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The induced homomorphism of the Bott map on K-theory

Dedicated to Professor Yasutoshi Nomura on his 60th Birthday

By

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1. Introduction

The original Bott map is a map from the unreduced suspension of a compact symmetric space into another compact symmetric space (see [5]). The complex K-theory of a compact symmetric space has been studied well (see [6], [12], [13] and [14]). The purpose of this paper is to describe the behavior of the homomorphism induced by the Bott map on complex K-theory.

Throughout this paper, G denotes a compact connected Lie group and σ an involutive automorphism of G. Then the fixed point set

$$G^\sigma = \{x \in G \mid \sigma(x) = x\}$$

of σ forms a closed subgroup of G. Let $(G^\sigma)_1$ be its identity component and H a closed subgroup of G such that $(G^\sigma)_1 \subset H \subset G^\sigma$. Then the pair (G, H) is called a compact symmetric pair, and the coset space G/H is called a compact symmetric space. If G is simply connected, then G^σ is connected, so $(G^\sigma)_1 = G^\sigma$, and G/G^σ is simply connected. Conversely, every compact, simply connected symmetric space can be expressed as a homogeneous space of a simply connected group G. When G^σ is not connected and a coset space G^σ/H is under consideration, we will use $(G^\sigma)_1$ instead of G^σ and abbreviate $(G^\sigma)_1$ to G^σ unless otherwise stated.

Associated with a symmetric space G/G^σ, there is a fibre sequence

$$G^\sigma \xrightarrow{i} G \xrightarrow{\pi} G/G^\sigma \xrightarrow{j} BG^\sigma \xrightarrow{Bi} BG$$

and a map $\xi_\sigma\colon G/G^\sigma \to G$ defined by

$$\xi_\sigma(xG^\sigma) = x\sigma(x)^{-1} \qquad \text{for } xG^\sigma \in G/G^\sigma.$$

Let $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{K}^n$ where $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$. An $n \times n$ matrix $A = (a_{ij}) \in M(n, \mathbf{K})$ with coefficients in $\mathbf{K}$ acts on $\mathbf{K}^n$ by $(A\mathbf{x})_i = \sum a_{ik}x_k$. Let $1 = I_n$ denote the $n \times n$ unit matrix and put

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$$I_{p,q} = (-I_p) \oplus I_q = \begin{pmatrix} -I_p & O \\ O & I_q \end{pmatrix}; \qquad J_n = \begin{pmatrix} O & -I_n \\ I_n & O \end{pmatrix}.$$

Note that $I_{p,q}^2 = I_{p+q}$ and $J_n^2 = -I_{2n}$. The transpose and (for $\mathbf{K} = \mathbf{C}$ or $\mathbf{H}$) the conjugate of A are denote by ${}^tA$ and $c(A) = \bar{A}$. Let G be a closed subgroup of the general linear group $GL(n, \mathbf{K})$. If $a \in G$, we have an inner automorphism Ad a of G defined by Ad $a(x) = axa^{-1}$ for $x \in G$.

Let $U(n, \mathbf{K})$ denote the group of all matrices leaving the Hermitian inner product on $\mathbf{K}^n$ invariant. We have $U(n, \mathbf{R}) = O(n)$, $U(n, \mathbf{C}) = U(n)$, $U(n, \mathbf{H}) = Sp(n)$, called the *orthogonal*, *unitary*, and *symplectic groups* respectively. $O(n)$ has two connected components, while $U(n)$ and $Sp(n)$ are connected. The orthogonal (resp. unitary) matrices of determinant 1 are denoted by $SO(n)$ (resp. $SU(n)$). The groups $SU(n)$ and $Sp(n)$ are simply connected. The group $SO(n)$ has a simply connected 2-fold covering $Spin(n)$ for $n \geq 3$. We denote by G_2, F_4, E_6, E_7 and E_8 the 1-connected exceptional Lie groups having the corresponding simple Lie algebras respectively.

The compact, 1-connected, simple Lie groups G and their centers $Z(G)$ are given as follows:

G	$Z(G)$
$SU(n+1)$	$\mathbf{Z}_{n+1} = \{\alpha I_{n+1} \mid \alpha \in \mathbf{C}, \alpha^{n+1} = 1\}$
$Spin(2n+1)$	$\mathbf{Z}_2 = \{\pm 1\}$
$Sp(n)$	$\mathbf{Z}_2 = \{\pm I_n\}$
$Spin(4n)$	$\mathbf{Z}_2 \times \mathbf{Z}_2 = \{\pm 1, \pm e_1 \cdots e_{4n}\}$
$Spin(4n+2)$	$\mathbf{Z}_4 = \{\pm 1, \pm e_1 \cdots e_{4n+2}\}$
G_2	$\{1\}$
F_4	$\{1\}$
E_6	$\mathbf{Z}_3 = \{1, \omega 1, \omega^2 1\}$ where $\omega = (-1 + \sqrt{3}i)/2 \in \mathbf{C}$
E_7	$\mathbf{Z}_2 = \{\pm 1\}$
E_8	$\{1\}$

where $\mathbf{Z}_n$ is the cyclic group of order n. Any compact connected Lie group G can be regarded as a compact symmetric space in the following manner. The product space $G \times G$ has an involutive automorphism τ' given by interchanging the factors: $\tau'(x, y) = (y, x)$ for $(x, y) \in G \times G$. Let $\Delta: G \to G \times G$ be the diagonal map. Then $(G \times G)^{\tau'} = \Delta(G)$ and the homogeneous space $G \times G/\Delta(G)$ may be identified with G through the homeomorphism $\varphi: G \times G/\Delta(G) \to G$ defined by

$$\varphi((x, y)\Delta(G)) = xy^{-1} \qquad \text{for } (x, y)\Delta(G) \in G \times G/\Delta(G).$$

Notice that rank $\Delta(G) <$ rank $G \times G$ provided $G \neq \{1\}$.

The classification of the compact 1-connected irreducible symmetric spaces $M = G/G^\sigma$ is known (e.g., see [15]). They are the compact 1-connected simple Lie groups G and the following:

$\underline{M}$	$\underline{G/G^\sigma}$
AI	$SU(n)/SO(n)$ $(n > 2)$
AII	$SU(2n)/Sp(n)$ $(n > 1)$
$AIII$	$U(m+n)/U(m) \times U(n)$ $(1 \le m \le n)$
BDI	$SO(m+n)/SO(m) \times SO(n)$ $(2 \le m \le n,\ m+n \ne 4)$
$BDII$	$SO(n+1)/SO(n)$ $(n \ge 2)$
$DIII$	$SO(2n)/U(n)$ $(n \ge 4)$
CI	$Sp(n)/U(n)$ $(n \ge 3)$
CII	$Sp(m+n)/Sp(m) \times Sp(n)$ $(1 \le m \le n)$
EI	$E_6/[Sp(4)/\mathbf{Z}_2]$ where $\mathbf{Z}_2 = \{I_4, -I_4\}$
EII	$E_6/[(S^3 \times SU(6))/\mathbf{Z}_2]$ where $\mathbf{Z}_2 = \{(1, I_6), (-1, I_6)\}$
$EIII$	$E_6/[(T^1 \times Spin(10))/\mathbf{Z}_4]$ where $\mathbf{Z}_4 = \{(\pm 1, \phi(\pm 1)), (\pm i, \varphi(\mp 1))\}$
EIV	E_6/F_4
EV	$E_7/[SU(8)/\mathbf{Z}_2]$ where $\mathbf{Z}_2 = \{I_8, -I_8\}$
EVI	$E_7/[(S^3 \times Spin(12))/\mathbf{Z}_2]$ where $\mathbf{Z}_2 = \{(1,1), (-1, -e_1 \cdots e_{12})\}$
$EVII$	$E_7/[(T^1 \times E_6)/\mathbf{Z}_3]$ where $\mathbf{Z}_3 = \{(1,1), (\omega, \phi(\omega^2)), (\omega^2, \phi(\omega))\}$
$EVIII$	$E_8/Ss(16)$ where $Ss(16) = Spin(16)/\{1, e_1 \cdots e_{16}\}$
EIX	$E_8/[(S^3 \times E_7)/\mathbf{Z}_2]$ where $\mathbf{Z}_2 = \{(1,1), (-1,-1)\}$
FI	$F_4/[(S^3 \times Sp(3))/\mathbf{Z}_2]$ where $\mathbf{Z}_2 = \{(1, I_3), (-1, -I_3)\}$
FII	$F_4/Spin(9)$
G	$G_2/SO(4)$

where we have used the notation of [16].

Let LG denote the Lie algebra of a compact simple Lie group G. The group of outer automorphisms $\mathrm{Out}\,(LG) = \mathrm{Aut}\,(LG)/\mathrm{Inn}\,(LG)$ is trivial except in the cases

A_n, $n > 1$: $\mathrm{Out}\,(LG) = \mathbf{Z}_2$;
D_n, $n > 4$: $\mathrm{Out}\,(LG) = \mathbf{Z}_2$;
D_4: $\mathrm{Out}\,(LG) = \Sigma_3$, the symmetric group on 3 letters;
E_6: $\mathrm{Out}\,(LG) = \mathbf{Z}_2$.

Each generator has a representative $\tau \in \mathrm{Aut}\,(LG)$ of order 2, given as follows. A_n is the Lie algebra of $G = SU(n+1)$, and $\tau = c$, the complex conjugation. The fixed point set on $SU(n+1)$ is $SO(n+1)$, so $SU(n+1)/SO(n+1)$ is the corresponding symmetric space. D_n is the Lie algebra of $G = SO(2n)$, and $\tau = \mathrm{Ad}\, I_{1,2n-1}$. It has fixed point set $SO(2n) \cap [O(1) \times O(2n-1)]$, so the sphere $S^{2n-1} = SO(2n)/SO(2n-1)$ is the corresponding symmetric space. For E_6, one constructs $\tau = \lambda$ to have fixed point set F_4, so E_6/F_4 is the corresponding symmetric space.

Consider involutive automorphisms σ on compact 1-connected simple Lie groups G. According to [15, p. 287], there are two cases: either

(a) σ is an outer automorphism and $\mathrm{rank}\, G^\sigma < \mathrm{rank}\, G$,

or

(b) σ is an inner automorphism and rank G^σ = rank G.

The irreducible symmetric spaces which belong to the case (a) are the compact 1-connected simple Lie groups G and

G/G^σ	σ
$SU(n)/SO(n)$	c
$SU(2n)/Sp(n)$	$\mathrm{Ad}\, J_n \circ c$
$SO(2m+2n+2)/SO(2m+1) \times SO(2n+1)$	$\mathrm{Ad}\, I_{2m+1,2n+1}$
$S^{2n-1} = SO(2n)/SO(2n-1)$	$\mathrm{Ad}\, I_{1,2n-1}$
E_6/F_4	λ
$E_6/[Sp(4)/\mathbf{Z}_2]$	$\lambda \circ \mathrm{Ad}\, \gamma$

Consider the case (b). Since $\sigma: G \to G$ is inner, there is an element $x_\sigma \in G$ such that $\sigma = \mathrm{Ad}\, x_\sigma$, and G^σ is the centralizer $C_G(x_\sigma) = \{x \in G | x_\sigma x = x x_\sigma\}$. ($x_\sigma \in G$ is explicitly given in terms of an element $X_\sigma \in LG$; for details, see [11] or [15, Chapter 8].) Looking over the two tables below, we see that $x_\sigma^2 = 1$ or $x_\sigma^2 = -1$. Since G is connected, there is a (unique) one-parameter subgroup $v_\sigma: \mathbf{R} \to G$ such that $v_\sigma(1) = x_\sigma$. (It is defined by $v_\sigma(t) = \exp tX_\sigma$ for $t \in \mathbf{R}$, where $\exp: LG \to G$ is the exponential map.) Clearly

$$C_G(\mathrm{Im}\, v_\sigma) \subset C_G(x_\sigma) = G^\sigma.$$

Let $s_\sigma: \mathbf{R} \to G$ be the map defined by

$$s_\sigma(t) = \begin{cases} v_\sigma(2t) & \text{if } x_\sigma^2 = 1 \\ v_\sigma(4t) & \text{if } x_\sigma^2 = -1 \end{cases} \tag{1}$$

for $t \in \mathbf{R}$. Then $s_\sigma(0) = s_\sigma(1) = 1$. So s_σ induces a homomorphism $S^1 = \mathbf{R}/\mathbf{Z} \to G$ of Lie groups, and $C_G(\mathrm{Im}\, s_\sigma) = C_G(\mathrm{Im}\, v_\sigma)$.

Notice that $C_G(x_\sigma)$ is not always connected. Recalling our convention in the second paragraph of this paper, we put $H = G^\sigma$. Then it is a connected subgroup of maximal rank in a compact connected Lie group G. A complete list of such inclusions $H \subset G$ is given in [3]; to discuss it, we may take $\bar{G} = G/Z(G)$ instead of G and $\bar{H} = H/(Z(G) \cap H)$ instead of H, because $\bar{G}/\bar{H} \approx G/H$. These inclusions $\bar{H} \subset \bar{G}$ may be divided into two cases (see [2, §13]):

(b1) $\bar{H}$ is the connected centralizer $C_{\bar{G}}(\bar{x})_0$ of an element $\bar{x}$ of order 2, and $\bar{x}$ generates the center of $\bar{H}$.

(b2) $\bar{H}$ is the centralizer $C_{\bar{G}}(T^1)$ (which is known to be connected) of a one dimensional torus T^1, and T^1 is the identity component of the center of $\bar{H}$.

The coset spaces G/G^σ which belong to the cases (b1) and (b2) are called *Riemannian* and *Hermitian symmetric spaces* respectively. The irreducible Riemannian symmetric spaces are given by

G/G^σ	$\sigma = \mathrm{Ad}\, x_\sigma$	x_σ^2
$SO(m+n)/SO(m) \times SO(n)$	$\mathrm{Ad}\, I_{m,n}$	$I_{m,n}^2 = I_{m+n}$
$SO(n+1)/SO(n)$	$\mathrm{Ad}\, I_{1,n}$	$I_{1,n}^2 = I_{n+1}$
$Sp(m+n)/Sp(m) \times Sp(n)$	$\mathrm{Ad}\, I_{m,n}$	$I_{m,n}^2 = I_{m+n}$
$E_6/[(S^3 \times SU(6))/\mathbf{Z}_2]$	$\mathrm{Ad}\, \gamma$	$\gamma^2 = 1$
$E_7/[SU(8)/\mathbf{Z}_2]$	$\mathrm{Ad}\, \lambda\gamma$	$(\lambda\gamma)^2 = -1$
$E_7/[(S^3 \times Spin(12))/\mathbf{Z}_2]$	$\mathrm{Ad}\, \sigma$	$\sigma^2 = 1$
$E_8/Ss(16)$	$\mathrm{Ad}\, \tilde{\lambda}\gamma$	$(\tilde{\lambda}\gamma)^2 = 1$
$E_8/[(S^3 \times E_7)/\mathbf{Z}_2]$	$\mathrm{Ad}\, v$	$v^2 = 1$
$F_4/[(S^3 \times Sp(3))/\mathbf{Z}_2]$	$\mathrm{Ad}\, \gamma$	$\gamma^2 = 1$
$F_4/Spin(9)$	$\mathrm{Ad}\, \sigma$	$\sigma^2 = 1$
$G_2/SO(4)$	$\mathrm{Ad}\, \gamma$	$\gamma^2 = 1$

where $\gamma \in G_2$, $\lambda\gamma \in E_7$, $\sigma \in F_4$, $\tilde{\lambda}\gamma \in E_8$ and $v \in E_8$. The irreducible Hermitian symmetric spaces are given by

G/G^σ	$\sigma = \mathrm{Ad}\, x_\sigma$	x_σ^2
$U(m+n)/U(m) \times U(n)$	$\mathrm{Ad}\, I_{m,n}$	$I_{m,n}^2 = I_{m+n}$
$SO(2n)/U(n)$	$\mathrm{Ad}\, J_n$	$J_n^2 = -I_{2n}$
$Sp(n)/U(n)$	$\mathrm{Ad}\, jI_n$ (where $j \in \mathbf{H}$)	$(jI_n)^2 = -I_n$
$SO(n+2)/SO(2) \times SO(n)$	$\mathrm{Ad}\, I_{2,n}$	$I_{2,n}^2 = I_{n+2}$
$E_6/[(T^1 \times Spin(10))/\mathbf{Z}_4]$	$\mathrm{Ad}\, \sigma$	$\sigma^2 = 1$
$E_7/[(T^1 \times E_6)/\mathbf{Z}_3]$	$\mathrm{Ad}\, \iota$	$\iota^2 = -1$

The one dimensional torus T^1 in (b2) is the image of s_σ of (1).

Summarizing the above, the symmetric spaces G/G^σ which belong to the case (b2) satisfy

$$C_G(\mathrm{Im}\, s_\sigma) = G^\sigma,$$

and the symmetric spaces G/G^σ which belong to the case (b1) satisfy

$$C_G(\mathrm{Im}\, s_\sigma) \subsetneqq G^\sigma$$

(see [11]).

For a space X let $SX = (X \times I/X \times 1)/X \times 0$, the unreduced suspension of X. Let $\dot{I} = \{0, 1\} \subset I = [0, 1]$. For a space X with base point $x_0 \in X$, let $\Sigma X = X \times I/(X \times \dot{I} \cup x_0 \times I)$, the reduced suspension of X, and $\Omega X = \{l | l\colon I \to X, l(0) = l(1) = x_0\}$, the loop space on X. If $[X, Y]_0$ denotes base point preserving homotopy classes of maps $X \to Y$, there is a natural isomorphism

$$[\Sigma X, Y]_0 \underset{\cong}{\to} [X, \Omega Y]_0.$$

Throughout this paper, we will assume that two involutive automorphisms σ, τ of a compact connected Lie group G satisfy

$$\sigma \circ \tau = \tau \circ \sigma \tag{2}$$

where σ is inner, but τ may be outer. It follows from (2) that

$$(G^{\sigma})^{\tau} = (G^{\tau})^{\sigma} = G^{\sigma} \cap G^{\tau},$$

for which we write $G^{\sigma\tau}$ (or $G^{\tau\sigma}$), and

$$\sigma(G^{\tau}) \subset G^{\tau}, \qquad \tau(G^{\sigma}) \subset G^{\sigma}.$$

To be precise, we suppose the following two situations. The first situation is:

(I-i) $\sigma: G \to G$ is inner;

(I-ii) $\tau: G \to G$ is inner, but $\tau: G^{\sigma} \to G^{\sigma}$ is outer;

So there is an element $x_{\tau} \in G$ such that $\tau = \mathrm{Ad}\, x_{\tau}$, and there is a one-parameter subgroup $v_{\tau}: \mathbf{R} \to G$ with $v_{\tau}(1) = x_{\tau}$.

(I-iii) If $x \in G^{\sigma\tau}$, then $xv_{\tau}(t) = v_{\tau}(t)x$ for all $t \in I$, i.e.,

$$G^{\sigma\tau} \subset C_G(\mathrm{Im}\, v_{\tau}).$$

(This condition is automatically satisfied if G/G^{τ} belongs to the case (b2). But, if G/G^{τ} belongs to the case (b1), that condition may not be satisfied. One way to satisfy it is to replace $G^{\sigma\tau} = G^{\sigma} \cap G^{\tau}$ by $G^{\sigma} \cap C_G(\mathrm{Im}\, v_{\tau})$. However, the coset space $G^{\sigma}/(G^{\sigma} \cap C_G(\mathrm{Im}\, v_{\tau}))$ cannot be a symmetric space any longer.) Following [8, §3], we define

$$\hat{b}: S(G^{\sigma}/G^{\sigma\tau}) \to G/G^{\sigma}$$

by

$$\hat{b}(xG^{\sigma\tau}, t) = xv_{\tau}(t)G^{\sigma} \qquad \text{for } xG^{\sigma\tau} \in G^{\sigma}/G^{\sigma\tau}.$$

This is the Bott map [5]. Note that $\hat{b}$ does not preserve the base point, i.e., by (I-iii), $\hat{b}(G^{\sigma\tau}, t) = v_{\tau}(t)G^{\sigma}$ for all $t \in I$. Following Harris [9], we define a reduced version

$$\hat{b}_0: \Sigma(G^{\sigma}/G^{\sigma\tau}) \to G/G^{\sigma}$$

of $\hat{b}$ by

$$\hat{b}_0(xG^{\sigma\tau}, t) = v_{\tau}(t)^{-1}xv_{\tau}(t)G^{\sigma} \qquad \text{for } xG^{\sigma\tau} \in G^{\sigma}/G^{\sigma\tau}.$$

This map preserves the base point.

The second situation is:

(II-i) $\sigma: G \to G$ is inner;

So there is an element $x_{\sigma} \in G$ such that $\sigma = \mathrm{Ad}\, x_{\sigma}$.

(II-ii) $\tau: G \to G$ is outer;

Let $v_{\sigma}: \mathbf{R} \to G$ be a one-parameter subgroup with $v_{\sigma}(1) = x_{\sigma}$.

(II-iii) If $x \in G^{\tau\sigma}$, then $xv_{\sigma}(t) = v_{\sigma}(t)x$ for all $t \in I$, i.e.,

$$G^{\tau\sigma} \subset C_G(\mathrm{Im}\, v_{\sigma});$$

(We keep on the same comment as that after (I-iii).)

(II-iv) $\tau: G \to G$ satisfies

$$\tau(v_\sigma(t)) = v_\sigma(t)^{-1} \qquad \text{for all } t \in \mathbf{R}. \tag{3}$$

By this with $t = 1$, $\tau(x_\sigma) = x_\sigma$ if $x_\sigma^2 = 1$, and $\tau(x_\sigma) = -x_\sigma$ if $x_\sigma^2 = -1$. Hence, in either case, the condition (3) is satisfied. A typical example of the second situation is: $G = U(2)$, $\sigma = \operatorname{Ad} I_{1,1}$, $\tau = c$; then $G^\sigma = U(1) \times U(1)$, $G^\tau = O(2)$, and v_σ: $\mathbf{R} \to U(2)$ is given by

$$v_\sigma(t) = \begin{pmatrix} e^{\pi i t} & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{for } t \in \mathbf{R}.$$

Now we define

$$\hat{b}: S(G^\tau/G^{\tau\sigma}) \to G/G^\tau$$

by

$$\hat{b}(xG^{\tau\sigma}, t) = \begin{cases} xv_\sigma(t)G^\tau & \text{if } x_\sigma^2 = 1 \\ xv_\sigma(2t)G^\tau & \text{if } x_\sigma^2 = -1. \end{cases}$$

for $xG^{\tau\sigma} \in G^\tau/G^{\tau\sigma}$. This is the Bott map [5]. A reduced version

$$\hat{b}_0: \Sigma(G^\tau/G^{\tau\sigma}) \to G/G^\tau$$

of this $\hat{b}$ is defined by

$$\hat{b}_0(xG^{\tau\sigma}, t) = \begin{cases} v_\sigma(t)^{-1} xv_\sigma(t)G^\tau & \text{if } x_\sigma^2 = 1 \\ v_\sigma(2t)^{-1} xv_\sigma(2t)G^\tau & \text{if } x_\sigma^2 = -1 \end{cases}$$

for $xG^{\tau\sigma} \in G^\tau/G^{\tau\sigma}$.

For a loop $s \in \Omega G$, let $\Omega_s G = \{l | l \in \Omega G, l \simeq s\}$, the component of ΩG containing s. In particular, when $s = 0_1$, the constant loop, it is denoted by $\Omega_0 G$. The loop product with $s^{-1} \in \Omega G$ yields a homotopy equivalence $\Omega_s G \simeq \Omega_0 G$. If $\pi: \tilde{G} \to G$ is the universal covering of G, then $\Omega\pi$ gives a homeomorphism $\Omega\tilde{G} \approx \Omega_0 G$.

Suppose given a symmetric space G/G^σ which belongs to the case (b). (In the case (b1), we have to replace $G^\sigma = C_G(x_\sigma)$ by $C_G(\operatorname{Im} s_\sigma) = C_G(\operatorname{Im} v_\sigma)$.) Then we have a homomorphism $s_\sigma: S^1 \to G$ defined as in (1). Hereafter, for simplicity, we abbreviate s_σ to s. Following Bott [4], we define a *generating map*

$$g_s: G/G^\sigma \to \Omega G$$

by

$$[g_s(xG^\sigma)](t) = xs(t)x^{-1}s(t)^{-1} \qquad \text{for } xG^\sigma \in G/G^\sigma.$$

Its image is contained in $\Omega_0 G$. We also define its unreduced version

$$f_s: G/G^\sigma \to \Omega G$$

by

$$[f_s(xG^\sigma)](t) = xs(t)x^{-1} \qquad \text{for } xG^\sigma \in G/G^\sigma.$$

Its image is contained in $\Omega_s G$. This map can be viewed as a variant of the Bott map as follows. Let $\sigma': G \times G \to G \times G$ be the involutive automorphism defined by

$$\sigma'(x, y) = (\sigma(x), \sigma(y)) \qquad \text{for } (x, y) \in G \times G.$$

Then $(G \times G)^{\sigma'} = G^\sigma \times G^\sigma$. Let $v_{\sigma'}: \mathbf{R} \to G \times G$ be the one-parameter subgroup defined by

$$v_{\sigma'}(t) = \begin{cases} (v_\sigma(t), v_\sigma(t)^{-1}) & \text{if } x_\sigma^2 = 1 \\ (v_\sigma(2t), v_\sigma(2t)^{-1}) & \text{if } x_\sigma^2 = -1 \end{cases}$$

for $t \in \mathbf{R}$. Then σ', τ', $v_{\sigma'}$ satisfy the condition (3). In this case, the adjoint $b: \Delta(G)/\Delta(G^\sigma) \to \Omega(G \times G/\Delta(G))$ of $\hat{b}: \Sigma(\Delta(G)/\Delta(G^\sigma)) \to G \times G/\Delta(G)$ may be identified with $f_s: G/G^\sigma \to \Omega G$. In effect,

$$\begin{aligned} \varphi\{[b(xG^\sigma, xG^\sigma)](t)\} &= \varphi\{(x, x)v_{\sigma'}(t)\Delta(G)\} \\ &= \varphi\{(x, x)(v_\sigma(t), v_\sigma(t)^{-1})\Delta(G)\} \\ &= \varphi\{(xv_\sigma(t), xv_\sigma(t)^{-1})\Delta(G)\} \\ &= xv_\sigma(t)(xv_\sigma(t)^{-1})^{-1} \\ &= xv_\sigma(t)v_\sigma(t)x^{-1} \\ &= xv_\sigma(2t)x^{-1} \\ &= xs(t)x^{-1} \\ &= [f_s(xG^\sigma)](t). \end{aligned}$$

2. Main results

For a compact Lie group G, there is a standard inclusion $\hat{\kappa}: \Sigma G \to BG$. Let $\kappa: G \to \Omega BG$ be its adjoint. If G is connected, κ becomes a homotopy equivalence. The following lemma was proved by Harris [9, §4].

Lemma 1. *In the first situation, the diagram*

$$\begin{array}{ccc} G^\sigma/G^{\sigma\tau} & \xrightarrow{b_0} & \Omega(G/G^\sigma) \\ \downarrow{\scriptstyle \xi_\tau} & & \downarrow{\scriptstyle \Omega j} \\ G^\sigma & \xrightarrow{\kappa} & \Omega BG^\sigma. \end{array}$$

is homotopy-commutative.

Examples of Lemma 1 are as follows.

(I-1) $G = Sp(n)$, $\sigma = \operatorname{Ad} jI_n$, $\tau = \operatorname{Ad} iI_n$ (where $i \in \mathbf{C} \subset \mathbf{H}$):

$$\begin{array}{ccc} U(n)/O(n) & \longrightarrow & \Omega(Sp(n)/U(n)) \\ \downarrow & & \downarrow \\ U(n) & \longrightarrow & \Omega BU(n). \end{array}$$

(I-2) $G = SO(4n)$, $\sigma = \mathrm{Ad}\, J_{2n}$, $\tau = \mathrm{Ad}\,(J_n \oplus J_n^{-1})$:

$$\begin{array}{ccc} U(2n)/Sp(n) & \longrightarrow & \Omega(SO(4n)/U(2n)) \\ \downarrow & & \downarrow \\ U(2n) & \longrightarrow & \Omega BU(2n). \end{array}$$

(I-3) $G = SO(2n+1)$, $\sigma = \mathrm{Ad}\, I_{2,2n-1}$ and $\tau = \mathrm{Ad}\, I_{1,2n}$:

$$\begin{array}{ccc} SO(2n)/SO(2n-1) & \longrightarrow & \Omega(SO(2n+1)/SO(2n)) \\ \downarrow & & \downarrow \\ SO(2n) & \longrightarrow & \Omega BSO(2n). \end{array}$$

(I-4) $G = G_2$, $\sigma = \mathrm{Ad}\, \gamma$ (in the notation of [16]) and $\tau = \mathrm{Ad}\, I_{1,3}$:

$$\begin{array}{ccc} S^3 = SO(4)/SO(3) & \longrightarrow & \Omega(G_2/SO(4)) \\ \downarrow & & \downarrow \\ SO(4) & \longrightarrow & \Omega BSO(4). \end{array}$$

(I-5) $G = E_7$, $\sigma = \mathrm{Ad}\, \sigma$ and $\tau = \mathrm{Ad}\, \iota$ (in the notation of [16]):

$$\begin{array}{ccc} [(T^1 \times E_6)/\mathbf{Z}_3]/F_4 & \longrightarrow & \Omega(E_7/[(T^1 \times E_6)/\mathbf{Z}_3]) \\ \downarrow & & \downarrow \\ (T^1 \times E_6)/\mathbf{Z}_3 & \longrightarrow & \Omega B((T^1 \times E_6)/\mathbf{Z}_3). \end{array}$$

The diagrams (I-1) and (I-2) appeared in the proof of the Bott periodicity theorem for KO-theory (see [4] or [7]).

Lemma 2. *In the second situation, the diagram*

$$\begin{array}{ccc} G^\tau/G^{\tau\sigma} & \xrightarrow{b_0} & \Omega(G/G^\tau) \\ {\scriptstyle \tilde{i}}\downarrow & & \downarrow{\scriptstyle \Omega\xi_\tau} \\ G/G^\sigma & \xrightarrow{g_s} & \Omega G. \end{array}$$

is homotopy-commutative.

Proof. It suffices to prove that the diagram

$$\begin{array}{ccc} \Sigma(G^\tau/G^{\tau\sigma}) & \xrightarrow{\hat{b}_0} & G/G^\tau \\ {\scriptstyle \Sigma\tilde{i}}\downarrow & & \downarrow{\scriptstyle \xi_\tau} \\ \Sigma(G/G^\sigma) & \xrightarrow{\hat{g}_s} & G \end{array}$$

is homotopy-commutative. We have

$$\begin{aligned}(\hat{g}_s \circ \Sigma\tilde{i})(xG^{\tau\sigma}, t) &= xs(t)x^{-1}s(t)^{-1} \\ &= xs(t)x^{-1}v_\sigma(t)^{-1}v_\sigma(t)^{-1}.\end{aligned}$$

On the other hand, in case $x_\sigma^2 = 1$, we have

$$\begin{aligned}(\xi_\tau \circ \hat{b}_0)(xG^{\tau\sigma}, t) &= v_\sigma(t)^{-1}xv_\sigma(t)\tau(v_\sigma(t)^{-1}xv_\sigma(t))^{-1} \\ &= v_\sigma(t)^{-1}xv_\sigma(t)(v_\sigma(t)xv_\sigma(t)^{-1})^{-1} \\ &\qquad \text{by (3) and since } x \in G^\tau \\ &= v_\sigma(t)^{-1}xv_\sigma(t)v_\sigma(t)x^{-1}v_\sigma(t)^{-1} \\ &= v_\sigma(t)^{-1}xv_\sigma(2t)x^{-1}v_\sigma(t)^{-1} \\ &= v_\sigma(t)^{-1}xs(t)x^{-1}v_\sigma(t)^{-1}.\end{aligned}$$

So, if we define $H: \Sigma(G^\tau/G^{\tau\sigma}) \times I \to G$ by

$$H(xG^{\tau\sigma}, t, u) = v_\sigma(t(1-u))^{-1}xs(t)x^{-1}v_\sigma(t)^{-1}v_\sigma(tu)^{-1},$$

then H is a base point preserving homotopy between $\xi_\tau \circ \hat{b}_0$ and $\hat{g}_s \circ \Sigma\tilde{i}$. The case $x_\sigma^2 = -1$ can be proved similarly, and the proof is completed.

Examples of Lemma 2 are as follows.

(II-1) $G = SU(2n)$, $\sigma = \mathrm{Ad}\, I_{n,n}$, and $\tau = c$:

$$\begin{array}{ccc} SO(2n)/SO(n) \times SO(n) & \longrightarrow & \Omega(SU(2)/SO(2n)) \\ \downarrow & & \downarrow \\ SU(2n)/[SU(2n) \cap (U(n) \times U(n))] & \longrightarrow & \Omega SU(2n). \end{array}$$

(II-2) $G = SU(4n)$, $\sigma = \mathrm{Ad}\, I_{2n,2n}$, $\tau = \mathrm{Ad}\, J_{2n}$:

$$\begin{array}{ccc} Sp(2n)/Sp(n) \times Sp(n) & \longrightarrow & \Omega(SU(4n)/Sp(2n)) \\ \downarrow & & \downarrow \\ SU(4n)/[SU(4n) \cap (U(2n) \times U(2n))] & \longrightarrow & \Omega SU(4n). \end{array}$$

(II-3) $G = SO(2n)$, $\sigma = \mathrm{Ad}\, I_{2,2n-2}$ and $\tau = \mathrm{Ad}\, I_{1,2n-1}$:

$$\begin{array}{ccc} SO(2n-1)/SO(2n-2) & \longrightarrow & \Omega(SO(2n)/SO(2n-1)) \\ \downarrow & & \downarrow \\ SO(2n)/SO(2) \times SO(2n-2) & \longrightarrow & \Omega SO(2). \end{array}$$

(II-4) $G = E_6$, $\sigma = \mathrm{Ad}\,\sigma$ and $\tau = \lambda$ (for notation see Yokota [16]):

$$\begin{array}{ccc} F_4/Spin(9) & \longrightarrow & \Omega(E_6/F_4) \\ \downarrow & & \downarrow \\ E_6/[(T^1 \times Spin(10))/\mathbf{Z}_4] & \longrightarrow & \Omega E_6. \end{array}$$

The diagrams (II-1) and (II-2) appeared in the proof of the Bott periodicity theorem for KO-theory (see [5] or [7]).

Let $R(G)$ be the complex representation ring of G. It has an augmentation ε: $R(G) \to \mathbf{Z}$ given by assigning to each representation ρ: $G \to U(n)$ its dimension n, and $\mathrm{Ker}\,\varepsilon$ is usually denoted by $I(G)$. For $\rho \in R(G)$, we put $\tilde{\rho} = \rho - \varepsilon(\rho) \in I(G)$. Let α: $R(G) \to K^0(BG)$ be the homomorphism of [1, §5]. Then $\alpha(I(G)) \subset \tilde{K}^0(BG)$. Let β: $R(G) \to K^{-1}(G)$ be the map of [10, Chapter 1, §4]. Then β is (up to sign) the suspension of α. That is, if $\boldsymbol{\sigma}^*$: $K^0(BG) \to K^{-1}(G)$ is the cohomology suspension in K-theory, by [10, Proposition 4.1], we have $\boldsymbol{\sigma}^* \circ \alpha = \beta$. Since

$$\hat{\kappa}^*: K^*(BG) \to K^*(\Sigma G) = K^{*-1}(G)$$

is just $\boldsymbol{\sigma}^*$, it follows that

$$\hat{\kappa}^*(\alpha(\rho)) = \beta(\rho)$$

for all $\rho \in R(G)$.

Consider now a symmetric space G/G^τ such that τ: $G \to G$ is outer, and an honest representation ρ: $G \to U(n)$. Let U be the infinite unitary group and ι_n: $U(n) \to U$ the canonical inclusion. Then the map $\xi_{\tau,\rho}$: $G/G^\tau \to U(n)$ defined by

$$\xi_{\tau,\rho}(xG^\tau) = \rho(x)\rho(\tau(x))^{-1} \qquad \text{for } xG^\tau \in G/G^\tau$$

gives rise to an element $\beta(\rho - \tau^*\rho) = [\iota_n \circ \xi_{\tau,\rho}] \in \tilde{K}^{-1}(G/G^\tau)$ (see [9, p. 325]). Note that $\xi_{\tau,\rho} = \rho \circ \xi_\tau$.

Our main result consists of two theorems, one of which is stated as follows.

Theorem 3. *In the first situation,*

$$\hat{b}_0^*: \tilde{K}^0(G/G^\sigma) \to \tilde{K}^0(\Sigma(G^\sigma/G^{\sigma\tau})) = \tilde{K}^{-1}(G^\sigma/G^{\sigma\tau})$$

satisfies

$$\hat{b}_0^*(j^*\alpha(\tilde{\rho})) = \beta(\rho - \tau^*\rho)$$

for all $\rho \in R(G^\sigma)$, *where*

$$\tilde{K}^0(BG^\sigma) \xrightarrow{j^*} \tilde{K}^0(G/G^\sigma).$$

Proof. It suffices to prove this for each representation $\rho: G^{\sigma} \to U(n)$. By definition

$$\hat{b}_0^*(j^*\alpha(\tilde{\rho})) = [B\iota_n \circ B\rho \circ j \circ \hat{b}_0] \in [\Sigma(G^{\sigma}/G^{\sigma\tau}), BU \times \mathbf{Z}]_0.$$

Under the isomorphism

$$[\Sigma X, BU \times \mathbf{Z}]_0 \cong [X, \Omega(BU \times \mathbf{Z})]_0 = [X, \Omega BU]_0,$$

it corresponds to

$$\begin{aligned}[\Omega B\iota_n \circ \Omega B\rho \circ \Omega j \circ b_0] &= [\Omega B\iota_n \circ \Omega B\rho \circ \kappa \circ \xi_{\tau}] \qquad \text{by Lemma 1} \\ &= [\kappa \circ \iota_n \circ \rho \circ \xi_{\tau}] \in [G^{\sigma}/G^{\sigma\tau}, \Omega BU]_0.\end{aligned}$$

Under the isomorphism $\kappa^*: [X, U]_0 \cong [X, \Omega BU]_0$, it corresponds to

$$[\iota_n \circ \rho \circ \xi_{\tau}] = \beta(\rho - \tau^*\rho) \in [G^{\sigma}/G^{\sigma\tau}, U]_0.$$

To state the other of our theorems, we need to recall the work of Clarke [6]. Suppose that the second situation is given; in particular, G/G^{σ} is a symmetric space which belongs to the case (b). (In the case (b1), we have to replace G^{σ} by $C_G(\mathrm{Im}\, s_{\sigma})$.) Then we have a circle $s = s_{\sigma}: S^1 \to G$ defined as in (1), and G^{σ} coincides with the centralizer of the image of s in G. As is well known, if $\pi: \tilde{G} \to G$ is the universal covering, $\pi_1(G) = Z(\tilde{G})$ is finite. Let d be the smallest positive integer such that $d[s] = 0$ in $\pi_1(G)$. Then there is a unique homomorphism $\tilde{s}: S^1 \to \tilde{G}$ such that $s \circ d = \pi \circ \tilde{s}$ (see [6, Proposition (2.6)]). Let $\tilde{G}^{\sigma}$ be the centralizer of the image of $\tilde{s}$ in $\tilde{G}$. Then $\tilde{G}^{\sigma} = \pi^{-1}(G^{\sigma})$ and $\tilde{G}/\tilde{G}^{\sigma} \approx G/G^{\sigma}$ (see [6, Proposition (2.7)]). We define $f_{\tilde{s}}: \tilde{G}/\tilde{G}^{\sigma} \to \Omega\tilde{G}$ by

$$[f_{\tilde{s}}(x\tilde{G}^{\sigma})](t) = x\tilde{s}(t)x^{-1}.$$

Let $\theta_s: R(G) \to R(G^{\sigma})$ and $\theta_{\tilde{s}}: R(\tilde{G}) \to R(\tilde{G}^{\sigma})$ be the derivations of [6, Definition (2.1) and Proposition (2.5)] associated with s and $\tilde{s}$, respectively. Then

$$\theta_{\tilde{s}} \circ \pi^* = \pi^* \circ (d \cdot \theta_s). \tag{4}$$

Let $\boldsymbol{\sigma}^*: \tilde{K}^{-1}(\tilde{G}) \to \tilde{K}^{-2}(\Omega\tilde{G})$ be the cohomology suspension in K-theory and g: $\tilde{K}^0(\tilde{G}/\tilde{G}^{\sigma}) \to \tilde{K}^{-2}(\tilde{G}/\tilde{G}^{\sigma})$ the Bott isomorphism, i.e., multiplication by the Bott generator $g \in \tilde{K}^{-2}(S^0)$. Then, by [6, Proposition (2.8)],

$$f_{\tilde{s}}^* \circ \boldsymbol{\sigma}^* \circ \beta = g \circ j^* \circ \alpha \circ \theta_{\tilde{s}}. \tag{5}$$

Moreover, as in [6, Lemma (2.10)], for the induced homomorphisms

$$f_{\tilde{s}}^*,\ g_s^*: K^*(\Omega\tilde{G}) = K^*(\Omega_0 G) \to K^*(\tilde{G}/\tilde{G}^{\sigma}) = K^*(G/G^{\sigma}),$$

we may identify $f_{\tilde{s}}^*$ with $d \cdot g_s^*$. Combining (4) and (5), we conclude that

$$g_s^* \circ \boldsymbol{\sigma}^* \circ \beta = g \circ j^* \circ \alpha \circ \theta_s. \tag{6}$$

Thus

$$\hat{g}_s^*: K^*(G) \to K^*(\Sigma(G/G^{\sigma})) = K^{*-1}(G/G^{\sigma})$$

satisfies

$$\hat{g}_s^*(\beta(\rho)) = g(j^* \circ \alpha)(\theta_s \rho)$$

for all $\rho \in R(G)$.

The other of our theorems is stated as follows.

Theorem 4. *In the second situation,*

$$\hat{b}_0^*\colon \tilde{K}^{-1}(G/G^\tau) \to \tilde{K}^{-1}(\Sigma(G^\tau/G^{\tau\sigma})) = \tilde{K}^{-2}(G^\tau/G^{\tau\sigma})$$

satisfies

$$\hat{b}_0^*(\beta(\rho - \tau^*\rho)) = g(j^* \circ Bi^* \circ \alpha)(\theta_s \rho)$$

for all $\rho \in R(G)$, *where*

$$\tilde{K}^0(BG^\sigma) \xrightarrow{Bi^*} \tilde{K}^0(BG^{\tau\sigma}) \xrightarrow{j^*} \tilde{K}^0(G^\tau/G^{\tau\sigma}).$$

Proof. It suffices to prove this for each representation $\rho\colon G \to U(n)$. We have

$$\begin{aligned}\hat{b}_0^*(\beta(\rho - \tau^*\rho)) &= [\iota_n \circ \rho \circ \xi_\tau \circ \hat{b}_0] && \text{by definition}\\ &= [\iota_n \circ \rho \circ \hat{g}_s \circ \Sigma\tilde{i}] && \text{by Lemma 2}\\ &\in [\Sigma(G^\tau/G^{\tau\sigma}), U]_0 = \tilde{K}^{-1}(\Sigma(G^\tau/G^{\tau\sigma})).\end{aligned}$$

Under the isomorphism $[\Sigma X, U]_0 \cong [X, \Omega U]_0$, it corresponds to

$$[\Omega\iota_n \circ \Omega\rho \circ g_s \circ \tilde{i}] = (\tilde{i}^* \circ g_s^* \circ \boldsymbol{\sigma}^* \circ \beta)(\rho).$$

By (6), it is equal to

$$\begin{aligned}(\tilde{i}^* \circ g \circ j^* \circ \alpha \circ \theta_s)(\rho) &= g(\tilde{i}^* \circ j^* \circ \alpha)(\theta_s \rho)\\ &= g(j^* \circ Bi^* \circ \alpha)(\theta_s \rho)\\ &\in [G^\tau/G^{\tau\sigma}, \Omega U]_0 = \tilde{K}^{-2}(G^\tau/G^{\tau\sigma}).\end{aligned}$$

There remains the following situation:

(III-i) $\sigma\colon G \to G$ is inner;

(III-ii) $\tau\colon G \to G$ is inner, and $\tau\colon G^\sigma \to G^\sigma$ is inner.

In this case, by the result of [13] or [14], $K^*(G/G^\sigma)$, $K^*(G^\sigma/G^{\sigma\tau})$ are generated by 0-dimensional generators. Therefore, for dimensional reasons, $\hat{b}_0^*\colon K^*(G/G^\sigma) \to K^*(G^\sigma/G^{\sigma\tau})$ is trivial.

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