal{U});\ \begin{array}{l}((M)_{ij})\text{ is strictly upper triangular.}\\ \text{i.e., }(M)_{ij}=0\text{ for }j\geq i\end{array}\right\}\tag{4.1}$$

$$U_{\langle 1\rangle}(\mathcal{U})=1+U_{\langle 0\rangle}(\mathcal{U})=\{1+M;\ M\in U_{\langle 0\rangle}(U)\}\ ,\tag{4.2}$$

$$U_{*}(\mathcal{U})=\left\{M=(m_{ij})\in E(\mathcal{U});\ \begin{array}{l}((M)_{ij})\text{ is upper triangular.}\\ \text{i.e., }(M)_{ij}=0\text{ for }j>i\end{array}\right\}.\tag{4.3}$$

Here we add some remarks. Each element of $U_{\langle 1\rangle}(\mathcal{U})$ is invertible in $E(\mathcal{U})$ and the inverse is again an element of $U_{\langle 1\rangle}(\mathcal{U})$. An element of $U_{*}(\mathcal{U})$ is invertible if and only if its diagonal entries are all invertible elements in $\mathcal{B}$, and in that case the inverse is again an element of $U_{*}(\mathcal{U})$. The condition of a matrix being continuous is automatic for upper triangular one. In contrast, the following definition of an algebra of lower triangular matrices requires some conditions concerning topology.

(4.4)
$$L_{\langle 0\rangle}(\mathcal{U})=\left\{M\in E(\mathcal{U});\ \begin{array}{l}((M)_{ij})\text{ is strictly lower triangular and}\\ \text{there exists a topologically nilpotent ideal }J\text{ of }\mathcal{B}\\ \text{such that }(M)_{ij}\in J\text{ for all }i\text{ and }j\end{array}\right\},$$

$$L_{\langle 1\rangle}(\mathcal{U})=1+L_{\langle 0\rangle}(\mathcal{U})=\{1+M;\ M\in L_{\langle 0\rangle}(U)\}\ ,\tag{4.5}$$

(4.6)
$$L_{*}(\mathcal{U})=\left\{M\in E(\mathcal{U});\ \begin{array}{l}((M)_{ij})\text{ is lower triangular and}\\ \text{there exists a topologically nilpotent ideal }J\text{ of }\mathcal{B}\\ \text{such that }(M)_{ij}\in J\text{ for all }i\neq j\end{array}\right\}.$$

Note each element of $L_{\langle 1\rangle}(\mathcal{U})$ has an inverse in itself.

As in the operator algebra theory, it is convenient to consider elements which are in the above sets "modulo compact operators". For every subset S of $E(\mathcal{U})$, we define,

$$\widetilde{S}=S+\mathcal{K}(\mathcal{U})=\{M+K;\ M\in S,\ K\in\mathcal{K}(\mathcal{U})\}\tag{4.7}$$

The following lemma is an easy consequence of the fact that each element of $L_{\langle 1\rangle}(\mathcal{U})$ and $U_{\langle 1\rangle}(\mathcal{U})$ is invertible.

Lemma 4.1.

$$\widetilde{L_{\langle 1\rangle}(\mathcal{U})}\cap\widetilde{U_{\langle 1\rangle}(\mathcal{U})}=1+\mathcal{K}(\mathcal{U})$$

4.3. Denfinition of "finite" determinants. In this subsection we define determinants for matrices of elements in $1+\mathcal{K}(\mathcal{U})$.

For each positive integer n, we define $\mathcal{U}_n=\sum_{i>-n}\mathcal{B}e_i$. Furthemore, we define the injection ι_n: $\mathcal{U}_n\rightarrow\mathcal{U}$ and a projection p_n: $\mathcal{U}\rightarrow\mathcal{U}_n$ as follows.

$$\iota_n\Big(\sum_{i>-n} x_i e_i\Big)=\sum_{i>-n} x_i e_i$$

$$p_n\Big(\sum_{i} x_i e_i\Big)=\sum_{i>-n} x_i e_i \ .$$

It corresponds to the following decomposition of $\mathcal{U}$.

$$\mathcal{U}=\mathrm{Image}\,\iota_n \oplus \ker p_n=\mathcal{U}_n \oplus \sum_{i\leq -n} x_i e_i \ . \tag{4.8}$$

Note that $\mathcal{U}_n$ is a free $\mathscr{B}$-module of finite rank. We have the following lemma, which is a consequence of the definition of $\mathscr{K}(\mathcal{U})$.

Lemma 4.2. *Let $K\in\mathscr{K}(\mathcal{U})$. Then* $\det(p_n\circ(1+\mathscr{K})\circ\iota_n)$ *converges as $n\to\infty$. It is in fact independent of the choice of the topological base $\{e_i\}$ of $\mathcal{U}$. We denote it by* $\det_f(1+K)$.

It is easy to see that $\det_f$ commutes with multiplication. Moreover, we have the following

Lemma 4.3. *Let $K\in\mathscr{K}(\mathcal{U})$. Then for any $E\in(\mathcal{U})$, we have,*

$$\det{}_f(EK)=\det{}_f(KE) \ .$$

Proof. If E is invertible, then replacing K in the above formula by $E^{-1}K$ we see that it is equivalent to

$$\det{}_f(K)=\det{}_f(E^{-1}KE) \ .$$

This holds because the definition of $\det_f$ is independent of a choice of a base. In case E is not invertible, the proof of the lemma is obtained by taking a limit of the invertible case.

4.4. LU- and UL-decomposition of matrices. In this subsection we define determinant for a wider class of matrices. We first decompose a matrix into a product of an upper half matrix and a lower half matrix. Then we define the determinant of the original matrix by the product of determinant of the two components. Our first task is to define a class of matrices which are decomposable into product of upper- and lower half matrices.

$$M(\mathcal{U})=U_{\langle 1\rangle}(\mathcal{U})\ .\ \ L_{\langle 1\rangle}(\mathcal{U})$$

$$N(\mathcal{U})=L_{\langle 1\rangle}(\mathcal{U})\ .\ \ U_{\langle 1\rangle}(\mathcal{U})$$

Agin it is important to consider the above class modulo "compact" operators (4.7). The follwing lemma is fundamental.

Lemma 4.4. *For $M\in\widetilde{M(\mathcal{U})}$, let us may decompose M in the following way.*

$$M=UL(1+K) \ ,$$

where $U \in U_{\langle 1\rangle}(\mathcal{U})$, $L \in L_{\langle 1\rangle}(\mathcal{U})$, and $K \in \mathcal{K}(\mathcal{U})$. *We refer to this decomposition as a UL-decomposition of M.*

Then we have the following facts.

(1) *Both L and U are determined uniquely modulo* $\mathcal{K}(\mathcal{U})$.

(2) $\det_{UL} M = \det_f(1+K)$ *depends only on M. We call it the determinant of N according to a UL-decomposition, or UL-determinant.*

We note that $\det_{UL}(M)$ is invertible if M is invertible in $E(\mathcal{U})$. Indeed, $(1+K)$ is invertible in that case. Similarly, we have also the following lemma.

Lemma 4.5. *For* $N \in \widetilde{N(\mathcal{U})}$, *let us decompose M in the following as a LU-decomposition of N.*

Then we have the following facts.

(1) *Both L and U are determined uniquely modulo* $\mathcal{K}(\mathcal{U})$.

(2) $\det_{LU} M = \det_f(1+K)$ *delends only on N. We call it the determinant of N according to a LU-decomposition, or LU-determinant.*

As in the case of UL-determinant, we note that $\det_{LU}(N)$ is invertible if N is invertible in $E(\mathcal{U})$.

There is a useful formula for a calculation of LU-determinant.

Lemma 4.6.

$$\det_{LU}(N) = \lim_{n\to+\infty} \det(p_n \circ N \circ \iota_n)$$

Proof. Let

$$N = LU(1+K)$$

a LU-decomposition of N. We may rewrite the above formula as

$$N = L(1+K_U)U\ ,$$

where $K_U = UKU^{-1}$ is an element of $\mathcal{K}(\mathcal{U})$. Now, for any operator $E \in E(\mathcal{U})$, let us denote the operator $p_n \circ E \circ \iota_n$ on $\mathcal{U}^{(n)} = p_n \mathcal{U}$ by $E^{(n)}$. Then the triangularity of L and U enables us to decompose the matrix $N^{(n)} = p_n \circ N \circ \iota_n$ as,

$$N^{(n)} = L^{(n)} \circ (1+K_U)^{(n)} \circ U^{(n)}\ .$$

We therefore see that the right hand side of the statement of the lemma is equal to $\det_f(1+K_U)$, which is equal to $\det_f(1+K)$ (Lemma 4.3).

We should note that although this formula is the same as in the calculation of $\det_f$, the LU-determinant does depend on the choice of the topological base $\{e_i\}$.

Unfortunately, we can not obtain a UL-decomposition in an easy way as above. We denote the difference between the UL-and LU- determinant as ρ_1. In precise, we give the following definition.

Definition 4.2. Let A be an invertible element of $X(\mathcal{U})=\widetilde{M(\mathcal{U})}\cap\widetilde{N(\mathcal{U})}$. $\rho_1(A)$ is given by

$$\rho_1(A)=\frac{\det_{\mathrm{LU}}(A)}{\det_{\mathrm{UL}}(A)}\ .$$

It is clear from this definition that for two elements A, B of $X(\mathcal{U})$ whose difference $A-B$ is in $\mathcal{K}(\mathcal{U})$, $\rho_1(A)$ is equal to $\rho_1(B)$.

For each element M of $E(\mathcal{U})$ and for each pair of non positive integers c and d with $c\leq d$, we denote by $M_{(d;c)}$ the transpose of the $(((-\infty,\ d]\backslash c)\times(-\infty,\ d-1]))$-portion of M. In other words, $M_{(d;c)}$ is an element of $E(\mathcal{U})$ defined by the following formula.

$$(M_{(d;c)})_{i,j}=\begin{cases}(M)_{j+d,i+d-1} & \text{if}\quad j+d-1\geq c\\ (M)_{j+d-1,i+d-1} & \text{if}\quad j+d-1<c\end{cases}$$

Lemma 4.7. *Let N be an element of $\widetilde{N}(\mathcal{U})$. Then,*

(1) *$N_{(d;c)}$ is an element of $\widetilde{N}(\mathcal{U})$ for all d, c.*

(2) *There exists an open ideal J of $\mathcal{B}$ and a non positive integer j_0 such that,*

$$\det_{\mathrm{LU}}(N_{(j;j)})\equiv 1\qquad \mathrm{mod} J\qquad \textit{for all}\quad j\leq j_0\ .$$

In particular, $N_{(j;j)}$ is invertible for all $j\leq j_0$.

(3) *For all non positive integers k, l with $k\leq l$, we have,*

$$\sum_{i\leq l}(-1)^{(i+l)}(N)_{i,k}\det_{\mathrm{LU}}(N_{(l;i)})=\delta_{kl}\det_{\mathrm{LU}}(N_{(l-1;l-1)})\ .$$

Proof. We note that each element E in $E(\mathcal{U})$ admits a representation in block form,

$$E=\begin{bmatrix}E_{11} & E_{12}\\ E_{21} & E_{22}\end{bmatrix}\ ,$$

corresponding to the decomposition 4.8. (1) and (2) follows from the block form representation of each factors in the LU-decomposition of N. On the other hand, by the Lemma 4.6, (3) reduces to an easy exericise of finite dimensional case.

Corollary 4.8. *Let N, j_0 be the same as in the above lemma. Let U be the element of $U_{\langle 1\rangle}(\mathcal{U})$ determined by*

$$(U)_{ij}=\begin{cases}(-1)^{i+j}\dfrac{\det_{\mathrm{LU}}(N_{(j;i)})}{\det_{\mathrm{LU}}(N_{(i;i)})} & \text{if } i\leq j \text{ and } \leq j_0\\ \delta_{ij} & \textit{otherwise}\end{cases}$$

Then UN is an element of $(\widetilde{L})$ and we have,

$$(UN)_{jj}=\frac{\det_{\mathrm{LU}}(N_{(j+1;j+1)})}{\det(N_{(j;j)})} \qquad \text{if} \quad j\leq j_0 \ .$$

We have the following lemma which gives an explicit description of ρ_1.

Lemma 4.9. *For $M\in\widetilde{N(\mathcal{U})}$, M is an element of $\widetilde{M(\mathcal{U})}$, if and only if the seuence $\{\det_{\mathrm{LU}}M_n\}$ converges to an element r of $\mathcal{B}$ as $n\to\infty$. If this is the case, r is equal to $\rho_1(M)$.*

Proof. The first statement results from the corollary 4.8. For generic N, we may put the number j_0 in the corollary 4.8 as 0 and see that the second statement is true in this case. Now an argument of specialization proves the lemma.

The following lemma states that LU- and UL- determinants do not commute with multiplication.

Lemma 4.10. *Let A, B be elements of $X(\mathcal{U})$. Assume also that $AB\in X(\mathcal{U})$. Let*

$$A=U_AL_A(1+K_A) \ ,$$

$$B=U_BL_B(1+K_B)$$

be LU-decompositions of A and B, respectively. Then, we have

$$\det_{\mathrm{LU}}(AB)=\det_{\mathrm{LU}}(A)\det_{\mathrm{LU}}(B)\rho(A,B) \ ,$$

$$\det_{\mathrm{UL}}(AB)=\det_{\mathrm{UL}}(A)\det_{\mathrm{UL}}(B)\rho(A,B)^{-1} \ ,$$

where $\rho(A,B)$ is defined by

$$\rho(A,B)=\rho_1(L_AU_B) \ .$$

$\rho(A,B)$ satisfies the following cocycle condition.

$$\rho(A,BC)\rho(B,C)=\rho(AB,C)\rho(A,B) \ . \tag{4.9}$$

Proof. This is an easy consequence of the definition of ρ_1.

We can prove the Lemma 4.6 by another method. For each element E of $E(\mathcal{U})$ and for each pair of non positive integers c and d such that $c\leq d$, we denote by $E^{(d;c)}$ the transpose of the $([j+1,0]\times([j,0]\setminus\{i\}))$-portion of E. That is, $E^{(d;c)}$ is a $(-d)$-square matrix defined by

$$(E^{(d;c)})_{i,j}=\begin{cases}(E)_{j,i} & \text{if} \quad i\geq c\\ (E)_{j,i-1} & \text{if} \quad i<c\end{cases} .$$

Then we have the following.

Lemma 4.11. *For all integers i, k such that $i>k$, we have,*

$$\sum_{j\geq i}(E)_{ij}(-1)^{j+k}\det(E^{(k;j)})=0 \ .$$

Now, a similar arguments as in the proof of 4.9 gives the following

Lemma 4.12. *For any element E of $E(\mathcal{U})$, the following two statements are equivalent.*

(1) *The limit*

$$\lim_{n\to\infty}\det(E^{(-n;-n)})$$

exists and is invertible.

(2) *E is an element of $\widetilde{N}$ and is invertible.*

Suppose the index set $-\mathbf{N}$ of $\mathcal{U}$ is decomposed into a disjoint union,

$$-\mathbf{N}=\mathfrak{J}_1\amalg\mathfrak{J}_2\amalg\ldots\amalg\mathfrak{J}_k\ ,$$

and assume that each $\mathfrak{J}_j$ is isomorphic to $-\mathbf{N}$ as an ordered set. Then, we have a direct sum decomposition

$$C_0(-\mathbf{N};\mathscr{B})=C_0(\mathfrak{J}_0;\mathscr{B})\oplus C_0(\mathfrak{J}_1;\mathscr{B})\oplus\ldots\oplus C_0(\mathfrak{J}_k;\mathscr{B})\ .$$

Lemma 4.13. *Let N, X be elements of $E(\mathcal{U})$ such that,*

(1) $N(C_0(\mathfrak{J}_j;\mathscr{B}))\subset\bigoplus_{l\geq j}C_0(\mathfrak{J}_l;\mathscr{B})$

(2) $X(C_0(\mathfrak{J}_j;\mathscr{B}))\subset\bigoplus_{l>j}C_0(\mathfrak{J}_l;\mathscr{B})$

Assume that N is an invertible element of $\widetilde{N}(\mathcal{U})$. Then $N+X$ is an element of $\widetilde{N}(\mathcal{U})$ and we have $\det_{\mathrm{LU}}(N+X)=\det_{\mathrm{LU}}(N)$.

Proof. We use Lemma 4.12. Then, the lemma reduces to the finite dimensional case.

Lemma 4.14. *Let $N=L(1+K)U$ be an element of $\widetilde{N}(\mathcal{U})$. For each non positive integers c, d, let $N^{\{c,d\}}$ be the $(((-\infty,0]\backslash c)\times((-\infty,0]\backslash d))$-portion of N. Then,*

(1) *$N^{\{c,d\}}$ is an element of $\widetilde{N}(\mathcal{U})$.*

(2) *Put $N^{[c,d]}=(-1)^{(c+d)}\det_{\mathrm{LU}}N^{\{d,c\}}$ (the "cofactors" of $N^{\{d,c\}}$). Then, the matrix $\widehat{N}$ defined by*

$$(\widehat{N})_{c,d}=N^{[c,d]}$$

is an element of $E(\mathcal{U})$, and we have,

$$N\widehat{N}=\widehat{N}N=\det_{\mathrm{LU}}(N)\ .$$

(3) *We have*

$$U\widehat{N}L=\widehat{1+K}\ .$$

In other words, we have

$$\widehat{N}=U^{-1}\widehat{1+K}L^{-1}\ ,$$

4.5. Matrices related to power series. In this subsection we restrict ourselves to a smaller class of matrices, corresponding to "multiplications" by elements of $\exp(\mathscr{B}\{\{z\}\})$ on $\mathscr{B}\{\{z\}\}/\mathscr{B}[[z]]$. (The "multiplications" are defined if we fix a direct sum decomposition,

$$\mathscr{B}\{\{z\}\}\cong\mathscr{B}\{\{z\}\}/\mathscr{B}[[z]]\otimes\mathscr{B}[[z]]\ .$$

A change of choice of this decomposition is not important in our argument, since difference of choices is given by a compact operator and we are concerned with the properties which are not affected by perturbations by compact operators.)

We first introduce "z" and "z^{-1}".

$$(\Gamma)_{ij}=\delta_{i,j+1}$$

$$(Ч)_{ij}=\delta_{i,j-1}$$

We see immediately that Γ is an element of $U_{\langle 0\rangle}(\mathscr{U})$. It is not, on the other hand, true that $Ч$ is an element of $L_{\langle 0\rangle}(\mathscr{U})$ due to the topological condition in the definition of $L_{\langle 0\rangle}(\mathscr{U})$. It is of course true that $tЧ\in L_{\langle 0\rangle}(\mathscr{U})$ for any topologically nilpotent element t of $\mathscr{B}$. We further observe that,

$$\Gamma Ч=1\ , \tag{4.10}$$

$$Ч\Gamma=1-\text{diagonal}\{1, 0, 0, \ldots\}\ . \tag{4.11}$$

In other words, Γ and $Ч$ are the inverse to each other modulo $\mathscr{K}(\mathscr{U})$. Next we define

$$\sum=\left\{\sum_{i\geq 0}t_i\Gamma^i+\sum_{j>0}s_jЧ^j;\ \begin{array}{l}t_i,\ s_j\in\mathscr{B},\ s_j\to 0 \text{ as } j\to\infty \text{ and the ideal}\\ J \text{ generated by } s_j\text{'s is topologically nilpotent.}\end{array}\right\}\ .$$

Again we put $\tilde{\Sigma}=\Sigma+\mathscr{K}(\mathscr{U})$. Looking at the multiplication rule 4.10 and 4.11, we see that $\Sigma^{\sim}$ is closed under multiplication, although Σ itself is not. The following lemma states that modulo "compact" operators, $(\tilde{\Sigma})^{\times}$ forms a commutative group.

Lemma 4.15. *For all non negative integers i, j, define f_{ij} by*

$$f_{ij}=[\Gamma^i, Ч^j]\ .$$

Then, f_{ij}; is an element of $\mathscr{K}(\mathscr{U})$, and we have the following commutation relation,

$$\left[\sum_{i>0}a_i\Gamma^i, \sum_{j>0}b_jЧ^j\right]=\sum_{i>0,j>0}a_ib_{ij}f_{ij}\ ,$$

for all $\{a_i\}$, $\{b_j\}$ such that the ideal generated by $\{b_j\}$ is topologically nilpotent. The right hand side of the above equation is in $\mathscr{K}(\mathscr{U})$, so $\tilde{\Sigma}/\mathscr{K}$ forms a commutative ring. In fact, it is isomorphic to a subring of $\mathscr{B}\{\{z\}\}$ defined as

$$\mathscr{B}\{\{z\}\}_0=\left\{\sum_i (t_i z^i)\,;\ \begin{array}{l} t_0=0,\ t_i\to 0 \text{ as } i\to-\infty, \\ \textit{the ideal } J \textit{ generated by} \\ \{t_i\}_{i<0} \textit{ is topologically} \\ \textit{nilpotent.}\end{array}\right\}.$$

Thus $\widetilde{\Sigma}$ is an extension of a function ring by the space of "compact" operators. It may be interesting to note that C^*-analogue of this extension appears as a deformation of the space of bounded holomorphic functions on the unit disk. [5]

We may define the exponential of an element of $\widetilde{\Sigma}$ by

$$\exp(A)=\sum_{i\geq 0}\frac{1}{i\,!}A^i\ .$$

We see easily that the sum converges in $E(\mathscr{U})$ and defines an element of $\widetilde{\Sigma}$.

We set,

$$(\widetilde{\Sigma_{\langle 1\rangle}})^\times=$$

$$=\left\{k+\exp(\sigma)\,;\ k\in\mathscr{K}(\mathscr{U}),\ \sigma\in\Sigma;\ \begin{array}{l} k+\exp(\sigma) \text{ is an invertible element of } E(\mathscr{U}), \\ \sigma \text{ has no constant term.}\end{array}\right\}$$

It forms a subfgroup of the semigroup $(\widetilde{\Sigma},\ \times)$. We have the following lemma.

Lemma 4.16.

$$(\widetilde{\Sigma_{\langle 1\rangle}})\subset X(\mathscr{U})=\widetilde{N(\mathscr{U})}\cap\widetilde{M(\mathscr{U})}\ .$$

So we may consider both LU- and UL- determinant of an element of $\widetilde{\Sigma}$.

Put

$$(1+\mathscr{K}(\mathscr{U})_1^\times=\{1+K;\ K\in\mathscr{K}(\mathscr{U}),\ \det_f(1+K)=1\}\ ,$$

$$G=(\widetilde{\Sigma})^\times/(1+\mathscr{K}(\mathscr{U}))_1^\times\ .$$

Then, using the isomophism

$$(1+\mathscr{K}(\mathscr{U}))^\times/(1+\mathscr{K}(\mathscr{U}))_1^\times\cong\mathscr{B}^\times\ ,$$

defined by the finite determinant, we have the following exact sequence.

$$1\to\mathscr{B}^\times\to G\to\exp(\mathscr{B}\{\{z\}\})\to 1\ ,$$

which shows that G is a central extension of the function space $\exp(\mathscr{B}\{\{z\}\})$. It is not a trivial extension. In fact, the Lie algebra of G is the $U(1)$-current algebra. We will study this in the next subsection.

4.6. Relation with current algebras. In order to describe the multiplication table of G, it is convenient to introduce the Fock space and second quantization. First let us define the set of Maya diagrams [4]. It is divided into infinitley many sectors labeled by integers called charges:

$$(\mathrm{Maya}) = \cup_k (\mathrm{Maya})_k \ .$$

Each sector consists of order preserving maps α: $(-\infty, k) \cap \mathbf{Z} \rightarrow \mathbf{Z}$ with the following property.

(*) For all but finite x, we have $\alpha(x) = x$.

The Fock space is defined by

$$F = \prod_{\alpha \in (\mathrm{Maya})} \mathscr{B} z^{\alpha} \ .$$

Each vector z^{α} is interpreted as a semi-infinite form,

$$z^{\alpha} = z^{\alpha(k)} \wedge z^{\alpha(k-1)} \wedge z^{\alpha(k-2)} \wedge \ldots.$$

and the Fock space is also called the space of semi-infinite forms. It is an infinite version of exterior algebra. We may extend various operations on exterior algebras of finite dimensional vector spaces to this infinite dimensional case. Among them is the second quantization of linear operators on $\mathscr{B}\{\{z\}\}$. For any linear operator L on $\mathscr{B}\{\{z\}\}$, we put (heuristically)

$$\begin{aligned} & q(L).\ (z^{\alpha(k)} \wedge z^{\alpha(k-1)} \wedge z^{\alpha(k-2)} \wedge \ldots.) \\ & = L(z^{\alpha(k)}) \wedge z^{\alpha(k-1)} \wedge z^{\alpha(k-2)} \wedge \ldots. \\ & + z^{\alpha(k)} \wedge L(z^{\alpha(k-1)}) \wedge z^{\alpha(k-2)} \wedge \ldots. \\ & + z^{\alpha(k)} \wedge z^{\alpha(k-1)} \wedge L(z^{\alpha(k-2)}) \wedge \ldots. \\ & + \ldots \end{aligned}$$

This lifts an action of a Lie algebra on $\mathscr{B}\{\{z\}\}$ to the Fock space in principle. But here we added the word "heuristically", because the right hand side of the above formula does not converge in general. For example, $q(1)$ can not defined by the above formula. But when we deal with the case

$$L(f) = z^i \times f \quad (f \in \{\{z\}\})$$

for some i which is not equal to zero, the formula 4.6 defines a continuous transformation $q(z^i)$ on the Fock space. We put

$$J_i = q(z^i)$$

for non zero i. We (re)define J_0 to be a linear operator on the Fock space which gives the "charge", or the "index" ([4]). That is,

$$J_0(z^{\alpha}) = p z^{\alpha} \quad (\text{if } \alpha \in (\mathrm{Maya})_p)$$

It is easy to calculate the following commutation relations. [4]

$$[J_i, J_j] = -i\delta_{i+j,0} \tag{4.12}$$

Contrary to the finite dimensional case, to lift an action of a Lie algebra on $\mathscr{B}\{\{z\}\}$ by means of 4.6 we need a central extension of the Lie algebra. (This fact is called an anomaly). We will define the $U(1)$-cuarrent algebra (with the central charge. 1) to be the Lie algebra $\widehat{\mathrm{u}(1)}$ with a basis $\{J_i\}_{i\in\mathbf{Z}}$ with the commutation relation 4.12.

We sometimes denote each function z^{-m} by e^{-m} to express that we regard the function as a vector in $\mathscr{V}=\mathscr{B}\{\{z\}\}$ and not as an operator on $\mathscr{V}$.

Let $\{\overline{e_i}\}$ be the dual basis of $\{e^i\}$ in the topological dual $\mathscr{V}^*$ of $\mathscr{V}$ with

$$(\overline{e_i}|e^j)=\delta_i^j\ .$$

We may construct a Fock space $\overline{F}$ based on the vector space $\mathscr{V}^*$ instead of $\mathscr{V}$. $\overline{F}$ is a vector space with a topological basis $\{\overline{e_\alpha}\ \alpha\in(\mathrm{Maya})\}$. It is the dual of F with the follwing pairing.

$$(\overline{e_\alpha}|e^\beta)=\delta_\alpha^\beta\ .$$

Lemma 4.17. *Let* $|0\rangle$ *denote the element* $z^0\wedge z^{-1}\wedge z^{-2}\wedge\ldots$ *in the Fock space. Let* $\langle 0|$ *denote the element* $\ldots\wedge\overline{e}_{-2}\wedge\overline{e}_{-1}\wedge\overline{e_0}$ *in the dual Fock space. Then we have,*

$$\begin{aligned}&\langle 0|\exp\Big(\sum_{i<0}t_iJ_i\Big)\exp\Big(\sum_{j>0}s_jJ_j\Big)|0\rangle\\&=(\textit{The coefficient of } |0\rangle \textit{ in } \exp\Big(\sum_{i<0}t_iJ_i\Big)\exp\Big(\sum_{j>0}s_jJ_j\Big)|0\rangle)\\&=\det{}_{LU}(\exp\Big(\sum_{j>0}s_j\Gamma^j\Big)\exp\Big(\sum_{i<0}t_i\Psi^i\Big)\ .\end{aligned}$$

Proof. We use the Sato theory of universal Grassmann manifold (see [4]). By checking the Hirota bilinear relation [4], we see that $|\{s\}\rangle=\exp(\sum_{j\geq 0}s_jJ_j)|0\rangle$ corresponds to a point of Sato's universal Grassmann manifold (UGM). Observing the result of action of vertex operators on $|\{s\}\rangle$ as in the argument of [4], we see that,

$$|\{s\}\rangle=\wedge_{m\geq 0}\Big(\exp\Big(\sum_{j>0}s_jz^j\Big)e^{-m}\Big)\ .$$

We represent the linear operator $\exp(\sum s_jz^j)$ on $\mathscr{V}$ by a block matrix form

$$\exp\Big(\sum s_jz^j\Big)=\begin{bmatrix}U' & M\\ 0 & U\end{bmatrix}$$

corresponding to the direct sum decomposition,

$$\mathscr{V}=\mathrm{span}\,\{e^i\}_{i>0}\oplus\mathrm{span}\,\{e^i\}_{i\leq 0}\ .$$

Then we see that U is an element of $U_{\langle 1\rangle}(\mathcal{U})$ and obtain,

$$\wedge_{m\geq 0}\Big(\exp\Big(\sum_{j\leq 0} s_j z^j\Big)\Big)=\wedge_{m\geq 0}(MU^{-1}\oplus 1)e^{-m}$$

We represent $\sum_i t_i z^i$ as

$$\begin{bmatrix} L' & 0 \\ N & L \end{bmatrix}$$

and obtain,

$$\langle 0|\exp\Big(\sum_{i<0} t_i J_i\Big)=\wedge_{m\geq 0}(L^{-1}N\oplus 1)^{\dagger}\bar{e}_{-m}\ ,$$

where the dagger sign indicates the adjoint of operators. We conclude therefore by definition, that,

$$\langle 0|\exp\Big(\sum_{i<0} t_i J_i\Big)\exp\Big(\sum_{j>0} s_j J_j\Big)|0\rangle$$

$$=(\wedge_{-m\geq 0}(L^{-1}N^{\dagger}\oplus 1)\bar{e}_{-m}|\wedge_{n\geq 0}(MU^{-1}\oplus 1)e^{-n})$$

$$=\det{}_f\Big(\begin{pmatrix} L^{-1}N \\ 1 \end{pmatrix}(MU^{-1}1)\Big)$$

$$=\det{}_f(L^{-1}NMU^{-1}+1)$$

$$=\det{}_{\mathrm{LU}}(NM+LU)$$

The commutatibity relation,

$$\begin{bmatrix} U' & M \\ 0 & U \end{bmatrix}\begin{bmatrix} L' & 0 \\ N & L \end{bmatrix}=\begin{bmatrix} U' & M \\ N & L \end{bmatrix}$$

shows that $NM+LU+UL$, so we prove the lemma.

We deduce from the lemma that a formal integration of the $U(1)$-current algebra is given by G. Let us explain this more precisely. For any topological algebra $\mathcal{B}$ which satisfies the condition (B1) and (B2) of section 4.1, we define a formal integration of the $U(1)$-current algebra over $\mathcal{B}$ as follows.

$$\exp(\widehat{\mathfrak{u}(1)})=\left\{ c\exp\Big(\sum_{i\in\mathbf{Z}} x_i J_i\Big);\ \begin{array}{l} c\in\mathcal{B}^{\times},\ x_i\in\mathcal{B},\ x_i\to 0 \text{ as } i\to-\infty, \\ \text{there exists a topologically nilpotent ideal } J \text{ of } \mathcal{B} \\ \text{such that } x_i\in J \text{ for all } i<0. \end{array}\right\}$$

Note that each element of $\exp(\widehat{\mathfrak{u}(1)})$ can be regarded as an operator on the Fock space. The multiplication law of $\exp(\widehat{\mathfrak{u}(1)})$ is given by the following formula.

$$\exp(\sum_i x_i J_i)\exp(\sum_j y_j J_j)$$

$$=\exp(\frac{1}{2}[\sum_i x_i J_i, \sum_j y_j J_j])\exp(\sum_i (x_i+y_i)J_i)$$

$$=\exp(-\frac{1}{2}[\sum_i x_i y_{-i}))\exp(\sum_i (x_i+y_i)J_i)$$

We can now state the following proposition.

Proposition 4.18. *We have an isomorphism,*

$$G\cong\exp(\widehat{\mathrm{u}(1)}) \quad ,$$

given by the following formula.

$$(\exp(\sum_{i>0} t_i\Gamma^i)\exp(\sum_{j<0} t_j Ч^{-i}),\ a)\mapsto a\exp(\sum_{i>0} t_i J_i)\exp(\sum_{j<0} t_j J_j)$$

So we may regard G as a formal integration of the $U(1)$-current algebra.

Proof. The only thing to prove is the fact that the correspondence is actually an homomorphism between the groups. We need to check that the correspondence preserves the commutation relation of a lower half matrix and an upper half matrix. Let $\widetilde{C}$ be an element of $U_{\langle 1\rangle}(\mathcal{U})\cap\Sigma$ and $\widetilde{B}$ be an element of $L_{\langle 1\rangle}(\mathcal{U})\cap\Sigma$. Let us denote as follows.

$$B=\sum s_j z^j,\ \widetilde{B}=\sum s_j Ч^{-i},\ J(B)=\sum s_j J_j \tag{4.13}$$

$$C=\sum u_k z^k,\ \widetilde{C}=\sum u_k\Gamma^k,\ J(C)=\sum u_k J_k \tag{4.14}$$

First we notice that

$$\mathrm{Ad}(\exp(J(B)))J(C)=\exp(adJ(C))=\exp(J(C)+[J(B),J(C)]) \quad ,$$

since $[J(B),J(C)]$ is a constant. This implies,

$$\mathrm{Ad}(\exp(J(B))).\exp(J(C))=\exp(J(C)+[J(B),J(C)]) \quad ,$$

which in turn gives rise to

$$(\exp J(C))^{-1}\exp J(B)\exp J(C)\,(\exp J(B))^{-1}=\exp([J(B),J(C)])$$

The constant $\exp([J(B),J(C)])$ may be computed by taking its vacuum expectation value.

$$\exp([J(B),J(C)])$$

$$= \langle 0|\exp([J(B), J(C)]|0\rangle$$

$$= \langle 0|\exp J(C))^{-1}\exp J(B)\exp J(C)\ (\exp J(B))^{-1}|0\rangle$$

$$= \langle 0|\exp J(B)\exp J(B)|0\rangle$$

$$= \det_{LU}(\exp(\widetilde{C})\exp(\widetilde{B}))\quad \text{(by Lemma 4.17)}$$

$$= \rho(\exp(\widetilde{C}), \exp(\widetilde{B}))$$

This completes the proof.

4.7. A lemma. We have used exponentials which would cause a problem when we extend our theory to positive characteristics. In this subsection we will show the following proposition, which enables us to avoid exponentials. The proposition is also used later in the statement of the proposition on formal periodicity.

Lemma 4.19. *Let $A=\sum_i a_i\Gamma_i$ be an element of $\sum(\mathcal{U})$. Assume $z_0=a_0-1$ is topologically nilpotent. Then, there exist a topologically nilpotent element c of $\mathcal{B}$ such that*

$$A+c1 \in (\Sigma\cap U_{\langle 1\rangle}(\mathcal{U}))(\Sigma\cap L_{\langle 1\rangle}(\mathcal{U})) \ . \tag{4.15}$$

Since triangular matrices have their logarithms well-defined, the right hand side of 4.15 is equal to $\widetilde{\Sigma}_{\langle 1\rangle}$.

Corollary 4.20. *Let A be an element of $(\widetilde{\Sigma})^{\times}$. Then there exists an invertible element c of $\mathcal{B}$ such that cA is an element of $(\widetilde{\Sigma}_{\langle 1\rangle})^{\times}$.*

The proof of this lemma is fairly long. It occupies the rest of this subsection.

Lemma 4.21. *Let A be as in Lemma 4.19. We put*

$$\mathcal{L}(A)=\{1+\sum_{i<0}(\frac{a_i}{1+z_0})\Gamma_i)\}^{-1}=\sum_{k=0}^{\infty}\{-\sum_{i<0}(\frac{a_i}{1+z_0})\Gamma_i)\}^{k}\ .$$

Let I_m be the ideal of $\mathcal{B}$ generated by $\{a_{i_0}\dots a_{i_k};\ i_0,\ \dots,\ i_k<0,\ i_0+\dots i_k\le 0\}$. Then, $\{I_m\}_m$ is a decreasing sequence of ideals converging to 0. We have the following.

(1) *$\mathcal{L}(A)=1+\sum_{l<0}b_l\Gamma_l$ for some element $b_l(z_0)$ of $I_{-l}[[z_0]]$.*

(2) *$A\mathcal{L}(A)=\sum_j c_j(z_0)\Gamma_j$* with some element *$c_j(z_0)$ of $\mathcal{B}[[z_0]]$.*

(3) *$c_0(z_0)=1+z_0+d_0(z_0)$ for some element $d_0(z_0)$ of $I_1[[z_0]]$.*

(4) *$c_i(z_0)$ is an element of $I_{-i+1}[[z_0]]$ for all $i<0$*

By a successive use of the above lemma we obtain the following:

Corollary 4.22. *Let A be as in Lemma 4.21. Then we way express A uniquely as*

$$A=UL\ ,$$

with L an element of $L_{\langle 1\rangle}(\mathcal{U})\cap\sum(\mathcal{U})$, U an element of $U_*(\mathcal{U})$. The diagonal component of U is of the form $1+z_0+f(z_0)$, where $f(z_0)$ is an element of $\mathcal{B}[[z_0]]$ whose coefficients are all in the ideal generated by $\{a_{-1}, a_{-2}, \ldots\}$.

The proof of the Lemma 4.19 is completed by the following result of implicit function theorem.

Lemma 4.23. *Let $f(z_0)=z+\sum_{k>1}a_k z_0^k$ be an element of $\mathcal{B}[[z_0]]$. Let I be a topologically nilpotent open ideal of $\mathcal{B}$. Then there exists a sequence $\{c_k\}_{k=0}^{\infty}$ such that,*

(1) $c_0=0$.

(2) $f(\sum_{k=0}^{l}c_k)-b\in(b^{2^l})$ *for all* $l\leq 0$.

(3) $c_k\in(b^{2^{k-1}})$ *for all* $k>0$.

Proof. Put

$$c_{k+1}=\frac{b-f(s_k)}{f'(s_k)}\ ,$$

with $s_l=\sum_{k=0}^{l}c_k$.

Corollary 4.24. (*implicit function theorem*) *Under the assumption of the lemma, if we put $c=\sum_{k=0}^{\infty}c_k$, then we have*

$$f(c)=0$$

4.8. Infinite Berezinians. We describe here a theory of infinite Berezinians (super determinant). The definition of Berezinian involves a division in its definition. So we have to localize our function space. We first introduce such localization.

Definition 4.3. Let $\mathcal{B}$ be a $\mathbf{N}$-graded commutative ring which is complete with respect to the topology given by the gradation. Let $(\mathcal{B}[\mathbf{I}])^\wedge$ be the completion of the polynomial ring $\mathcal{B}[\mathbf{I}]$ with the grading determined by letting the degree of $\mathbf{I}$ to be 0. Then we introduce the following set.

$$Q^0(\mathcal{B})=\{(F(\mathbf{I};\,Д);\,F\in(\mathcal{B}[\mathbf{I}])^\wedge,\,Д\in\mathcal{B},\,d \text{ is not a sero divisor of } \mathcal{B}\}\ .$$

We let the monoid $(\mathcal{B}^r,\times)$ of non zero divisors in $\mathcal{B}$ act on $Q^0(\mathcal{B})$ as follows.

$$(F(\mathbf{I});\,Д)^a=(F(a\mathbf{I});\,aД)\quad(\text{for any } a \text{ in } \mathcal{B}^r)\ .$$

We define the equivalence relation $\sim$ as follows.

$$(F;\,Д)\sim(G;\,Д')\Leftrightarrow\exists\, a,\,b\in\mathcal{B}^r \text{ such that } (F;\,Д)^a=(G;\,Д')^b$$

We denote the set of equivalence class in $Q^0(\mathcal{B})$ as $Q(\mathcal{B})$. The equivalence class of $(F;\,Д)$ is denoted as $[F;\,Д]$. The set $Q^0(\mathcal{B})$ has a natural structure of ring with the following addition and multiplication rules.

$$(F;\,д)+(G;\,д')=(F(д'\mathbf{I})+G(д\mathbf{I});\,дд')$$

$$(F;\,д)\times(G;\,д')=(F(д'\mathbf{I})\times G(д\mathbf{I});\,дд')$$

The operations descend to $Q(\mathscr{B})$ and make it also a ring.

We call $Q(\mathscr{B})$ the total quotient ring of $\mathscr{B}$. (This definition coincides with the usual one if the ring $\mathscr{B}$ is discrete.)

Definition 4.4. Let $\mathscr{B}$ be a super commutative super topological algebra, with its even part $\mathscr{B}_0$ satisfying the conditions (B1) and (B2) of section 4.1. We assume furthermore that $\mathscr{B}$ is so chosen that it contains an element $\mathbf{I}$ of degree 0. Let M be a $(-\mathbf{N}|-\mathbf{N})$-square matrix of the standard format. That is, M is decomposed as

$$M=\begin{pmatrix} M^{00} & M^{01} \\ M^{10} & M^{11} \end{pmatrix},$$

with each matrices M^{pq} having entries $(M^{pq})^{ij}$ indexed by $(-\mathbf{N})\times(-\mathbf{N})$. We assume that M is even, that means, each entry of (M^{pq}) has its parity $p+q$. If furthermore M satisfies the following three conditions,

(1) M^{11} is an element of $\widetilde{N}$,

(2) $\det_{LU}(M^{11})$ (certified to be defined by (1)) is a non zero divisor in $\mathscr{B}_0$.

(3) $M^{00}-\mathbf{I}M^{01}\widehat{M^{11}}M^{10}$ is an element of $\widetilde{N}$,

then we define the LU-Berezinian $\mathrm{Ber}_{LU}\,M$ of M as an element of $Q(\mathscr{B}_0)$ (4.3) corresponding to the following "pre LU-Berezinian" in $Q^0(\mathscr{B}_0)$.

$$\mathrm{preBer}_{LU}(M)=\det{}_{LU}(M^{00}-\mathbf{I}M^{01}M^{11}M^{10});\,\det{}_{LU}(M^{11}))$$

Proposition 4.25. *(absence of anomaly) Suppose the index set $-\mathbf{N}$ is decomposed into a disjoint union,*

$$-\mathbf{N}=\mathfrak{J}_1\amalg\mathfrak{J}_2\amalg\ldots\amalg\mathfrak{J}_k\ ,$$

and assume that each $\mathfrak{J}_j$ is isomorphic to $-\mathbf{N}$ as an ordered set. We define subspaces $\widetilde{\Sigma}'$, E'_0 of E as follows.

$$\widetilde{\Sigma}'=\{N\in(\widetilde{\Sigma})^{\times};\,N(C_0(\mathfrak{J}_j;\,\mathscr{B}_0))\subset\bigoplus_{l\geq j}C_0(\mathfrak{J}_l;\,\mathscr{B}_0)\}$$

$$E'_0=\{X\in E;\,X(C_0(\mathfrak{J}_j;\,\mathscr{B}_0))\subset\bigoplus_{l>j}C_0(\mathfrak{J}_l;\,\mathscr{B}_0)\}$$

Let S be the set of $(-\mathbf{N}|-\mathbf{N})$-square matrices of the standard format which watisfy the following conditions.

(1) $M^{00},\,M^{11}\in\widetilde{\Sigma}'+E'_0+\mathscr{K}$,

(2) $M^{00}-M^{11}\in\mathscr{K}$,

(3) $M^{01},\,M^{10}\in E'_0+\mathscr{K}$,

Then we have the following.

(1) *The LU-super Berezinian exists for any element of S.*

(2) *For any elements A, B of S, we have,*

$$(\mathbf{I};1)\,\mathrm{preBer}_{\mathrm{LU}}(AB)=(\mathrm{preBer}_{\mathrm{LU}}(A)\,\mathrm{preBer}_{\mathrm{LU}}(B))^{\rho(A^{11},B^{11})}\ . \tag{4.16}$$

Proof. The first claim is a consequence of Lemma 4.13. To prove the second claim, we essentially follow the argument in [8]. The cocycle condition 4.9 of ρ enables us to conclude that the set of all B satisfying the condition 6.8 for all A is closed under multiplication. The same is true in case the roll of A and B are interchanged. This fact reduces our claim to the following two special cases.

$$A\text{: general, } B=\begin{pmatrix} B^{00} & 0 \\ 0 & B^{00} \end{pmatrix} \tag{4.17}$$

$$A=\begin{pmatrix} 1 & 0 \\ A^{10} & 1 \end{pmatrix},\quad B=\begin{pmatrix} 1 & B^{01} \\ 0 & 1 \end{pmatrix} \tag{4.18}$$

The first case is handled by use of Lemmas 4.13 and 4.10. The Lemma 4.13 reduces the second case furthermore to the case where A^{10} and B^{01} are elements of $\mathcal{K}$. The claim in this case is proved by taking the limit of the finite dimensional case.

5. Definition of super theta function

5.1. Definition. In this section we define a super tau function associated to a family of Viraforo uniformized super Riemann surface $(X, \pi, (S, s_0), q, (z, \zeta_1, \dots, \zeta_N))$ of dimension $(1|N)$ (definition 3.1). From the proposition of absence of anomaly (Proposition 4.25), we may interpret the tau function as a (Taylor expansion of) super theta function.

We first define the set of "super Maya diagrams", an analogue of the set of Maya diagrams 4.6. It is divided into infinitely many sectors labeled by sets of 2^N-tuple of integers $\{c_I;\ I\subset[1,N]\}$;

$$(\mathrm{Maya})=\bigcup_{\{c_I;I\subset[1,N]\}}(\mathrm{Maya})_{\{c_I\}}\ .$$

Each sector consists of sets $\alpha=\{\alpha_I\}$ of 2^N-tuple of order preserving maps α_I: $(-\infty, c_I)\in\mathbf{Z}\to\mathbf{Z}$ with the following property.

(*) For all except finite x, we have $\alpha_I(x)=x$.

Each element $\{\alpha_I\}$ determines a direct sum decomposition of $\mathscr{H}=\mathscr{B}\{\{z\}\}[\zeta_1,\dots,\zeta_N]$,

$$\mathscr{H}=\mathscr{H}_\alpha^+\oplus\mathscr{H}_\alpha^-\ , \tag{5.1}$$

where the closed subspaces $\mathcal{H}^+_\alpha$ and $\mathcal{H}^-_\alpha$ of $\mathcal{H}$ is spanned respectively by

$$\{z^i\zeta^I;\ i\notin \mathrm{Image}\alpha_I\}\quad ,$$

and

$$\{z^i\zeta^I;\ i\in \mathrm{Image}\alpha_I\}\quad .,$$

Lemma 5.1. *For any family $(X, \pi, (S, s_0), q, (z, \zeta_1, \dots, \zeta_N))$ of Virasoro uniformized super Riemann surface, there exists an open neighborhood S_0 of s_0 in S and a super Maya diagram α_0 so that the natural inclusion map gives an direct sum decomposition*

$$\mathcal{H}^+_{\alpha_0}\oplus\mathcal{O}_X(\pi^{-1}(S_0)\cap\dot{X})=\mathcal{H}\ . \tag{5.2}$$

Proof. This is a direct consequence of the Proposition 3.9.

We fix a super Maya diagram α_0 with the above property. We shrink S if necessary and assume that $S=S_0$. We note that the direct sum decompositions 5.1 and 5.2 gives us an isomorphism Φ_0 between $A=\mathcal{O}_X(\dot{X})$ and $\mathcal{H}^-_{\alpha_0}$. We denote by ι the inclusion of $\mathcal{O}_X(\dot{X})\to\mathcal{H}$, and by ϕ_0: $\mathcal{H}^-_{\alpha_0}\to\mathcal{H}$ the composition map $\phi_0=\iota\circ\phi_0$. More generally, we may take a super Maya diagram α with the same index as α_0, that means, for sufficiently large integer M, we have,

$$\dim(\mathcal{H}/(\mathcal{H}^-_{\alpha_0}+\mathcal{B}[[z]][\zeta_1, \dots, \zeta_N]z^M))=\dim(\mathcal{H}/(\mathcal{H}^-_\alpha+\mathcal{B}[[z]][\zeta_1, \dots, \zeta_N]z^M))\ .$$

We denote by p the projection map p^α: $\mathcal{H}\to\mathcal{H}^-_\alpha$ given by 5.1 and we choose a linear map ϕ: $\mathcal{H}^-_\alpha\to\mathcal{H}$ with the following properties.

(1) The image of ϕ is equal to A

(2) The matrix $(p\circ\phi)-1$ is of finite rank.

Definition 5.1. Let our topological super algebra $\mathcal{B}$ of coefficient so chosen that it contains formal variables $\{x_{i,I}\}_{i\in\mathbf{Z},I\subset[1,N]}$ such that

$$\deg(x_{i,I})=\max(-i,1)\ ,$$

$$\mathrm{parity}(x_i, I)=|I|\quad \mathrm{mod}2\ .$$

Put

$$F=F(x)=\sum_{\substack{i,I\\(i,I)\neq(0,0)}} x_{i,I}z^i\zeta^I\ .$$

We call F a "formal function".

Then we define the super tau function associated to $(X, \pi, (S, s_0), q, (z, \zeta))$ and α by.

$$\tau_\alpha=\tau_\alpha(F)=\mathrm{Ber}_{\mathcal{H}^-_\alpha,\mathrm{LU}}(p_\alpha\circ M_{\exp(F)}\circ\phi)\ . \tag{5.3}$$

We have to check,

Theorem 5.2. *The Berezinian in the right hand side of* 5.3 *is a* *well-defined object, so we have our tau function well defined.*

Proof. According to the direct sum decomposition 5.1, we express p, $M_{\exp(F)}$, ϕ in a block matrix form in the following way.

$$p=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \quad M_{\exp(F)}=\begin{pmatrix}M_{++} & M_{+-}\\ M_{-+} & M_{--}\end{pmatrix}, \quad \phi=\begin{pmatrix}K_1\\ 1+K_2\end{pmatrix}$$

where 1 denote an identity matrix, and the matrix K_2 is, by the choice of ϕ, a matrix of finite rank. Then, we have

$$p\circ M_{\exp(F)}\circ\phi=M_{--}+M_{-+}K_1+M_{--}K_2 .$$

The last two terms of the right hand side are elements of $\mathcal{K}$. By taking modulo "compact operators", the matrix $p\circ M_{\exp(F)}\circ\phi$ is congruent to M_{--}. We then represent M_{--} in a block matrix form by the even and odd decomposition of $\mathcal{H}_\alpha^-$:

$$M_{--}=\begin{pmatrix}M_{--}^{00} & M_{--}^{01}\\ M_{--}^{10} & M_{--}^{11}\end{pmatrix}.$$

When $N=1$, we see that M_{--}^{00}, M_{--}^{11} are matrices of LU- (and UL-) type, and that M_{--}^{01}, M_{--}^{10} are "compact" matrices. Thus in this case the Berezinian is well-defined. When $N\geq 2$, these facts do not hold in general so we have to be more careful. Since a product by a function will increase the odd degree of functions, we apply the first part of the Proposition 4.25 and conclude that the LU-super determinant in the definition of the super tau function is well-defined.

We note here that the choice of the isomorphism ϕ in the above definition is not important, since we have

$$\tau_\alpha(\text{with } \phi_1)=(\text{const.})\times\tau_\alpha(\text{with } \phi_2)$$

Recall that a super manifold M is said to be decomposable ([8]), if there exists a vector bundle $\mathbf{E}$ on M_{rd} such that we have an isomorphism,

$$(M, \mathcal{O}_M)\cong(M_{\mathrm{rd}}, S_{\mathcal{O}_{M,\mathrm{rd}}}(\Pi\mathbf{E})) .$$

A decomposable super manifold with an isomorphism as above specified is called a decomposed super manifold. We note that there is a distinguished $\mathbf{Z}$-grading on the structure sheaf on a decomposed super manifold which is compatible with the above isomorphism. We call a family ω: $X\to S$ of super manifolds (that is, a submersion) decomposable, if there exist vector bundles $\mathbf{E}$ and $\mathbf{F}$ on X_{rd} and S_{rd}, respectively such that there are isomorphisms:

$$\begin{aligned}(X, \mathcal{O}_X)&\cong(X_{\mathrm{rd}}, S_{\mathcal{O}_{X,\mathrm{rd}}}\Pi\mathbf{E}\oplus\Pi\omega^*\mathbf{F})) ,\\ (S, \mathcal{O}_S)&\cong(S_{\mathrm{rd}}, S_{\mathcal{O}_{S,\mathrm{rd}}}(\Pi\mathbf{F})) .\end{aligned} \tag{5.4}$$

We can compute the super tau function for a decomposable family of super Riemann surfaces.

Proposition 5.3. *Let $(X, \pi, (S, s_0), q, (z, \zeta_1, \ldots, \zeta_N))$ be a decomposable family of Virasoro uniformized super Riemann surfaces satisfying the condition 5.4. Let α be a super Maya diagram. For each non negative integer k, we denote by $\alpha[k]$ the set of maps $\{\alpha_I; |I|=k\}$. Then we have the following formula for the super tau function.*

$$\tau_\alpha(X)=\frac{\prod_{k:\mathrm{even}}\tau_{\alpha[k]}(a;\ (X_{\mathrm{rd}}, Q_{\mathrm{rd}}, z_{\mathrm{even}}, \wedge^k\mathbf{E}, t))}{\prod_{k:\mathrm{odd}}\tau_{\alpha[k]}(a;\ (X_{\mathrm{rd}}, Q_{\mathrm{rd}}, z_{\mathrm{even}}, \wedge^k\mathbf{E}, t))}$$

where Q_{rd} is the point of X_{rd} obtained by the reduction of X, z_{even} is a formal coordinate of X_{rd} obtained by the reduction of z, and t_{even} and t_{odd} are formal trivializations of the $\mathcal{O}_{X_0}$ and $\mathcal{O}_{X_1}$, respectively obtained by the formal coordinate (z, ζ). (See the proof of this proposition for the definition of a tau function associated to vector bundles.) If, furthermore, the vector bundle $\mathbf{E}$ is a direct sum of line bundles, say,

$$\mathbf{E}\cong\oplus_{i=1}^N\mathbf{L}_i\ ,$$

then, the tau function is discribed as a quotient of pruducts of the theta function of the Riemann surface X_{rd},

$$\tau_\alpha=\frac{\prod_{k:\mathrm{even}}\prod_{i_1<\cdots<i_k}\tau_{\alpha_{i_1,\ldots,i_k}}(X, \mathbf{L}_{i_1}\otimes\ldots\otimes\mathbf{L}_{i_k})}{\prod_{k:\mathrm{odd}}\prod_{j_1<\cdots<j_k}\tau_{\alpha_{j_1,\ldots,j_k}}(X, \mathbf{L}_{j_1}\otimes\ldots\otimes\mathbf{L}_{j_k})}$$

$$=\frac{\prod_{k:\mathrm{even}}\prod_{i_1<\cdots<i_k}\vartheta(X, I_F+c(\mathbf{L}_{i_1})+\ldots+c(\mathbf{L}_{i_k}))}{\prod_{k:\mathrm{odd}}\prod_{j_1<\cdots<j_k}\vartheta(X, I_F+c(\mathbf{L}_{j_1})+\ldots+c(\mathbf{L}_{j_k}))}$$

Proof. The structure sheaf $\mathcal{O}_X$ of X is decomposed as a direct sum of $N+1$ subsheaves, with respect to the number of factors of odd coordinates. We decompose $\mathcal{H}$, $\mathcal{H}^-$, $\mathcal{H}^+$ accordingly and divide the matrices p, $M_{\exp(F)}$, ϕ into block form with respect to this decomposition. Since these matrices do not increase the number of factors of odd coordinates, we may use Lemma 4.13 and find that

$$\mathrm{Ber}_{\mathrm{LU}}(p\circ M_{\exp(F)}\circ\phi)=\prod_{i=0}^N\mathrm{Ber}_{\mathrm{LU}}((p\circ M_{\exp(F)}\circ\phi)|_{\mathcal{H}_{\alpha,i}^-})$$

$$=\prod_{i=0}^N(\det_{\mathrm{LU}}((p\circ M_{\exp(F)}\circ\phi)|_{\mathcal{H}_{\alpha,i}^-}))^{(-1)^i}\ ,$$

where $\mathcal{H}_{\alpha,i}^-=\mathrm{span}\{z^p\zeta^I;\ p\notin\mathrm{Image}\alpha_I,\ |I|=i\}$.

The factors of the last line of the above equality may be taken as a definition of a "tau function associated to vector bundles." This proves the first part of the proposition. The second part of the proposition may be proved in a similar manner, with the help of Theorem 2.3.

It goes without saying that the theory of decomposable super schemes is essentially equivalent to the theory of vector bundles on Riemann surfaces. (Although the former may shed new lights onto the latter.)

The above proposition tells us two things. First, we notice that super theta functions do not carry the full information of the original Virasoro uniformized data. In fact, let us consider a super theta function corresponding to an $N=1$ decomposable super Riemann surface with trivial $\mathcal{O}_{X_1}$. There are many such super Riemann surfaces, but the above proposition tells us that the theta function in this case is 1. On the other hand we have the following lemma.

Lemma 5.4. *If $N=1$ and if we restrict ourselves to the decomposable super Riemann surfaces, the pair of super tau function $\tau(X)$ and the tau function of the reduction $\tau(X_{\mathrm{rd}})$ recovers the original super Riemann surface.*

Proof. This is an easy consequence of the fact that the line bundle on a Riemann surface is determined by its tau finction [4].

This lemma applies in particular to a family with pure even base, that is, a family with S a usual scheme.

Second, we see that the tau function thus defined is, in the decomposable case, a section of a tensor product of determinant line bundles over the orbit of the Jacobian flows (flows made by the tensor product by elements of the Jacobian) on the moduli space of vector bundles of rank N. (Here we mean by "determinant line bundles on the moduli space of vector bundle" the bondle whose fiber at each point $[\mathbf{E}]$ of moduli space the line $\det \mathbf{R}\pi_* (\wedge^k \mathbf{E})$.) In order to obtain a definition of a theta function which is related to the full moduli space rather than the Jacobian flows, it is better to consider a determinant of differential operators.

5.2. Formal periodicity. For each multi index I and for each integer $i \leq c_I$, let us dnote by $f_{i,I}$ and element $\phi(z^{\alpha(i,I)}\zeta^I)$ which is a Laurent expansion of an element of A. Then we have the following.

Proposition 5.5. *Assume $N \geq 1$. Let our topological super algebra $\mathcal{B}$ be so chosen that it contains besides formal variables $\{x_{i,I}\}$ (needed to define super tau function), other super variables $\{y_{i,I}; I \subset [1, N], i \leq c_I\}$ which satisfy*

$$\deg(y_{i,I}) = \max(-i, 1)$$

$$\mathrm{parity}(y_{i,I}) = |I| \bmod 2$$

Let G be a formal function $G = \sum_{i,I} y_{i,I} f_{i,I}$. We denote by M_G the matrix on $\mathcal{H}$ which represents the effect of multiplication by G. We choose $e \in \mathbf{C}[[\{y_{i,I}\}]]$ such that $1 + c + p \circ M_G \circ \phi$ is an element of $\widetilde{\sum}_{\langle 1 \rangle}$ (Lemma 4.19). Then we have the following identity.

$$(5.5)\qquad \begin{aligned}&\mathrm{Ber}_{\mathscr{H}_{\bar{\alpha}},\mathrm{LU}}(p_\alpha\circ M_{\exp}(F)\circ\phi\circ(1+c+p_\alpha\circ M_G\circ\phi))\\ &=\mathrm{Ber}_{\mathscr{H}_{\bar{\alpha}},\mathrm{LU}}(p_\alpha\circ M_{\exp(F)}\circ\phi\circ\mathrm{Ber}_{\mathscr{H}_{\bar{\alpha}},\mathrm{LU}}(1+c+p_\alpha\circ M_G\circ\phi))\end{aligned}$$

where $M_{\exp(F)}$ *is the matrix of multiplication by* $\exp(F)$. (*This definion is valid only when* α *is a Maya diagram with which the denominator of the above Berezinian is not zero* (*for example*, α_0.)

Proof. This is a direct consequence of the Proposition 4.25.

6. Super Jacobians

In this section, we study the moduli space of line bundles on a super Riemann surface. The space may be called as a super Jacobian of the super Riemann surface. We encounter several unpleasant behavior of the super Jacobian. (We should note that the name "(super) Jacobian" should be preserved for an object which behaves better than the moduli space as Manin states in [9]. For example, Jacobian of an elliptic curve may be defined to be the elliptic curve itself, but it is not true in our definition (see subsection 6.1). In this paper, however, we temporary use this terminology.) In this section we fully use the fact that our ground field is **C**.

We may begin with the cohomological interpretation of the super Jacobian. For any super manifold M, line bundles on M is parametrized by the cohomology group

$$H^1(M;\mathscr{O}_{M,0}^{\times})$$

the set of transition functions modulo equivalence. The "tangent space" of this cohomology group is given by,

$$(6.1)\qquad H^1(M;\mathscr{O}_{M,0})$$

The first problem is that the dimension of this tangent space is not invariant under deformations of the space M. Consider for example a decomposable family $\pi\colon X\to S$ of super Riemann surfaces with odd dimension $N=1$, where the structure sheaf $\mathscr{O}_X$ is decomposed as follows:

$$\mathscr{O}_X\cong(\mathscr{O}_{X\mathrm{rd}}\oplus\Pi\mathbf{L})\otimes\mathscr{O}_S\ .$$

A relative counterpart $R^1\pi_*(\mathscr{O}_{X_0})$ of the cohomology group 6.1 is expressed in terms of the usual highter direct image sheaves as

$$(R^1\pi_*(X_{\mathrm{rd}};\mathscr{O}_{X\mathrm{rd}})\otimes\mathscr{O}_{S,0})\oplus\Pi R^1\pi_*(X_{\mathrm{rd}};\mathbf{L})\otimes\mathscr{O}_{S,1})$$

which is not necessarily flat, To overcome this difficulty, we may consider the moduli space of line bundles with "trivializations of jets".

Definition 6.1. Let n be a non negative integer. An n-trivialized line bundle (L, t) on a Virasoro uniformized super Riemann surface (X, π, S, q, z) is a line bundle L on X with isomorphism

$$t\colon L/L((n+1)D)\cong\mathcal{O}_S[[z]][\zeta_1, \dots, \zeta_N]/\mathcal{O}_S[[z]][\zeta_1, \dots, \zeta_N]z^{n+1} \ ,$$

where the divisor D is determined by the equation $z=0$.

We again express the moduli of n-trivialized line bundles in cohomological terms. The objects are parametrized by the sheaf of groups

$$R^1\pi_*(X;\ (1+\mathcal{O}_X(-(n+1)D))_0^\times) \ ,$$

and the tangent space of this sheaf at the origin is expressed as

$$R^1\pi_*(X;\ \mathcal{O}_X(-(n+1)D))_0) \ .$$

Lemmas 3.7 and 3.10 assure us that locally on S there exists an integer n_0 such that the sheaf is flat for all $n\geq n_0$.

Let us use analytic geometry. The exponential sequence

$$0\to\mathbf{Z}_{;n}\to\mathcal{O}_{X,\mathrm{an}}(-(n+1)D)_0\xrightarrow{\exp(2\pi i\cdot)}(1+\mathcal{O}_{X,\mathrm{an}}(-(n+1)D))_0^\times\to 1$$

gives rise to an exact sequence of cohomologies

$$\begin{aligned}(6.3)\qquad &0\to R^1\pi_*(\mathbf{Z}_{;n})\to R^1\pi_*(\mathcal{O}_{X,\mathrm{an}}(-(n+1)D)_0)\\ &\quad\to R^1\pi_*(1+\mathcal{O}_{X,\mathrm{an}}(-(n+1)D)_0^\times)\to R^2\pi_*(\mathbf{Z}_{;n}),\end{aligned}$$

where

$$\mathbf{Z}_{;n}=\mathbf{Z}\cap\mathcal{O}_{X,\mathrm{an}}(-(n+1)D)_0=j_!(\mathbf{Z}_{\dot{X}}),\quad j\colon\dot{X}\to X\text{: the inclusion.}$$

We deduce on the other hand from Proposition 3.5 that

$$(6.4)\qquad R^1\pi_*(\mathcal{O}_{X,\mathrm{an}}(-(n+1)D)_0\cong(R^1\pi_*(X;\ \mathcal{O}_X(-(n+1)D)_0))_{\mathrm{an}} \ ,$$

$$(6.5)\quad R^1\pi_*((1+\mathcal{O}_{X,\mathrm{an}}(-(n+1)D))_0^\times)\cong(R^1\pi_*(X;\ (1+\mathcal{O}_X(-(n+1)D))_0^\times))_{\mathrm{an}} \ .$$

We have the follosing

Lemma 6.1. *Assume that $R^1\pi_*(\mathcal{O}_X(-(n+1)D)$ is locally free. Then, there exists an analytic variety $Jac_n(X/S)_{\mathrm{an}}$ such that the sheaf of sections to $Jac_n(X/S)_{\mathrm{an}}$ is isomorphic to*

$$(R^1\pi_*(\mathcal{O}_X(-(n+1)D)_0))_{\mathrm{an}}/R^1\pi_*(\mathbf{Z}_{;n}) \ .$$

Proof. Put $E=\mathrm{Specan}(S((R^1\pi_*(\mathcal{O}_{X,\mathrm{an}}(-(n+1)D)^\vee))$. Since the image of the composition of maps

$$R^1\pi_*(\mathbf{Z})\to R^1\pi_*(\mathcal{O}_{X,\mathrm{an}}(-(n+1)D)\to R^1\pi_*(\mathcal{O}_{X,\mathrm{rd,an}}(-(n+1)D_{\mathrm{rd}}))$$

gives a sheaf of discrete subgroups, we see that the local system L on S corresponding to $R^1\pi_*(\mathbf{Z})$ forms a discrete subgroups of E along each fibers. We put $Jac_n(X/S)=E/L$.

Under the assumption of the preceding lemma, sheaves $R^1\pi_*(\mathcal{O}_X(-(n+

$1)D)_0)$ and $R^0\pi_*(\omega_{X/S}(+(n+1)D)_0)$ are the dual to each other under the residue pairing at Q. Namely, we may represent each section of $R^1\pi_*(\mathcal{O}_X(-(n+1)D)_0$ by an element f of $\mathcal{O}_X(-(n+1)D)(U)_0$. Then the dual pairing is defined by

$$(f, \omega)=\mathrm{ress}_Q(f\omega) \quad (\omega\in\omega_{X/S}(+(n+1)D) \ . \tag{6.6}$$

The residue theorem implies that this pairing is in fact well defined and perfect. This duality extends naturally to the analytic counterpart via the GAGA relations 6.4 and 6.5. Lemma 3.7 is valid and duality between $R^1\pi_*(\mathcal{O}_X(-(n+1)D)_0)_{\mathrm{an}}$ and $R^0\pi_*(\omega_{X/S}(+(n+1)D)_0)_{\mathrm{an}}$ is defined by 6.6, where we replace $\mathcal{O}_X(\dot{X})$ by $\mathcal{O}_{X,\mathrm{an}}(\dot{X})$ and $\mathcal{O}_X(-(n+1)D)(U)_0$ by

$$\mathcal{O}_{X,\mathrm{an}}(-(n+1)D)(\dot{U})_0=\lim_{U:\mathrm{nbd\ of\ }Q}(\mathcal{O}_{X,\mathrm{an}}(N\backslash Q)_0)+\mathcal{O}_X(U)_0 \ .$$

We shrink S if necessary and fix a free basis $f_1, \dots, f_r, \phi_1, \dots, \phi_\rho$ of $R^0\pi_*(\omega_{X/S}(+(n+1)D)_0)$.

Lemma 6.2. *Let $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ be embedded smooth super paths on X whose reduction consists of a symplectic homology bass of $X_{\mathrm{rd,an}}$ on each fiber, Then the n-super Jacobian $Jac_n(X/S)$ is expressed as,*

$$\mathbf{A}^{(r|\rho)}/(\vec{p_1}, \dots, \vec{p_g}, \vec{q_1}, \dots, \vec{q_g}) \ ,$$

where the period vectors $\vec{p_1}, \dots, \vec{p_r}, \vec{q_1}, \dots, \vec{q_\rho}$ are defined as follows.

$$\vec{p_1}=\left(\int_{\alpha_1}f_1, \dots, \int_{\alpha_1}f_r, \int_{\alpha_1}\phi_1, \dots, \int_{\alpha_1}\phi_\rho\right) \ ,$$

$$\vdots$$

$$\vec{p_g}=\left(\int_{\alpha_g}f_1, \dots, \int_{\alpha_g}f_r, \int_{\alpha_g}\phi_1, \dots, \int_{\alpha_g}\phi_\rho\right) \ ,$$

$$\vec{q_1}=\left(\int_{\beta_1}f_1, \dots, \int_{\beta_1}f_r, \int_{\beta_1}\phi_1, \dots, \int_{\beta_1}\phi_\rho\right) \ ,$$

$$\vdots$$

$$\vec{q_g}=\left(\int_{\beta_g}f_1, \dots, \int_{\beta_g}f_r, \int_{\beta_g}\phi_1, \dots, \int_{\beta_g}\phi_\rho\right) \ .$$

Proof. First we notice that the residue theorem implies that for any $\omega\in\omega_{X/S}(+(n+1)D)$ we have

$$\int_C\omega=0 \quad (C\text{: closed super curve on } \dot{X} \text{ such that } C_{\mathrm{rd}} \text{ is a circle around } Q.)$$

and that for any closed super curve γ on $\dot{X}$, the value $\int_\gamma\omega$ is determined only by the homotopy class of γ_{rd}. The only problem here is to calculate the periods. This problem is local on S_{an}, so we may assume that S_{an} is a polydisc. This enables us to identify higher direct images $R^i\pi_*$ and cohomologies H^i in an obvious sense. We deduce from the exact sequence 6.3 the following description of $H^1(\mathcal{O}_{X\mathrm{an}}(-(n+1)D))$.

(6.7) $$H^1(\mathcal{O}_{X\mathrm{an}}(-(n+1)D))_0$$
$$\cong \mathcal{O}_{X\mathrm{an}}(\dot{U})_0/[\{(\log\mathcal{O}_{X\mathrm{an}}(\dot{X})^{\times})_0 \cap \mathcal{O}_{X\mathrm{an}}(\dot{U})_0\} + \mathcal{O}_{X\mathrm{an}}(-(n+1)D)(U)_0]$$

We need to explain "log" in the above formula. $2\pi i f$ is an element of $\{(\log\mathcal{O}_{X\mathrm{an}}(\dot{X})^{\times})_0 \cap \mathcal{O}_{X\mathrm{an}}(\dot{U})_0\}$ if and only if it is a logarithm (defined on an open neighbourhood of Q) of an analytic function h satisfying the following condition.

(1) h is an even analytic function on $\dot{X}$ whose reduction has no zeros on $\dot{X}$.

(2) $\log(h_{\mathrm{rd}})$ is well defined in a small neighbourhood of Q.

Let $\dot{X}_{\mathrm{an}}$ be the covering of $|X_{\mathrm{an}}|$ which is universal along each fibers. By pulling back the structure sheaf we may introduce on $\dot{X}_{\mathrm{an}}$ a natural structure of smooth analytic super space over S_{an}. $f=\log h$ may be interpreted as a univalent analytic function on $\dot{X}_{\mathrm{an}}$.

It is well known that $|X_{\mathrm{an}}|$ is obtained by gluing appropriate pairs of "wedges" of a "polygon" P (a fundamental region of X) in $|X_{\mathrm{an}}|$. Namely, we have

$$\partial P=\prod_{i=1}^{g}(|\alpha_i^{(1)}||\beta_i^{(1)}|(|\alpha_i^{(2)}|)^{-1}(|\beta_i^{(2)}|)^{-1}) \quad ,$$

where $|\alpha_i^{(1)}|$, $|\alpha_i^{(2)}|$ (respectively, $|\beta_i|$, $|\beta_i^{(2)}|$) are two lifts of $|\alpha_i|$ (respectively, $|\beta_i|$) to $|\tilde{X}|$. It is easy to verify that these paths lift to smooth super paths $\alpha_i^{(1)}$, $\alpha_i^{(2)}$, $\beta_i^{(1)}$, $\beta_i^{(2)}$ on $\widetilde{X}$ such that

(1) their projections to X are closed;

(2) the super path

$$C=\prod_{i=1}^{g}(\alpha_i^{(1)}\beta_i^{(1)}(\alpha_i^{(2)})^{-1}(\beta_i^{(2)})^{-1})$$

is also closed.

We may assume that Q is in the interior of the polygon P so that the super residue along Q is equal to the integral along the super path C. Thus, we have

$$(\omega|f)=\mathrm{res}_Q(\omega f)=\frac{1}{2\pi i}\int_C(\omega\log h)$$
$$=\sum_{i=1}^{g}\frac{1}{2\pi i}\left(\int_{\alpha_i^{(1)}}\omega\log h-\int_{\alpha_i^{(2)}}\omega\log h+\int_{\beta_i^{(1)}}\omega\log h-\int_{\beta_i^{(2)}}\omega\log h\right)$$

To calculate the last line, we may move the super paths $\alpha_i^{(1),(2)}$ and assume that the projections of these paths to X coincide with α_i. The same is true for beta paths. Then the difference of $\log h$ between (1) and (2) paths are integers and we obtain

$$(\omega|f)=\sum_{i=1}^{g}(n_i\int_{\alpha_i}\omega+m_i\int_{\beta_1}\omega) \ ,$$

where we have

$$\{n_i, m_i\}_{i=1}^{g}=\{(\log g|_{\alpha_i^{(1)}}-\log g|_{\alpha_i^{(2)}}),\ (\log g|_{\beta_i^{(1)}}-\log g|_{\beta_i^{(2)}})\}_{i=1}^{g} \ . \tag{6.8}$$

The only problem now is that elements 6.8 for various h span the whole $\mathbf{Z}^{2g}$. But this is equivalent to the same claim for the reduced family, which is well known.

We call the variety $Jac_n(X/S)_{\mathrm{an}}$ the n-super Jacobian of X over S. We give an example in the next subsection.

6.1. An example of super Jacobian. In this subsection we compute the n-super Jacobian of the odd family of $N=1$ super elliptic curves introduced in [7]. See also [9]. Put

$$S_{\mathrm{an}}=\{(T, \Upsilon);\ T\in\mathbf{C},\ \Re(T)<0,\ \Upsilon\text{: odd}\}$$

$$X_{\mathrm{an}}=\{(z,\zeta;\ T,\ \Upsilon);\ z\in\mathbf{C},\ \zeta\text{: odd},\ (T,\ \Upsilon)\in S\}/\sim$$

where the equivalence relation "$\sim$" is generated by the following relation.

$$(z,\ \zeta;\ T,\ \Upsilon)\sim(z+2\pi i,\ \zeta,\ T,\ \Upsilon) \ ,$$

$$(z,\ \zeta;\ T,\ \Upsilon)\sim(z+T+\zeta\Upsilon,\ \zeta,\ T,\ \Upsilon) \ .$$

We define q to be the (equivalence class of the) origin. Then $(X_{\mathrm{an}}, S_{\mathrm{an}}, \pi, q, (z, \zeta))$ forms (an analytic) family of Virasoro uniformized super Riemann surfaces.

Lemma 6.3. *The affine coordinate ring* $A=\varinjlim_k H^0(X_{\mathrm{an}};\ \mathcal{O}_{X\,\mathrm{an}}(+kQ))$ *of* $\dot{X}$ *is generated by the following functions.*

$$\wp(z,\ \zeta;\ T,\ \Upsilon)=\wp(z;\ T)+\frac{\partial}{\partial T}\wp(z;\ T)\zeta\Upsilon \ ,$$

$$\wp(z,\ \zeta;\ T,\ \Upsilon)=\frac{\partial}{\partial z}\wp(z,\ \zeta;\ T,\ \Upsilon) \ ,$$

$$\omega(z,\ \zeta;\ T,\ \Upsilon)=\zeta+(\mathfrak{z}(z,\ T)-\frac{1}{2\pi i}\eta_i(T)z)\Upsilon \ ,$$

where $\wp$, $\mathfrak{z}$, *respectively is the Weierstrass* $\wp$- *and zeta functions respectively. Namely,*

$$\mathfrak{z}(z;T)=\frac{1}{z}+\sum_{\substack{\omega\in(2\pi i\mathbf{Z}+T\mathbf{Z})\\ \omega\neq 0}}\left(\frac{1}{z-\omega}+\frac{1}{\omega}+\frac{z}{\omega^2}\right),$$

$$\mathfrak{p}(z;T)=\frac{\partial}{\partial z}\mathfrak{z}(z;T).$$

Proof. It is easy to verify that $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{z}$ are elements of A and that

$$A/\Upsilon A\cong\mathcal{O}_S(S)[\mathfrak{p},\mathfrak{p}',\mathfrak{z}]/\Upsilon\mathcal{O}_S(S)[\mathfrak{p},\mathfrak{p}',\mathfrak{z}].$$

This implies the lemma.

Corollary 6.4. *$R^1\pi_*(\mathcal{O}_X(-(n+1)D))$ is locally free for $n\geq 0$.*

Thus, we have n-super Jacobian for any non negative integer n.

There is a non vanishing section $\mathrm{Ber}(dz, d\zeta)$ of the dualizing sheaf $\omega_{X/S}=\mathrm{Ber}_{X/S}$. So the dualizing sheaf is trivial in this case. A basis of $\omega_{X/S}(+D)(\dot{X})$ is given as follows.

(6.9) $\quad \mathfrak{f}=\mathrm{Ber}(dz, d\zeta)$,

(6.10) $\quad \phi=\mathrm{Ber}(dz, d\zeta)\,\omega$.

The periods of X are as follows.

(6.11) $\quad \vec{p}=(0, 2\pi i)$

(6.12) $\quad \vec{q}=(\Upsilon, T)$.

7. Analytic theory of super theta function

Let $(X, S, \pi, q, (z, \zeta_1, \ldots, \zeta_N))$ be a Virasoro uniformized family (definition 3.1) with an affine base schems $S=\mathrm{Spec}B$.

We already know that a super tau function τ is defined in an algebraic way by using the theory of infinite determinants (definition 5.1). In this section we use analytic methods to show that the formal super tau function so defined is regarded as a holomorphic section of a line bundle on a Zariski open set (in the sense of analytic geometry) of the "super Jacobian" of the original family of super Riemann surfaces. This fact and calculations of examples given in the preceding section suggest that we may call it a super theta function.

Our work in this section may be regarded as a comparison between the formal topology introduced inthis paper and uniform topology on each compact set.

7.1. Additional conditions we need in this section. We fix in this section a Virasoro normalized family of super Riemann surfaces $(X, S, \pi, q, (z, \zeta_1, \ldots, \zeta_N))$, $N\geq 1$. We assume S to be affine, $S=\mathrm{Spec}B$. In this section we assume that the coordinate system $(z, \zeta_1, \ldots, \zeta_N)$ is an "actual" coordinate system on an open neighbourhood of $Q=q(S)$. We fix a compact neighbourhood K of the base point s_0 in S_{an} such that its interior K_0 is Stein. We equip B with a supremum norm on K. (A supremum norm is given as follows. We first cover K with finite coordinate open sets $(V_n;\ (s_1^{V_n}, \ldots, s_r^{V_n}, \sigma_1^{V_n}, \ldots, \sigma_\rho^{V_n}))$ in S. On V_n, any function f may be written in coordinate terms as

$$f=\sum_{J\subset[1,\rho]} f_J^{(V_n)}(s_1, \ldots, s_r)\,\sigma^J\ ,$$

where each "component" $f_J^{(V_n)}$ is an analytic function of r-variables. Put

$$\|f\|_K=\max_{n,J}\ \sup_{(s_1,\ldots,s_r)\in(V_n\cap K)_{\mathrm{rd}}} |f_J^{(V_n)}(s_1, \ldots, s_r)|\ .$$

The definition of the norm obviously depends on the choice of a coodinate covering, but the topology given by the norm is intrinsically defined.) Let B_{an} be the completion of B. It forms a Banach algebra. Every element of the algebra gives an analytic function on the interior of K. We take an positive number R_K such that $(z, \zeta_1, \ldots, \zeta_N)$ are actual coodinates for $\{|z|\le R_K\}\times\mathbf{A}^{(0|N)}\times K$.

7.2. Capability of substitution: announcement of theorems. The main purpose of this section is to see existence of a representative $(\tilde{\tau}(F(x);\ \mathbf{I});\ \varDelta(F(x)))$ of the tau function so that we may substitute the formal variables $\{x\}$ in $\tilde{\tau}(F(x);\ \mathbf{I})$, $\varDelta(x)$ by any sequence $\{a\}$ of coefficients of a convergent power series. We first define the space of "substitutes".

Definition 7.1. Let the space $\mathscr{C}$ of convergent power series be defined as follows

$$\mathscr{C}=\{a=\sum_{i,I} a_{i,I}z^i\zeta^I;\ \text{for all } \epsilon\in(0, R_K],\ \text{we have } \mathfrak{N}_\epsilon(a):=\sum_{i,I}\|a_{i,I}\|\epsilon^i<\infty\}$$

We are also interested in the convergent power series with no constant term. We define

$$\mathscr{C}_{\mathrm{nct}}=\{a=\sum_{i,I} a_{i,I}z^i\zeta^I\in\mathscr{C};\ a_{0,0}=0\}$$

We will estimate the coefficients of the tau function and show that the tau function is an element of the following class of functionals on $\mathscr{C}$.

Definition 7.2. We define a space of functionals on $\mathscr{C}$ as follows.

$$B_{\mathrm{an}}\langle\langle x\rangle\rangle=\{c=\sum_{w\in W} c_w x^w;\ \text{for all } a\in\mathscr{C},\ \text{we have } \|c\|_a:=\|c_w\|\|a^w\|<\infty\}\ .$$

Here the index set W is defined as

$$W=\{w: \Lambda \to \mathbf{N};\ \sum_{i,I} w(i,I)<\infty\}$$

and the symbols x^w, a^w denote respectively the monomials $\prod_{i,I} x_{i,I}^{w(i,I)}$, $\prod_{i,I} a_{i,I}^{w(i,I)}$. We also define the following space of functionals on $\mathscr{C}\times B_{\mathrm{an}}$.

$$B_{\mathrm{an}}\langle\langle x, \mathbf{I}\rangle\rangle=\left\{c=\sum_{w\in W, k\in\mathbf{N}} c_{w,k}x^w\mathbf{I}^k;\ \begin{array}{l}\text{for all } a\in\mathscr{C}, \text{ and for any } M>0,\\ \text{we have } \|c\|_{a,M}:=\|c_{w,k}\|\|a^w\|M^k<\infty\end{array}\right\}.$$

We now state the following main theorems of the present paper.

Theorem 7.1. *There exists a representative* $(\tilde{\tau};\ \Delta)$ *of the super tau function such that* Δ *is an element of* $B_{\mathrm{an}}\langle\langle x\rangle\rangle$ *and that* $\tilde{\tau}$ *is an element of* $B_{\mathrm{an}}\langle\langle x, \mathbf{I}\rangle\rangle$.

So the super tau function is a holomorphic function on $\{a\in\mathbf{C}_{\mathrm{nct}};\ \Delta(a)\neq 0\}$. By the consideration in the preceding section we see immediately that the n-super Jacobian is a quotient of $\mathscr{C}$.

Theorem 7.2. *The super tau function* τ *has periods such that the super tau function is regarded as a holomorphic section of a line bundle* L *on a Zariski open subset (in the sense of analytic geometry) of the super* n*-super Jacobian for some* n.

We shall give proofs of the above two theorems in the rest of this section.

7.3. Analytic theory of infinite determinants. We first note that our space of functionals $B_{\mathrm{an}}\langle\langle x\rangle\rangle$ is an intersection of Banach spaces

$$B_{\mathrm{an}}\langle\langle x\rangle\rangle_a=\{c=\sum_{w\in W} c_w x^w;\ \|c\|_a:=\|c_w\|\|a^w\|<\infty\}\ .$$

where a is an element of $\mathscr{C}$. Therefore, to show that an element d of $B_{\mathrm{an}}[[x]]$ belongs to the algebra $B_{\mathrm{an}}\langle\langle x\rangle\rangle$ is equivalent to proving that $d\in B_{\mathrm{an}}\langle\langle x\rangle\rangle_a$ for all $a\in\mathscr{C}$. So we fix such a in the rest of this section and will show that our elements have finite a-norms. Several objects defined in the following depend on the choice of a, but we will not mention it explicitly.

We first define the following algebras of operators.

Definition 7.3.

$$E_{\mathrm{an}}=\left\{(b_{ij})_{i\le 0, j\le 0};\ \begin{array}{l}\text{for all } \epsilon>0, \text{ there exit } M_\epsilon>0,\ A_\epsilon>0\\ \text{such that } \|b_{ij}\|_a<A_\epsilon\epsilon^{|i|}M_\epsilon^{|j|}\end{array}\right\}$$

$$K_{\mathrm{an}}=\left\{(k_{ij})_{i\le 0, j\le 0};\ \begin{array}{l}\text{there exists } M>0\\ \text{such that for all } \epsilon>0, \text{ there exits } A_\epsilon>0\\ \text{such that } \|k_{ij}\|_a<A_\epsilon\epsilon^{|i|}M^{|j|}\end{array}\right\}$$

It is easy to see that the set E_{an} forms an algebra and that set K_{an} forms an ideal of the algebra E_{an}. The following lemma is fundamental.

Lemma 7.3. *Let K be an element of K_{an}, i_0, i_1 be non positive integers with $|i_0|\leq|i_1|$ and let σ_0 be the transposition of i_0 and i_1, an permulation of $(-\mathbf{N})$. Let D be a matrix such that*

$$(D)_{ij}=0 \text{ unless } i=\sigma_0(j)\ ,$$

$$(D)_{\sigma(j),j}=0 \text{ or } 1 \text{ for all } j\ .$$

Then, for any positive number ϵ there exist constants C_1, C_2, independent of i_0, i_1 such that we have

$$\|\det(p_n(D+K)\iota_n)\|_a\leq C_1+C_2\epsilon^{|i_0|}M^{|i_1|}$$

for any positive integer n.

Proof. We note first that the product $C_0=\prod_{i\leq 0}(1+\|(P)_{ii}\|)$ converges.

For any element M of E_{an}, we denote the matrix $p_n\circ M\circ\iota_n$ by $M^{(n)}$. Then we have

$$\begin{aligned}
&\|\det(D+P)^{(n)}\|\\
&\leq C_0\sum_{m\geq 0}\ \sum_{\substack{J\subset[0,-n]\\|J|=m}}\ \sum_{\substack{\sigma\in\mathfrak{S}([0,-n])\\ J=\{j;\tau(j)=\sigma(j)\}}}\ \prod_{j\notin J}\|(P)_{\sigma(j),j}\|\\
&=C_0\sum_{m\geq 0}\ \sum_{\substack{J\subset[0,-n]\\|J|=m}}\ \sum_{\substack{\tilde\sigma\in\mathfrak{S}([0,-n])\\ J=\{j;j(j)=\sigma(j)\}}}\ \prod_{j\notin J}\|(P)_{\tau(\tilde\sigma(j)),j}\|\qquad(\tilde\sigma=\tau^{-1}\sigma)\\
&=C_0\sum_{m\geq 0}\ \sum_{\substack{J\subset[0,-n]\\|J|=m}}\ \sum_{\tilde\sigma\in\mathfrak{S}([0,-n]\setminus J)}\ \prod_{j\notin J}\|(P)_{\tau(\tilde\sigma(j)),j}\|\\
&\leq C_0\sum_{m\geq 0}\ \sum_{\substack{J\subset[0,-n]\\|J|=m}}\ \sum_{\tilde\sigma\in\mathfrak{S}([0,-n]\setminus J)}\ \prod_{j\notin J}A_\epsilon\epsilon^{|\tau(\tilde\sigma(j))|}M^{|j|}\\
&=C_0\sum_{m\geq 0}\ \sum_{\substack{I\subset[0,-n]\\|I|=m}}\ \sum_{\tilde\sigma\in\mathfrak{S}(I)}A_\epsilon^{|I|}\epsilon^{\mathrm{ht}(\tau(I))}M^{\mathrm{ht}(I)}\qquad(I=[0,\ -n]\setminus J)\\
&=C_0\sum_{m\geq 0}\ \sum_{\substack{I\subset[0,-n]\\|I|=m}}\ \sum_{\tilde\sigma\in\mathfrak{S}(I)}m!A_\epsilon^{|I|}\epsilon^{\mathrm{ht}(\tau(I))}M^{\mathrm{ht}(I)}\qquad(I=[0,\ -n]\setminus J)\\
&=C_0\sum_{m\geq 0}\ \sum_{\substack{|I|=m\\ i_1\notin I}}+\sum_{m\geq 0}\ \sum_{\substack{|I|=m\\ i_1\notin I}})
\end{aligned}$$

$$=C_0\sum_{m\geq 0} m!A_\epsilon^m(\epsilon M)^{m(m-1)/2}(1-\epsilon M)^m$$

$$+C_0\sum_{\tilde m\geq 1}(\tilde m+1)!A_\epsilon^{\tilde m+1}(\epsilon M)^{\tilde m(\tilde m-1)/2}(1-\epsilon M)^{\tilde m}\epsilon^{|i_0|}M^{|i_1|}\ ,$$

where we have used the notation "ht" to mean "the height of a set I" defined as $\mathrm{ht}(I)=\sum_{j\in J}|j|$. This completes the proof.

The lemma leads to the following proposition.

Proposition 7.4. *Let K be the same as in Lemma 7.3. Then the sequence*

$$\{\det(p_n(1+K)\iota_n)\}$$

converges as n tends to infinity.

Proof. We need to show that $\{\det(1+K)^{(n)}\}_{n\geq 0}$ forms a Cauchy sequence. This is an easy exercise of application of the above lemma.

Corollary 7.5. *Let K be an element of $K_{\mathrm{an}}\cap\mathcal{K}$. Then the finite determinant of K (as an element of $\mathcal{K}$) belongs to $B_{\mathrm{an}}\langle\langle x\rangle\rangle|_a$.*

7.4. Existence of a family of normalized function with good estimate. To define the formal super tau function, we have used a normalized family of functions in $A=\mathcal{O}_X(\dot X)$. A change of choice of such a family differs τ by a constant multiplication. In this subsection, we show that we have a nice choice of such a family suitable for estimations.

Proposition 7.6. *There exist a family of normalized functions $f_{i,I}\in A$ such that*

(1) $f_{i,I}=z^i\zeta^I+\sum_{j,J}a_{i,I,j,J}z^j\zeta^J$,

(2) *There exist positive numbers $C>0$, $M>0$ such that*

$$\|a_{i,I,j,J}\|_K\leq CM^{|i|}$$

For the proof, we need the following lemma.

Lemma 7.7. *There exists a section f of $p_2^*(\omega_{X/S})(+\Delta)$ on $\dot X\times\dot X$ with the following property.*

For all local coordinate system $(z^{(V)},\zeta_1^{(V)},\ldots,\zeta_N^{(V)})$ defined on an open set V on $\dot X$, we have,

$$f=\frac{\mathrm{Ber}(d\bar z^{(V)},d\bar\zeta_1^{(V)},\ldots,d\bar\zeta_N^{(V)})}{\bar z-z}\prod_{i=1}^N(\bar\zeta_i-\zeta_i)+homolomorphic\ terms \tag{7.1}$$

around Δ, where $(z^{(V)},{}_1^{(V)},\ldots,\zeta_N^{(V)},\bar z^{(V)},\bar\zeta_1^{(V)},\ldots,\bar\zeta_N^{(V)})$ is a local coordinate system *of $\dot{X}\times\dot{X}$ defined on $V\times V$.*

Proof. Either by using the super residue theorem or by doing a direct computation, we see that the condition 7.1 is independent of the choice of local coordinate systems. The lemma is then clear because $\dot{X}\times\dot{X}$ is affine.

Proof of the Proposition 7.6. We set

$$f_{i,I}=(-1)^{i+|I|}\mathrm{ress}_Q(f(z,\zeta_1,\dots,\zeta_N,\bar{z},\bar{\zeta}_1,\dots,\bar{\zeta}_N)\bar{z}^{-i}\bar{\zeta}^I)\ ,$$

or sufficiently large i. (See super residue theorem (Proposition 3.13) for the definition of super residue (ress).) Then the super residue above may be written in terms of a contour integral. The estimation follows easily from that.

7.5. Proof of the main theorems.

Definition 7.4.

$$(\mathscr{B}\{\{z\}\}[\zeta_1,\dots,\zeta_N])^\times_{\langle 1\rangle}\overset{\mathrm{def}}{=}(\mathscr{B}\{\{z\}\}[\zeta_1,\dots,\zeta_N])^\times_-\times(\mathscr{B}\{\{z\}\}[\zeta_1,\dots,\zeta_N])^\times_-\ ,$$

where

$$(\mathscr{B}\{\{z\}\}[\zeta_1,\dots,\zeta_N])^\times_-=\{\sum_{i\geq 0}a_{i,I}z^{i,I};\ a_{0,0}=0\}\ ,$$

$$(\mathscr{B}\{\{z\}\}[\zeta_1,\dots,\zeta_N])^\times_+=\{1+\sum_{i<0}a_{i,I}z^i\zeta^I\in\mathscr{B}\{\{z\}\}[\zeta_1,\dots,\zeta_N])\}\ .$$

Lemma 7.8. *Let K be an element of $\mathscr{K}_{\mathrm{an}}$. Let us denote by $(1+K)_{[i_0j_0]}$ the matrix obtained by omitting i_0-th row and j_0-th column from the matrix $(1+K)$. Then we have the following estimate.*

$$\|\det_{\mathrm{LU}}(1+K)_{[i_0,j_0]}\|\leq\sum_{\substack{G\subset -\mathbf{N}\\ |G|:finite}}(|G|+1)!A_\epsilon^{|G|}\epsilon^{\mathrm{ht}(G)}M^{\mathrm{ht}(G)}\epsilon^{|j_0|}M^{|i_0|}$$

The proof is quite similar to that of Lemma 7.3. and is omitted.

Definition 7.5. *We define the following algebras of operators on $\mathscr{H}^-_{\alpha,0}$.*

$$\widetilde{\Sigma}_{\mathrm{super,an}}=\left\{M;\ \begin{matrix}\exists f=\sigma(M)\in\mathscr{B}\{\{z\}\}[\zeta_1,\dots,\zeta_N]\ (\text{necessarily unique})\\ \text{such that }(M-p\circ M_f\circ\phi)\in\mathscr{K}\end{matrix}\right\}$$

$$\widetilde{\Sigma}_{\mathrm{super},\langle 1\rangle}=\{M\in\widetilde{\Sigma_{\mathrm{super}}};\ \sigma(M)\text{ is an element of }(\mathscr{B}\{\{z\}\}[\zeta_1,\dots,\zeta_N])^\times_1\}$$

Proof of Theorem 7.1. First of all, we recall we have to choose a basis $\{f_{i,I}\}$ of $A=\mathscr{O}_X(\dot{X})$ to define the pre super tau function. We fix them to satisfy the condition stated in Theorem 7.6. We then let

$$p\circ M_{\exp F}\circ\phi=\begin{pmatrix}T^{00} & T^{01}\\ T^{10} & T^{11}\end{pmatrix}$$

be the even-odd decomposition. It is easy to see that both T^{00} and T^{11} are the element of the set $\widetilde{\sum}_{\mathrm{super,an}}$. The Lemma 7.8 shows that $\widehat{T^{11}}$ is also an element of the set and so therefore is $T^{00}-IT^{01}\widehat{T^{11}}T^{10}$. So the LU-determinants of them are well defined.

Proof of Theorem 7.2. We apply the formal periodicity (Proposition 5.5). We see that both hand sides of the equation 5.5 are continuous functionals on $(x, y)\in\mathscr{C}\times\mathscr{C}$. The equation means that Taylor expansion of both hand sides coincides term by term so that it is also valid as an equation of functionals. This computes an effect of multiplications by A_{an}, the completion of $A=\mathscr{O}_X(\dot{X})$ in $\mathscr{C}$. By the maximum principle, the algebra A_{an} coincides with the algebra of analytic functions on $\dot{X}\cap\pi^{-1}(K)$. We conclude that the super tau function may be thus considered as an function on the following space.
$\mathscr{C}_{\mathrm{nct}}/\log(A^{\times}_{\mathrm{an},\langle 1\rangle})$ $(A^{\times}_{\mathrm{an},\langle 1\rangle}=\{f\in A_{\mathrm{an}};\ \mathrm{Ber}_{\mathrm{LU}}(p\circ M_f\circ\phi)$ is well defined and is equal to $1\})$.
The Lemma 4.19. shows us that

$$A^{\times}_{\mathrm{an}}/A^{\times}_{\mathrm{an},*}\cong B^{\times}_{\mathrm{an}} \quad (A^{\times}_{\mathrm{an},*}=\{f\in A_{\mathrm{an}};\ \mathrm{Ber}_{\mathrm{LU}}(p\circ M_f\circ\phi) \text{ is well defined}\})\ ,$$

and the homomorphism theorem shows that $A^{\times}_{\mathrm{an},*}/A_{\mathrm{an},\langle 1\rangle}$ is a subset of $(B^{\times}_{\mathrm{an}})$. We thus see that $A_{\mathrm{an},\langle 1\rangle}$ is "large enough".

We prove in a simlar way that there exists an integer n such that,

$$\tau_{\alpha}(F\times\log(G))=\tau_{\alpha}(F)\ ,$$

for any element G of

$$\mathscr{C}_{+,n}=\{f\in\mathscr{C};\ f=1+\sum_{i\geq n}a_{i,I}z^i\zeta^I\}\ .$$

This shows that the super tau function descends furthermore to the space,

$$\mathscr{C}_{\mathrm{nct}}/\log(\mathscr{C}_{+,n}\times A_{\mathrm{an},\langle 1\rangle})$$

This space is a subset of a line bundle of over the following space.

$$\mathscr{C}_{\mathrm{nct}}/\log(\mathscr{C}_{+,n}\times A_{\mathrm{an},*})$$

But the above space is n-Jacobian. We state it in a form of lemma.

Lemma 7.9.

$$\mathscr{C}_{\mathrm{nct}}/\log(\mathscr{C}_{+,n}\times A_{\mathrm{an},*})\cong\mathrm{Jac}_n(X/S)_{\mathrm{an}}\ .$$

Proof of Theorem 7.3. Analogous to the proof of equation 6.7 in Lemma 6.2. This completes the proof of the theorem.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

References

[1] A. A. Beilinson and V. V. Schechtman, Determinant bundles and Virasoro algebras, Commun Math. Phys., **118** (1988), 651-701.

[2] R. Hartshorne, Algebraic geometry, Springe Verlag, 1977.

[3] N. J. Hitchin, Flat connections and geometric quantization, Commun, Math. Phys., **131**-2 (1990), 347-2180.

[4] N. Kawamoto, and Y. Namikawa, A. Tsuchiya, and Y. Yamada, Geometric realization of conformal field theory on Riemann surfaces, Commun. Math. Phys., **116** (1988), 247-308.

[5] S. Klimek and A. Lesniewski, Quantum Riemann surfaces I. the unit disc, Commun. Math. Phys., **146** (1992), 103-122.

[6] F. F. Knudsen and D. Mumford, The projectivity of the moduli space of stable curves. I. preliminaries on "det" and "div", Math. Scand., **39** (1976), 19-55.

[7] A. Levin, Supersymmetric algebraic curves, Funke. Analiz i ego Priloz., **21** (1987), 83-84.

[8] Yuri I. Manin, Gauge field theory and complex geometry, Springer Verlag, 1988.

[9] Yuri I. Manin, Topics in non commutative geometry, Princeton University Press, 1991.

[10] A. S. Schwarz, Fermionic string and universal moduli space, Nucl. Phys., **B317** (1989), 323-343.

[11] J. P. Serre, Géométrie algébrique et géométrie analytique, Ann. Inst. Fourier., **6** (1956), 1-42.

[12] Y. Tsuchimoto, On the coordinate-free description of the conformal blocks, J. Math. Kyoto Univ., **33** (1993), 29-49.

[13] A. Tsuchiya, K. Ueno, and Y. Yamada, Conformal field theory on universal family of stable curves with gauge symmeties, Advanced Studies in Pure Math., **19** (1989), 459-566.