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Elliptic cohomology of classifying spaces of cyclic groups and higher level modular forms

By

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0. Introduction

The subject of elliptic cohomology $Ell^*(-)$ defined by P. S. Landweber, D. C. Ravenel and R. E. Stong (see [14]) is one of the most important subjects in algebraic topology. They used the elliptic curve defined by the Jacobi quartic $y^2=1-2\delta x^2+\varepsilon x^4$ in projective 3-space and the associated formal group law (so called the Euler formal group law):

$$F(x, y)=\frac{x\sqrt{R(y)}+y\sqrt{R(x)}}{1-\varepsilon x^2y^2},$$

where $R(x)=1-2\delta x^2+\varepsilon x^4$. The coefficient ring Ell_* is identified with $\mathbf{Z}[\frac{1}{2}][\delta, \varepsilon, \Delta^{-1}]$ the ring of meromorphic modular forms on Γ_θ over $\mathbf{Z}[\frac{1}{2}]$.

Later A. Baker [1] has defined elliptic cohomology based on the modular forms on $SL_2(\mathbf{Z})$ over $\mathbf{Z}[\frac{1}{6}]$ and the elliptic cohomologies of higher level have been defined by J-L. Brylinski (cf. [3]). A. Baker [2] has shown that given a prime $p>3$, the supersingular reduction of $Ell^*(-)$ at p, namely reduction with respect to the ideal I_2 generated by p and v_1, is essentially isomorphic to the Morava $K(2)$-theory.

On the other hand, T. Torii [17] has shown the following. Let $B\mathbf{Z}/(p^n)$ be the classifying space of the cyclic group $\mathbf{Z}/(p^n)$ for a prime p and $\overline{K(r)}^*(-)$ be the p-adic Morava K-theory, then the ring $\overline{K(r)}^*(B\mathbf{Z}/(p^n))$ is described as the totally ramified extension of $\overline{K(r)}_*\cong\mathbf{Z}_p[\zeta_{p^n-1}]$, obtained by adding the roots of the equation $[p^n](x)=0$ for the p^n-sequence of the Lubin-Tate formal group law of height r. Where we denote by ζ_l a primitive l-th root of unity.

From now on, we assume that p is an odd prime. By the above result, we may expect that the elliptic cohomology $Ell^*(B\mathbf{Z}/(p^n))$ of $B\mathbf{Z}/(p^n)$ can be described by level $2p^m$ modular forms for $0\leq m\leq n$. The purpose of this paper is to show that this is true after certain completion of Ell_*. We shall study the level 2 elliptic cohomology. Now the main result is stated as follows.

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Theorem 0.1. *There is an algebra isomorphism:*

$$K_*\otimes_{Ell_*}(Ell^*(B\mathbf{Z}/(p^n))[\zeta_{p^n}])^{\wedge}_{\mathfrak{M}} \cong \prod_{0\le m\le n} K_*\otimes_{Ell_*}(Ell_*(\Gamma_1(2p^m))[\zeta_{p^n}])^{\wedge}_{\mathfrak{M}}.$$

Here $\mathfrak{M}$ is an arbitrary maximal graded ideal of Ell_* containing $I_2=(p, v_1)$, K_* is the quotient field of Ell_* and $Ell(\Gamma_1(2p^m))_*$ is the graded ring of modular forms on $\Gamma_1(2p^m)$ over $\mathbf{Z}[\frac{1}{2}]$.

Now we state some points in proof. It is well-known (cf. [11]) that there is an isomorphism:

$$Ell^*(B\mathbf{Z}/(p^n))\cong Ell_*[[x]]/([p^n](x)).$$

The ring $Ell_*[[x]]/([p^n](x))$ is a complete topological ring in a natural way. In section 3 we shall show an equality:

$$[p^n](x)=u(x)\prod_{0\le m\le n}\phi_{p^m}(x),$$

where $\phi_{p^m}(x)$ are monic irreducible polynomials and $u(x)\in Ell_*[[x]]$ is a unit. This means the ideal generated by $[p^n](x)$ is seen as an ideal generated by the polynomial $\prod_{0\le m\le n}\phi_{p^m}(x)$. Moreover we shall show that the polynomials $\phi_{p^m}(x)$ are described by using the Jacobi sine (Proposition 3.2). Therefore the study of $Ell^*(\mathrm{B}\mathbf{Z}/(p^n))$ can be reduced to the algebraic one $Ell_*[x]/(\phi_{p^m}(x))$. If we consider the quotient field K_* of Ell_*, then we can show the following isomorphism

$$\sigma : K_*(\zeta_{p^n})[x]/(\phi_{p^n}(x))\cong K_*(\zeta_{p^n})\otimes_{Ell_*[\zeta_{p^n}]}Ell_*(\Gamma_1(2p^n)).$$

This implies Theorem 0.1.

This paper is organized as follows. In section 1, we study the graded rings of modular forms of higher level. In particular the structure of those as modules over the ring of level 2 modular forms will be studied. Section 2 is devoted to the study of the Jacobi sine and associated modular forms. In sections 3, p^n-sequence $[p^n](x)$ of the formal group law is described in terms of polynomials associated with Jacobi sine. The main theorem is proved in sections 4, 5.

1. Modular forms of higher level

First we recall some definitions of modular forms and graded rings of modular forms (cf. [16]). Let $\gamma\in SL_2(\mathbf{R})$ and k be an integer. For a meromorphic function f defined on the upper half plane $\mathfrak{H}$, we put

$$f|_{[\gamma]k}(\tau)=f(\gamma\cdot\tau)j(\gamma,\tau)^{-k},$$

where $\gamma\cdot\tau=\frac{a\tau+b}{c\tau+d}$ and $j(\gamma,\tau)=c\tau+d$, for $\tau\in\mathfrak{H}$ and $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$.

Definition 1.1. Let Γ be a subgroup of $SL_2(\mathbf{Z})$. We define an automorphic form on Γ of weight k as a meromorphic function f on $\mathfrak{H}$ and at each cusps of Γ such that $f_{|[\gamma]_k}(\tau)=f(\gamma)$ for all $\gamma\in\Gamma$. The $\mathbf{C}$-vector space of all such functions is denoted by $A_{2k}(\Gamma)$, and define $A_*(\Gamma)=\bigoplus_{k\in\mathbf{Z}}A_{2k}(\Gamma)$. Then it is a graded field under a natural ring structure. We also define graded subrings $M_*(\Gamma)\subset\mathscr{E}_*(\Gamma)$ of $A_*(\Gamma)$ as follows; $\mathscr{E}_*(\Gamma)$ consists of elements of $A_*(\Gamma)$ which are holomorphic on $\mathfrak{H}$ and $M_*(\Gamma)$ consists of elements of $\mathscr{E}_*(\Gamma)$ which are holomorphic at cusps of Γ. An element of $\mathscr{E}_*(\Gamma)$ (resp. $M_*(\Gamma)$) is called a meromorphic (resp. holomorphic) modular form.

Remark. We give an automorphic form a grading twice of its weight, so that the coefficient ring Ell_* of the elliptic cohomology theory is isomorphic to $\mathscr{E}_*(\Gamma_1(2))$ as a graded rings for a congruence subgroup $\Gamma_1(2)$ of $SL_2(\mathbf{Z})$ defined below.

For example, it is well-known that for a principal congruence subgroup $\Gamma_1(2)$ (defined later), $M_*(\Gamma_1(2))$ is isomorphic to $\mathbf{C}[\delta,\varepsilon]$, where δ and ε are modular forms on $\Gamma_1(2)$ of weight 2 and 4, respectively. Also we define $\Delta=4096\varepsilon(\delta^2-\varepsilon)^2$, then it is a cusp form on $SL_2(\mathbf{Z})$ of weight 12. Moreover their q-expansions at $i\infty$ are calculated as

$$\delta(\tau)=-\frac{1}{8}-3\sum_{n\geq 1}\Big(\sum_{d|n,2\nmid d}d\Big)q^n=-\frac{1}{8}-3q-3q^2+\cdots \tag{1.2}$$

$$\varepsilon(\tau)=\sum_{n\geq 1}\Big(\sum_{d|n,2\nmid\frac{n}{d}}d^3\Big)q^n=q+8q^2+28q^3+\cdots \tag{1.3}$$

$$\Delta(\tau)=q\prod_{n=1}^{\infty}(1-q^n)^{24}=q-24q^2+252q^3+\cdots, \tag{1.4}$$

(cf. [7, 19, 10, 16]).

Remark. P. S. Landweber [12] considered the modular forms on the theta group Γ_θ which is conjugate to our $\Gamma_1(2)$. So our modular forms δ, ε are conjugate to his δ,ε respectively, however ours do not coinside with his.

We can regard $A_*(\Gamma)$ and $\mathscr{E}_*(\Gamma)$ as $\mathbf{C}$-algebras. Let Γ be a Fuchsian group of the first kind. Then the subring $A_0(\Gamma)\subset A_*(\Gamma)$ is the function field of the closed Riemann surface $\bar{\Gamma}\backslash\mathfrak{H}^*$, where $\bar{\Gamma}=\Gamma/(\{\pm 1\}\cap\Gamma)$ and $\mathfrak{H}^*=\mathfrak{H}\cup\{\text{cusps}\}$. Moreover $A_*(\Gamma)$ becomes a graded field, so $A_*(\Gamma)$ is isomorphic to $A_0(\Gamma)[x,x^{-1}]$, where $\deg x=4$ or 2 according to Γ contains $\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ or not. (cf. [16, Proposition 2.15]) Let Γ be a normal subgroup of Γ'. Then $A_*(\Gamma)$ is a Galois extension of graded fields $A_*(\Gamma')$ with the Galois group $G=\Gamma'/\Gamma$, and the usual Galois theory can be applied. For example, for any group Λ such as $\Gamma\subset\Lambda\subset\Gamma'$, one has $A_*(\Lambda)=A_*(\Gamma)^{\Lambda/\Gamma}$.

For any automorphic function $f\in A_0(\Gamma)$, one can choose a polynomial of j-function $g(j)\in\mathbf{C}[j]$ such that $f\cdot g(j)$ has no pole on $\mathfrak{H}$. Thus we see that $A_0(\Gamma)$ is a quotient field of $\mathfrak{H}_0(\Gamma)$ and $A_*(\Gamma)\cong A_0(\Gamma)\otimes_{\mathscr{E}_0(\Gamma)}\mathscr{E}_*(\Gamma)$.

Definition 1.5. Let us define the following subgroups of $SL_2(\mathbf{Z})$ for $n\in\mathbf{N}$.

$$\Gamma(n)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})\ \middle|\ \begin{pmatrix} a & b\\ c & d\end{pmatrix}\equiv\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\quad \mathrm{mod}\ n\right\}, \tag{1.6}$$

$$\Gamma_1(n)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})\ \middle|\ \begin{pmatrix} a & b\\ c & d\end{pmatrix}\equiv\begin{pmatrix} 1 & *\\ 0 & 1\end{pmatrix}\quad \mathrm{mod}\ n\right\}, \tag{1.7}$$

A subgroup Γ of $SL_2(\mathbf{Z})$ containing $\Gamma(n)$ for some n is called a congruence subgroup, and the smallest number n is called the level of Γ.

Let Γ be a modular subgroup and R be a subring of $\mathbf{C}$. We define $\mathscr{E}_*(\Gamma)^R$ and $M_*(\Gamma)^R$ to be subrings of $M_*(\Gamma)$ and $\mathscr{E}_*(\Gamma)$, consisting of modular forms whose q-expansion coefficients are contained in the ring R at each cusps of Γ. For a congruence subgroup Γ of level $2n$, let us define a graded ring $Ell_*(\Gamma)=\mathscr{E}_*(\Gamma)^{\mathbf{Z}[\frac{1}{2},\zeta_n]}$. We will be concerned mainly with modular groups contained in $\Gamma_1(2)$. Therefore we use the following notation:

$$\Gamma_{(2)}=\Gamma\cap\Gamma_1(2).$$

The following proposition is easily proved from the q-expansions of δ and ε.

Proposition 1.8. *Let R be a subring of $\mathbf{C}$ containing $\mathbf{Z}[\frac{1}{2}]$. Then*

$$M_*(\Gamma_1(2))^R\cong R[\delta,\varepsilon],\quad \mathscr{E}_*(\Gamma_1(2))^R\cong R[\delta,\varepsilon,\Delta^{-1}].$$

In particular we have $Ell_*(\Gamma_1(2))\cong\mathbf{Z}[\delta,\varepsilon,\Delta^{-1}]$, denoted simply by Ell_* Now a key theorem for our argument is the following (cf. [4, Chapter VII] and also see [3]).

Theorem 1.9. (P. Deligne and M. Rapoport, and J-L. Brylinski). *Let Γ be a congruence subgroup of level n, then we have*

(i) *$Ell_{4*}(\Gamma_{(2)})$ is a finitely generated $\mathbf{Z}[\frac{1}{2},\zeta_n]$-algebra*

(ii) *$\mathscr{E}_{4*}(\Gamma_{(2)})\cong Ell_{4*}(\Gamma_{(2)})\otimes_{\mathbf{Z}[\frac{1}{2},\zeta_n]}\mathbf{C}$,*

where A_{4*} means $\bigoplus_{k\in\mathbf{Z}}A_{4k}$, for a graded module A_*.

Corollary 1.10. *$Ell_{4*}(\Gamma_{(2)})$ is a finitely generated Ell_*-module.*

Proof. Since by Theorem 1.9 $Ell_{4*}(\Gamma_{(2)})$ is finitely generated as an Ell_*-algebra. So to show that $Ell_{4*}(\Gamma_{(2)})$ is finitely genarated as an Ell_*-module, it is enough to show that all elements in $Ell_{4*}(\Gamma_{(2)})$ are integral over Ell_*. Since $Ell_*[\zeta_n]$ is a finitely generated Ell_*-module, so we have only to prove that all elements of $Ell_{4*}(\Gamma_{(2)}$ are integral over $Ell_*[\zeta_n]$. Now given an $f\in\mathrm{Ell}_{4n}(\Gamma_{(2)})$, consider a polynomial:

$$\phi(X) = \prod_g (X - f|_{[g]2n}),$$

where the product is taken over a representatives g of $\Gamma_1(2)/\Gamma_{(2)}$. Then we easily see that all of the coefficients of the polynomial are fixed under the action of the group $\Gamma_1(2)$, i.e. they are modular forms on $\Gamma_1(2)$. By the assumption on the q-expansions of f, we also see that their q-expansions at each cusps are contained in $\mathbf{Z}[\frac{1}{2}, \zeta_n]((\mathrm{q}))$. So they are contained in $\mathscr{E}_*^{\mathbf{Z}[\frac{1}{2}, \zeta_n]} \cong Ell^*[\zeta_n]$. This shows that f is integral over $Ell_*[\zeta_n]$.

We denote a graded quotient field of Ell_* *by* K_* and that of $Ell_*(\Gamma)$ by $K_*(\Gamma)$.

Theorem 1.11. *Let Γ be a congruence subgroup $\Gamma(n)$ or a modular subgroup conjugate to $\Gamma_1(n)$. Then*

(i) $K_*(\zeta_n) \otimes_{Ell_*[\zeta_n]} Ell_*(\Gamma_{(2)}) \cong K_*(\Gamma_{(2)})$.

(ii) $[K_*(\Gamma_{(2)}) : K_*(\zeta_n)] = [\Gamma_1(2) : \Gamma_{(2)}]$.

Proof. Note that $Ell_*(\Gamma(n)_{(2)})$ is a graded integral domain with the action of the group $G = \Gamma_1(2)/\Gamma(n)_{(2)}$ and satisfies the following equality;

$$Ell_*[\zeta_n] = Ell_*(\Gamma(n)_{(2)})^G.$$

Also for a subgroup $\Gamma_{(2)}$ of $\Gamma_1(2)$ containing $\Gamma(n)_{(2)}$, we put $H = \Gamma_{(2)}/\Gamma(n)_{(2)}$. Then we have the following equality;

$$Ell_*(\Gamma_{(2)}) = Ell_*(\Gamma(n)_{(2)})^H.$$

We define a ring homomorphism $\phi : K_*(\zeta_n) \otimes_{Ell_*[\zeta_n]} Ell_*(\Gamma_{(2)}) \to K_*(\Gamma_{(2)})$ by

$$\phi\left(\frac{a}{\mathrm{b}} \otimes \mathrm{c}\right) = \frac{\mathrm{ac}}{\mathrm{b}},$$

where a, $b \in Ell_*[\zeta_n]$ and $c \in Ell_*(\Gamma_{(2)})$. It is easy to see that this is well-defined. For $\forall y \in Ell_*(\Gamma_{(2)})$ we put $z = \prod g \cdot y$, where the product is taken over the cosets G/H except eH. This defines well since y is fixed under the action of H. Moreover it is easy to see that yz is in $Ell_*[\zeta_n]$. Then we define an inverse ring homomorphism ψ by

$$\psi\left(\frac{x}{y}\right) = \frac{1}{yz} \otimes xz,$$

where x, $y \in Ell_*(\Gamma_{(2)})$. By definition, it is easy to verify that $\phi \circ \psi = id$ and $\psi \circ \phi = id$. This shows the assertion (i).

For the proof of (ii), we define $E_{ev} = \bigoplus_{k \in \mathbf{Z}} K_*(\zeta_n) \otimes_{Ell_*[\zeta_n]} Ell_{4k}(\Gamma_{(2)})$ and $E_{odd} = \bigoplus_{k \in \mathbf{Z}} K_*(\zeta_n) \otimes_{Ell_*[\zeta_n]} Ell_{4k+2}(\Gamma_{(2)})$. In Theorem 2.14 we will see that $E_{odd} \neq \{0\}$. So we can choose a non-zero element $x \in E_{odd}$, and multiplying it with elements of E_{odd} we have an isomorphism of $K_*(\zeta_n)$-modules E_{odd} and E_{ev}. Then we are enough to check that the rank of E_{ev} is equal to half of the number $[\Gamma_1(2) : \Gamma_{(2)}]$. While by Theorem 1.9, we have

$$A_0(\Gamma_1(2)) \otimes_{K_*(\zeta)} E_{ev} \cong A_0(\Gamma_1(2)) \otimes_{\mathscr{E}_*} \mathscr{E}_{4*}(\Gamma_{(2)}).$$

Since it is isomorphic to $A_{2*}(\Gamma_{(2)})$, its rank as an $A_*(\Gamma_1(2))$-vector space is equal to the covering degree of $\overline{\Gamma_{(2)}}\backslash\mathfrak{H}^*\to\overline{\Gamma_1(2)}\backslash\mathfrak{H}^*$. It is known to be equal to the desired value $[\Gamma_1(2):\Gamma_{(2)}]/2$.

Remark. Let F be a subfield of $\mathbf{C}$ containing ζ_n. Then we can prove that $F\otimes_{\mathbf{Z}[\zeta_n]}Ell_*(\Gamma_{(2)})$ is a free $F\otimes_{\mathbf{Z}[\zeta_n]}\mathrm{Ell}_*[\zeta_n]$-module with rank $[\Gamma_1(2):\Gamma_{(2)}]$. The point of the proof is that $F\otimes_{\mathbf{Z}[\zeta_n]}Ell_*[\zeta_n]$ is a graded principal ideal domain. So $F\otimes_{\mathbf{Z}[\zeta_n]}Ell_*(\Gamma_{(2)})$ is a free $F\otimes_{\mathbf{Z}[\zeta_n]}Ell_*[\zeta_n]$-module, since it has no torsion elements. The rank of it can be calculated similarly to the case of Theorem 1.11.

2. The Jacobi sine and its property

In this section, we shall define Jacobi sines and associated higher level modular forms. We first recall the definition of Jacobi forms (cf. [7]). For a function f defined on $\mathfrak{H}\times\mathbf{C}$, $\gamma\in SL_2(\mathbf{Z})$ and $k\in\mathbf{Z}$, we put

$$f|_{[\gamma]_k}(\tau,z)=f(\gamma\cdot(\tau,z))j(\gamma,\tau)^{-k},$$

where $\gamma\cdot(\tau,z)=\left(\gamma\cdot\tau,\frac{z}{j(\gamma,\tau)}\right)$.

Definition 2.1. Let $k\in\mathbf{Z}$ be an integer and $\Gamma\subset SL_2(\mathbf{Z})$ be a modular subgroup. A function $f:\mathfrak{H}\times\mathbf{C}\to\mathbf{C}$ satisfying the following conditions is called a Jacobi form on Γ of weight k (of index 0).

(i) f is meromorphic in both variables and meromorphic at cusps of Γ.

(ii) $f(\tau,-)$ is periodic relative to the lattice $L_\tau=\langle 4\pi i\tau, 4\pi i\rangle$.

(iii) $f|_{[\gamma]_k}(\tau,z)=f(\tau,z)$, for all $\gamma\in\Gamma$.

The following proposition is obvious by definition.

Proposition 2.2. *Let f a Jacobi form on Γ of weight k. Then*

(i) *$\frac{d}{dz}(f(\tau,z))$ is a Jacobi form on Γ of weight k.*

(ii) *For an element $\gamma\in SL_2(\mathbf{Z})$, $f|_{[\gamma]_k}(\tau,z)$ is a Jacobi form of weight k on the conjugate subgroup $\gamma\Gamma\gamma^{-1}$.*

Now define an action of the group $SL_2(\mathbf{Z})$ on $\mathbf{C}/L_\tau$ as

$$4\pi i(\alpha\tau+\beta)\cdot\begin{pmatrix}a&b\\c&d\end{pmatrix}=4\pi i((\alpha a+\beta c)\tau+(\alpha b+\beta d)).$$

Let $T(n)$ be a subset of $\mathbf{C}/L_\tau$ consisting of n-torsion points. Then it is easy to see that the action of $SL_2(\mathbf{Z})$ is restricted to $T(n)$ and $T(n)$ is decomposed into the orbits $T(n)=\cup_{m|n}T_*(m)$, where $T_*(m)$ is a subset of $\mathbf{C}/L_\tau$ consisting of exact m-torsion points. Note that $T_*(1)=\{0\}$. Then we recall a basic property of Jacobi forms.

Theorem 2.3. ([5]). *Let f be a Jacobi form on Γ of weight k. For an interger $n\in\mathbf{Z}$ and $z_0\in\mathbf{C}/L_\tau$, let $g_n(\tau, z_0)$ be the n-th coefficient in the Laurent expansion of $f(\tau, z)$ at $z=z_0$. Then $g_n(\tau, z_0)$ is a modular form on $\Gamma_{[z_0]}$ of weight* $k+n$, *where* $\Gamma_{[z_0]}=\{\gamma\in\Gamma \mid z_0\cdot\gamma=z_0\}$.

Now we define Jacobi sines by using the Weierstraß $\mathfrak{H}$-functions. Let η be one of the half period points $\omega_1=2\pi i, \omega_2=2\pi i\tau$ and $\omega_3=\omega_1\omega+\omega_2$ of $\mathbf{C}/L_\tau$. Then the Jacobi sine with respect to η is defined as

$$s_\eta(\tau, z)=-\frac{2(\mathfrak{H}(\tau, z)-\mathfrak{H}(\tau, \eta))}{\mathfrak{H}'(\tau, z)},$$

where $\mathfrak{H}(\tau, z)$ is the Weierstraß $\mathfrak{H}$-function relative to L_τ.

Theorem 2.4. *The Jacobi sine satisfies the following properties:*

(i) $s_\eta(\tau, z+\eta)=-s_\eta(\tau, z)$

(ii) $s_\eta(\tau, z+4\pi i)=s_\eta(\tau, z)$ *and* $s_\eta(\tau, z+4\pi i\tau)=s_\eta(\tau, z)$

(iii) $s_\eta(\tau, z)\neq 0$ *if* $z\notin <4\pi i\tau, 4\pi i, \eta>\subset\mathbf{C}$

(iv) $s_\eta(\tau, z)=z+O(z^2)$ *as* $z\to 0$.

Proof. Easy from the definition, see [19].

It's easy to check that a function satisfies the above 4 conditions is uniquely determined. D. Zagier has shown in [19] that the function s_η has the following expansion. Let $q=e^{2\pi i\tau}$ and $\zeta=e^z$. Then for $|q|<\min(|\zeta|^{\frac{1}{2}}, |\zeta|^{-\frac{1}{2}})$, we have

$$\frac{1}{s_{\omega_1}(\tau, z)}=\frac{1}{\zeta^{\frac{1}{2}}-\zeta^{-\frac{1}{2}}}-\sum_{r=1}^{\infty}a_r(\zeta^{\frac{1}{2}}-\zeta^{-\frac{1}{2}})^{2r-1}, \tag{2.5}$$

where a_r are defined as follows.

$$a_r=\sum_{n=1}^{\infty}\sum_{2\nmid d\mid n}\left(\binom{\frac{1}{2}(d-1)+r}{2r-1}+\binom{\frac{1}{2}(d-3)+r}{2r-1}\right)q^n\in q^{2r-1}\mathbf{Z}[[q]].$$

Also we have

$$\frac{1}{s_{\omega_2}(\tau, z)}=\frac{1}{4}\frac{\zeta^{\frac{1}{2}}+1}{\zeta^{\frac{1}{2}}-1}\left(1-2\sum_{r=1}^{\infty}b_r(\zeta^{\frac{1}{4}}-\zeta^{-\frac{1}{4}})^{2r}\right), \tag{2.6}$$

where b_r are defined as follows.

$$b_r=\sum_{n=1}^{\infty}\sum_{d\mid n}(-1)^{\frac{n}{d}}\binom{r+d-1}{2r-1}q^{\frac{n}{2}}\in q^{\frac{r}{2}}\mathbf{Z}[[q^{\frac{1}{2}}]].$$

Proposition 2.7. *Let $\gamma\in SL_2(\mathbf{Z})$. Then we have*

$$s_\eta|_{[\gamma]^{-1}}(\tau, z)=s_{\eta\gamma}(\tau, z),$$

where $\eta\cdot\gamma$ is the action of $SL_2(\mathbf{Z})$ on the set $T_*(2)=\{\omega_1, \omega_2, \omega_3\}$.

Proof. Since $SL_2(\mathbf{Z})$ is generated by $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$, it suffices to prove the proposition in cases of $\gamma=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $\gamma=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$. But it is easy to check that the left hand side satisfies the properties in Theorem 2.4 for $\eta'=\eta\cdot\gamma$. So by the uniquencess of the Jacobi sine, we obtain the assertion.

Let Γ_η be an isotropy subgroup of $SL_2(\mathbf{Z})$ fixing η. Then by Proposition 2.7, it is easily verified that the Jacobi sine $s_\eta(\tau, z)$ is a Jacobi form on Γ_η of weight -1. Also as some of other properties of Jacobi sines;

Proposition 2.8. (i) *$s_\eta(\tau, z)$ is an odd function.*

(ii) *$s_\eta(\tau, z)$ satisfies the following differential equation.*

$$\left(\frac{ds_\eta(\tau, z)}{dz}\right)^2=1-2\delta_\eta(\tau)s_\eta(\tau, z)^2+\varepsilon_\eta(\tau)s_\eta(\tau, z)^4, \tag{2.9}$$

where $\delta_\eta(\tau)$ and $\varepsilon_\eta(\tau)$ are modular forms on Γ_η of weight 2 and 4, respectively.

(iii) *Suppose that $\eta'\neq\eta$. Then the function $\kappa_\eta(\tau)=s_\eta(\tau, z)s_\eta(\tau, z+\eta')$ is constant with respect to z, and satisfies the equality $\kappa_\eta(\tau)^2=\varepsilon_\eta(\tau)^{-1}$.*

Proof. (i) is easy. For (ii), see for example [12, 19]. The property (iii) is easily verified by the differential equation (2.9).

Especially, the modular forms $\delta_{\omega_1}(\tau)$ and $\varepsilon_{\omega_1}(\tau)$ correspond to the ones $\delta(\tau)$ and $\varepsilon(\tau)$ in the previous section, respectively. On the other hand $\delta_{\omega_2}(\tau)$ and $\varepsilon_{\omega_2}(\tau)$ are modular forms on $\Gamma^1(2)=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\Gamma_1(2)\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ of weight 2 and 4, respectively. Since $\begin{pmatrix}0&-1\\1&0\end{pmatrix}0=i\infty$, the q-expansions of $\delta_{\omega_2}(\tau)$ and $\varepsilon_{\omega_2}(\tau)$ at $i\infty$ are equal to that of $\delta_{\omega_2}(\tau)$ at 0, respectively. They are listed below (cf. [19]).

$$\delta_{\omega_2}(\tau)=\frac{1}{16}+\frac{3}{2}\sum_{n=1}^{\infty}\Big(\sum_{d|n, 2\nmid d} d\Big)q^{\frac{n}{2}}=\frac{1}{16}+\frac{3}{2}q^{\frac{1}{2}}+\cdots, \tag{2.10}$$

$$\varepsilon_{\omega_2}(\tau)=\frac{1}{256}+\frac{1}{16}\sum_{n=1}^{\infty}\Big(\sum_{d|n}(-1)^d d^3\Big)q^{\frac{n}{2}}=\frac{1}{256}+\cdots. \tag{2.11}$$

Remark. δ_{ω_3} and ε_{ω_3} are modular forms on Γ_θ and P. S. Landweber used those forms in his paper as δ and ε respectively ([12]).

Now we define modular forms associated with Jacobi sines. Let λ be an element of the group $\mathbf{C}/L_\tau$. We put

$$e_\lambda(\tau)=s_{\omega_1}(\tau,\lambda). \tag{2.12}$$

Then we have the following proposition.

Proposition 2.13. *Let* $\gamma\in\Gamma_1(2)$. *Then* $e_\lambda|_{[\gamma]^{-1}}=e_{\lambda\cdot\gamma}$.

Proof. The proof is easy.

Let λ be a torsion point of $\mathbf{C}/L_\tau$ with order n. Then there exists $\gamma\in SL_2(\mathbf{Z})$ such that $\lambda\cdot\gamma=\frac{4\pi i}{n}$.

Theorem 2.14. *Let* λ, n *and* γ *be as above. Then we have*

(i) $e_\lambda\in Ell_{-2}((\gamma\Gamma_1(n)\gamma^{-1})_{(2)})$

(ii) e_λ *is non-vanishing on* $\mathfrak{H}$ *and cusps equivalent to* $i\infty$ *over* $\Gamma_1(2)$.

Proof. Let $\lambda=4\pi i(a\tau/n+b/n)$. Since the modularity of e_λ on $(\gamma\Gamma_1(n)\gamma^{-1})_{(2)}$ follows by Theorem 2.3, so it is enough to check the q-expansions at cusps. Note that any cusp s of $\gamma\Gamma_1(n)\gamma^{-1}$ is equivalent to $i\infty$ or 0 by an element σ of $\Gamma_1(2)$. Then by Proposition 2.13 the q-expansion of e_λ at s is equal to that of $e_{\lambda\cdot\sigma}$ at $i\infty$ or 0. So it suffices to check the q-expansions at $i\infty$ and 0. By the expansion (2.5), easily we have that the following q-expansions.

$$\frac{1}{e_\lambda(\tau)}=\begin{cases}(\zeta_n^b-\zeta_n^{-b})^{-1}+q_nu_0(q_n), & a=0,\\ -q_n^a\zeta_n^b+q_n^{a+1}u_a(q_n), & 1\leq a\leq\frac{n-1}{2},\\ -q_n^{n-a}\zeta_n^b+q_n^{n-a+1}u_a(q_n), & \frac{n+1}{2}\leq a\leq n-1.\end{cases}$$

where $q_n=e^{\frac{2\pi i\tau}{n}}$ and $u_a(q_n)\in\mathbf{Z}[\frac{1}{2},\zeta_n][[q_n]]$. While by the expansion (2.6) the q-expansions at 0 are seen as follows.

$$\frac{1}{s_{\omega_2}(\tau,\lambda)}=\begin{cases}4^{-1}(\zeta_n^b+1)(\zeta_n^b-1)^{-1}(1+q_{2n}v_0(q_{2n})), & a=0,\\ -4^{-1}+q_{2n}v_a(q_{2n}), & 1\leq a\leq\frac{n-1}{2},\\ 4^{-1}+q_{2n}v_a(q_{2n}), & \frac{n+1}{2}\leq a\leq n-1.\end{cases}$$

where $v_a(q_{2n})\in\mathbf{Z}[\frac{1}{2},\zeta_n][[q_{2n}]]$. By this expression, we obtain the assertion for $a\neq0$. For $a=0$, note that ζ_n^b+1 is a unit in $\mathbf{Z}[\zeta_n]$.

3. Formal group law of elliptic curves

In this section we shall study multiplication-by-n sequence of the formal group law associated with elliptic curves. The results in this section are essentially given by J. Igusa ([8]). In the previous section, we defined the

Jacobi sine $s(\tau, z) = s_{\omega_1}(\tau, z)$. By the differential equation (2.9) and the expansion $s(\tau, z) = z + O(z^2)$ at $z=0$, we see that $s(\tau, z)$ parametrizes an elliptic curve:

$$E/\mathbf{C} : y^2 = 1 - 2\delta(\tau)x^2 + \varepsilon(\tau)x^4.$$

Thus we have a local isomorphism from the additive group $\mathbf{C}/L_\tau$ to E. Hence we obtain a formal group law F associated with the elliptic curve E with the formal inverse of $s(\tau, z)$ as its logarithm:

$$s^{-1}(\tau, z) = \int_0^z \frac{dx}{\sqrt{1 - 2\delta(\tau)x^2 + \varepsilon(\tau)x^4}}.$$

The formal group law F is called the Euler formal group law and is seen to be defined over $\mathbf{Z}[\frac{1}{2}][\delta, \varepsilon] \cong M_*(\Gamma_1(2))^{\mathbf{Z}[\frac{1}{2}]}$. We regard the formal group F defined over $\mathbf{Z}[\frac{1}{2}][\delta, \varepsilon, \Delta^{-1}] \cong Ell_*$.

For a non-negative integer n, define an n-sequence $[n](x) \in Ell_*[[x]]$ inductively by

$$[0](x) = 0, \quad [n](x) = x +_F [n-1](x).$$

Before studying the n-sequence we state the following lemma:

Lemma 3.1. (i) $e_\lambda{}^2 e_{\lambda+\omega_2}{}^2 = \varepsilon^{-1}$.

(ii) *Let n be odd. Then* $\Pi_{0 \neq \lambda \in T(n)} e_\lambda = n(-1)^{\frac{n-1}{2}} \varepsilon^{-\frac{n^2-1}{4}}$

Proof. By Proposition 2.8 (iii) the assertion (i) is easily obtained. Let $c(\tau) = \Pi_{0 \neq \lambda \in T(n)} e_\lambda$. Then $c(\tau)$ is fixed under the action of the group $\Gamma_1(2)$, we see that $c(\tau)$ is a modular form on $\Gamma_1(2)$ of weight $-(n^2-1)$. By the q-expansion of e_λ, we have

$$c(\tau) = \begin{cases} n(-1)^{\frac{n-1}{2}} q^{-\frac{n^2-1}{4}} + \cdots & \text{at } i\infty, \\ n(-1)^{\frac{n-1}{2}} q^{n^2-1} + \cdots & \text{at } 0. \end{cases}$$

Hence by the q-expansions of $\varepsilon(\tau)$, we see that $\varepsilon(\tau)^{\frac{n^2-1}{4}} c(\tau)$ is a holomorphic modular form on $\Gamma_1(2)$ i.e. $\varepsilon(\tau)^{\frac{n^2-1}{4}} c(\tau) \in M_*(\Gamma_1(2)) \cong \mathbf{C}[\delta, \varepsilon]$. But the weight of $\varepsilon(\tau)^{\frac{n^2-1}{4}} c(\tau)$ is equal to 0, so $\varepsilon(\tau)^{\frac{n^2-1}{4}} c(\tau)$ must be constant. Then we obtain th assertion from the above q-expansions.

Remark. The property (ii) is also seen by calculating the divisor of $s(\tau, nz + \omega_2)$ relative to the lattice L_τ and by Proposition 2.8 (iii) (cf. [13]).

Theorem 3.2. *Let n be odd. Then there exist $f_n, g_n \in \mathbf{Z}[\frac{1}{2}][\delta, \varepsilon][x]$ satisfying the following conditions:* (i) $[n](x) = \dfrac{f_n(x)}{g_n(x)} \in \mathbf{Z}[\frac{1}{2}][\delta, \varepsilon][[x]]$.

(ii) *$f_n(x)$ is an odd polynomial:* $f_n(x) = nx + \cdots + (-1)^{\frac{n-1}{2}} \varepsilon^{\frac{n^2-1}{4}} x^{n^2}$.

(iii) *$g_n(x)$ is an odd polynomial with constant term $g_n(0)=1$.*

Proof. First define polynomials over $Ell_*(\Gamma(n)_{(2)})$ as

$$f_n(x)=(-1)^{\frac{n-1}{2}}\varepsilon^{\frac{n^2-1}{4}}\prod_{\lambda\in T(n)}(x-e_\lambda), \tag{3.3}$$

$$g_n(x)=(-1)^{\frac{n-1}{2}}n\prod_{0\neq\lambda\in T(n)}(x-e_{\lambda+\omega_2}) \tag{3.4}$$

In the previous section we saw the elliptic function $s(\tau, z)$ has the divisor $\mathrm{div}(s)=(0)+(\omega_1)-(\omega_2)-(\omega_3)$. So we can calculate the divisors of $s(\tau, nz)$, $f_n(s(\tau, z))$ and $g_n(s(\tau, z))$ considered as elliptic functions relative to the lattice L_τ

$$\mathrm{div}\ s(\tau, nz)=\sum_{\lambda\in T(n)}((\lambda)+(\omega_1+\lambda)-(\omega_2+\lambda)-(\omega_3+\lambda)),$$

$$\mathrm{div}\ f_n(s(\tau, z))=\sum_{\lambda\in T(n)}((\lambda)+(\omega_1+\lambda))-n^2(\omega_2)-n^2(\omega_3),$$

$$\mathrm{div}\ g_n(s(\tau, z))=\sum_{\lambda\in T(n)}((\omega_2+\lambda)+(\omega_3+\lambda))-n^2(\omega_2)-n^2(\omega_3),$$

Also regard the Taylor expansion of $s(\tau, z)$ at $z=0$ as a power series over $\mathbf{Q}[\delta, \varepsilon]$ with the leading term z. Then comparing the leading terms of the Taylor expansions of these elliptic functions and by Lemma 3.1, we have

$$s(\tau, nz)=\frac{f_n(s(\tau, z))}{g_n(s(\tau, z))}. \tag{3.5}$$

On the other hand by the definition of the Euler formal group law we have

$$[n](s(\tau, z))=s(\tau, nz)$$

as power series. This shows (i).

We consider the coefficients of $f_n(x)$ and $g_n(x)$ as modular forms on $\Gamma(n)_{(2)}$. Then by Theorem 2.14 and Lemma 3.1, we see that they are holomorphic modular forms on $\Gamma(n)_{(2)}$ defined over $\mathbf{Z}[\frac{1}{2}, \zeta_n]$. Moreover we see that they are actually holomorphic modular forms on $\Gamma_1(2)$. Let σ be an element of the Galois group $G(\mathbf{Z}[\frac{1}{2}, \zeta_n]/\mathbf{Z}[\frac{1}{2}])\cong(\mathbf{Z}/n\mathbf{Z})^\times$ and act as $\sigma:\zeta_n\to\zeta_n^m$, where $(n, m)=1$. Then σ acts the q-expansion of the modular form $e_\lambda(\tau)$ as $\sigma(e_\lambda(\tau))=e_{\sigma\cdot\lambda}(\tau)$, where for $\sigma\cdot(4\pi i(\alpha\tau+1/n))=4\pi i(\alpha\tau+ml/n)$. On the other hand the coefficients of $f_n(x)$ are symmetric functions of $\{e_\lambda\}_\lambda$ multiplied by $(-1)^{\frac{n-1}{2}}\varepsilon^{\frac{n^2-1}{4}}$. So their q-expansions are fixed under the action of σ. Then by Galois theory we see that the coefficients of the q-expansions are contained in the subring $\mathbf{Z}[\frac{1}{2}]$. This shows that all of the coefficients of the polynomial $f_n(x)$ are holomorphic modular forms on $\Gamma_1(2)$ over $\mathbf{Z}[\frac{1}{2}]$. The rest of the assertion follows from Lemma 3.1.

Definition 3.6. Let n be a non-negative integer and p an odd prime. Then we define a polynomial $\phi_{p^n}(x)$ by

$$\phi_{p^n}(x) = \prod_{\lambda\in T_*(p^n)} (x-e_\lambda). \tag{3.7}$$

Theorem 3.8. (i) $f_{p^n}(x) = (-1)^{\frac{n-1}{2}}\varepsilon^{\frac{n^2-1}{4}}\Pi_{i=0}^{n}\phi_{p^i}(x)$.
(ii) $\phi_{p^n}(x)$ is a monic irreducible polynomial over Ell_*.

Proof. The property (i) follows from the definition of $f_n(x)$. By Proposition 2.13 we see that the Galois group of the polynomial $\phi_{p^i}(x)$ is $\Gamma_1(2)/\Gamma_1(2p^i)\cong SL_2(\mathbf{Z}/(p^i))$ and it acts transitively on the set of the solutions of $\phi_{p^i}(x)$. So the polynomial $\phi_{p^i}(x)$ must be irreducible.

Let R be a graded ring containing $\mathbf{Z}[\frac{1}{2}][\delta, \varepsilon][\varepsilon^{-1}]$ (for example, Ell_* or $Ell_*(\Gamma)$) and p be an odd prime. We define $I_2=(p, v_1)\subset R$ be a graded ideal of R generated by p and v_1, where v_1 is the image of Hazewinkel generator v_1 under the elliptic genus φ. (cf. [6, 15, 18]).

Proposition 3.9. $\phi_{p^n}(x)\equiv x^{p^{2n}-p^{2(n-1)}} \bmod I_2$

Proof. It is enough to prove that $\theta_{p^n}(x) = (-1)^{\frac{p^n-1}{2}}\varepsilon^{\frac{p^{2n}-1}{4}}f_{p^n}(x)\equiv x^{p^{2n}} \bmod I_2$. For $n = 0$, there is nothing to prove. We prove this assertion by induction on n for $n\geq 1$. Since we think of graded ideals, we may assume $\varepsilon=1$. For the case of $n=1$, by [8, 9] we have

$$\theta_p(x)\equiv x^{p^2} \mod (p, P_{\frac{p-1}{2}}(\delta)),$$

where $P_n(x)$ is the n-th Legendre polynomial. On the other hand it is well-known that $v_1 = P_{(p-1)/2}(\delta, \varepsilon)$ (cf. [6] and [13]). So we obtain the assertion for $n = 1$. Next we suppose that the assertion is correct for $n-1$, then by the proof of Theorem 3.2, we have

$$[p^n](x) = \theta_{p^n}(x)h_n(x),$$

where $h_n(x)$ is a unit of $Ell_*[[x]]$. Using the result of $n=1$ case, we have $[p](x)\equiv ux^{p^2} \bmod I_2+(x^{p^2+1})$, where u is a unit in Ell_*. Inductively we have

$$[p^n](x)\equiv u_n x^{p^{2n}} \mod I_2+(x^{p^{2n}+1}),$$

where u_n is a unit in Ell_*. By these two equalities we have

$$\theta(x)\equiv v_n x^{p^{2n}} \mod I_2+(x^{p^{2n}+1}),$$

where u_n is a unit in Ell_*. Since $\theta_{p^n}(x)$ is a polynomial with the highest term $x^{p^{2n}}$, the assertion follows.

4. Graded field of automorphic forms of higher level

In this section, we shall study $Ell_*(\Gamma_1(2p^n)) = Ell_*(\Gamma_1(p^n)_{(2)})$. We define an Ell_*-algebra homomorphism:

$$\sigma : Ell_*[x]/(\phi_{p^n}(x)) \to Ell_*(\Gamma_1(2p^n))$$

by $\sigma(x) = e_{\lambda_1}$, where $\lambda_1 = \frac{4\pi i}{p^n}$. In Theorem 1.11 we saw that $K_*(\zeta_{p^n}) \otimes_{Ell_*[\zeta_{p^n}]} Ell_*(\Gamma_1(2p^n))$ is a graded quotient field of $Ell_*(\Gamma_1(2p^n))$. We denote it as $K_*(\Gamma_1(2p^n))$. Then we have

Theorem 4.1. *σ is extended as an isomorphism of graded fields.*

$$\sigma : K_*(\zeta_{p^n})[x]/(\phi_{p^n}(x)) \longrightarrow K_*(\Gamma_1(2p^n)).$$

Proof. It is easy to see that σ can be extended as K_*-algebra homomorphism. By Theorem 3.8 the left hand side is a graded field. So σ is injective. Then it is enough to check that the dimension of the both sides are equal. The dimension of $K_*(\zeta_{p^n})[x]/(\phi_{p^n}(x))$ over $K_*(\zeta_{p^n})$ *is* equal to the degree of the polynomial $\phi_{p^n}(x)$. By the definition of $\phi_{p^n}(x)$, it is equal to $p^{2n} - p^{2(n-1)}$. On the other hand by Theorem 1.11, we saw that of the right hand side is equal to $[\Gamma_1(2) : \Gamma_1(2p^n)] = p^{2n} - p^{2(n-1)}$.

5. A topological interpretation

In this section, we shall state our main result. First we begin with some definitions. Let

$$t : B\mathbf{Z}/(p^n)_+ \longrightarrow B\mathbf{Z}/(p^{n-1})_+,$$

be a transfer of the covering $Bi : B\mathbf{Z}/(p^{n-1}) \to B\mathbf{Z}/(p^n)$, where $n \in \mathbf{N}$ and p is an odd prime, and $i : \mathbf{Z}/(p^{n-1}) \to \mathbf{Z}/(p^n)$ is a natural inclusion. We define a spectrum $\mathbf{T}(p^n)$ as a stable fiber of the transfer t.

$$\mathbf{T}(p^n) \longrightarrow B\mathbf{Z}/(p^n)_+ \xrightarrow{t} B\mathbf{Z}/(p^{n-1})_+.$$

Let $E^*(-)$ be a complex oriented cohomology theory. Let $[n](x) \in E_*[[x]]$ be the n-sequence of the formal group law of E_*. Since $[p](x)$ is divisible by x and $[p^n](x) = [p]([p^{n-1}](x))$, we see that there is a power series, $\lambda_{p^n}(x) \in E_*[[x]]$ such that $[p^n](x) = \lambda_{p^n}(x)\,[p^{n-1}](x)$. Hence we have $[p^n](x) = \Pi_{i=0}^n \lambda_{p^i}(x)$. Then T. Torii [17] has shown the following.

Proposition 5.1. *There is a unit $u \in E^*(B\mathbf{Z}/(p^n))$* such that

$$t^*(1) = u\lambda_{p^n}(x),$$

where $t^* : E^*(B\mathbf{Z}/(p^{n-1})) \to E^*(B\mathbf{Z}/(p^n))$ is the induced homomorphism.

Corollary 5.2. *Let $j : \mathbf{T}(p^n) \to B\mathbf{Z}/(p^n)_+$ be the inclusion map. Then the homomorphism:*

$$j^* : E^*(B\mathbf{Z}/(p^n)_+) \longrightarrow E^*(\mathbf{T}(p^n))$$

identified with the reduction map:

$$E_*[[x]]/([p^n](x)) \longrightarrow E_*[[x]]/(\lambda_{p^n}(x)).$$

In the case of $E^*(-)=Ell^*(-)$, we have shown that $[p^n](x) = f_{p^n}(x)/g_{p^n}(x)$ in Theorem 3.2. Therefore we have an equality of ideals $(\lambda_{p^n}(x)) = (\phi_{p^n}(x))$. We fix a maximal ideal $\mathfrak{M}$ containing I_2 and $\wedge$ denoted a graded completion with respect to $\mathfrak{M}$. Then we have

Theorem 5.3. *There are isomorphisms of Ell_*-algebras;*

$$Ell^*(B\mathbf{Z}/(p^n))^\wedge \cong Ell_*^\wedge[x]/(f_{p^n}(x)),$$

$$Ell^*(\mathbf{T}(p^n))^\wedge \cong Ell_*^\wedge[x]/(\phi_{p^n}(x)).$$

Proof. By Proposition 3.9, the representatives of $Ell_*^\wedge[[x]]/(\phi_{p^n}(x))$ are given by polynomials. So we have a homomorphism of Ell_*-algebra:

$$Ell_*^\wedge[[x]]/(\phi_{p^n}(x)) \to Ell_*^\wedge[x]/(\phi_{p^n}(x)).$$

It is easy to see that this homomorphism gives an isomorphism.

Since $f_{p^n}(x) = (-1)^{\frac{p^n-1}{2}} \varepsilon^{\frac{p^{2n}-1}{4}} \Pi_{0 \le m \le n} \phi_{p^m}(x)$ and $\phi_{p^m}(x)$ are irreducible, tensoring the quotient field K_* we obtain the following K_*-algebra isomorphism;

Theorem 5.4. $K_* \otimes_{Ell_*} Ell^*(B\mathbf{Z}/(p^n))^\wedge \cong \Pi_{m=0}^n K_* \otimes_{Ell_*} Ell^*(\mathbf{T}(p^m))^\wedge.$

Now our final result follows by Theorem 4.1 and Theorem 5.3.

Theorem 5.5. *We have an isomorphism of graded fields:*

$$K_*(\zeta_{p^m}) \otimes_{Ell_*} Ell^*(\mathbf{T}(p^m))^\wedge \cong K_*(\Gamma_1(2p^m))^\wedge,$$

where $K_*(\Gamma_1(2p^m))$ is the graded quotient field of $Ell_*(\Gamma_1(2p^m))$.

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