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Abelian conformal field theory and $N = 2$ supercurves

By

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§. Introduction

Among models of conformal field theory, one of the simplest is the one with abelian gauge symmetry or what we call abelian conformal field theory. It has been studied by many people and from several view-points. We refer only a few of them [KNTY, ACKP, IMO, KSU 1-2], where one finds the notion of dressed moduli spaces and the so-called Krichever maps.

More recently, Ueno [U] studied it in parallel with conformal field theory with nonabelian gauge symmetry [TUY]. In particular, he adjoined the gauge condition by vertex operators among Fock spaces following a suggestion of Tsuchiya.

The aim of this paper is to give a geometric interpretation of this result by Ueno on abelian conformal field theory.

The basic strategy is as follows. First, the gauge condition on conformal blocks or correlators can be interpreted as the effect of localization à la Beilinson-Bernstein of representations of some infinite dimensional Lie algebra on certain "dressed" moduli space, cf. [B, BS].

Second, we want to consider the direct sum of infinitely many Fock spaces together with vertex operators. This amounts to consider the irreducible highest weight representations of Clifford algebra generated by free fermions corresponding to those vertex operators.

This suggests that we should consider geometric objects with fermionic degree of freedom. Thus we are led to consider an analog of Picard (or Jacobian) variety for algebraic supercurves with odd dimension $N = 2$ (abbreviated as $N = 2$ supercurves). We study here the space (Π-Picard variety) of locally free sheaves of rank $1|1$ with Π-symmetry (Π-invertible sheaves) introduced by Skornyakov [VMP, §4, M2, Ch. 2, §8].

Then one of the main results is the following:

Theorem (cf. 5.2.1). *The space of conformal blocks $\mathscr{V}(\mathfrak{X})$ equals a fiber of the localization of the given representation on the Π-Picard scheme of the supercurve $X^{(1)}$ of odd dimension $N = 2$ associated with an ordinary curve C.*

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For the definition of the space of conformal blocks, see 5.1. The supercurve $X^{(1)}$ is the first infinitesimal neighbourhood of C in the supercurve $X = (C, \Lambda \cdot (\omega_C^{1/2} \oplus \omega_C^{1/2}))$ where $\omega_C^{1/2}$ is a theta characteristic on C.

As a corollary to the above theorem and a theorem of Ueno [U], we can conclude:

Theorem (cf. 5.3.4). *We have a canonical isomorphism*

$$H^0(Pic_C^{g-1}, \mathcal{O}(\Theta)) \simeq \mathscr{V}^{\dagger}(\mathfrak{X}).$$

Here $\mathcal{O}(\Theta)$ is the theta divisor on the Jacobian Pic_C^{g-1}.

The theory of superschemes has been given some foundation by Deligne, Manin and his collaborators [D, M1, M2, VMP, Va], at least among mathematicians. $N = 2$ supercurves are considered in this article upon this foundation in §2, 3.

We prove, in particular, the following

Theorem (cf. 3.1.4). *The Π-Picard functor for a proper smooth supercurve of odd dimension $N = 1$ or 2 is representable by some smooth superscheme.*

We will have more concrete description of the representing superscheme in 3.2.

This paper is organized as follows. First we recall $U(1)$-currents or the Heisenberg algebra and vertex operators on its Fock representations briefly in §1. $N = 2$ supercurves and locally free sheaves on them are recalled in §2 with some digression on superconformal curves. Π-Picard varieties for $N = 1, 2$ supercurves are studied in §3 including their structure. After a brief review of the above-mentioned result on abelian conformal field theory, its geometric interpretation is given in §5, using "dressed" version of Π-Picard varieties which is examined in §4. Some discussion on the result is also given in §5.

The result in [U] is stated only for the level M-version for even integer M. So we need an extension to the case $M = 1$, which we will spell out in a future publication using $N = 1$ stable superconformal curve [S1].

In the first version of this work, the result is stated also for the level M version for arbitrary odd integer M. But the calculation of the operator algebra in that case was false except for the case $M = 1$. We would like to return to that situation in near future.

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§1. Free fermions and Heisenberg algebras

Let us recall the associative algebra of free fermions and the Heisenberg algebra which are required in abelian conformal field theory in the form given by Ueno [U].

We mean by abelian conformal field theory the one with gauge symmetry of $U(1)$-currents, or equivalently the Heisenberg algebra. Ueno studied it in parallel with [TUY] introducing a further gauge condition by vertex operators among boson Fock spaces. In retrospect, it was hidden under the boson-fermion correspondence cf. [KNTY].

1.1. $U(1)$ current algebra is the infinite dimensional Heisenberg algebra *Heis* generated by $a(n)(n \in \mathbf{Z})$ and 1 (central element) satisfying the relations

$$[a(m), a(n)] = m\delta_{m+n,0}. \tag{1.1.1}$$

By abuse of language, we also mean by this Heisenberg algebra its universal enveloping (associative) algebra.

The Fock space $\mathscr{F}(p)(p \in \mathbf{Z})$ is the irreducible representations (over $\mathbf{C}$) of *Heis* with a generator (highest weight vector) $|p\rangle$ such that

$$a(m)|p\rangle = 0 \ (p \geq 1), \ a(0)|p\rangle = p|p\rangle.$$

For each $k \in \mathbf{Z}$, put

$$V_k(z) = \exp\left\{k \sum_{n \geq 1} \frac{a(-n)}{n} z^n\right\} \exp\{ka(0)\log z\}\, T_k \exp\left\{-k \sum_{n \geq 1} \frac{a(n)}{n} z^{-n}\right\}, \tag{1.1.2}$$

where T_k is the translation operator

$$T_k \colon \mathscr{F}(p) \to \mathscr{F}(p+k)$$

defined by the condition

$$T_k|p\rangle = |p+k\rangle, \ a(n)T_k = T_k a(n) \qquad (n \neq 0).$$

This is called the vertex operator, which maps $\mathscr{F}(p)$ into the completion of $\mathscr{F}(p+k)$ with respect to the degree, cf. (1.2.1).

The totality of the vertex operators as well as *Heis* is equivalently described by the Clifford algebra *Clif* through the boson-fermion correspondence cf. [DJKM, KNTY].

Clif is an associative algebra (over $\mathbf{C}$) with generators $\psi_\mu, \psi_\mu^\dagger (\mu \in \frac{1}{2} + \mathbf{Z})$ satisfying the anti-commutation relation

$$\begin{aligned} &[\psi_\mu, \psi_\nu]_+ = 0, \ [\psi_\mu^\dagger, \psi_\nu^\dagger]_+ = 0 \\ &[\psi_\mu, \psi_\nu^\dagger]_+ = \delta_{\mu+\nu,0}. \end{aligned} \tag{1.1.3}$$

Under the boson-fermion correspondence, the fermion operators

$$\psi(z) = \sum_{\mu \in \frac{1}{2} + \mathbf{Z}} \psi_\mu z^{-\mu - \frac{1}{2}}, \ \psi^\dagger(z) = \sum_{\mu \in \frac{1}{2} + \mathbf{Z}} \psi_\mu^\dagger z^{-\mu - \frac{1}{2}} \tag{1.1.4}$$

are identified with $V_{-1}(z)$, $V_1(z)$ respectively. The $U(1)$-current is given in turn by

(1.1.5) $$a(z) =: \psi^\dagger(z)\psi(z):$$

Here : : means taking the normally ordered product cf. [KNTY].

1.2. It is instructive to know a realization of the representation $\mathscr{F}(p)$. The Heisenberg commutation relation is satisfied by $\partial/\partial t$ and t for an indeterminate t. So one has the realization

$$\mathscr{F}(p) = \mathbf{C}[t_1, t_2, \cdots] e^{pt_0}$$

with the following action:

$$a(n) = \partial/\partial t_n \qquad (n \geq 0)$$
$$a(-n) = nt_n \qquad (n \geq 1)$$

Then the translation operator T_k is given by "multiplication by e^{kt_0}". Note that the multiplication operator $q = t_0$ satisfies the Heisenberg commutation relation with $a(0) = \partial/\partial t_0$.

By the well-known manner ("Sugawara construction"), representations of *Heis* (or $U(1)$-current) produces representations of Virasoro algebra *Vir*. In the above notation, if we put

$$T(z) = \sum_{n\in\mathbf{Z}} L_n z^{-n-2} = \frac{1}{2} : a(z)a(z):,$$

then L_n's satisfy the commutation relation of *Vir*. Here we put

$$a(z) = \sum_{n\in\mathbf{Z}} a(n) z^{-n-1}.$$

$T(z)$ is usually called the energy-momentum tensor. Note that the conformal dimension of $V_k(z)$ (with respect to $T(z)$) is $k^2/2$.

Let us put

(1.2.1) $$\mathscr{H} = \bigoplus_{p\in\mathbf{Z}} \mathscr{F}(p).$$

The vertex operators act as homomorphisms from this space to the space

$$\mathscr{H}^\dagger := Hom_{\mathbf{C}}(\mathscr{H}, \mathbf{C}).$$

Then we have a natural pairing

(1.2.2) $$\mathscr{H}^\dagger \times \mathscr{H} \to \mathbf{C}; \qquad (\langle\psi|, |\phi\rangle) \mapsto \langle\psi|\phi\rangle,$$

The operator product expansion (OPE) of $V_{\pm 1}(z)$ is given by the following

Proposition 1.2.3.

$$V_{+1}(z)V_{-1}(w) \sim \frac{1}{z-w}.$$

Since the order of pole is 1, this is equivalent to

$$\langle 0 | V_{+1}(z) V_{-1}(w) | 0 \rangle = \frac{1}{z-w} \qquad (|z| > |w| > 0).$$

This is standard using, for example,

$$e^{\lambda a(n)} e^{\mu a(-n)} = e^{\mu a(-n)} e^{\lambda a(n)} e^{\lambda \mu n} \quad (n > 0)$$

$$e^{a(0)\log z} e^{-q} = e^{-q} e^{a(0)\log z} e^{-\log z}.$$

We refer the reader to [KNTY, §3, C)] for a proof. There the second quantized operators are used in the course. For the calculation of the normally ordered product, one uses Wick's theorem.

§2. Supercurves of odd dimension $N = 2$

We recall $N = 2$ algebraic supercurves and its properties in this section. Basic references are Manin [M1, 2].

Schemes will be over the field of complex numbers $\mathbf{C}$. The following discussion is valid for a schemes over an algebraically closed field.

2.1. A smooth supercurve is a smooth superscheme (i.e. supermanifold) of dimension $1 | N$. So it is a ringed space $X = (C, \mathcal{O}_X)$ with a sheaf of supercommutative ($\mathbf{Z}/2\mathbf{Z}$-graded) rings as structure sheaf and $C = X_{red} = X^{(0)}$ is a smooth (ordinary) curve. Here we put

$$X^{(i)} = (C, \mathcal{O}_X / \mathcal{N}^{i+1})$$

$$\mathcal{N} = \{\text{nilpotents}\} (= \mathcal{O}_{X,1} + \mathcal{O}_{X,1}^2)$$

for $i \geq 0$. "Smooth of dimension $1 | N$" means that $\mathcal{O}_X$ is locally of the form $\Lambda^{\bullet}(\mathcal{E})$ where $\mathcal{E}$ is a locally free $\mathcal{O}_C$-module of rank N. In the supergeometry, one might prefer the notation $Sym^{\bullet}(\Pi\mathcal{E})$ for $\mathcal{O}_X$, where Π is the parity changing functor.

One recovers the ordinary (i.e. even) curve in the case $N = 0$. cf. [M2, Ch. 2].

The case of $N = 1$ is rather simple. Since $N = 1$, $\mathcal{N}$ is an invertible $\mathcal{O}_C$-module (with odd parity) and

$$\mathcal{O}_X = \mathcal{O}_C + \mathcal{N}, \; \mathcal{N}^2 = 0$$

$$\mathcal{O}_{X,0} = \mathcal{O}_C, \; \mathcal{O}_{X,1} = \mathcal{N}.$$

Thus $N = 1$ curves are always split. The ring structure on $\mathcal{O}_X$ reduces to the $\mathcal{O}_C$-module structure on $\mathcal{N}$. This consideration is valid for the case of higher even dimension.

The case of $N = 2$ is slightly complicated. Since $N = 2$, $\mathcal{O}_{X,1} \simeq \mathcal{N}/\mathcal{N}^2 =: \mathcal{F}$ is locally free of rank 2 over $\mathcal{O}_C$. The even part $\mathcal{O}_{X,0}$ is described by an extension

$$0 \to \Lambda^2(\mathscr{F}) \to \mathcal{O}_{X,0} \to \mathcal{O}_C \to 0$$

The extension class belongs to the group $H^1(C, T_C \otimes \Lambda^2(\mathscr{F}))$. In this situation, we have $\mathcal{N}^3 = 0$ and the 2-step infintesimal thickening

$$X^{(0)} = X_{red} = C \subset X^{(1)} = C \subset X^{(1)} = (C, \mathcal{O}_C \oplus \mathscr{F}) \subset X^{(2)} = X$$

We will need this first infinitesimal thickening in §5.

2.2. We digress on superconformal curves which are ones of the most important supercurves.

Let $\pi\colon X \to S$ be a smooth morphism of superschemes of relative dimension $1 \mid N$, namely a family of smooth supercurves. A superconformal structure or $SUSY_N$-structure on π is (a choice of) a locally free and locally direct $\mathcal{O}_X$-submodule $\mathscr{T}^1$ of the relative tangent sheaf $\mathscr{T}_{X/S}$ of rank $0 \mid N$ which locally has an isotropic direct $\mathcal{O}_X$-submodule of maximal possible rank for $N = 2k$ or $2k + 1$ with respect to the Frobenius form

$$\Lambda^2 \mathscr{T}^1 \to \mathscr{T}^0 := \mathscr{T}_{X/S}/\mathscr{T}^1;\ t_1 \wedge t_2 \mapsto [t_1, t_2] \bmod \mathscr{T}^1.$$

Supercurves with $N = 1, 2$ superconformal structure (or $N = 1, 2$ superconformal curves) are also called $N = 1, 2$ super Riemann surfaces or $SUSY_N$-curves in the literature.

Let us specify the above definition in the case $S = pt$ and $N = 1, 2$. So let X be a smooth supercurve of dimension $(1 \mid N)$.
Case $N = 1$: A ($N = 1$) *superconformal structure* on X or a $SUSY_1$-*structure* is (a choice of) a locally free $\mathcal{O}_X$-submodule $\mathscr{T}^1$ of rank $0 \mid 1$ of the tangent sheaf $\mathscr{T}_X$ which is a locally direct summand and gives rise to an isomorphism

$$(\mathscr{T}^1)^{\otimes 2} \simeq \mathscr{T}_X/\mathscr{T}^1 =: \mathscr{T}^0;\ t_1 \otimes t_2 \mapsto [t_1, t_2] \bmod \mathscr{T}^1.$$

Case $N = 2$: A ($N = 2$) *superconformal structure* on X or a $SUSY_2$-*structure* is (a choice of) a pair of locally free $\mathcal{O}_X$-submodules $\mathscr{T}', \mathscr{T}''$ of rank $0 \mid 1$ of the tangent sheaf $\mathscr{T}_X$ such that i) $\mathscr{T}' \oplus \mathscr{T}''$ is direct in $\mathscr{T}_X$, ii) $[\mathscr{T}', \mathscr{T}'] \subset \mathscr{T}'$, $[\mathscr{T}'', \mathscr{T}''] \subset \mathscr{T}''$, and that iii) one has an isomorphism

$$\mathscr{T}' \otimes \mathscr{T}'' \simeq \mathscr{T}_X/(\mathscr{T}' \oplus \mathscr{T}'') =: \mathscr{T}^0 : t' \oplus t'' \mapsto [t', t''] \pmod{\mathscr{T}' \oplus \mathscr{T}''}.$$

Here $\mathscr{T}' \oplus \mathscr{T}'' = \mathscr{T}^1$.

2.3. Over a purely even base S, one has a simple description of superconformal curves.

So let $\pi_0\colon X_0 \to S$ be an ordinary S-curve. Then we have

Proposition [M2, 2.7]. *There is a one-to-one correspondence between*
Case $N = 1$:
a) $\{N = 1$ *suerconformal* S-*curve* $\pi\colon X \to S$ *with* $X_{red} = X_{0,red}$ *and* $\mathcal{O}_{X,0} = \mathcal{O}_{X_0}\}$ *up to isomorphism identical on* X_0

b) $\{(I;\alpha) \mid I \in Pic(X_0/S),\ \alpha: I \otimes I \simeq \Omega^1_{X_0/S}\}$ *up to isomorphism of I transforming* α

Case $N = 2$:

a) $\{$*oriented* $N = 2$ *superconformal S-curve* $\pi: X \to S$ *with* $X_{red} = X_{0,red}$ *and* $\mathcal{O}_{X,0} = \mathcal{O}_{X_0^{(1)}}\}$

b) $\{(I', I''; \beta) \mid I', I'' \in Pic(X_0/S),\ \beta: I' \otimes I'' \simeq \Omega^1_{X_0/S}\}$

Here "oriented" means that the sheaves I', I'' are globally distinguishable, cf. [M2, 2.6]. $X_0^{(1)}$ is the first infinitesimal neighbourhood of X_0 in $X_0 \times {}_S X_0$. A pair $(I', I''; \beta)$ is called as a relative theta pair.

The above correspondences are given as follows. In the case $N = 1$,

$$X_{red} = X_{0,red},\ \mathcal{O}_{X,0} = \mathcal{O}_{X_0},\ \mathcal{O}_{X,1} = \Pi I$$

and

$$\mathcal{T}^1 = \mathcal{O}_X \otimes_{\mathcal{O}_{X_{red}}} I^{\otimes -1}.$$

α gives rise to the Frobenius form. The pair $(I;\alpha)$ is a theta characteristic of the family.

In the case $N = 2$,

$$X_{red} = X_{0,red},\ \mathcal{O}_{X,0} = \mathcal{O}_{X_0} \oplus I' \otimes I'',\ \mathcal{O}_{X,1} = \Pi(I' \oplus I'')$$

and

$$\mathcal{T}' = \mathcal{O}_X \otimes_{\mathcal{O}_{X_{red}}} I'^{-1},\ \mathcal{T}'' = \mathcal{O}_X \otimes_{\mathcal{O}_{X_{red}}} I''^{\otimes -1}$$

β gives rise to the Frobenius form.

$N = 1$ superconformal curves are sudied by many authors. In particular, their moduli is studied by LeBrun-Rothstein [LBR] and Deligne [D].

One of the basic fact is

Proposition. *One has a natural isomorphism*

$$\mathcal{T}^{sc}_{X/S} \simeq (\mathcal{T}^1)^{\otimes 2},$$

where $\mathcal{T}^{sc}_{X/S} := \{\xi \in \mathcal{T}_{X/S} \mid [\xi, \mathcal{T}^1] \subset \mathcal{T}^1\}$ *is the sheaf of infinitesimal S-automorphism of a* $N = 1$ *superconformal S-curve.*

We have an analog of this fact in the $N = 2$ case.

Proposition. *One has a natural isomorphism*

$$\mathcal{T}^{sc}_{X/S} \simeq \mathcal{T}' \otimes \mathcal{T}'',$$

where $\mathcal{T}^{sc}_{X/S}$ *denotes the sheaf* $\{\xi \in \mathcal{T}_{X/S} \mid [\xi, \mathcal{T}'] \subset \mathcal{T}',\ [\xi, \mathcal{T}''] \subset \mathcal{T}''\}$.

The proof is a direct calculation which can be found in [S2], where a study of the moduli space of $N = 2$ superconformal curves is given.

One can associate to any supercurve X of dimension $(1 \mid N)$ a superconformal curve $\tilde{X} = Gr(0 \mid N, T_X)$ of dimension $(1 \mid 2N)$. This was known to Deligne (at least for the case $N = 1$) and Dolgikh-Rosly-Schwarz [DRS]. In the case of $N = 1$, the essential image of this functorial construction is oriented $N = 2$ superconformal curves. These phenomenon might be relevant to observations on the hierarchy of superconformal symmetry of Berkovitz-Vafa, etc.

Another remark is that semi-rigid geometry considered by Distler-Nelson [DN] means to restrict to the superconformal curves of the form $(C, \wedge(\mathcal{O} \oplus \omega))$ (C an ordinary curve) among arbitrary $N = 2$ superconformal curves.

2.4. Let us briefly review sheaves on a superscheme X.

2.4.1. First all (supercommutative) rings and modules are $\mathbf{Z}/2\mathbf{Z}$-graded. $\mathcal{O}_{X,1}$ is a coherent $\mathcal{O}_{X,0}$-module and $\mathcal{O}_X$ is a coherent ring.

A left $\mathcal{O}_X$-module has a natural right $\mathcal{O}_X$-module structure consistent with the left one, cf. [M1, Ch. 3. §1, 4]. Thus one can form tensor products of $\mathcal{O}_X$-modules freely.

The group of homomorphisms is $\mathbf{Z}/2\mathbf{Z}$-graded:

$$Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) = Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})_0 \oplus Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})_1$$

The first (resp. second) factor consists of even (resp. odd) homomorphisms. An automorphism is an even endomorphism which is isomorphic.

A locally free $\mathcal{O}_X$-module of rank $p \mid q$ is a coherent $\mathcal{O}_X$-module which is locally isomorphic to $\mathcal{O}_X^{p|q} = \mathcal{O}_X^p \oplus \Pi\mathcal{O}_X^q$. Here Π is the parity changing functor: $(\Pi(\mathcal{E}))_i = \mathcal{E}_{i+1}$, $i \in \mathbf{Z}/2\mathbf{Z}$.

The set of locally free $\mathcal{O}_X$-modules of rank $p \mid q$ up to isomorphism is in bijection with the set $H^1(X, GL(p \mid q; \mathcal{O}_X))$ as usual. Here $GL(p \mid q; \mathcal{O}_X)$ denotes the sheaf of germs of (even) automorphisms of such an $\mathcal{O}_X$-module.

2.4.2. We return to the situation (2.1) here. So X is a smooth supercurve of dimension $1 \mid N$.

A special feature in the case of supercurves in general is that for any locally free $\mathcal{O}_{X_{red}}$-module $\mathcal{E}_0$ there exists a locally free $\mathcal{O}_X^{(i)}$-module $\mathcal{E}^{(i)}$ such that $\mathcal{E}^{(i)} \otimes \mathcal{O}_X^{(i)}/(\mathcal{N}/\mathcal{N}^{i+1}) \simeq \mathcal{E}_0$. In particular, $\mathcal{E}_0$ extends always to to X, cf. [VMP, 3.12].

In the case $N = 1$, the splitness of X implies that to know a locally free $\mathcal{O}_X$-module amounts to know the exact sequence of locally free $\mathcal{O}_{X_{red}}$-modules:

$$0 \to \mathcal{N} \otimes \mathcal{E}_{red} \to \mathcal{E} \to \mathcal{E}_{red} \to 0$$

$$\mathcal{E}_{red} = \mathcal{E} \otimes \mathcal{O}_X/\mathcal{N}.$$

In the case of $N = 2$, a locally free $\mathcal{O}_X$-module $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$ can be reconstructed from its even part $\mathcal{E}_0$. This is because

$$\mathcal{O}_{X,1} = \mathcal{F} = \mathcal{F} \otimes \mathcal{O}_{X,0}$$

$$\mathcal{E}_1 = \mathcal{F} \otimes_{\mathcal{O}_{X,0}} \mathcal{E}_0.$$

2.4.3. A Π-symmetry on a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank $r|s$ is an odd endomorphism $p: \mathcal{E} \to \mathcal{E}$ with $p^2 = -id$. Then $r = s$. We call such a pair $(\mathcal{E}, p)$ a locally free $\mathcal{O}_X$-Π-module. A morphism $f: (\mathcal{E}, p_{\mathcal{E}}) \to (\mathcal{F}, p_{\mathcal{F}})$ is understood to preserve the Π-symmetry: $p_{\mathcal{F}} \cdot f = f \cdot p_{\mathcal{E}}$. Denote by $Hom_\Pi(\mathcal{E}, \mathcal{F})$ the group of morphisms. Similarly for End_Π, Aut_Π, etc.

We will be particularly interested in the case rank $1|1$ later. Then we call a Π-invertible $\mathcal{O}_X$-module. It is easy to see the following:

Lemma. *For a Π-invertible $\mathcal{O}_X$-module $(\mathcal{E}, p)$, one has*

$$\mathcal{E}nd_\Pi(\mathcal{E}) \simeq \mathcal{O}_X \oplus \Pi\mathcal{O}_X$$

$$\mathcal{A}ut_\Pi(\mathcal{E}) \simeq \mathcal{O}_X^*.$$

Note that End_Π is a subgroup of $GL(1|1)$. We refer the reader to I. A. Skornyakov's detailed study of $\mathcal{O}_X$-modules of rank $1|1$ cf. [VMP, §4].

§3. Π-Picard schemes: a superanalog of Jacobians

We study Π-Picard schemes for supercurves with $N = 1, 2$ in this §. Its dressed version and localization (à la Beilinson-Bernstein) on them will be studied in the next §.

3.1. We first recall Skornyakov's theory of the Π-Picard group of locally free sheaves of rank $1|1$ with Π-symmetry.

Let X be a superscheme. We can define two analogs of the Picard group in supergeometry.

Definition 3.1.1. 1) Let $Pic_0(X)$ denote the set of isomorphism classes of locally free $\mathcal{O}_X$-modules of rank $1|0$.

This set has a group structure as usual and is naturally isomorphic to the group $H^1(X, \mathcal{O}_{X,0}^*)$.

2) Let $Pic_\Pi(X)$ denote the set of isomorphism classes of locally free $\mathcal{O}_X$-modules of rank $1|1$ with Π-symmetry. Here a *Π-symmetry* on a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank $1|1$ is an odd endomorphism $p: \mathcal{E} \to \mathcal{E}$ with $p^2 = -id$. An isomorphism of $\mathcal{E}$ is understood to preserve the Π-symmetry.

This is merely a pointed set and is naturally isomorphic to the set $H^1(X, \mathcal{O}_X^*)$. We call such a pair $(\mathcal{E}, p)$ a Π-invertible sheaf or Π-invertible $\mathcal{O}_X$-module.

Thus we have two functors from the category of superschemes *Ssch* to the category of sets *Set*.

3.1.2. In the supergeometry, one consideres the following functors:

$$\mathbf{G}_m^{1|0}: S \mapsto \mathcal{O}_{S,0}^*$$

$$\mathbf{G}_m^{1|1}: S \mapsto \mathcal{O}_S^*$$

$$\mathbf{G}_a^{0|1}: S \mapsto \mathcal{O}_{S,1}$$

These are sheaves in the Zariski topology.

From the exact sequence

$$1 \to \mathbf{G}_m^{1|0} \to \mathbf{G}_m^{1|1} \to \mathbf{G}_a^{0|1} \to 1,$$

one has the following exact sequence

$$Pic_0(X) \to Pic_\Pi(X) \to H^1(X, \mathcal{O}_{X,1}) \to H^2(X, \mathcal{O}_{X,0}^*),$$

where the first map associates $\mathscr{L} \oplus \Pi\mathscr{L}$ to $\mathscr{L} \in Pic_0(X)$.

3.1.3. Fix a supercurve X of dimension $(1 \mid N)$. Then consider the following functors from *Ssch* to *Set*.

$$Pic_0 : S \mapsto Pic_0(X \times S)/p_S^* \, Sic_0(S)$$
$$Pic_\Pi : S \mapsto H^1(S, p_{S,*}(\mathcal{O}_{X \times S}^*))$$

Here $p_S: X \times S \to S$ is the second projection.

These are sheaves for the fppf topology and the values at S coincide with $H^0(S, R^1_{fppf} p_{S,*} \mathbf{G}_m^{1|0})$, $H^0(S, R^1_{fppf} p_{S,*} \mathbf{G}_m^{1|1})$ respectively. To see this, we use the edge exact sequence of the Leray spectral sequence for p_S in non-abelian cohomology and the fact that X is a supercurve over a point. We may replace *fppf* by *ét*. We refer the reader to FGA, n^o 232 and [R].

From this definition, we can justify localizing the situation.

Now the main result in §3 is the following

Theorem 3.1.4. *Let X be a proper smooth supercurve of dimension $1 \mid N$. For $N = 1, 2$, the above functors are representable by some superschemes denoted as $Pic_{0,X}$, $Pic_{\Pi,X}$.*

We'll have more precise statement about the structure of $Pic_{0,X}$, $Pic_{\Pi,X}$ in the course of its proof.

3.2. Proof of Theorem 3.1.4. The main technical tool is the obstruction theory for extending sheaves to infinitesimal neighbourhoods (in the odd direction) (the so-called "component analysis"). Let us recall it in the form we need, cf. [VMP, 3.10].

Let X be a smooth superscheme and $\mathscr{E}_0$ a locally free $\mathcal{O}_{X_{red}}$-module. $X^{(i)}$ denote the i-th infinitesimal neighbourhood of X_{red} in X: $\mathcal{O}_{X^{(i)}} = \mathcal{O}_X/\mathscr{N}^{i+1}$ ($\mathscr{N}$ is the ideal of nilpotents in $\mathcal{O}_X$).

Suppose that $\mathscr{E}_0$ is endowed with a Π-symmetry p_0. Denote the Lie superalgebra of endomorphisms of $\mathscr{E}_0$ preserving the Π-symmetry by $\mathscr{E}nd_\Pi(\mathscr{E}_0)$. Then the rank of $\mathscr{E}_0$ is $r|r$. Let us call such a pair $(\mathscr{E}_0, p_0)$ a locally free $\mathcal{O}_{X_{red}}$-Π-module.

Given a locally free $\mathcal{O}_{X^{(i)}}$-Π-module $(\mathscr{E}^{(i)}, p^{(i)})$ which extends $(\mathscr{E}_0, p_0)$.

Theorem 3.2.1. *a) There exists a unique cohomology class*

$$c(\mathscr{E}^{(i)}, p^{(i)}) \in H^2(X, (\mathscr{E}nd_\Pi(\mathscr{E}_0) \otimes_{\mathcal{O}_{X_{red}}} \mathscr{N}^{i+1}/\mathscr{N}^{i+2})_0)$$

such that a locally free $\mathcal{O}_{X^{(i+1)}}$-Π-module $(\mathscr{E}^{(i+1)}, p^{(i+1)})$ with

$$\mathscr{E}^{(i+1)}/(\mathscr{N}^{i+1}/\mathscr{N}^{i+2})\mathscr{E}^{(i+1)} = \mathscr{E}^{(i)}$$

exists and $p^{(i+1)}$ extends $p^{(i)}$ if and only if $c(\mathscr{E}^{(i)}, p^{(i)}) = 0$.

b) If $c(\mathscr{E}^{(i)}, p^{(i)}) = 0$, then the group $H^1(X, (\mathscr{E}nd_\Pi(\mathscr{E}_0) \otimes_{\mathcal{O}_{X_{red}}} \mathscr{N}^{i+1}/\mathscr{N}^{i+2})_0)$ acts transitively on the set of isomorphism classes of locally free $\mathcal{O}_{X^{(i+1)}}$-Π-modules which extends $(\mathscr{E}^{(i)}, p^{(i)})$.

Moreover, if there exists an extension $(\tilde{\mathscr{E}}^{(i+1)}, \tilde{p}^{(i+1)})$ such that the restriction map $H^0(\mathscr{E}nd_\Pi(\tilde{\mathscr{E}}^{(i+1)})) \to H^0(\mathscr{E}nd_\Pi(\mathscr{E}^{(i)}))$ is surjective, then the action of

$$H^1(X, (\mathscr{E}nd_\Pi(\mathscr{E}_0) \otimes_{X_{red}} \mathscr{N}^{i+1}/\mathscr{N}^{i+2})_0)$$

is effective.

The proof is parallel to the argument in the purely even case, cf. [M1, Ch. 2, §6] and [VMP, 3.10].

Variant 3.2.2. *Let X be a smooth superscheme and S a superscheme. Then the relative version of the statements for $X \times S \to S$ in* Theorem (3.2.1) *as well as those of* [VMP, 3.10] *hold.*

Namely, similar statements hold replacing cohomology groups by higher direct images on S, a cohomology class by a section of the direct image of first degree, $\mathscr{N}$ by $\mathscr{N}_{X \times S/S} = \mathscr{N}_X \boxtimes \mathcal{O}_S$, taking the even part by taking the even part along the fiber, etc.

The author doesn't know what happens if one considers an arbitrary family of smooth superschemes.

We now apply the theorem to our situation (2.1). Note that there is no obstruction for extension because of $H^2 = 0$ for coherent sheaves on a curve.

3.2.3. Case: $N = 2$ and Pic_0. We treat Pic_0 first. Take an element $\mathscr{L} \in Pic_0(X \times S)$ where S is an arbitrary superscheme. Then $\mathscr{L}_{red/S} := \mathscr{L} \otimes \mathcal{O}_{X \times S/S}/\mathscr{N}_{X \times S/S}$ is an element of $Pic_0(X_{red} \times S)$.

Then we have the following:

Theorem 3.2.4. *Let X_0 be a purely even scheme. Then Pic_0 is representable by the ordinary Picard scheme Pic_{X_0}. Pic_Π is also representable by Pic_{X_0}.*

For Pic_0, this simply follows from the fact

$$\mathcal{O}_{X_0,0} = \mathcal{O}_{X_0}.$$

For Pic_Π, the assertion reduces to that for Pic_0 from the fact

$$Pic_\Pi(X_0) \simeq Pic_0(X_0).$$

for the same reason.

From this theorem, $\mathscr{L}_{red/S}$ mod $p_2^* Pic_0(S)$ corresponds to a morphism $S \to Pic_{X_{red}}$ and is induced by a locally free $\mathcal{O}$-module of rank $1|0$ on $X_{red} \times Pic_{X_{red}}$.

So the remaining task is to parametrize all extensions of $\mathscr{L}_{red/S}$ to $X \times S$. We use the variant 3.2.2 on locally free $\mathcal{O}$-modules. Let $p_2 \colon X_{red} \times S \to S$ denote the second projection. The extensions to $X^{(1)}$ are in one-to-one correspondence with the group

$$R^1 p_{2,*}((\mathscr{E}_{red/S}) \otimes \mathscr{N}_{X \times S/S}/\mathscr{N}^2_{X \times S/S})_0) = 0.$$

This is because the fiber over a point of S

$$H^1((\mathscr{E}nd\,(\mathscr{L}_{red}) \otimes \mathscr{N}/\mathscr{N}^2)_0) = 0,$$

for $\mathscr{E}nd\,(\mathscr{L}_{red}) = \mathcal{O}_{X_{red}}$ and $\mathscr{N}/\mathscr{N}^2 = \Pi\mathscr{F}$ is purely odd.

Once one choose such an extension, its extensions to $X^{(2)} = X$ are in one-to-one correspondence with the group

$$\begin{aligned} &R^1 p_{2,*}((\mathscr{E}nd\,(\mathscr{L}_{red/S}) \otimes \mathscr{N}^2_{X \times S/S}/\mathscr{N}^3_{X \times S/S})_0) \\ &\simeq R^1 p_{2,*}(\mathscr{N}^2_{X \times S/S}). \end{aligned}$$

Considering this group as the value of a sheaf, we see that it is represented by a vector group

$$H^1(X, \mathscr{N}_X^2) = H^1(X, \Lambda^2(\mathscr{F})).$$

In conclusion, the sheaf Pic_0 is represented by $Pic_{0,X} = Pic_{X_{red}} \times H^1(X, \Lambda^2(\mathscr{F}))$. This is even superscheme.

3.2.5. Case: $N = 2$ and Pic_Π. The strategy is the same as for Pic_0.

Let us take an element $\mathscr{E} \in Pic_\Pi(X \times S)$. Here S is an arbitrary superscheme. Then $\mathscr{E}_{red/S} = \mathscr{E} \otimes \mathcal{O}_{X \times S/S}/\mathscr{N}_{X \times S/S}$ is of the form $\mathscr{L} \oplus \Pi\mathscr{L}$ for $\mathscr{L} \in Pic_0(X_{red} \times S)$, since

$$Pic_\Pi(X_{red}) \simeq Pic_0(X_{red})$$

because of $\mathcal{O}_{X_{red},0} = \mathcal{O}_{X_{red}}$, cf. 3.2.4.

So the remaining task is to parametrize all extensions of $\mathscr{E}_{red/S}$ to $X \times S$. Again we use the variant 3.2.2 on locally free Π-$\mathcal{O}$-modules.

The extensions to $X^{(1)}$ are in one-to-one correspondence with the group

$$\begin{aligned} &R^1 p_{2,*}(\mathscr{E}nd_\Pi(\mathscr{E}_{red/S}) \otimes \mathscr{N}_{X \times S/S}/\mathscr{N}^2_{X \times S/S})_0) \\ &\simeq R^1 p_{2,*}(\mathscr{F} \boxtimes \mathcal{O}_S) \\ &= H^1(X, \mathscr{F}) \otimes \mathcal{O}_S \end{aligned}$$

because of

$$\begin{aligned} &\mathscr{E}nd_\Pi(\mathscr{E}_{red/S}) \simeq (\mathcal{O}_{X_{red}} \oplus \Pi\mathcal{O}_{X_{red}}) \boxtimes \mathcal{O}_S, \\ &\mathscr{N}_{X \times S/S}/\mathscr{N}^2_{X \times S/S} = \Pi\mathscr{F} \otimes \mathcal{O}_S. \end{aligned}$$

Once one choose such an extension, its extensions to $X^{(2)} = X$ are in one-to-one correspondence with the group

$$\begin{aligned} & R^1 p_{2,*}((\mathscr{E}nd_\Pi(\mathscr{E}_{red/S}) \otimes \mathscr{N}^2_{X \times S/S}/\mathscr{N}^3_{X \times S/S})_0) \\ \simeq\ & R^1 p_{2,*}(\Lambda^2(\mathscr{F}) \boxtimes \mathscr{O}_S) \\ =\ & H^1(X, \Lambda^2(\mathscr{F})) \otimes \mathscr{O}_S. \end{aligned}$$

The successive extensions are controlled by the exact sequence

$$1 \to 1 + \mathscr{N}^2 \to 1 + \mathscr{N} \to \mathscr{N}/\mathscr{N}^2 \to 0$$

which doesn't split. Taking the cohomology, we obtain

$$0 \to H^1(X, 1 + \mathscr{N}^2) \to H^1(X, 1 + \mathscr{N}) \to H^1(X, \mathscr{N}/\mathscr{N}^2) \to 0.$$

The whole extension from X_{red} to X is controlled by the non-commutative cohomology $H^1(X, 1 + \mathscr{N}) \otimes \mathscr{O}_S$.

In conclusion, the sheaf Pic_Π is represented by $Pic_{\Pi,X} = Pic_{X_{red}} \times H^1(X, 1 + \mathscr{N})$.

As a corollary of the proof, we have the following:

Corollary 3.2.6. *The Π-Picard group for $X^{(1)}$ has a structure of a scheme represented by $Pic_{\Pi,X^{(1)}} = Pic_{X_{red}} \times H^1(X, \mathscr{F})$.*

3.2.7. Case: $N = 1$. Basic strategy is the same as in the case $N = 2$ and the situation is much simpler; one has to extend the sheaves only once. We briefly state the results.

As to Pic_0, we have

$$Pic_{0,X} = Pic_{X_{red}},$$

since the group of extensions of a locally free $\mathscr{O}_{X_{red}}$-module of rank 1 $\mathscr{L}_0$ is trivial:

$$H^1(X, (\mathscr{E}nd(\mathscr{L}_0) \otimes \mathscr{N})_0) = 0.$$

As to Pic_Π, we have

$$Pic_{\Pi,X} = Pic_{X_{red}} \times H^1(X, \mathscr{N}).$$

This is because

$$\begin{aligned} & H^1(X, (\mathscr{E}nd_\Pi(\mathscr{E}_0) \otimes \mathscr{N})_0) \\ =\ & H^1(X, \mathscr{N}) \end{aligned}$$

for a Π-invertible sheaf on X_{red}.

Remark 3.2.8. 1) The superanalog of Jacobians are considered by several authors. Pic_0 is mostly considered.

In [RSV], it is defined as a complex supertorus and coincides with $Pic_{\Pi,X}$ in our notation. But the statement that it coincides with the group of line bundles (of rank $1 \mid 0$) is false in general.

In [Mu], Mulase uses $Pic_{\Pi,X}$ defined as a complex supertorus in order to characterize some finite dimensional super KP flows, cf. [Ra].

2) $Pic_{0,X}$, $Pic_{\Pi,X}$, $Pic_{\Pi,X^{(1)}}$ are smooth as we saw above.

§4. Dressed Π-Picard schemes for $N = 2$ supercurves

We introduce some dressed version of Π-Picard scheme for certain $N = 2$ supercurves and study their tangential structure. We construct a homomorphism from the Clifford algebra to the ring of differential operators on the dressed spaces in order to express the gauge condition as (dual of) localization on such a space (i.e. taking coinvariants), cf. §5.

4.1. Let $X = (C, \mathcal{O}_X)$ be a proper smooth supercurve of odd dimension $N = 2$. We use the notation in 2.1. Thus $X^{(1)}$ is the first infinitesimal neighbourhood of $X^{(0)} = C$ in X.

We have Π-Picard schemes for X as well as for C and $X^{(1)}$, cf. 3.2. There are the restriction maps

$$Pic_\Pi(X) \to Pic_\Pi(X^{(1)}) \to Pic_\Pi(C) \simeq Pic\,(C)$$

and the corresponding morphisms of the representing schemes.

Recall that we have the universal (Poincaré) bundle $\mathcal{L}_{univ}$ on $C \times Pic_C$. Then we have the determinant line bundle $\det R\pi_*(\mathcal{L}_{univ}) = d(\mathcal{L}_{univ})$ on Pic_C where $\pi\colon C \times Pic_C \to Pic_C$ is the second projection, cf. [KM, Sz].

We have also the universal Π-invertible sheaves $\mathcal{P}_{univ}$ on $X \times Pic_{\Pi,X}$. The restriction of $\mathcal{P}_{univ}$ to $C \times Pic_{\Pi,X}$ is isomorphic to $\mathcal{L}_{univ} \oplus \Pi\mathcal{L}_{univ}$. We use the same notation $d(\mathcal{L}_{univ})$ for the pull-back to $Pic_{\Pi,X}$, $Pic_{\Pi,X^{(1)}}$.

Note that the Berezinian bundle (i.e. superanalog of determinant line bundle) or Π is trivial.

Let us introduce the dressed Π-Picard schemes which classify dressed Π-invertible sheaves, i.e. Π-invertible sheaves with trivialization at given points.

From now on, we assume that the supercurve X is split, i.e. the even part of $\mathcal{O}_X$ is split:

$$\mathcal{O}_{X,0} = \mathcal{O}_C \oplus \Lambda^2(\mathcal{F})$$

Let $Q \in C$ be a (closed) point and $Z = (z, \theta_1, \theta_2)$ be formal local coordinates at Q, i.e. z is a formal local coordinate at Q and θ_1, θ_2 are local generating sections of $\mathcal{F}$:

$$\hat{\mathcal{O}}_{C,Q} \simeq \mathbf{C}[[z]],\ \hat{\mathcal{F}}_Q \simeq \mathbf{C}[[z]]\theta_1 \oplus \mathbf{C}[[z]]\theta_2 \tag{4.1.1}$$

Thus we have

$$\hat{\mathcal{O}}_{X,Q} \simeq \Lambda(\mathbf{C}[[z]]\theta_1 \oplus \mathbf{C}[[z]]\theta_2)$$

Let $\mathfrak{m}_Q$ denote the maximal ideal of the (supercommutative) local ring $\mathcal{O}_{X,Q}$, which is generated by the maximal ideal of $\mathcal{O}_{C,Q}$ and odd generators θ_1, θ_2.

Definition 4.1.2. Let k be an integer ≥ 1.

A trivialization of k-th order at a point $Q \in C$ of a Π-invertible sheaf $\mathcal{L}$ is an $\mathcal{O}_{X,Q}/\mathfrak{m}_Q^{k+1}$-isomorphism

$$\alpha : \mathcal{L}/\mathfrak{m}_Q^{k+1}\mathcal{L} \simeq \mathcal{O}_{X,Q}/\mathfrak{m}_Q^{k+1} \oplus \Pi(\mathcal{O}_{X,Q}/\mathfrak{m}_Q^{k+1})$$

which transforms the Π-symmetry on the left to the obvious one on the right.

Considering the projective limit of such an isomorphism, we define the formal trivialization of a Π-invertible sheaf.

A remark at this point: we don't have to assume that X is split at least for the definition.

Let $\mathfrak{X} = (X; Q_1, \cdots, Q_n; Z_1, \cdots, Z_n)$ be a datum consisting of the supercurve X, its points and formal local coordinates (cf. 4.1.1) at these points.

Denote by $Pic_\Pi^{(k)}(\mathfrak{X})$ (resp. $Pic_\Pi^{(\infty)}(\mathfrak{X})$) the set of isomorphism classes of all Π-invertible sheaves with trivialization of k-th order (resp. formal trivialization) at given points.

Then it is easy to see the following

Proposition 4.1.3. *$Pic_\Pi^{(k)}(\mathfrak{X})$ (resp. $Pic_\Pi^{(\infty)}(\mathfrak{X})$) has a structure of superscheme which is a $\Pi_i \mathbf{G}_m^{1|1}(\mathcal{O}_{X,Q_i}/\mathfrak{m}_{Q_i}^{k+1})$-torsor (resp. $\Pi_i \mathbf{G}_m^{1|1}(\hat{\mathcal{O}}_{X,Q_i})$-torsor) over $Pic_{\Pi,X}$.*

Let us denote these superschemes by $Pic_{\Pi,\mathfrak{X}}^{(k)}$, $Pic_{\Pi,\mathfrak{X}}^{(\infty)}$.

A variant 4.1.4. We can define in complete similar way trivialization of Π-invertible sheaves on $X^{(1)}$. Then for a datum $\mathfrak{X}^{(1)} = (X^{(1)}; Q_1, \cdots, Q_n; Z_1, \cdots, Z_n)$ as above, we have groups $Pic_\Pi^{(k)}(\mathfrak{X}^{(1)})$, $Pic_\Pi^{(\infty)}(\mathfrak{X}^{(1)})$ and $\Pi_i \mathbf{G}_m^{1|1}(\mathcal{O}_{X^{(1)},Q_i}/\mathfrak{m}_{Q_i}^{k+1})$-torsor $Pic_{\Pi,\mathfrak{X}^{(1)}}^{(k)}$ (resp. $\Pi_i \mathbf{G}_m^{1|1}(\hat{\mathcal{O}}_{X^{(1)},Q_i})$-torsor $Pic_{\Pi,\mathfrak{X}^{(1)}}^{(\infty)}$) over $Pic_{\Pi,X^{(1)}}$ correspondingly.

There are natural restriction maps

$$Pic_\Pi^{(k)}(\mathfrak{X}) \to Pic_\Pi^{(k)}(\mathfrak{X}^{(1)})$$
$$Pic_\Pi^{(\infty)}(\mathfrak{X}) \to Pic_\Pi^{(\infty)}(\mathfrak{X}^{(1)})$$

4.2. We study infinitesimal structure of the dressed Π-Picard schemes introduced above.

Proposition 4.2.1. 1) *Let $(\mathcal{L}, \alpha)$ be an element of $Pic_\Pi^{(k)}(\mathfrak{X})$. Then we have a canonical exact sequence:*

$$H^0(X, \mathcal{O}_X(*\sum_i Q_i)) \to z^k\mathbf{C}[z^{-1}] \otimes \Lambda^\bullet(\mathbf{C}\theta_1 \oplus \mathbf{C}\theta_2) \to T_{(\mathcal{L},\alpha)} Pic_{\Pi,\mathfrak{X}}^{(k)} \to 0$$

2) *Let $(\mathcal{L}, \alpha)$ be an element of $Pic_\Pi^{(k)}(\mathfrak{X}^{(1)})$. Then we have a canonical exact sequence:*

$$H^0(X^{(1)}, \mathcal{O}_{X^{(1)}}(*\sum_i Q_i)) \to z^k\mathbf{C}[z^{-1}] \otimes (\mathbf{C} \oplus \mathbf{C}\theta_1 \oplus \mathbf{C}\theta_2) \to T_{(\mathcal{L},\alpha)} Pic_{\Pi,\mathfrak{X}^{(1)}}^{(k)} \to 0$$

Letting k tend to ∞, we obtain the tangent space of the dressed Π-Picard schemes:

$$T_{(\mathscr{L},\alpha)}\, Pic^{(\infty)}_{\Pi,\mathfrak{X}} \simeq \mathbf{C}((z)) \otimes \Lambda^{\cdot}(\mathbf{C}\theta_1 \oplus \mathbf{C}\theta_2)/H^0(X, O_X(*\sum_i Q_i))$$

$$T_{(\mathscr{L},\alpha)}\, Pic^{(\infty)}_{\Pi,\mathfrak{X}^{(1)}} \simeq \mathbf{C}((z)) \otimes (\mathbf{C} \oplus \mathbf{C}\theta_1 \oplus \mathbf{C}\theta_2)/H^0(X^{(1)}, \mathcal{O}_{X^{(1)}}(*\sum_i Q_i))$$

Proof of 4.2.1. The tangent space is given by H^1 of the group of infinitesimal automorphisms of the object in question. In the above situation, this group is given by

$$\mathscr{E}nd_{\Pi}(\mathscr{L})(-(k+1)\sum_i Q_i) \simeq \mathcal{O}_X(-(k+1)\sum_i Q_i)$$

Then we have only to use the exact sequence associated with the exact sequence

$$0 \to \mathcal{O}_X(-(k+1)\sum_i Q_i) \to \mathcal{O}_X((l-k-1)\sum_i Q_i) \to \bigoplus_{i=-k}^{l} \mathbf{C}z^{-i} \otimes \Lambda^{\cdot}(\mathbf{C}\theta_1 \oplus \mathbf{C}\theta_2) \to 0$$

(Take the limit $l \to \infty$.) cf. [KNTY, KSU1].

Similarly for $\mathfrak{X}^{(1)}$.

Remark 4.2.2. 1) The tangent space in 4.2.1 reflects the parity of the space. For $k=0$, we have

$$T_{\mathscr{L}}\, Pic_{\Pi,X} \simeq H^1(C, \mathcal{O}_C \oplus \Lambda^2(\mathscr{F})) \oplus H^1(C, \mathscr{F})$$

$$T_{\mathscr{L}}\, Pic_{\Pi,X^{(1)}} \simeq H^1(C, \mathcal{O}_C) \oplus H^1(C, \mathscr{F})$$

As we have seen in 3.2.5, $Pic_{\Pi,X}$ (resp. $Pic_{\Pi,X^{(1)}}$) is the product of Pic_C and $H^1(X, 1+\mathscr{N})$ (resp. $H^1(X^{(1)}, \mathscr{F})$. So the part

$$\mathbf{C}[z^{-1}] \otimes C/H^0(C, \mathcal{O}_C(*\sum_i Q_i))$$

is just the tangent space of Pic_C in the case $k=0$. For $k \geq 1$, the tangent space of $\mathbf{G}_m^{1|1}(\Lambda^{\cdot}(\mathbf{C}\theta_1 \oplus \mathbf{C}\theta_2))$ (resp. $\mathbf{G}_m^{1|1}(\mathbf{C} \oplus \mathbf{C}\theta_1 \oplus \mathbf{C}\theta_2)))$ takes part in that of $Pic^{(k)}_{\Pi,X}$ (resp. $Pic^{(k)}_{\Pi,X^{(1)}}$).

2) The tangent space of $Pic_{0,X}$ is similarly given by $H^1(X, \mathcal{O}_{X,0})$ and in the situation 4.1 is equal to $H^1(C, \mathcal{O}_C \oplus \Lambda^2(\mathscr{F}))$. It is a purely even space.

We can equally develop the dressed version of Pic_0.

4.3. The calculation of tangent spaces of dressed Π-Picard schemes in the previous paragraph implies that they are homogeneous spaces of the loop groups $\Pi_i \mathbf{G}_m^{1|1}(\hat{\mathcal{O}}_{X,Q_i})$ and $\Pi_i \mathbf{G}_m^{1|1}(\hat{\mathcal{O}}_{X^{(1)},Q_i})$, infinitesimally.

In view of application to conformal field theory 5.2, we want to relate representations of certain algebra pertaining to fermionic fields with the space of conformal blocks of this symmetry algebra. This is the localization procedure.

In the case of the Virasoro algebra, Beilinson-Schechtman utilized it on the moduli space of curves [BS, BFM]. In the case of $U(1)$-current algebra, Ueno and the author tried it on the (dressed) Picard scheme of the universal family of curves [SU].

Here we develop its analog with some supersymmetry. We specialize the sheaf $\mathscr{F}$ to the case $(\omega_C^{1/2})^{\oplus 2}$ where $\omega_C^{1/2}$ is a theta characteristic on C, i.e. an invertible $\mathcal{O}_C$-module $\mathscr{I}$ such that

$$\mathscr{I}^{\otimes 2} \simeq \Omega_C^1 =: \omega_C.$$

This choice means that the relevant fermionic fields in this situation have conformal dimension $1/2$. More precisely, they are vertex operators $V_{\pm 1}(z)$ (1.1.2).

Recall that $V_{\pm 1}(z)$ generate the algebra *Clif*. Denote its degree 1 part by $Clif_1$ (cf. § 1):

$$Clif_1 = \oplus_\mu \mathbf{C}\psi_\mu \oplus \bigoplus_\mu \mathbf{C}\psi_\mu^+.$$

Put also

$$Heis_1 = \bigoplus_n \mathbf{C}a(n),$$

where $a(z) = \sum_n a(n)z^{-n-1}$ is the $U(1)$-current operator (1.1.5). So $Heis_1 \oplus \mathbf{C}$ is the Heisenberg (Lie) algebra. Then Proposition 4.2.1 implies that we have a Lie superalgbra homomorphism

$$\Pi_{i=1}^n (Heis_1 \oplus Clif_1) \to T_{(\mathscr{L},\alpha)} P_{\mathfrak{X}}^\infty$$

for $(\mathscr{L}, \alpha) \in Pic_\Pi^{(\infty)}(\mathfrak{X}^{(1)})$, where we put $Pic_{\Pi,\mathfrak{X}^{(1)}}^{(\infty)} = P_{\mathfrak{X}}^\infty$ and $Heis_1 \oplus Clif_1$ is considered as a Lie superalgebra with trivial center. Its kernel is $H^0(C, \mathcal{O}_{X^{(1)}}(*\sum_i Q_i))$. This is a Lie sub-superalgebra of $\Pi_i(Heis_1 \oplus Clif_1)$ via the formal trivialization. Recall that

$$Lie\, \mathbf{G}_m^{1|1}(\mathcal{O}_{X^{(1)}}) = \mathcal{O}_C \oplus \omega^{1/2} \oplus \omega^{1/2}.$$

Then $H^0(C, \mathcal{O}_C(*\sum_i Q_i))$ (resp. $H^0(C, \omega^{1/2} \oplus \omega^{1/2}(*\sum_i Q_i))$ injects into $\Pi_i(Heis_1)$ (resp. $\Pi_i(Clif)$) through

$$g \mapsto (Res_{z_i=0}(g(z_i)a(z_i)dz_i))_{i=1,\dots,n}$$

$$(h_1, h_2) \mapsto (Res_{z_i=0}(h_j(z_i)V_{(-1)^j}(z_i)dz_i))_{i=1,\dots,n,\, j=1,2}$$

The above homomorphism amounts to the homomorphism

$$\mathcal{O}_{P_{\mathfrak{X}}^\infty} \otimes \Pi_{i=1}^n (Heis_1 \oplus Clif_1) \to T_{P_{\mathfrak{X}}^\infty}.$$

Proposition 4.3.1. *The above homomorphism lifts to*

$$\mathcal{O}_{P_{\mathfrak{X}}^\infty} \otimes (\Pi_{i=1}^n (Heis_1 \oplus Clif_1) \oplus \mathbf{C}) \to \mathscr{D}_{d(\mathscr{L}_{univ})}^{\leq 1}$$

where $d(\mathscr{L}_{univ})$ is the determinant line bundle pulled back to $P_{\mathfrak{X}}^\infty$. $\mathscr{D}_{d(\mathscr{L}_{univ})}^{\leq 1}$ is the degree ≤ 1 part of the ring of differential operators $\mathscr{D}_{d(\mathscr{L}_{univ})}$ acting on the sections of the invertible sheaf $d(\mathscr{L}_{univ})$.

The kernel of this homomorphism equals $\mathcal{O}_{P_{\mathfrak{X}}^{\infty}} \otimes H^0(C, \mathcal{O}_{X^{(1)}}(*\sum_i Q_i))$.

The factor $\mathbf{C}$ in the left term is the (common) center of the Lie superalgebra.

The idea of the proof is to regard $\mathcal{D}^{\leq 1}_{\tilde{d}(\mathcal{L}_{univ})}$ as an Atiyah algebra $(p_2)_*({}^{tr}\mathcal{A}_{\mathcal{L}_{univ}})$ where $p_2 \colon C \times P_{\mathfrak{X}}^{\infty} \to P_{\mathfrak{X}}^{\infty}$ is the second projection. Then calculate the direct image by Čech covering along the fiber. Since $P_{\mathfrak{X}}^{\infty}$ is over Pic_C, one can construct a Lie superalgebra homomorphism

$$\Pi_{i=1}^{n}(Heis_1 \oplus Clif_1) \oplus \mathbf{C} \to (p_2)_*({}^{tr}\mathcal{A}_{\mathcal{L}_{univ}})$$

This proof is parallel to [BS, 4.1] and for the part concerning only *Heis* and Pic_C, cf. [SU].

From this proposition, we obtain a ring homomorphism

$$\rho \colon \mathcal{O}_{P_{\mathfrak{X}}^{\infty}} \otimes (\otimes_i (Heis \otimes Clif)) \to \mathcal{D}_{d(\mathcal{L}_{univ})}$$

Now we can define the localization of representations of $Heis \otimes Clif$ as follows.

Given representations of $Heis \otimes Clif$ $M_i (i = 1, \cdots, n)$ with the same center, we can extend the scalars by ρ

$$\Delta(\otimes_i M_i) = \mathcal{D}_{d(\mathcal{L}_{univ})} \otimes_{\rho} \otimes_i M_i. \tag{4.3.2}$$

For the meaning of this construction, we have the following

Lemma 4.3.3.

$$\Delta(\otimes_i M_i) \otimes \mathcal{O}_{P_{\mathfrak{X}}^{\infty}}/\mathfrak{m}_{(\mathcal{L},\alpha)} \simeq \otimes_i M_i / H^0(C, \mathcal{O}_{X^{(1)}}(*\sum_i Q_i)) \otimes_i M_i$$

where $\mathfrak{m}_{(\mathcal{L},\alpha)}$ *is the ideal sheaf of* $P_{\mathfrak{X}}^{\infty}$ *at the point* $(\mathcal{L}, \alpha)$.

Thus the fiber of localization 4.3.2 is equal to the coinvariants with respect to the subalgebra $H^0(C, \mathcal{O}_{X^{(1)}}(*\sum_i Q_i))$, or the space of conformal blocks in conformal field theory.

For the original localization construction, see [B]. Its adaptation to CFT can be found in [BFM, BS, SU].

§5. Applications to abelian conformal field theory

We give a natural geometric framework for abelian conformal field theory with $U(1)$ gauge symmetry using the Π-Picard scheme and its dressed version of certain $N = 2$ supercurves.

5.1. Let us recall the definition of the space of conformal blocks for the Fock space $\mathcal{H}^{\otimes n}$ according to Ueno [U]. We use the same notation as in § 1.

Let $\mathfrak{X} = (C; Q_1, \cdots, Q_n; z_1, \cdots, z_n)$ be an n-pointed smooth curve of genus g with formal local parameters z_i at Q_i. Here we consider the curve C over the field of complex numbers $\mathbf{C}$ for simplicity.

Remember that we have a natural pairing

$$\mathscr{H}^{\dagger\otimes n} \otimes \mathscr{H}^{\otimes n} \to \mathbf{C}$$

induced from the pairing between $\mathscr{H}^{\dagger}$ and $\mathscr{H}$ (1.2.2).

$a(z)$ denotes the current operator (1.1.5) and $V_{\pm 1}(z)$ denotes the vertex operator (1.1.4) and has conformal dimension 1/2, cf. § 1.

Definition 5.1.1. The space of conformal blocks $\mathscr{V}^{\dagger}(\mathfrak{X})$ is the subspace of $\mathscr{H}^{\dagger\otimes n}$ consisting of vectors $\langle\psi|$ satisfying the following conditions for any $|\phi\rangle \in \mathscr{H}^{\otimes n}$:

(1) $$\sum_{j=1}^{n} Res_{z_j=0}(\langle\psi|\rho_j(a(z_j))|\phi\rangle g(z_j)dz_j) = 0$$

for the Laurent expansion $g(z_j)$ of any $g \in H^0(C, \mathscr{O}_C(*\sum Q_j))$ and

(2) $$\sum_{j=1}^{n} Res_{z_j=0}(\langle\psi|\rho_j(V_{\pm 1}(z_j))|\phi\rangle h(z_j)dz_j) = 0$$

for any $h \in H^0(C, \omega_C^{\otimes\frac{1}{2}}(*\sum Q_j))$ and $h(z_j)$ is its Laurent expansion of h at Q_j.

Here $\rho_j(?)$ means that ? acts on the j-th factor in the tensor product $\mathscr{H}^{\otimes n}$. ω_C denotes the dualizing sheaf of C and we have chosen its square root. To put it the other way, we have chosen an $N = 1$ superconformal structure on C, cf. 2.3.

We define the dual of the above space of conformal blocks to be

$$\mathscr{V}(\mathfrak{X}) = Hom_{\mathbf{C}}(\mathscr{V}^{\dagger}(\mathfrak{X}), \mathbf{C}).$$

So it can be identified with the quotient of $\mathscr{H}^{\otimes n}$ modulo the relation generated by $H^0(C, \mathscr{O}_C(*\sum Q_j))$ via $a(z)$ and $H^0(C, \omega_C^{\otimes\frac{1}{2}}(*\sum Q_j))$ via $V_{\pm 1}(z)$. This is nothing but the space of coinvariants.

Remark 5.1.2. 1) The above condition (2) means that $\{\langle\psi|\rho_j(V_{\pm 1}(z_j))|\phi\rangle dz_j\}$, $(j = 1, \cdots, n)$ are the Laurent expansions of an element of $H^0(C, \omega_C^{\otimes\frac{1}{2}}(*\sum Q_j))$ at Q_j with respect to the formal local parameter z_j.

Similarly for the condition (1).

2) The condition (1) is the usual gauge condition, cf. [KNTY, 7.1.2)].

3) One can adapt the above definition in the case C is assumed to be n-pointed stable as well.

The main theorem in [U] is the following:

Theorem [U, § 1].

$$\dim_{\mathbf{C}} \mathscr{V}^{\dagger}(\mathfrak{X}) = 1$$

Remark 5.1.3. The above theorem generalizes [KNTY, 7.7]. This was possible due to the factorization property for the conformal blocks [U, 2.5]. The formulation parallels the one in [TUY] and it should construct a projectively flat connection on the sheaf of conformal blocks on the moduli space of stable curves (at least over smooth curves), cf. [BK, BFM].

There was also an attempt [SU] to generalize [KNTY] and was partially successful because only the gauge condition by $U(1)$-current was considered there.

The formulation in [U] is stated for an arbitrary level M (even integer). So the above result is not included in [U] strictly speaking. We will treat this in a forthcoming paper [S1].

5.2. Let $\omega_C^{1/2}$ be a theta characteristic on C. It amounts to choose a superconformal structure overlying C. We consider the supercurve of odd dimension $N = 2$

$$X = (C, S^{\cdot}\,\Pi(\omega_C^{1/2} \oplus \omega_C^{1/2}))$$

as well as

$$X^{(1)} = (C, \mathcal{O}_C \oplus \Pi(\omega_C^{1/2} \oplus \omega_C^{1/2})).$$

We follow the notation in §4.

Then we have the localization homomorphism 4.3

$$\rho\colon \bigoplus_i (Heis_1 \oplus Clif_1) \oplus \mathbf{C} \to \mathcal{D}^{\leq 1}_{d(\mathcal{L}_{univ})}$$

where $\mathcal{D}_{d(\mathcal{L}_{univ})}$ denote the ring of (twisted) differential operators acting on the local sections of $d(\mathcal{L}_{univ})$ on $Pic_{\Pi, X^{(1)}}$.

Using this homomorphism, we have the localization functor Δ (4.3.2). We now apply Δ to the Fock space representations $\mathcal{H}$. Namely, these representations have the common scalar action of the center of *Heis* and they become representations of *Clif* through vertex operators $V_{\pm 1}(z_j)$. Thus $\mathcal{H}^{\otimes n}$ becomes a representation of $\bigoplus_i (Heis_1 \oplus Clif_1) \oplus \mathbf{C}$.

Then we can state one of the main results in this article.

Theorem 5.2.1. *The space of conformal blocks* $\mathcal{V}(\mathfrak{X})$ 5.1.1 *equals the fiber of the localization* $\Delta(\mathcal{H}^{\otimes n})$ *at any point of* $Pic^{(\infty)}_{\Pi, \mathfrak{X}^{(1)}}$.

This theorem is almost clear from the construction of the localization on $Pic^{(\infty)}_{\Pi, \mathfrak{X}^{(1)}}$ 4.3.1 and 4.3.3. The key point is to consider the supercurve $X^{(1)}$ associated with the theta characteristic $\omega_C^{1/2}$ and its Π-Picard scheme in order to express the gauge condition by vertex operators as well as the current operator.

Remark 5.2.2. 1) We used Pic_Π for the first infinitesimal neighbourhood $X^{(1)}$ in order to realize the operator content of abelian CFT. If we use Pic_Π for X itself, we have to introduce another field whose meaning the author does not know.
2) The space of conformal blocks is realized as a fiber of a module over the ring $\mathcal{D}_{d(\mathcal{L}_{univ})}$, which is a twisted ring of differential operators. But we used mostly data concerning the underlying curve C, e.g. $\mathcal{L}_{univ}$. This can be understood as follows.

An obvious extension of Penkov's theorem [P] on the equivalence of the category of $\mathscr{D}$-modules and that of $\mathscr{D}_{red}$-modules implies that it is enough to consider the restriction of the module $\Delta(\mathscr{H}^{\otimes n})$ to $(Pic^{(\infty)}_{\Pi,\mathfrak{X}^{(1)}})_{red} = Pic^{(\infty)}_{\{C;(Q_i);(z_i)\}}$. This last space is the dressed Picard scheme on the even part $\{C;(Q_i);(z_i)\}$ of $\mathfrak{X}^{(1)}$, cf. [SU].

5.3. Let us relate the space of conformal blocks with the space of global sections of the determinant line bundle. So we restrict $Pic_{\Pi,X^{(1)}}$ etc. over the component of degree $g-1$.

5.3.1. In [SU], we note that a fiber of the determinant line bundle $d(\mathscr{L}_{univ})$ injects into the space of conformal blocks. There were two points. The first is that we have the identification:

$$d(\mathscr{L}_{univ})_{(\mathscr{L},\alpha)} = d(\mathscr{L}) = d(\mathscr{L}(m\sum_i Q_i))\cdot \wedge^{\max}(\oplus_i \oplus_{k=-1}^{-m} \mathbf{C}z_i^k)^{-1}$$

derived from the exact sequence

$$0 \to \mathscr{L} \to \mathscr{L}(m\sum_i Q_i) \to \oplus_i \oplus_{k=-1}^{-m} \mathbf{C}z_i^k \to 0.$$

The second point is that the space $H^0(C, \mathscr{L}(*\sum_i Q_i))$ is stable under the multiplication by $H^0(C, \mathcal{O}_C(*\sum_i Q_i))$. This implies that its maximal exterior power $d(\mathscr{L})$ considered as a subspace of $\mathscr{H}^{\otimes n}$ satisfies the gauge condition 5.1.1, (1).

5.3.2. If we regard $\mathscr{H}^{\otimes n}$ as a representation of *Clif* by the procedure described in 1.3, the gauge conditions 5.1.1, (1) and (2) correspond to the invariance of $H^0(C, \mathscr{L}(*\sum_i Q_i))$ under the multiplication by $H^0(C, \mathcal{O}_{X^{(1)}}(*\sum_i Q_i))$, which holds because $\mathcal{O}_{X^{(1)}}$ acts on $\mathscr{L}$ through $\mathcal{O}_C$.

Let us descend the $\mathscr{D}_{d(\mathscr{L}_{univ})}$-module $\Delta(\mathscr{H}^{\otimes n})$ on $Pic^{(\infty)}_{\Pi,\mathfrak{X}^{(1)}}$ to Pic^{g-1}_C and then integrate on Pic^{g-1}_C. So let us denote the natural projection by $r\colon Pic^{(\infty)}_{\Pi,\mathfrak{X}^{(1)}} \to Pic_{\Pi,X^{(1)}}$. Take the invariants and restrict to Pic^{g-1}_C:

$$\Delta_n = r_*(\Delta(\mathscr{H}^{\otimes n}))^{\Pi_i \mathbf{G}_m^{1|1}(\hat{\mathcal{O}}_{X,Q_i})}|_{Pic^{g-1}_C} \tag{5.3.3}$$

This amounts to taking invariants with respect to the change of formal trivialization of $\mathscr{L}$. We also used the relation $(Pic_{\Pi,X^{(1)}})_{red} = Pic_C$ cf. Remark 5.2.2, 2).

Then $d(\mathscr{L}_{univ})$ injects into it because it comes from a similar one (with the same notation by abuse of notation) on $Pic_{\Pi,X^{(1)}}$:

$$d(\mathscr{L}_{univ}) \hookrightarrow \Delta_n$$

The dual of this is the following:

$$d(\mathscr{L}_{univ})^{-1} \leftarrow \Delta_n^*$$

Taking the global sections, we obtain

$$H^0(Pic_C^{g-1}, d(\mathscr{L}_{univ})^{-1}) \leftarrow H^0(Pic_C^{g-1}, \Delta_n^*)$$

Remember that $d(\mathscr{L}_{univ})$ on Pic_C^{g-1} is nothing but the dual $\mathcal{O}(-\Theta)$ of the theta divisor, cf. [Sz]. We also note that Δ_n^* is the module whose fiber is the space of conformal blocks $\mathscr{V}^\dagger(\mathfrak{X})$ since taking invariants is cancelled by taking r_*.

Taking into account of the dimension of the theta divisor, we obtain the following as a corollary of Theorem [U, § 1] and 5.2.1.

Theorem 5.3.4. *We have a canonical isomorphism*

$$H^0(Pic_C^{g-1}, \mathcal{O}(\Theta)) \simeq \mathscr{V}^\dagger(\mathfrak{X}).$$

Remark 5.3.5. We studied the space of conformal blocks considering only the gauge condition (1) in [SU]. We don't know the exact relation of the D-module in [SU] and the above one 5.3.3.

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