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An increasing union of q-complete manifolds whose limit is not q-complete

By

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In this short note, modeling on the beautiful example of Fornæss [2], we produce, for any given integer $q \geq 1$, a complex manifold M which *is an increasing union of q-complete open submanifolds $\{M_\nu\}_{\nu\in\mathbf{N}}$ such that M itself fails to be q-complete.*

For $q=1$, we regain Fornæss' example, however, with a different proof.

To proceed, we recall some definitions [1].

Let M be a complex manifold (always countable at infinity) of dimension n. A function $\varphi \in C^\infty(M, \mathbf{R})$ is said to be q-convex if the Levi form of φ computed in local coordinates has at least $n-q+1$ strictly positive eigenvalues.

M is said to be q-complete (resp. q-convex) if it carries a smooth exhaustion function φ (i. e., such that the sublevel set $\{x\in M \mid \varphi(x)<c\}$ is relatively compact in M for any $c\in\mathbf{R}$) which is q-convex on the whole space M (resp. outside a compact subset of M).

A typical situation in our set-up is :

Example 1. *Let L be a linear subspace of codimension q of the complex projective space $\mathbf{P}^n$. Then $\mathbf{P}^n - L$ is q-complete.*

Proof. Indeed, without any loss of generality, we may take $L=\{w_{p+1}=\cdots=w_n=0\}$, $p:=n-q$, where $[w_0 : \ldots : w_n]$ are the homogeneous coordinates on $\mathbf{P}^n$. We check that $\varphi : \mathbf{P}^n - L \rightarrow \mathbf{R}$ given by

$$\varphi(w) := \log\frac{|w_0|^2+\cdots+|w_n|^2}{|w_{p+1}|^2+\cdots+|w_n|^2}, \qquad w\in\mathbf{P}^n - L$$

is q-convex and exhaustive.

Since the exhaustion property is obvious, it remains to show the q-convexity. To verify this pass to non-homogeneous coordinates and check that the function $\psi : \mathbf{C}^n \rightarrow \mathbf{R}$ by

$$\psi(z) = \log\frac{1+|z_1|^2+\cdots+|z_n|^2}{1+|z_{p+1}|^2+\cdots+|z_{n-1}|^2}, \qquad z\in\mathbf{C}^n$$

is q-convex. But this is quite simple! To see this, we let $F\subset\mathbf{C}^n$ be the complex

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vector subspace given by $F:=\{\xi\in\mathbf{C}^n \mid \xi_{p+1}=\cdots=\xi_{n-1}=0\}$. Then dim $F=n-q+1$ and for any $z\in\mathbf{C}^n$, $\xi\in F$, $\xi\neq 0$, one can verify readily that $L(\psi, z)\xi>0$.

Less trivial examples are produced by the next

Lemma 1. *Let $E\subset\mathbf{C}^n$ be a complex vector subspace of codimension $q\geq 1$ and $a^{(1)}, \ldots, a^{(k)}$ some points of E. Let X be the blowing-up of $\mathbf{C}^n$ at $a^{(1)}, \ldots, a^{(k)}$ and E' the proper transform of E. Then $X-E'$ is q-complete.*

Proof. Without losing the generality, we may suppose $E=\{z\in\mathbf{C}^n \mid z_{p+1}=\cdots=z_n=0\}$ where $p:=n-q\geq 1$. We divide the proof into two steps.

Step I. Here we deal with the case $k=1$. Set $a^{(1)}=(a_{11}, \ldots, a_{1p}, 0, \ldots, 0)$. Consequently, X has the following description:

$$X=\left\{(z, w)\in\mathbf{C}^n\times\mathbf{P}^{n-1} \mid \frac{z_1-a_{11}}{w_1}=\cdots=\frac{z_p-a_{1p}}{w_p}=\frac{z_{p+1}}{w_{p+1}}=\cdots=\frac{z_n}{w_n}\right\}.$$

Let $\pi: X\to\mathbf{C}^n$ be the canonical map induced by the projection $\mathbf{C}^n\times\mathbf{P}^{n-1}\to\mathbf{C}^n$. Moreover, $E'=\{(z, w)\in X \mid w_{p+1}=\cdots=w_n=0\}$. Now set $\varphi_1=\varphi_{a^{(1)}}: X-E'\to\mathbf{R}$ by

$$\varphi_1(z, w):=\log\frac{|w_1|^2+\cdots+|w_n|^2}{|w_{p+1}|^2+\cdots+|w_n|^2},\quad (z, w)\in X-E'.$$

Since $X-\pi^{-1}(a^{(1)})\simeq\mathbf{C}^n-\{a^{(1)}\}$, we have

$$\varphi_1(z, w)=\log\frac{\|z-a^{(1)}\|^2}{|z_{p+1}|^2+\cdots+|z_n|^2},\quad (z, w)\in X-\pi^{-1}(a^{(1)}).$$

By the above example, φ_1 is q-convex. Further, we let $\sigma: \mathbf{C}^n\to\mathbf{R}$, $\sigma(z)=\|z\|^2$, $z\in\mathbf{C}^n$. Then since, π is proper, $\varphi_1+\sigma\circ\pi$ defines the q-completeness of $X-E'$.

Step II. Let $k\geq 2$ and $\pi: X\to\mathbf{C}^n$ be the canonical proper map. Define $\theta\in C^\infty(\mathbf{C}^n-E, \mathbf{R})$ by $\theta(z):=\log(|z_{p+1}|^2+\cdots+|z_n|^2)$, $z\in\mathbf{C}^n-E$. Note that by definition $X-\pi^{-1}(\{a^{(1)}, \ldots, a^{(k)}\})\simeq\mathbf{C}^n-\{a^{(1)}, \ldots, a^{(k)}\}$. Put now $E'_j:=E'\cup\bigcup_{i\neq j}\pi^{-1}(a^{(i)})$ and $\theta_j\in C^\infty(\mathbf{C}^n-\{a^{(j)}\}, \mathbf{R})$ by $\theta_j(z):=\log\|z-a^{(j)}\|^2$, $z\in\mathbf{C}^n$, $z\neq a^{(j)}$, $1\leq j\leq k$. Then, as in Step I we can produce $\varphi_j\in C^\infty(X-E'_j, \mathbf{R})$ such that

$$\varphi_j=\theta_j\circ\pi-\theta\circ\pi \text{ on } X-\widetilde{E}$$

where $\widetilde{E}=\pi^{-1}(E)=E'\cup\bigcup_{i=1}^{k}\pi^{-1}(a^{(i)})$.

We shall show that the function $\varphi_1+\cdots+\varphi_k+(k-1)\theta\circ\pi$ extends over $\widetilde{E}-E'$, and moreover, it defines a function $\varphi\in C^\infty(X-E', \mathbf{R})$.

Indeed, this is a local question on $\pi^{-1}(a^{(j)})-E'$, $j=1, \ldots, k$. Thus fix an index $j_0\in\{1, \cdots, k\}$ and $(z^{(0)}, w^{(0)})\in\pi^{-1}(a^{(j_0)})-E'$, say $w_n^{(0)}\neq 0$. Suppose, without any loss of generality that $a^{(j_0)}=(0, \ldots, 0)$. Hence, on a suitable neighborhood of $(z^{(0)}, w^{(0)})$ in $X-E'$, we have a canonical coordinate system given by

$$(t_1, \ldots, t_n) \rightsquigarrow (t_1 t_n, \ldots, t_{n-1}t_n, t_n, [t_1 : \ldots : t_{n-1} : 1]) =: (z(t), w(t)).$$

Now, for $t_n \neq 0$ we get

$$(\clubsuit) \qquad \varphi(z(t), w(t)) = \log \frac{1+|t_1|^2+\cdots+|t_{n-1}|^2}{1+|t_{p+1}|^2+\cdots+|t_{n-1}|^2} + \sum_{i \neq j_0} \log\|z(t) - a^{(i)}\|^2 ,$$

which clearly extends over $t_n = 0$.

On the other hand, by $(\clubsuit)$, it can be checked that $\varphi + \sigma \circ \pi$ is q-convex and exhaustive (Here we use simple facts like: the function

$$\mathbf{C}^n \ni t \mapsto |t_n|^2 + \log \frac{1+|t_1|^2+\cdots+|t_{n-1}|^2}{1+|t_{p+1}|^2+\cdots+|t_{n-1}|^2} \in \mathbf{R}$$

is q-convex and that $t \mapsto \log\|z(t) - a^{(i)}\|^2$ are plurisubharmonic for $i \neq j_0$ for $\|t\|$ small enough).

Consequently $X-E'$ is q-complete.

Here we produce the above mentioned example.

Let $n \geq q+1$ and consider E a complex vector subspace of $\mathbf{C}^n$ of codimension q. Take an arbitrary sequence of mutually distinct points $\{a^{(\nu)}\}_{\nu \in \mathbf{N}}$ in E that converges to the origin $0 \in \mathbf{C}^n$, $a^{(\nu)} \neq 0$, $\forall \nu \in \mathbf{N}$.

Let X be the blowing-up of $\mathbf{C}^n - \{0\}$ at this sequence (that is a smooth 0-dimensional closed submanifold) and $\pi : X \to \mathbf{C}^n - \{0\}$ the blowing-up map. Thus, X is an n-dimensional connected complex manifold. Let E' be the proper transform of E.

We make the following

CLAIM. *$M := X - E'$ is an increasing union of q-complete open submanifolds $\{M_\nu\}_{\nu \in \mathbf{N}}$, and M is not q-complete.*

Indeed, let X_ν be the blowing-up of $\mathbf{C}^n$ at the points $\{a^{(1)}, \ldots, a^{(\nu)}\}$, $\nu \geq 1$, and $\pi_\nu : X_\nu \to \mathbf{C}^n$ the blowing-up maps. Set $E'_\nu :=$ the proper transform of E. Then, $X_\nu - E'_\nu$ can be viewed, canonically, as an open subset M_ν of M, that, by Lemma 1, is q-complete. Further, it is evident that the sequence $\{M_\nu\}$ increases to M.

To conclude the claim, it remains to show that M itself is *not* q-complete (As a matter of fact it is not even q-convex).

In order to do this, assume that M carries a q-convex exhaustion function $\psi : M \to \mathbf{R}$. Let $F \subset \mathbf{C}^n$ be a q-dimensional complex vector subspace, $E \cap F = \{0\}$. (Hence $E \oplus F = \mathbf{C}^n$)

Put $F_\nu := \{a^{(\nu)}\} + F$, $\nu \geq 1$. It gives a sequence of affine parallel q-dimensional linear subspaces of $\mathbf{C}^n$ that converges to F. Fix also $K \subset F$ a compact neighborhood of the origin in F. Let $F'_\nu \subset X$ be the proper transform of F_ν through π, $\nu \geq 1$. Then $F'_\nu \cap E' = \emptyset$ and F'_ν is a closed q-dimensional complex submanifold of $X - E' = M$. Note that π induces $\pi|_{F'_\nu}: F'_\nu \to F_\nu$ the blowing-up of F_ν at the point $a^{(\nu)}$.

Now, define in M two sequences of compact sets $\{K_\nu\}$ and $\{\Gamma_\nu\}$, $\nu \in \mathbf{N}$, by

$$K_\nu := \pi^{-1}(\{a^{(\nu)}\} + K) \cap F'_\nu$$

and

$$\Gamma_\nu := \pi^{-1}(\{a^{(\nu)}\} + \partial K).$$

Here, the boundary of K is taken in F. Thus, in F'_ν, $\partial K_\nu = \Gamma_\nu$.

Now, we have that $\cup \Gamma_\nu$ is relatively compact in M; since $\cup (\{a^{(\nu)}\} + \partial K)$ is relatively compact in $\mathbf{C}^n - E$, and $\pi : X \to \mathbf{C}^n - \{0\}$ is proper.

On the other hand, since ϕ is q-convex and F'_ν is a q-dimensional (non-compact) complex manifold, $\phi|_{F'_\nu}$ fulfils the maximum principle. We get for any $\nu \in \mathbf{N}$ that

$$\phi|_{K_\nu} \leq \max_{\Gamma_\nu} \phi.$$

Therefore, $\cup K_\nu$ is also relatively compact in M since ϕ is exhaustive.

But this is ridiculous! In fact, any sequence of points $\{x_\nu\}_\nu$ with $x_\nu \in K_\nu \cap \pi^{-1}(a^{(\nu)}) \simeq \mathbf{P}^{q-1}$ is discrete in M.

Remark. It was proved in [3] that an arbitrary complex manifold which is an increasing union of q-complete open subsets is always $2q$-complete.

However, in our example, it can be checked that M is $(q+1)$-complete.

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References

[1] A. Andreotti and H. Grauert, Theorémès de finitude pour la cohomologie des espaces complexes, Bull. Soc. Math. France, **90** (1962), 193-259.

[2] J. -E. Fornæss, 2-dimensional counterexamples to generalizations of the Levi problem, Math. Ann., **230** (1977), 169-174.

[3] V. Vâjâitu, q-completeness and q-concavity of the union of complex subspaces, Preprint IMAR Nr. 2/1994 to appear in Math. Z., 1995.