

Solutions in Besov spaces of a class of abstract parabolic equations of higher order in time

By

Yoshitaka YAMAMOTO

1. Introduction

We consider a linear evolution equation in a Banach space X_0 of order n :

$$(1.1) \quad \begin{cases} \sum_{j=0}^n A_j D_t^{n-j} u(t) = f(t), & 0 < t < T, \\ D_t^{n-j} u(0) = x_j, & j = 1, \dots, n, \end{cases}$$

where $A_j, j=1, \dots, n$, are linear continuous operators from Banach spaces X_j to X_0 , respectively and A_0 is the identity operator in X_0 . X_j are assumed to be continuously embedded into X_{j-1} for $j=1, \dots, n$.

In this paper we intend to solve (1.1) in the intersection of X_j -valued Besov spaces with exponents $n-j+\theta, j=0, 1, \dots, n$, which we denote by

$$(1.2) \quad \bigcap_{j=0}^n B_{p,q}^{n-j+\theta}(0, T; X_j).$$

In [25] we showed that when $n=1$, the equation (1.1) has a unique solution in (1.2) for every data $(f, x_1) \in B_{p,q}^\theta(0, T; X_0) \times X_0$ with suitable compatibility relations if and only if (1.1) is a parabolic equation in X_0 . A similar result on solvability is expected for a class of parabolic equations of order n . In this paper we impose a condition for parabolicity on the operator pencil $\sum_{j=0}^n \lambda^{n-j} A_j$. See the hypothesis (H) in Section 6. Such a condition was introduced by Dubinskii [8] to study the equation with zero initial condition.

Under the assumptions above (1.1) is solved as follows. By introducing unknown functions $u_j = D_t^{n-j} u, j=1, \dots, n$, the equation (1.1) is written as a system of first order equations. We show that in a certain Banach space the system is realized as a parabolic equation. Then the above result for the case $n=1$ is applicable to the system. The Banach space that we introduce is given in a definite way by using the functionals of Brézis and Fraenkel [3]. We shall study in detail the functionals, the operator pencil and the relation between them.

Our result applies to the initial boundary value problem of a parabolic equation in the sense of Petrovskii with boundary condition not containing derivatives in t . With a suitable choice of the spaces X_j our condition on the operator pencil holds true of certain partial differential operators. See Tanabe [22], [23]. Lagnese [13] handled this problem in a way similar to ours. It seems that in [13] the system has not been reduced to a parabolic equation in an appropriate space yet. For other applications we refer the reader to Favini and Obrecht [9], and Favini and Tanabe [10].

There are several papers treating an equation of the form (1.1) with operators depending on t . Methods of constructing a fundamental solution to such a problem are discussed by Obrecht [18], [19] and Tanabe [22], [23]. In [23] a partial differential equation with boundary condition variable in t is considered. Our method of reduction to a system would not apply to this problem, since the variation of boundary operators causes complicated relations of the derivatives of solutions.

The plan of this paper is as follows. In Section 2 after giving the notation we shall collect definitions and certain properties of Besov spaces. In the next three sections we shall study the functionals of Brézis and Fraenkel and the operator pencil. In Section 6 we shall describe compatibility relations between data of (1.1) and then state our main result. In Section 7 we shall reduce (1.1) to a parabolic equation of first order and solve the problem in (1.2).

2. Notation and preliminaries

\mathbf{R} and \mathbf{C} denote the fields of real and complex numbers, respectively. \mathbf{Z}_+ is the set of nonnegative integers. Let E and F be Banach spaces. $\mathcal{L}(E, F)$ is the space of bounded linear operators from E to F with uniform operator norm $\|\cdot\|_{\mathcal{L}(E, F)}$. We write simply $\mathcal{L}(E, E) = \mathcal{L}(E)$. For a linear operator A in E we denote the domain of A by $\mathcal{D}(A)$.

For $1 \leq p \leq \infty$, $0 \leq a < b \leq \infty$ and $l \in \mathbf{Z}_+$, set E -valued function spaces as follows. $\mathcal{D}'(a, b; E)$ is the space of distributions on (a, b) . The derivatives of $f \in \mathcal{D}'(a, b; E)$ are denoted by $D_t^l f$. $L^p(a, b; E)$ and $L_*^p(a, b; E)$ are the L^p spaces with respect to the Lebesgue measure dt and the measure $t^{-1}dt$ on (a, b) , respectively. For an interval $I = (a, b)$, $(a, b]$, $[a, b)$ or $[a, b]$, $C^l(I; E)$ is the space of l times continuously differentiable functions on I . In the notation above we omit the symbol E when $E = \mathbf{R}$ or \mathbf{C} .

Assume $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $0 < \theta < \infty$, $0 < T \leq \infty$. Set $m = [\theta] + 1$, where $[\theta]$ is the largest integer which does not exceed θ . For a strongly measurable function f on $(0, T)$ with values in E put

$$[f]_{B_{p,q}^\theta(0,T;E)} = \left| h^{-\theta} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} f(\cdot + kh) \right|_{L^p(0, T - mh; E)} \Big|_{L_*^q(0, T/m)}$$

We define the subspaces $B_{p,q}^\theta(0, T; E)$ and $\dot{B}_{p,q}^\theta(0, T; E)$ of $L^p(0, T; E)$, called the Besov spaces, as follows:

- (1) $f \in L^p(0, T; E)$ belongs to $B_{p,q}^\theta(0, T; E)$ if $[f]_{B_{p,q}^\theta(0, T; E)} < \infty$.
- (2) $f \in L^p(0, T; E)$ belongs to $\dot{B}_{p,q}^\theta(0, T; E)$ if

$$f \in B_{p,q}^\theta(0, T; E) \quad \text{and} \quad |h^{-\theta}|f|_{L^p(0,h;E)}|_{L^q(0,T)} < \infty.$$

The spaces $B_{p,q}^\theta(0, T; E)$ and $\dot{B}_{p,q}^\theta(0, T; E)$ are equipped with the norms

$$\begin{aligned} |f|_{B_{p,q}^\theta(0, T; E)} &= |f|_{L^p(0, T; E)} + [f]_{B_{p,q}^\theta(0, T; E)}, \\ |f|_{\dot{B}_{p,q}^\theta(0, T; E)} &= |h^{-\theta}|f|_{L^p(0, h; E)}|_{L^q(0, T)} + |f|_{B_{p,q}^\theta(0, T; E)}, \end{aligned}$$

respectively and become Banach spaces. We shall collect several properties of Besov spaces. For details, we refer the reader to Triebel [24]. In the following proposition $B(\theta, p, q)$ stands for $B_{p,q}^\theta(0, T; E)$ or $\dot{B}_{p,q}^\theta(0, T; E)$.

Proposition 2.1. *Assume $1 \leq p, p' \leq \infty, 1 \leq q, q' \leq \infty, 0 < \theta, \theta' < \infty, 0 < T \leq \infty$. Let us express θ as $l + \sigma$ with $l \in \mathbf{Z}_+$ and $\sigma \in (0, 1]$. Then (1)–(5) hold.*

- (1) $B(\theta, p, q) \subset B(\theta', p, q')$, when $\theta > \theta'$.
- (2) $B(\theta, p, q) \subset B(\theta - p^{-1} + p'^{-1}, p', q')$, when $p' \geq p, q' \geq q, \theta > p^{-1} - p'^{-1}$.
- (3) For $f \in B(\theta, p, q)$, if $T < \infty$, we have $\int_0^t f(s) ds \in B(\theta + 1, p, q)$.
- (4) For $f \in B(\theta, p, q)$ we have $D_t^k f \in B(\theta - k, p, q), k = 0, \dots, l$.
- (5) $\dot{B}_{p,q}^\theta(0, T; E) = \{f \in B_{p,q}^\theta(0, T; E);$

$$D_t^k f(0) = 0 \text{ for } k = 0, \dots, l - 1, D_t^l f \in \dot{B}_{p,q}^\sigma(0, T; E)\}.$$

When $\sigma - p^{-1}$ is not an integer, we have

$$\begin{aligned} &\dot{B}_{p,q}^\sigma(0, T; E) \\ &= \begin{cases} B_{p,q}^\sigma(0, T; E), & \text{when } \sigma - p^{-1} < 0, \\ \{f \in B_{p,q}^\sigma(0, T; E); f(0) = 0\}, & \text{when } \sigma - p^{-1} > 0. \end{cases} \end{aligned}$$

Finally, we give two notations of frequent use. A sum $\sum_{i=j}^l a_i$ of a sequence in a vector space always means zero if $i > j$. We use the notation even if some of the terms a_i are not defined in advance. Let f and g be two functions defined on a set \mathcal{E} with values in $[0, \infty) \cup \{\infty\}$. By “ $f(x) \sim g(x)$ ” we mean the equivalence relation between f and g that there exist positive constants c_1 and c_2 independent of $x \in \mathcal{E}$ such that $c_1 f(x) \leq g(x) \leq c_2 f(x)$ holds for all $x \in \mathcal{E}$.

3. Functionals L_j

Let X_0 be a complex vector space and $X_j, j = 1, \dots, n$, linear subspaces of X_0 satisfying the inclusion relations $X_j \subset X_{j-1}, j = 1, \dots, n$. Let $|\cdot|_j$ be seminorms on X_j .

In this section we introduce a functional on X_j with a positive parameter and study certain intermediate spaces between X_0 and X_n relative to the functional.

For $t > 0$, $x \in X_j$, $j = 0, 1, \dots, n$, and $\phi \in X_n$ put

$$L_j(t, x; \phi) = \sum_{l=0}^j t^{l-j} |x - \phi|_l + \sum_{l=j+1}^n t^{l-j} |\phi|_l.$$

The functional $L_j(t, x)$ on X_j with a positive parameter t is given by taking the infimum of $L_j(t, x; \phi)$ over $\phi \in X_n$:

$$L_j(t, x) = \inf_{\phi \in X_n} L_j(t, x; \phi).$$

Obviously, we have $L_n(t, x) \equiv 0$. For each $t > 0$, $L_j(t, x)$ gives a seminorm on X_j . The functional L_j , together with the functional L given in Section 4, was introduced by Brézis and Fraenkel [3], although the definition of L_j was not explicitly presented there. They used the functionals to describe the spaces of the traces at $t=0$ of functions belonging to $\cap_{j=0}^n C^{n-j}([0, 1]; X_j)$, where X_j , $j = 0, 1, \dots, n$, were assumed to be Banach spaces with respective norms $|\cdot|_j$. We shall give a brief comment on their result later on.

The following two lemmas contain some basic properties of the functional L_j .

Lemma 3.1. (1) For $t > 0$, $x \in X_j$, $j = 0, \dots, n-1$, and $\phi \in X_n$ we have

$$(3.1_j) \quad L_j(t, x) \leq \sum_{l=0}^j t^{l-j} |x|_l,$$

$$(3.2_j) \quad L_j(t, \phi) \leq \sum_{l=j+1}^n t^{l-j} |\phi|_l.$$

(2) For $t > 0$ and $x \in X_j$, $0 \leq k < j \leq n$, we have

$$(3.3_{j,k}) \quad L_k(t, x) \leq t^{j-k} \left(L_j(t, x) + \sum_{l=k+1}^j t^{l-j} |x|_l \right).$$

(3) For $t > 0$ and $x \in X_k$, $0 \leq j < k \leq n$, we have

$$(3.4_{j,k}) \quad L_k(t, x) \leq t^{j-k} \left(L_j(t, x) + \sum_{l=j+1}^k t^{l-j} |x|_l \right).$$

Proof. For $\psi \in X_n$ in Case (1) we have

$$L_j(t, x; \psi) \leq \sum_{l=0}^j t^{l-j} |x|_l + \sum_{l=0}^n t^{l-j} |\psi|_l,$$

$$L_j(t, \phi; \psi) \leq \sum_{l=j+1}^n t^{l-j} |\phi|_l + \sum_{l=0}^n t^{l-j} |\phi - \psi|_l.$$

In Case (2) we have

$$\begin{aligned} L_k(t, x; \psi) &\leq \sum_{l=0}^j t^{l-k} |x - \psi|_l + \sum_{l=j+1}^n t^{l-k} |\psi|_l + \sum_{l=k+1}^j t^{l-k} |x|_l \\ &= t^{j-k} \left(L_j(t, x; \psi) + \sum_{l=k+1}^j t^{l-j} |x|_l \right) \end{aligned}$$

and in Case (3) we have

$$\begin{aligned} L_k(t, x; \psi) &\leq \sum_{l=0}^j t^{l-k} |x - \psi|_l + \sum_{l=j+1}^n t^{l-k} |\psi|_l + \sum_{l=j+1}^k t^{l-k} |x|_l \\ &= t^{j-k} \left(L_j(t, x; \psi) + \sum_{l=j+1}^k t^{l-j} |x|_l \right). \end{aligned}$$

Taking the infimum over $\psi \in X_n$, we obtain the inequalities (3.1)–(3.4).

Lemma 3.2. For $t > 0$, $s > 0$ and $x \in X_j$, $j = 0, \dots, n - 1$, we have

$$(3.5_j) \quad s^j L_j(s, x) \leq \max\{1, s/t\}^n t^j L_j(t, x).$$

Proof. Notice that for $\phi \in X_n$ we have an inequality

$$\begin{aligned} s^j L_j(s, x; \phi) &= \sum_{l=0}^j (s/t)^l t^l |x - \phi|_l + \sum_{l=j+1}^n (s/t)^l t^l |\phi|_l \\ &\leq \max\{1, s/t\}^n t^j L_j(t, x; \phi). \end{aligned}$$

It follows from (3.5_j) that for each $x \in X_j$, $j = 0, \dots, n - 1$, the function $t \mapsto t^j L_j(t, x)$ is nondecreasing and continuous.

Using the functional L_j , we define the subspace Y_j of X_j as follows:

$$(3.6_j) \quad Y_j = \left\{ x \in X_j; \sup_{0 < t < \infty} L_j(t, x) < \infty \right\}, \quad j = 0, 1, \dots, n.$$

The seminorm $|\cdot|_j$ on Y_j is given by

$$(3.7_j) \quad |x|_j = |x|_j + [x]_j, \quad \text{where} \quad [x]_j = \sup_{0 < t < \infty} L_j(t, x).$$

Obviously, we have $Y_n = X_n$ and $|x|'_n = |x|_n$ for $x \in Y_n$. The inequality (3.1₀) implies that $Y_0 = X_0$ holds with equivalent seminorms. By (3.1_{j-1}) and (3.3_{j,j-1}) we obtain the inclusion relations $Y_j \subset Y_{j-1}$, $j = 1, \dots, n$.

Let us now mention the result of Brézis and Fraenkel [3]. They proved that there exists a function $u \in \bigcap_{j=0}^n C^{n-j}([0, 1]; X_j)$ with $D_t^{n-j} u(0) = x_j \in X_j$, $j = 0, 1, \dots, n$, if and only if $\lim_{t \rightarrow 0} L_j(t, x_j) = 0$, $j = 0, 1, \dots, n$, holds. This condition has another

expression that for each $j=0, 1, \dots, n$, x_j belongs to the closure of X_n in Y_j . We remark that when $X_j, j=0, 1, \dots, n$, are Banach spaces, so are Y_j with respective norms $|\cdot|_j$.

We have just presented the procedure for making from a set of subspaces $\{X_j, |\cdot|_j; j=0, 1, \dots, n\}$ of X_0 another set of subspaces $\{Y_j, |\cdot|_j; j=0, 1, \dots, n\}$ of $X_0 (= Y_0)$. An iteration of the procedure yields the same set of subspaces of X_0 as $\{Y_j, |\cdot|_j; j=0, 1, \dots, n\}$ with equivalent seminorms. Let us prove this fact. For $t > 0$, $x \in Y_j, j=0, 1, \dots, n$, and $\phi \in Y_n$ put

$$L'_j(t, x; \phi) = \sum_{l=0}^j t^{l-j} |x - \phi|_l + \sum_{l=j+1}^n t^{l-j} |\phi|_l.$$

Let us define the functional $L'_j(t, x)$ on Y_j by

$$L'_j(t, x) = \inf_{\phi \in Y_n} L'_j(t, x; \phi).$$

Then we have the following proposition.

Proposition 3.1. For $0 \leq \theta \leq 1$ and $x \in Y_j, j=0, \dots, n-1$, we have

$$(3.8) \quad |t^{-\theta} L'_j(t, x)|_{L^\infty(0, \infty)} \leq (2n+1) |t^{-\theta} L_j(t, x)|_{L^\infty(0, \infty)}.$$

The inequality (3.8) with $\theta=0$ shows what we have claimed above.

Proof. First let us prove the following inequalities, for $t > 0, x \in Y_j, j=0, \dots, n-1$, and $\phi \in Y_n$:

$$(3.9) \quad [x - \phi]_k \leq t^{j-k} (t^\theta |r^{-\theta} L_j(r, x)|_{L^\infty(0, \infty)} + L_j(t, x; \phi)), \quad 0 \leq k \leq j,$$

$$(3.10) \quad [\phi]_k \leq t^{j-k} (t^\theta |r^{-\theta} L_j(r, x)|_{L^\infty(0, \infty)} + L_j(t, x; \phi)), \quad j < k \leq n-1.$$

Proof of (3.9): For $s > 0$, by (3.3_{j,k}) and (3.2_j) we have

$$(3.11) \quad \begin{aligned} L_k(s, x - \phi) &\leq s^{j-k} \left(L_j(s, x - \phi) + \sum_{l=k+1}^j s^{l-j} |x - \phi|_l \right) \\ &\leq s^{j-k} \left(L_j(s, x) + L_j(s, \phi) + \sum_{l=k+1}^j s^{l-j} |x - \phi|_l \right) \\ &\leq s^{j-k} \left(L_j(s, x) + \sum_{l=j+1}^n s^{l-j} |\phi|_l + \sum_{l=k+1}^j s^{l-j} |x - \phi|_l \right). \end{aligned}$$

Therefore, for $0 < s \leq t$, we have

$$(3.12) \quad L_k(s, x - \phi)$$

$$\leq t^{j-k} \left(t^\theta |r^{-\theta} L_j(r, x)|_{L^\infty(0,t)} + \sum_{l=j+1}^n t^{l-j} |\phi|_l + \sum_{l=k+1}^j t^{l-j} |x - \phi|_l \right).$$

On the other hand, for $s \geq t$, by (3.1_k) we have

$$L_k(s, x - \phi) \leq \sum_{l=0}^k s^{l-k} |x - \phi|_l \leq \sum_{l=0}^k t^{l-k} |x - \phi|_l.$$

Combining the estimates above, we obtain for all $s > 0$,

$$L_k(s, x - \phi) \leq t^{j-k} (t^\theta |r^{-\theta} L_j(r, x)|_{L^\infty(0,t)} + L_j(t, x; \phi)).$$

The inequality (3.9) is a consequence of taking the supremum over $s > 0$.

Proof of (3.10): For $s > 0$, by (3.4_{j,k}) and (3.1_j) we have

$$\begin{aligned} (3.13) \quad L_k(s, \phi) &\leq s^{j-k} \left(L_j(s, \phi) + \sum_{l=j+1}^k s^{l-j} |\phi|_l \right) \\ &\leq s^{j-k} \left(L_j(s, x) + L_j(s, x - \phi) + \sum_{l=j+1}^k s^{l-j} |\phi|_l \right) \\ &\leq s^{j-k} \left(L_j(s, x) + \sum_{l=0}^j s^{l-j} |x - \phi|_l + \sum_{l=j+1}^k s^{l-j} |\phi|_l \right). \end{aligned}$$

Therefore, noting that $j - k + \theta \leq 0$, for $s \geq t$, we have

$$L_k(s, \phi) \leq t^{j-k} \left(t^\theta |r^{-\theta} L_j(r, x)|_{L^\infty(t,\infty)} + \sum_{l=0}^j t^{l-j} |x - \phi|_l + \sum_{l=j+1}^k t^{l-j} |\phi|_l \right).$$

On the other hand, for $0 < s \leq t$, by (3.2_k) we have

$$L_k(s, \phi) \leq \sum_{l=k+1}^n s^{l-k} |\phi|_l \leq \sum_{l=k+1}^n t^{l-k} |\phi|_l.$$

The subsequent argument is the same as that of the proof of (3.9).

Let us prove the inequality (3.8). For $t > 0$, $x \in Y_j$ and $\phi \in Y_n$, by (3.9) and (3.10) we have

$$L'_j(t, x; \phi) \leq n t^\theta |r^{-\theta} L_j(r, x)|_{L^\infty(0,\infty)} + (n+1) L_j(t, x; \phi).$$

Taking the infimum over $\phi \in Y_n (= X_n)$, we obtain (3.8).

When $0 < \theta < 1$, the inequality (3.8) is generalized in such a way that the L^∞ norms of $t^{-\theta} L_j(t, x)$ and $t^{-\theta} L'_j(t, x)$ are replaced by the L^q norms with respect to the measure dt/t . We shall prove a seemingly complicated version of this fact.

Proposition 3.2. *Assume $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $0 < \theta < 1$. For $x \in Y_j$, $j=0, \dots, n-1$, we have*

$$(3.14) \quad |h^{-\theta-p^{-1}}|L'_j(t, x)|_{L^p(0, h)}|_{L^q(0, \infty)} \leq C_j |h^{-\theta-p^{-1}}|L_j(t, x)|_{L^p(0, h)}|_{L^q(0, \infty)},$$

where C_j are constants depending only on n , θ and j .

The inequality (3.14) with $p=q$ is nothing but what we have claimed above.

Proof. As in the proof of (3.8) we first show the following inequalities analogous to (3.9) and (3.10), for $t > 0$, $x \in Y_j$, $j=0, \dots, n-1$, and $\phi \in Y_n$:

$$(3.15) \quad [x - \phi]_k \leq (n-k) \int_0^t r^{j-k} L_j(r, x) \frac{dr}{r} + t^{j-k} L_j(t, x; \phi), \quad 0 \leq k \leq j,$$

$$(3.16) \quad [\phi]_k \leq k \int_t^\infty r^{j-k} L_j(r, x) \frac{dr}{r} + t^{j-k} L_j(t, x; \phi), \quad j < k \leq n-1.$$

We first notice the estimates

$$(3.17) \quad s^{j-k} L_j(s, x) \leq (n-k) \int_0^s r^{j-k} L_j(r, x) \frac{dr}{r}, \quad k=0, \dots, n-1,$$

$$(3.18) \quad s^{j-k} L_j(s, x) \leq k \int_s^\infty r^{j-k} L_j(r, x) \frac{dr}{r}, \quad k=1, \dots, n.$$

These are obtained respectively by integrating over $(0, s)$ and (s, ∞) with respect to the measure dr/r the both sides of the following inequality due to (3.5):

$$\frac{(s/r)^k}{\max\{1, s/r\}^n} s^{j-k} L_j(s, x) \leq r^{j-k} L_j(r, x).$$

For $0 < s \leq t$, by (3.11) and (3.17) we have

$$L_k(s, x - \phi) \leq (n-k) \int_0^t r^{j-k} L_j(r, x) \frac{dr}{r} + \sum_{l=j+1}^n t^{l-k} |\phi|_l + \sum_{l=k+1}^j t^{l-k} |x - \phi|_l, \quad 0 \leq k \leq j.$$

Using this estimate in place of (3.12), we obtain the inequality (3.15) in the same manner as in the proof of (3.9). Similarly, (3.13) together with (3.18) implies the inequality (3.16).

Let us prove the inequality (3.14). For $t > 0$, $x \in Y_j$ and $\phi \in Y_n$, by (3.15) and (3.16) we have

$$L'_j(t, x; \phi) \leq \sum_{k=0}^j (n-k) \int_0^t (t/r)^{k-j} L_j(r, x) \frac{dr}{r}$$

$$+ \sum_{k=j+1}^{n-1} k \int_t^\infty (t/r)^{k-j} L_j(r, x) \frac{dr}{r} + (n+1)L_j(t, x; \phi).$$

Taking the infimum over $\phi \in Y_n (= X_n)$, we obtain

$$\begin{aligned} L'_j(t, x) &\leq \sum_{k=0}^j (n-k) \int_0^t (t/r)^{k-j} L_j(r, x) \frac{dr}{r} \\ &\quad + \sum_{k=j+1}^{n-1} k \int_t^\infty (t/r)^{k-j} L_j(r, x) \frac{dr}{r} + (n+1)L_j(t, x) \\ &= \sum_{k=0}^j (n-k) \int_0^1 r^{-(k-j)} L_j(tr, x) \frac{dr}{r} \\ &\quad + \sum_{k=j+1}^{n-1} k \int_1^\infty r^{-(k-j)} L_j(tr, x) \frac{dr}{r} + (n+1)L_j(t, x). \end{aligned}$$

Notice that $r^{-(k-j-\theta)} \in L^1_{\ast}(0, 1)$ for $k \leq j$, and that $r^{-(k-j-\theta)} \in L^1_{\ast}(1, \infty)$ for $k > j$. Then an application of Young's inequality gives (3.14).

The following proposition is useful for representing the spaces of the traces of functions belonging to (1.2) by means of the real interpolation method.

Proposition 3.3. For $t > 0$ and $x \in Y_j, j = 0, \dots, n-1$, put

$$(3.19_j) \quad K'_j(t, x) = \inf_{\phi \in Y_{j+1}} (|x - \phi|_j + t|\phi|'_{j+1}).$$

Assume $1 \leq p \leq \infty, 1 \leq q \leq \infty, 0 < \theta < 1$. For $x \in Y_j, j = 0, \dots, n-1$, we have

$$\begin{aligned} |h^{-\theta-p^{-1}}|L_j(t, x)|_{L^p(0, h)}|_{L^q(0, \infty)} &\sim |t^{-\theta}L_j(t, x)|_{L^q(0, \infty)} \\ &\sim |h^{-\theta-p^{-1}}|L'_j(t, x)|_{L^p(0, h)}|_{L^q(0, \infty)} \sim |t^{-\theta}L'_j(t, x)|_{L^q(0, \infty)} \\ &\sim |h^{-\theta-p^{-1}}|K'_j(t, x)|_{L^p(0, h)}|_{L^q(0, \infty)} \sim |t^{-\theta}K'_j(t, x)|_{L^q(0, \infty)}. \end{aligned}$$

If $p = q = \infty$, the assertion is valid also for $\theta = 0$ or $\theta = 1$.

Proof. We first prove the following lemma.

Lemma 3.3. For $t > 0, x \in Y_j, j = 0, \dots, n-1$, we have

$$K'_j(t, x) \sim L'_j(t, x).$$

Proof. It is easy to see that $K'_j(t, x) \leq L'_j(t, x)$, since we have an inequality

$$K'_j(t, x) \leq |x - \phi|_j + t|\phi|'_{j+1} \leq L'_j(t, x; \phi), \quad \phi \in Y_n.$$

To obtain an upper bound of $L'_j(t, x)$ we first notice that

$$c_j = \sup_{\substack{0 < t < \infty \\ |x|'_j \leq 1}} L'_j(t, x), \quad j=0, 1, \dots, n,$$

are finite by Proposition 3.1. For $\phi \in Y_{j+1}$, by (3.3_{j+1, j}) we have

$$\begin{aligned} L'_j(t, x) &\leq L'_j(t, x - \phi) + L'_j(t, \phi) \\ &\leq L'_j(t, x - \phi) + t(L'_{j+1}(t, \phi) + |\phi|'_{j+1}) \\ &\leq c_j |x - \phi|'_j + t(c_{j+1} + 1)|\phi|'_{j+1} \\ &\leq \max\{c_j, c_{j+1} + 1\}(|x - \phi|'_j + t|\phi|'_{j+1}). \end{aligned}$$

Taking the infimum over $\phi \in Y_{j+1}$, we obtain a desired estimate.

We return to a proof of Proposition 3.3. By Proposition 3.1, Proposition 3.2 and Lemma 3.3 we have only to show that

$$|h^{-\theta - p^{-1}}|K'_j(t, x)|_{L^p(0, h)}|_{L^q(0, \infty)} \sim |t^{-\theta}K'_j(t, x)|_{L^q(0, \infty)}.$$

This is a consequence of the inequality

$$\min\{1, t/h\}K'_j(h, x) \leq K'_j(t, x) \leq \max\{1, t/h\}K'_j(h, x),$$

due to Lemma 3.2 with $n=1$.

Remark 3.1. For $x \in X_j$, $j=0, \dots, n-1$, the condition

$$|h^{-\theta - p^{-1}}|L_f(t, x)|_{L^p(0, h)}|_{L^q(0, \infty)} < \infty$$

implies that $x \in Y_j$, since by (3.5_j) we have

$$s^{p^{-1}}L_f(s, x) \leq \{p(n-j)\}^{p^{-1}}|L_f(t, x)|_{L^p(0, h)}, \quad h \geq s > 0,$$

and hence we obtain

$$s^{-\theta}L_f(s, x) \leq \{p(n-j)\}^{p^{-1}}\{q(\theta + p^{-1})\}^{q^{-1}}|h^{-\theta - p^{-1}}|L_f(t, x)|_{L^p(0, h)}|_{L^q(s, \infty)}.$$

4. Functional L

Let $\{X_j, |\cdot|_j; j=0, 1, \dots, n\}$ be a set of complex vector spaces and seminorms satisfying the hypotheses of Section 3. For $t > 0$, $\zeta \in \mathbf{C}$ with $|\zeta|=1$, $x_j \in X_j$, $j=0, \dots, n-1$, and $\phi \in X_n$ put

$$L(t, \zeta, x_0, \dots, x_{n-1}; \phi) = \sum_{l=0}^n \left| \sum_{j=l}^{n-1} (t\zeta)^{l-j} x_j - (t\zeta)^{l-n} \phi \right|_l.$$

The functional $L(t, \zeta, x_0, \dots, x_{n-1})$ on the product space $X_0 \times \dots \times X_{n-1}$ with two parameters t and ζ is defined by taking the infimum of $L(t, \zeta, x_0, \dots, x_{n-1}; \phi)$ over $\phi \in X_n$:

$$L(t, \zeta, x_0, \dots, x_{n-1}) = \inf_{\phi \in X_n} L(t, \zeta, x_0, \dots, x_{n-1}; \phi).$$

The definition of $L(t, 1, x_0, \dots, x_{n-1})$ is due to Brézis and Fraenkel [3]. We introduce a new parameter ζ for technical reasons. The functional gives a seminorm on $X_0 \times \dots \times X_{n-1}$, while for each $(\zeta, x_0, \dots, x_{n-1})$ the function $t \mapsto L(t, \zeta, x_0, \dots, x_{n-1})$ is upper semicontinuous. In this section we shall observe several properties of the functional L . Also the relation between the functionals L and L_j is studied. Some of the results are given by Brézis and Fraenkel.

Lemma 4.1. For $t > 0, \zeta \in C$ with $|\zeta| = 1, \lambda > 0$ and $x_j \in X_j, j = 0, \dots, n-1$, we have

$$(4.1) \quad L(t, \zeta, x_0, \dots, \lambda^j x_j, \dots, \lambda^{n-1} x_{n-1}) \leq \max\{1, \lambda^n\} L(t/\lambda, \zeta, x_0, \dots, x_j, \dots, x_{n-1}).$$

Proof. For $\phi \in X_n$, by definition the left-hand side of (4.1) does not exceed

$$\sum_{l=0}^n \left| \lambda^l \left\{ \sum_{j=l}^{n-1} (t\zeta/\lambda)^{l-j} x_j - (t\zeta/\lambda)^l \lambda^{-n} \phi \right\} \right|_l.$$

This is bounded from above by

$$\max\{1, \lambda^n\} L(t/\lambda, \zeta, x_0, \dots, x_j, \dots, x_{n-1}; \lambda^{-n} \phi).$$

Taking the infimum over $\phi \in X_n$, we obtain the inequality (4.1).

Lemma 4.2. For $t > 0, \zeta \in C$ with $|\zeta| = 1$ and $x_j \in X_j, j = 0, \dots, n-1$, we have

$$L(t, \zeta, 0, \dots, x_j, \dots, 0) = L_j(t, x_j).$$

Proof. Notice that for $\phi \in X_n$ we have an equality

$$L(t, \zeta, 0, \dots, x_j, \dots, 0; \phi) = L_j(t, x_j; (t\zeta)^{j-n} \phi).$$

Let $\lambda_k, k = 0, \dots, n-1$, be distinct positive numbers. For $j = 0, \dots, n-1$, we define the numbers $\alpha_{jk}, k = 0, \dots, n-1$, by the roots of the system

$$\sum_{k=0}^{n-1} \lambda_k^l \alpha_{jk} = \delta_{jl}, \quad l = 0, \dots, n-1,$$

where δ_{jl} is Kronecker's symbol. By Lemma 4.2 we have

$$\begin{aligned} L_f(t, x_j) &= L\left(t, \zeta, \sum_{k=0}^{n-1} \alpha_{jk} x_0, \dots, \sum_{k=0}^{n-1} \lambda_k^j \alpha_{jk} x_j, \dots, \sum_{k=0}^{n-1} \lambda_k^{n-1} \alpha_{jk} x_{n-1}\right) \\ &\leq \sum_{k=0}^{n-1} |\alpha_{jk}| L(t, \zeta, x_0, \dots, \lambda_k^j x_j, \dots, \lambda_k^{n-1} x_{n-1}). \end{aligned}$$

Hence it follows from Lemma 4.1 that

$$(4.2) \quad L_f(t, x_j) \leq \sum_{k=0}^{n-1} \max\{1, \lambda_k^n\} |\alpha_{jk}| L(t/\lambda_k, \zeta, x_0, \dots, x_j, \dots, x_{n-1}).$$

On the other hand, Lemma 4.2 implies that

$$(4.3) \quad L(t, \zeta, x_0, \dots, x_{n-1}) \leq \sum_{j=0}^{n-1} L_f(t, x_j).$$

Combining (4.2) and (4.3), by Proposition 3.3 we obtain the following proposition.

Proposition 4.1. *Assume $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $0 < \theta < 1$. For $\zeta \in \mathcal{C}$ with $|\zeta| = 1$ and $x_j \in Y_j$, $j=0, \dots, n-1$, we have*

$$\begin{aligned} &|h^{-\theta-p-1} L(t, \zeta, x_0, \dots, x_{n-1})|_{L^p(0, h) | L^q(0, \infty)} \sim |t^{-\theta} L(t, \zeta, x_0, \dots, x_{n-1})|_{L^q(0, \infty)} \\ &\sim |h^{-\theta-p-1} L'(t, \zeta, x_0, \dots, x_{n-1})|_{L^p(0, h) | L^q(0, \infty)} \sim |t^{-\theta} L'(t, \zeta, x_0, \dots, x_{n-1})|_{L^q(0, \infty)} \\ &\sim \sum_{j=0}^{n-1} |h^{-\theta-p-1} L_f(t, x_j)|_{L^p(0, h) | L^q(0, \infty)} \sim \sum_{j=0}^{n-1} |t^{-\theta} L_f(t, x_j)|_{L^q(0, \infty)} \\ &\sim \sum_{j=0}^{n-1} |h^{-\theta-p-1} L'_f(t, x_j)|_{L^p(0, h) | L^q(0, \infty)} \sim \sum_{j=0}^{n-1} |t^{-\theta} L'_f(t, x_j)|_{L^q(0, \infty)} \\ &\sim \sum_{j=0}^{n-1} |h^{-\theta-p-1} K'_j(t, x_j)|_{L^p(0, h) | L^q(0, \infty)} \sim \sum_{j=0}^{n-1} |t^{-\theta} K'_j(t, x_j)|_{L^q(0, \infty)}, \end{aligned}$$

where $L'(t, \zeta, x_0, \dots, x_{n-1})$ is the functional on $Y_0 \times \dots \times Y_{n-1}$ given by

$$(4.4) \quad L'(t, \zeta, x_0, \dots, x_{n-1}) = \inf_{\phi \in Y_n} \sum_{l=0}^n \left| \sum_{j=l}^{n-1} (t\zeta)^{l-j} x_j - (t\zeta)^{l-n} \phi \right|.$$

If $p=q=\infty$, the assertion is valid also for $\theta=0$ or $\theta=1$.

Remark 4.1. For $x_j \in X_j$, $j=0, \dots, n-1$, by Remark 3.1 the condition

$$|h^{-\theta-p-1} L(t, \zeta, x_0, \dots, x_{n-1})|_{L^p(0, h) | L^q(0, \infty)} < \infty$$

implies that $x_j \in Y_j$, $j=0, \dots, n-1$.

5. Operator pencils

Let $\{X_j, |\cdot|_j; j=0, 1, \dots, n\}$ be a set of complex vector spaces and seminorms satisfying the hypotheses of Section 3. Let $A_j, j=1, \dots, n$, be linear operators from X_j to X_0 . For a complex parameter λ let us define the linear operators $P_j(\lambda), j=0, 1, \dots, n$, from X_j to X_0 by

$$P_j(\lambda) = \sum_{k=0}^j \lambda^{j-k} A_k,$$

where A_0 is the identity operator in X_0 . We call the operators $P_j(\lambda)$ operator pencils. In this section we make the following hypotheses.

Hypotheses. *There exists a constant $\zeta \in \mathbb{C}$ with $|\zeta|=1$ such that $P_n(\lambda)$ are bijective for $\lambda \in \mathbf{R}_+ \zeta = \{t\zeta; t \in \mathbf{R}_+\}$, where \mathbf{R}_+ is the set of positive real numbers. There exist positive constants M_j and N_j such that we have*

$$(5.1) \quad |P_n(\lambda)^{-1}x|_j \leq M_j |\lambda|^{j-n} |x|_0, \quad \lambda \in \mathbf{R}_+ \zeta, \quad x \in X_0, \quad j=0, 1, \dots, n,$$

$$(5.2) \quad |A_j x|_0 \leq N_j |x|_j, \quad x \in X_j, \quad j=0, 1, \dots, n.$$

In this section we shall first show that the functionals L_j and L are approximated well by means of the operator pencils. We shall then show that inequalities similar to (5.1) and (5.2) hold with the seminorms $|\cdot|_j$ given by (3.7). Finally, we shall study a matrix of operators describing a certain system of first order equations.

Lemma 5.1. *For $\lambda \in \mathbf{R}_+ \zeta$ and $x \in X_j, j=0, \dots, n-1$, we have*

$$\begin{aligned} L_j(|\lambda|^{-1}, x) &\sim \sum_{l=0}^j |\lambda|^{n-l} |P_n(\lambda)^{-1} P_j(\lambda)x - \lambda^{j-n}x|_l + \sum_{l=j+1}^n |\lambda|^{n-l} |P_n(\lambda)^{-1} P_j(\lambda)x|_l \\ &\sim \sum_{l=1}^j |\lambda|^{n-l} |P_n(\lambda)^{-1} P_j(\lambda)x - \lambda^{j-n}x|_l + \sum_{l=j+1}^n |\lambda|^{n-l} |P_n(\lambda)^{-1} P_j(\lambda)x|_l \\ &\sim \sum_{l=0}^j |\lambda|^{n-l} |P_n(\lambda)^{-1} P_j(\lambda)x - \lambda^{j-n}x|_l + \sum_{l=j+1}^{n-1} |\lambda|^{n-l} |P_n(\lambda)^{-1} P_j(\lambda)x|_l. \end{aligned}$$

Proof. We begin with a proof of the first equivalence relation. By definition it is clear that $L_j(|\lambda|^{-1}, x)$ does not exceed

$$\begin{aligned} &L_j(|\lambda|^{-1}, x; \lambda^{n-j} P_n(\lambda)^{-1} P_j(\lambda)x) \\ &= \sum_{l=0}^j |\lambda|^{n-l} |P_n(\lambda)^{-1} P_j(\lambda)x - \lambda^{j-n}x|_l + \sum_{l=j+1}^n |\lambda|^{n-l} |P_n(\lambda)^{-1} P_j(\lambda)x|_l. \end{aligned}$$

Since for $\phi \in X_n$ we have an equality

$$P_n(\lambda)^{-1}P_j(\lambda)\phi = \lambda^{j-n}\phi - \lambda^{j-n}P_n(\lambda)^{-1} \sum_{k=j+1}^n \lambda^{n-k}A_k\phi,$$

for $0 \leq l \leq j$ we obtain

$$\begin{aligned} & |P_n(\lambda)^{-1}P_j(\lambda)x - \lambda^{j-n}x|_l \\ & \leq |P_n(\lambda)^{-1}P_j(\lambda)(x - \phi)|_l + |\lambda^{j-n}|x - \phi|_l + |\lambda^{j-n} \left| P_n(\lambda)^{-1} \sum_{k=j+1}^n \lambda^{n-k}A_k\phi \right|_l, \end{aligned}$$

and for $j < l \leq n$ we obtain

$$\begin{aligned} & |P_n(\lambda)^{-1}P_j(\lambda)x|_l \\ & \leq |P_n(\lambda)^{-1}P_j(\lambda)(x - \phi)|_l + |\lambda^{j-n}|\phi|_l + |\lambda^{j-n} \left| P_n(\lambda)^{-1} \sum_{k=j+1}^n \lambda^{n-k}A_k\phi \right|_l. \end{aligned}$$

By (5.1) and (5.2) the right-hand sides of these estimates do not exceed

$$|\lambda|^{l-n} \left\{ \sum_{k=0}^j (\delta_{kl} + M_l N_k) |\lambda|^{j-k} |x - \phi|_k + \sum_{k=j+1}^n (\delta_{kl} + M_l N_k) |\lambda|^{j-k} |\phi|_k \right\}.$$

Therefore we have

$$\begin{aligned} & \sum_{l=0}^j |\lambda|^{n-l} |P_n(\lambda)^{-1}P_j(\lambda)x - \lambda^{j-n}x|_l + \sum_{l=j+1}^n |\lambda|^{n-l} |P_n(\lambda)^{-1}P_j(\lambda)x|_l \\ & \leq \left(1 + \sum_{l=0}^n M_l \max_{0 \leq k \leq n} N_k \right) L_f(|\lambda|^{-1}, x; \phi). \end{aligned}$$

Taking the infimum over $\phi \in X_n$, we obtain the first equivalence relation.

In order to prove the second equivalence relation it suffices to show that the first summand of the second term is estimated from above by the sum of the others. This follows from the equality

$$\begin{aligned} (5.3) \quad & P_n(\lambda)^{-1}P_j(\lambda)x - \lambda^{j-n}x = -\lambda^{-n} \sum_{k=1}^j \lambda^{n-k}A_k(P_n(\lambda)^{-1}P_j(\lambda)x - \lambda^{j-n}x) \\ & \quad - \lambda^{-n} \sum_{k=j+1}^n \lambda^{n-k}A_k P_n(\lambda)^{-1}P_j(\lambda)x. \end{aligned}$$

For $x \in X_n$, by (5.1) and (5.2) we have

$$(5.4) \quad |x|_n = |P_n(\lambda)^{-1}P_n(\lambda)x|_n \leq M_n \lim_{\lambda \rightarrow 0} |P_n(\lambda)x|_0 = M_n |A_n x|_0.$$

Hence the third equivalence relation follows from the equality

$$(5.5) \quad A_n P_n(\lambda)^{-1}P_j(\lambda)x = - \sum_{k=0}^j \lambda^{n-k}A_k(P_n(\lambda)^{-1}P_j(\lambda)x - \lambda^{j-n}x)$$

$$- \sum_{k=j+1}^{n-1} \lambda^{n-k} A_k P_n(\lambda)^{-1} P_j(\lambda) x.$$

Lemma 5.2. For $\lambda \in \mathbf{R}_+ \zeta$ and $x_j \in X_j, j=0, \dots, n-1$, we have

$$\begin{aligned} L(|\lambda|^{-1}, \zeta^{-1}, x_0, \dots, x_{n-1}) &\sim \sum_{l=0}^n |\lambda|^{n-l} \left| \sum_{j=0}^{n-1} P_n(\lambda)^{-1} P_j(\lambda) x_j - \sum_{j=l}^{n-1} \lambda^{j-n} x_j \right|_l \\ &\sim \sum_{l=1}^n |\lambda|^{n-l} \left| \sum_{j=0}^{n-1} P_n(\lambda)^{-1} P_j(\lambda) x_j - \sum_{j=l}^{n-1} \lambda^{j-n} x_j \right|_l \\ &\sim \sum_{l=0}^{n-1} |\lambda|^{n-l} \left| \sum_{j=0}^{n-1} P_n(\lambda)^{-1} P_j(\lambda) x_j - \sum_{j=l}^{n-1} \lambda^{j-n} x_j \right|_l. \end{aligned}$$

Proof. Let us prove the first equivalence relation. By definition we have

$$\begin{aligned} &L(|\lambda|^{-1}, \zeta^{-1}, x_0, \dots, x_{n-1}) \\ &\leq L\left(|\lambda|^{-1}, \zeta^{-1}, x_0, \dots, x_{n-1}; \sum_{j=0}^{n-1} P_n(\lambda)^{-1} P_j(\lambda) x_j\right) \\ &= \sum_{l=0}^n |\lambda|^{n-l} \left| \sum_{j=0}^{n-1} P_n(\lambda)^{-1} P_j(\lambda) x_j - \sum_{j=l}^{n-1} \lambda^{j-n} x_j \right|_l. \end{aligned}$$

Since for $\phi \in X_n$ we have an equality

$$\sum_{j=0}^{n-1} P_n(\lambda)^{-1} P_j(\lambda) x_j = \sum_{k=0}^n P_n(\lambda)^{-1} A_k \left(\sum_{j=k}^{n-1} \lambda^{j-k} x_j - \lambda^{n-k} \phi \right) + \phi,$$

we have

$$\begin{aligned} &\left| \sum_{j=0}^{n-1} P_n(\lambda)^{-1} P_j(\lambda) x_j - \sum_{j=l}^{n-1} \lambda^{j-n} x_j \right|_l \\ &\leq |\lambda|^{l-n} \left(M_l \sum_{k=0}^n N_k \left| \sum_{j=k}^{n-1} \lambda^{j-k} x_j - \lambda^{n-k} \phi \right|_k + \left| \sum_{j=l}^{n-1} \lambda^{j-l} x_j - \lambda^{n-l} \phi \right|_l \right), \end{aligned}$$

and hence we have

$$\begin{aligned} &\sum_{l=0}^n |\lambda|^{n-l} \left| \sum_{j=0}^{n-1} P_n(\lambda)^{-1} P_j(\lambda) x_j - \sum_{j=l}^{n-1} \lambda^{j-n} x_j \right|_l \\ &\leq \left(1 + \sum_{l=0}^n M_l \max_{0 \leq k \leq n} N_k \right) L(|\lambda|^{-1}, \zeta^{-1}, x_0, \dots, x_{n-1}; \phi). \end{aligned}$$

Taking the infimum over $\phi \in X_n$, we obtain the first equivalence relation.

By (5.3) and a simple calculation we have an equality

$$\begin{aligned} & \sum_{j=0}^{n-1} (P_n(\lambda)^{-1} P_j(\lambda) x_j - \lambda^{j-n} x_j) \\ &= - \sum_{k=1}^n \lambda^{-k} A_k \left(\sum_{j=0}^{n-1} P_n(\lambda)^{-1} P_j(\lambda) x_j - \sum_{j=k}^{n-1} \lambda^{j-n} x_j \right). \end{aligned}$$

This implies that the first summand of the second term is estimated from above by the sum of the others. Thus the second equivalence relation is proved. From (5.5) we can derive an equality

$$A_n \sum_{j=0}^{n-1} P_n(\lambda)^{-1} P_j(\lambda) x_j = - \sum_{k=0}^{n-1} \lambda^{n-k} A_k \left(\sum_{j=0}^{n-1} P_n(\lambda)^{-1} P_j(\lambda) x_j - \sum_{j=k}^{n-1} \lambda^{j-n} x_j \right).$$

Using this together with (5.4), we obtain the third equivalence relation.

Lemma 5.3. *There exist constants M'_j and N'_j , $j=0, 1, \dots, n$, such that we have*

$$(5.6) \quad |P_n(\lambda)^{-1} x'_j| \leq M'_j |\lambda|^{j-n} |x'_0|, \quad \lambda \in \mathbf{R}_+ \zeta, \quad x \in Y_0, \quad j=0, 1, \dots, n,$$

$$(5.7) \quad |A_j x'_0| \leq N'_j |x'_j|, \quad x \in Y_j, \quad j=0, 1, \dots, n.$$

Proof. The estimate (5.7) is obvious in view of (3.1₀) and the definition of $|\cdot|_j$. For $x \in Y_j$, $j=0, \dots, n-1$, by Lemma 5.1 we have an equivalence relation

$$[x]_j \sim \sup_{\mu \in \mathbf{R}_+ \zeta} \left(\sum_{l=0}^j |\mu|^{n-l} |P_n(\mu)^{-1} P_j(\mu) x - \mu^{j-n} x|_l + \sum_{l=j+1}^n |\mu|^{n-l} |P_n(\mu)^{-1} P_j(\mu) x|_l \right).$$

Therefore, to show (5.6) we have only to estimate each summand of

$$(5.8) \quad \begin{aligned} & \sum_{l=0}^j |\mu|^{n-l} |P_n(\mu)^{-1} P_j(\mu) P_n(\lambda)^{-1} x - \mu^{j-n} P_n(\lambda)^{-1} x|_l \\ & + \sum_{l=j+1}^n |\mu|^{n-l} |P_n(\mu)^{-1} P_j(\mu) P_n(\lambda)^{-1} x|_l. \end{aligned}$$

First, let us consider the case $0 \leq l \leq j$. Since we have an equality

$$P_n(\mu)^{-1} P_j(\mu) P_n(\lambda)^{-1} x - \mu^{j-n} P_n(\lambda)^{-1} x = - P_n(\mu)^{-1} \sum_{k=j+1}^n \mu^{j-k} A_k P_n(\lambda)^{-1} x,$$

for $|\mu| \geq |\lambda|$ we obtain

$$(5.9) \quad \begin{aligned} & |P_n(\mu)^{-1} P_j(\mu) P_n(\lambda)^{-1} x - \mu^{j-n} P_n(\lambda)^{-1} x|_l \\ & \leq |\mu|^{l-n} M_l \sum_{k=j+1}^n N_k |\mu/\lambda|^{j-k} |\lambda|^{j-k} |P_n(\lambda)^{-1} x|_k \end{aligned}$$

$$\leq |\mu|^{l-n} M_l \max_{j < k \leq n} N_k \sum_{k=j+1}^n |\lambda|^{j-k} |P_n(\lambda)^{-1} x|_k.$$

On the other hand, since we have an equality

$$\begin{aligned} & P_n(\mu)^{-1} P_j(\mu) P_n(\lambda)^{-1} x - \mu^{j-n} P_n(\lambda)^{-1} x \\ &= \sum_{k=1}^j \mu^{j-k} P_n(\mu)^{-1} A_k P_n(\lambda)^{-1} x + \mu^j P_n(\mu)^{-1} (P_n(\lambda)^{-1} x - \lambda^{-n} x) \\ &\quad - \mu^{j-n} (P_n(\lambda)^{-1} x - \delta_{0l} \lambda^{-n} x) + \mu^j \lambda^{-n} (P_n(\mu)^{-1} x - \delta_{0l} \mu^{-n} x), \end{aligned}$$

for $|\mu| \leq |\lambda|$ we obtain

$$\begin{aligned} (5.10) \quad & |P_n(\mu)^{-1} P_j(\mu) P_n(\lambda)^{-1} x - \mu^{j-n} P_n(\lambda)^{-1} x|_l \\ &\leq |\mu|^{l-n} M_l \sum_{k=1}^j N_k |\mu/\lambda|^{j-k} |\lambda|^{j-k} |P_n(\lambda)^{-1} x|_k \\ &\quad + |\mu|^{l-n} M_l |\mu/\lambda|^j |\lambda|^j |P_n(\lambda)^{-1} x - \lambda^{-n} x|_0 \\ &\quad + |\mu|^{l-n} |\mu/\lambda|^{j-l} |\lambda|^{j-l} |P_n(\lambda)^{-1} x - \delta_{0l} \lambda^{-n} x|_l \\ &\quad + |\mu|^{l-n} |\mu/\lambda|^j |\lambda|^{j-n} |\mu|^{n-l} |P_n(\mu)^{-1} x - \delta_{0l} \mu^{-n} x|_l \\ &\leq |\mu|^{l-n} M_l \max_{0 \leq k \leq j} N_k \sum_{k=0}^j |\lambda|^{j-k} |P_n(\lambda)^{-1} x - \delta_{0k} \lambda^{-n} x|_k \\ &\quad + |\mu|^{l-n} |\lambda|^{j-l} |P_n(\lambda)^{-1} x - \delta_{0l} \lambda^{-n} x|_l \\ &\quad + |\mu|^{l-n} |\lambda|^{j-n} |\mu|^{n-l} |P_n(\mu)^{-1} x - \delta_{0l} \mu^{-n} x|_l. \end{aligned}$$

Next, let us consider the case $j < l \leq n$. Since we have an equality

$$P_n(\mu)^{-1} P_j(\mu) P_n(\lambda)^{-1} x = \mu^{j-n} P_n(\lambda)^{-1} x - P_n(\mu)^{-1} \sum_{k=j+1}^n \mu^{j-k} A_k P_n(\lambda)^{-1} x,$$

for $|\mu| \geq |\lambda|$ we obtain

$$\begin{aligned} (5.11) \quad & |P_n(\mu)^{-1} P_j(\mu) P_n(\lambda)^{-1} x|_l \\ &\leq |\mu|^{l-n} |\mu/\lambda|^{j-l} |\lambda|^{j-l} |P_n(\lambda)^{-1} x|_l \\ &\quad + |\mu|^{l-n} M_l \sum_{k=j+1}^n N_k |\mu/\lambda|^{j-k} |\lambda|^{j-k} |P_n(\lambda)^{-1} x|_k \\ &\leq |\mu|^{l-n} |\lambda|^{j-l} |P_n(\lambda)^{-1} x|_l \\ &\quad + |\mu|^{l-n} M_l \max_{j < k \leq n} N_k \sum_{k=j+1}^n |\lambda|^{j-k} |P_n(\lambda)^{-1} x|_k. \end{aligned}$$

On the other hand, since we have an equality

$$\begin{aligned}
& P_n(\mu)^{-1}P_j(\mu)P_n(\lambda)^{-1}x \\
&= \sum_{k=1}^j \mu^{j-k}P_n(\mu)^{-1}A_kP_n(\lambda)^{-1}x \\
&\quad + \mu^jP_n(\mu)^{-1}(P_n(\lambda)^{-1}x - \lambda^{-n}x) + \mu^j\lambda^{-n}P_n(\mu)^{-1}x,
\end{aligned}$$

for $|\mu| \leq |\lambda|$ we obtain

$$\begin{aligned}
(5.12) \quad & |P_n(\mu)^{-1}P_j(\mu)P_n(\lambda)^{-1}x|_l \\
&\leq |\mu|^{l-n}M_l \sum_{k=1}^j N_k |\mu/\lambda|^{j-k} |\lambda|^{j-k} |P_n(\lambda)^{-1}x|_k \\
&\quad + |\mu|^{l-n}M_l |\mu/\lambda|^j |\lambda|^j |P_n(\lambda)^{-1}x - \lambda^{-n}x|_0 \\
&\quad + |\mu/\lambda|^j |\lambda|^{j-n} |P_n(\mu)^{-1}x|_l \\
&\leq |\mu|^{l-n}M_l \max_{0 \leq k \leq j} N_k \sum_{k=0}^j |\lambda|^{j-k} |P_n(\lambda)^{-1}x - \delta_{0k}\lambda^{-n}x|_k \\
&\quad + |\mu|^{l-n} |\lambda|^{j-n} |\mu|^{n-l} |P_n(\mu)^{-1}x|_l.
\end{aligned}$$

By virtue of (5.9) and (5.11), (5.8) dose not exceed

$$|\lambda|^{j-n} \left(1 + \sum_{l=0}^n M_l \max_{j < k \leq n} N_k \right) \sum_{l=j+1}^n |\lambda|^{n-l} |P_n(\lambda)^{-1}x|_l$$

when $|\mu| \geq |\lambda|$, and by virtue of (5.10) and (5.12), (5.8) dose not exceed

$$\begin{aligned}
& |\lambda|^{j-n} \left(1 + \sum_{l=0}^n M_l \max_{0 \leq k \leq j} N_k \right) \sum_{l=0}^j |\lambda|^{n-l} |P_n(\lambda)^{-1}x - \delta_{0l}\lambda^{-n}x|_l \\
&\quad + |\lambda|^{j-n} \sum_{l=0}^n |\mu|^{n-l} |P_n(\mu)^{-1}x - \delta_{0l}\mu^{-n}x|_l
\end{aligned}$$

when $|\mu| \leq |\lambda|$. Hence (5.8) is bounded from above by

$$\begin{aligned}
& |\lambda|^{j-n} \left(2 + \sum_{l=0}^n M_l \max_{0 \leq k \leq n} N_k \right) \sup_{\mu \in \mathbf{R} + \zeta} \sum_{l=0}^n |\mu|^{n-l} |P_n(\mu)^{-1}x - \delta_{0l}\mu^{-n}x|_l \\
&\sim |\lambda|^{j-n} [x]_0.
\end{aligned}$$

Thus we obtain a desired estimate of (5.8).

Let us now consider a matrix of operators

$$(5.13) \quad B = \begin{bmatrix} A_1 & A_2 & \dots & A_{n-1} & A_n \\ -I & 0 & \dots & 0 & 0 \\ 0 & -I & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & & -I & 0 \end{bmatrix}.$$

The matrix B is regarded as an operator in $E = X_0 \times \dots \times X_{n-1}$ with domain $\mathcal{D}(B) = X_1 \times \dots \times X_n$. The following lemma is proved by a simple calculation.

Lemma 5.4. *For $\lambda \in \mathbf{R}_+ \zeta$ the operator $\lambda + B$ is bijective. The inverse $(\lambda + B)^{-1}$ is expressed as a matrix of operators the (i, j) components of which are given by*

$$(5.14) \quad \begin{cases} \lambda^{n-i} P_n(\lambda)^{-1} P_{j-1}(\lambda) - \lambda^{j-i-1} I, & i < j, \\ \lambda^{n-i} P_n(\lambda)^{-1} P_{j-1}(\lambda), & i \geq j. \end{cases}$$

Let us give an approximation of the functional L by means of the operator B . For $(x_0, \dots, x_{n-1}) \in E$ put

$$|(x_0, \dots, x_{n-1})|_E = \sum_{j=0}^{n-1} |x_j|_j.$$

By Lemma 5.4 the l -th component of $(\lambda + B)^{-1} (x_0, \dots, x_{n-1})$ is

$$\lambda^{n-l} \left(\sum_{j=0}^{n-1} P_n(\lambda)^{-1} P_j(\lambda) x_j - \sum_{j=l}^{n-1} \lambda^{j-n} x_j \right).$$

Hence, for $\lambda \in \mathbf{R}_+ \zeta$ and $x_j \in X_j, j = 0, \dots, n-1$, by (5.4) we have

$$|B(\lambda + B)^{-1} (x_0, \dots, x_{n-1})|_E \sim \sum_{l=1}^n |\lambda|^{n-l} \left| \sum_{j=0}^{n-1} P_n(\lambda)^{-1} P_j(\lambda) x_j - \sum_{j=l}^{n-1} \lambda^{j-n} x_j \right|_l.$$

This together with Lemma 5.2 proves the following lemma.

Lemma 5.5. *For $\lambda \in \mathbf{R}_+ \zeta$ and $x_j \in X_j, j = 0, \dots, n-1$, we have*

$$|B(\lambda + B)^{-1} (x_0, \dots, x_{n-1})|_E \sim L(|\lambda|^{-1}, \zeta^{-1}, x_0, \dots, x_{n-1}).$$

Let us now consider a restriction of the operator B . Let B' be the operator in E given by

$$(5.15) \quad \mathcal{D}(B') = Y_1 \times \dots \times Y_n, \quad B'x = Bx.$$

As is easily seen, the range of B' is included in $E' = Y_0 \times \dots \times Y_{n-1}$. Therefore we

may regard B' as an operator in E' . By $A'_j, j=0, 1, \dots, n$, let us denote the restrictions of the operators A_j with domain $\mathcal{D}(A'_j)=Y_j$, respectively. Then the matrix of operators B' corresponds to the operator pencil $\Sigma_{j=0}^n \lambda^{n-j} A'_j$. As Lemma 5.3 shows, the operator pencil $\Sigma_{j=0}^n \lambda^{n-j} A'_j$ satisfies the same hypotheses as $\Sigma_{j=0}^n \lambda^{n-j} A_j$ dose. Hence, by Lemma 5.4 the operator $\lambda + B'$ is bijective for $\lambda \in \mathbf{R}_+ \zeta$. The inverse of $\lambda + B'$ is given by restricting the operator $(\lambda + B)^{-1}$ onto E' . For $(x_0, \dots, x_{n-1}) \in E'$ put

$$|(x_0, \dots, x_{n-1})|_{E'} = \sum_{j=0}^{n-1} |x_j|_j.$$

Let $L'(t, \zeta, x_0, \dots, x_{n-1})$ be as in (4.4). For $\lambda \in \mathbf{R}_+ \zeta$ and $x_j \in Y_j, j=0, \dots, n-1$, by Lemma 5.5 we have

$$|B'(\lambda + B')^{-1}(x_0, \dots, x_{n-1})|_{E'} \sim L'(|\lambda|^{-1}, \zeta^{-1}, x_0, \dots, x_{n-1}).$$

Recalling Proposition 4.1, we obtain the following lemma.

Lemma 5.6. *There exists a constant C such that for $\lambda \in \mathbf{R}_+ \zeta$ and $x_j \in Y_j, j=0, \dots, n-1$, we have*

$$(5.16) \quad |B'(\lambda + B')^{-1}(x_0, \dots, x_{n-1})|_{E'} \leq C \sum_{j=0}^{n-1} [x_j]_j.$$

In particular, $\sup_{\substack{|x|_{E'} \leq 1 \\ \lambda \in \mathbf{R}_+ \zeta}} |B'(\lambda + B')^{-1}x|_{E'}$ is finite.

Remark 5.1. The assertion of Lemma 5.6 is not always true of the operator B in E . Indeed, by Proposition 4.1, Remark 4.1 and Lemma 5.5, $\sup_{\substack{|x|_{E'} \leq 1 \\ \lambda \in \mathbf{R}_+ \zeta}} |B(\lambda + B)^{-1}x|_E$ is finite if and only if the following condition is satisfied:

$$Y_j = X_j \quad \text{and} \quad |x|'_j \sim |x|_j, \quad x \in Y_j, \quad \text{for } j=0, 1, \dots, n.$$

This provides a delicate relation of the spaces $X_j, |\cdot|_j, j=0, 1, \dots, n$. In fact, when X_j are function spaces such as Sobolev spaces, the relation is sensitive to the orders of the function spaces and the boundary conditions attached to the function spaces. This subject will be discussed in [26].

6. Main results

Let $X_j, j=0, 1, \dots, n$, be complex Banach spaces with norms $|\cdot|_j$. X_j are assumed to be continuously embedded into X_{j-1} for $j=1, \dots, n$. Let $A_j, j=1, \dots, n$, be linear continuous operators from X_j to X_0 . In the sequel we follow the notation given in the preceding sections. For instance, Y_j are the subspaces of X_j given by (3.6). With the hypotheses above $Y_j, j=0, 1, \dots, n$, become Banach spaces with

respective norms $|\cdot|'_j$.

Let us now consider a linear evolution equation in X_0 of order n :

$$(6.1) \quad \begin{cases} \sum_{j=0}^n A_j D_t^{n-j} u(t) = f(t), & 0 < t < T, \\ D_t^{n-j} u(0) = x_j, & j = 1, \dots, n, \end{cases}$$

where A_0 is the identity operator in X_0 . We are concerned with a solution of (6.1) satisfying

$$(6.2) \quad u \in \bigcap_{j=0}^n B_{p,q}^{n-j+\theta}(0, T; X_j).$$

In order that (6.1) has a solution u with (6.2) the data (f, x_1, \dots, x_n) must satisfy suitable compatibility relations. This is based on the following fact.

Lemma 6.1. *Assume $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $p^{-1} < \theta < 1 + p^{-1}$, $0 < T \leq \infty$. If $u \in \bigcap_{j=0}^n B_{p,q}^{n-j+\theta}(0, T; X_j)$, then we have $D_t^{n-j} u(0) \in (Y_j, Y_{j+1})_{\theta-p^{-1},q}$ (the real interpolation spaces between Y_j and Y_{j+1}) for $j=0, \dots, n-1$.*

Proof. Let $K'_j(t, x)$ be the functional on Y_j given by (3.19)_j. Recall the definition of the real interpolation space $(Y_j, Y_{j+1})_{\eta,q}$, $0 < \eta < 1$, $1 \leq q \leq \infty$, due to Peetre:

$$(Y_j, Y_{j+1})_{\eta,q} = \{x \in Y_j; |t^{-\eta} K'_j(t, x)|_{L^q(0, \infty)} < \infty\}$$

with norm

$$|x|'_{j+\eta,q} = |t^{-\eta} K'_j(t, x)|_{L^q(0, \infty)}.$$

By Proposition 4.1 and Remark 4.1 we have only to show that

$$\left| h^{-\theta} \left| L \left(t, 1, \frac{1}{n!} D_t^n u(0), \dots, D_t u(0) \right) \right| \right|_{L^p(0,h)} \Big|_{L^q(0,T)} < \infty.$$

In the definition of $L(t, 1, x_0, \dots, x_{n-1}; \phi)$ put

$$x_j = \frac{1}{(n-j)!} D_t^{n-j} u(0), \quad j=0, \dots, n-1, \quad \phi = u(t) - u(0).$$

Since for $l=0, \dots, n-1$, we have an equality

$$\sum_{j=l}^{n-1} t^{l-j} x_j - t^{l-n} \phi = - \frac{1}{(n-1-l)!} \int_0^1 (1-s)^{n-1-l} (D_t^{n-l} u(ts) - D_t^{n-l} u(0)) ds,$$

we obtain

$$L\left(t, 1, \frac{1}{n!} D_t^n u(0), \dots, D_t u(0)\right) \leq \sum_{l=0}^{n-1} \frac{1}{(n-1-l)!} \int_0^1 (1-s)^{n-1-l} |D_t^{n-l} u(ts) - D_t^{n-l} u(0)|_l ds + |u(t) - u(0)|_n.$$

Take the norm in $L^p(0, h)$ and then, after multiplying the both sides by $h^{-\theta}$, take the norm in $L^q_*(0, T)$. By Young's inequality the norm

$$\left| h^{-\theta} \left| L\left(t, 1, \frac{1}{n!} D_t^n u(0), \dots, D_t u(0)\right) \right|_{L^p(0, h)} \right|_{L^q_*(0, T)}$$

has an upper bound expressed as a linear combination of $|h^{-\theta} |D_t^{n-l} u(t) - D_t^{n-l} u(0)|_{L^p(0, h; X_l)}|_{L^q_*(0, T)}$, $l=0, 1, \dots, n$. By Proposition 2.1 (5) this gives a desired estimate.

Assume $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $0 < \theta < \infty$, $0 < T < \infty$. Further assume that $\theta - p^{-1}$ is not an integer. By \mathcal{D}_0 we denote the subspace of $B_{p,q}^\theta(0, T; X_0) \times X_0 \times \dots \times X_{n-1}$ which consists of elements (f, x_1, \dots, x_n) satisfying the conditions (1) and (2) below. Set $N = [\theta - p^{-1}] + 1$. Put $y_{0j} = x_j$, $j = 1, \dots, n$.

(1) If $k < N$, then $y_{kj} \in X_j$, $j = 1, \dots, n$. In this case put

$$y_{k+1j} = \begin{cases} D_t^k f(0) - \sum_{l=1}^n A_l y_{kl}, & j = 1, \\ y_{kj-1}, & j = 2, \dots, n. \end{cases}$$

(2) $y_{Nj} \in (Y_{j-1}, Y_j)_{\theta - p^{-1} + 1 - N, q}$, $j = 1, \dots, n$.

We see that

$$(6.3) \quad y_{kj} \in \begin{cases} (Y_{j-k+N-1}, Y_{j-k+N})_{\theta - p^{-1} + 1 - N, q}, & 1 \leq j \leq n - N + k, \\ Y_n (= X_n), & n - N + k < j \leq n, \end{cases}$$

because by definition we have

$$y_{kj} = \begin{cases} y_{Nj-k+N}, & N - k \leq n - j, \\ y_{k-j+nn}, & N - k > n - j. \end{cases}$$

Hence the space \mathcal{D}_0 is equipped with the norm

$$\|f\|_{B_{p,q}^\theta(0, T; X_0)} + \sum_{k=0}^{N-1} \|y_{kn}\|_n + \sum_{j=1}^n \|y_{Nj}\|_{j+\theta - p^{-1} - N, q}$$

and becomes a Banach space. For $u \in \cap_{j=0}^n B_{p,q}^{n-j+\theta}(0, T; X_j)$ put

$$f = \sum_{j=0}^n A_j D_t^{n-j} u \in B_{p,q}^\theta(0, T; X_0), \quad x_j = D_t^{n-j} u(0) \in X_{j-1}, \quad j = 1, \dots, n.$$

When $N=0$, that is, $0 < \theta < p^{-1}$, we apply Lemma 6.1 to $\int_0^t u(s) ds \in \cap_{j=0}^n B_{p,q}^{n-j+1+\theta}(0, T; X_j)$ and obtain $(f, x_1, \dots, x_n) \in \mathcal{D}_0$. When $N \geq 1$, differentiating f successively $N-1$ times in t and then taking the traces at $t=0$ of the derivatives, by Lemma 6.1 we obtain $(f, x_1, \dots, x_n) \in \mathcal{D}_0$. Thus \mathcal{D}_0 may be understood as a space of data with certain compatibility relations for (6.1) to have a solution u with (6.2).

We are now ready to state our main result. Define the linear operator P from $\mathcal{D}(P) = \cap_{j=0}^n B_{p,q}^{n-j+\theta}(0, T; X_j)$ to \mathcal{D}_0 by

$$Pu = \left(\sum_{j=0}^n A_j D_t^{n-j} u, D_t^{n-1} u(0), \dots, u(0) \right), \quad u \in \mathcal{D}(P).$$

Let us make the following hypothesis on the operator pencil $\sum_{j=0}^n \lambda^{n-j} A_j$:

(H). The linear operators $P_n(\lambda)$, $\lambda \in \mathbb{C}$, in X_0 given by

$$\mathcal{D}(P_n(\lambda)) = X_n, \quad P_n(\lambda)x = \sum_{j=0}^n \lambda^{n-j} A_j x$$

are bijective for $\lambda \in \Sigma \equiv \{ \lambda \in \mathbb{C}; |\arg \lambda| \leq \psi \}$ with a constant $\psi \in (\pi/2, \pi)$. The following estimates hold:

$$\sup_{\lambda \in \Sigma} \| \lambda^{n-j} P_n(\lambda)^{-1} \|_{\mathcal{L}(X_0, X_j)} < \infty, \quad j = 0, 1, \dots, n.$$

The following theorem is our main result.

Theorem. Under the hypothesis (H), the operator P is bijective.

In [25] we have studied in detail the operator P in the case $n=1$. For convenience to the reader we summarize the results.

Proposition 6.1. Assume $n=1$. The operator P is bijective if and only if $-A_1$ is a generator of an exponentially bounded analytic semigroup in X_0 .

A similar result is expected for an equation of order n . This subject will be discussed elsewhere.

Here we recall an exponentially bounded analytic semigroup in a complex Banach space and its generator. Such a semigroup has already been studied by [6] and [20]. Let X be a complex Banach space. A mapping $S: (0, \infty) \rightarrow \mathcal{L}(X)$ is called an exponentially bounded analytic semigroup in X , if S satisfies the following conditions (1)–(3):

- (1) S is analytic. S has an analytic continuation to a sectorial region $\{t \in \mathbf{C}; |\arg t| \leq \phi\}$ with a constant $\phi \in (0, \pi/2)$. The continuation, also denoted by S , satisfies the growth condition

$$\sup_{|\arg t| \leq \phi} \|e^{-\omega t} S(t)\|_{\mathcal{L}(X)} < \infty$$

with a constant $\omega \in \mathbf{R}$.

- (2) For $t_1, t_2 > 0$ we have $S(t_1 + t_2) = S(t_1)S(t_2)$.
- (3) For $x \in X$, if $S(t)x = 0, t > 0$, holds, then we have $x = 0$.

For S a linear operator in X is uniquely determined by the condition

$$\mathcal{D}(G) = \{x \in X; y \in X \text{ exists such that } D_t S(t)x - S(t)y = 0, t > 0, \text{ holds}\},$$

$$Gx = y.$$

The correspondence $S \mapsto G$ is shown to be one-to-one. The operator G is called the generator of S . Let \tilde{G} be a linear operator in X . There exists an exponentially bounded analytic semigroup in X with generator \tilde{G} , if and only if \tilde{G} satisfies the following conditions (1) and (2):

- (1) The operator $\lambda - \tilde{G}$ is bijective for $\lambda \in \mathbf{C}$ satisfying $|\arg(\lambda - \tilde{\omega})| \leq \tilde{\psi}$ with constants $\tilde{\omega} \in \mathbf{R}$ and $\tilde{\psi} \in (\pi/2, \pi)$.
- (2) The following estimate holds:

$$\sup_{|\arg(\lambda - \tilde{\omega})| \leq \tilde{\psi}} \|(\lambda - \tilde{\omega})(\lambda - \tilde{G})^{-1}\|_{\mathcal{L}(X)} < \infty.$$

In this case a semigroup with generator \tilde{G} is given by the inverse Laplace transform of $(\lambda - \tilde{G})^{-1}$.

We shall sketch a proof of the theorem. A detailed proof will be given in the next section. Introducing unknown functions $u_j = D_t^{n-j} u, j = 1, \dots, n$, we write (6.1) as a system of equations

$$(6.4) \quad \left\{ \begin{array}{l} D_t u_1(t) + \sum_{j=1}^n A_j u_j(t) = f(t), \quad 0 < t < T, \\ D_t u_j(t) - u_{j-1}(t) = 0, \quad j = 2, \dots, n, \quad 0 < t < T, \\ u_j(0) = x_j, \quad j = 1, \dots, n. \end{array} \right.$$

To prove that P is surjective we regard the system (6.4) as an equation in $E' = Y_0 \times \dots \times Y_{n-1}$ rather than in $E = X_0 \times \dots \times X_{n-1}$ and seek a solution of (6.1) in $\cap_{j=0}^n B_{p,q}^{n-j+\theta}(0, T; Y_j)$. The injectivity of P follows from the uniqueness of rather

weak solutions to (6.4) regarded as an equation in E .

7. Proof of Theorem

Proof of surjectivity of P . Let B' be the linear operator in $E' = Y_0 \times \dots \times Y_{n-1}$ given by (5.15). Since the inequality (5.1) holds for every $\zeta \in \mathbb{C}$ with $|\zeta| = 1, |\arg \zeta| \leq \psi$, so does (5.16). This implies that $-B'$ is a generator of an exponentially bounded analytic semigroup in E' .

By $\tilde{\mathcal{D}}_0$ we denote the subspace of $B_{p,q}^\theta(0, T; E') \times E'$ which consists of elements (\tilde{f}, \tilde{x}) satisfying the conditions (1) and (2) below. Put $\tilde{x}_0 = \tilde{x}$.

- (1) If $k < N$, then $\tilde{x}_k \in \mathcal{D}(B')$. In this case put $\tilde{x}_{k+1} = D_t^k \tilde{f}(0) - B' \tilde{x}_k$.
- (2) $\tilde{x}_N \in (E', \mathcal{D}(B'))_{\theta - p^{-1} + 1 - N, q}$.

Repeating the argument of Section 6 for the equation

$$\begin{cases} D_t \tilde{u} + B' \tilde{u} = \tilde{f}, & 0 < t < T, \\ \tilde{u}(0) = \tilde{x}, \end{cases}$$

we may define the linear operator \tilde{P} by

$$\tilde{P}: B_{p,q}^{1+\theta}(0, T; E') \cap B_{p,q}^\theta(0, T; \mathcal{D}(B')) \rightarrow \tilde{\mathcal{D}}_0, \quad \tilde{u} \mapsto (D_t \tilde{u} + B' \tilde{u}, \tilde{u}(0)).$$

By Proposition 6.1 we see that \tilde{P} is bijective.

For $(f, x_1, \dots, x_n) \in \mathcal{D}_0$ put $\tilde{f} = (f, 0, \dots, 0)$ and $\tilde{x} = (x_1, \dots, x_n)$. Since we have $(Y_0, Y_1)_{\theta - p^{-1} + 1 - N, q} \times \dots \times (Y_{n-1}, Y_n)_{\theta - p^{-1} + 1 - N, q} = (E', \mathcal{D}(B'))_{\theta - p^{-1} + 1 - N, q}$, it follows from (6.3) that $\tilde{x} \in E'$ and $(\tilde{f}, \tilde{x}) \in \tilde{\mathcal{D}}_0$. Hence there exists a unique function $\tilde{u} \in B_{p,q}^{1+\theta}(0, T; E') \cap B_{p,q}^\theta(0, T; \mathcal{D}(B'))$ such that $\tilde{P}\tilde{u} = (\tilde{f}, \tilde{x})$. Let u_n be the n -th component of \tilde{u} . It is easy to see that u_n belongs to $\cap_{j=0}^n B_{p,q}^{n-j+\theta}(0, T; Y_j)$ and satisfies $Pu_n = (f, x_1, \dots, x_n)$. We conclude that P is surjective.

Proof of injectivity of P . Let B be the linear operator in $E = X_0 \times \dots \times X_{n-1}$ given by (5.13). Writing (6.1) as (6.4), we may deduce the injectivity of P from the following proposition.

Proposition 7.1. *For $u \in C^0((0, T); E) \cap \mathcal{D}'(0, T; \mathcal{D}(B))$, if $D_t u + Bu = 0$ in $\mathcal{D}'(0, T; E)$ and if $\lim_{t \rightarrow 0} u(t) = 0$ in E , then we have $u = 0$ on $(0, T)$.*

Proof. Let γ be a contour running in Σ from $e^{-\sqrt{-1}\psi} \infty$ to $e^{\sqrt{-1}\psi} \infty$. For $t > 0$ we define the linear operator $S(t) \in \mathcal{L}(E)$ by

$$(7.1) \quad S(t) = \frac{1}{2\pi\sqrt{-1}} \int_{\gamma} e^{\lambda t} (\lambda + B)^{-1} d\lambda.$$

The restriction of $S(t)$ onto E' gives an exponentially bounded analytic semigroup in E' with generator $-B'$. Hence, for $x \in E'$, if $S(t)x=0$ holds for some $t>0$, then we have $x=0$. Put $X_j = X_n, j \geq n+1$. For $\mu \in \Sigma$ and $m \in \mathbf{Z}_+$, by (5.14) the operator $(\mu+B)^{-1}$ maps $X_m \times \cdots \times X_{m+n-1}$ into $X_{m+1} \times \cdots \times X_{m+n}$. Hence $(\mu+B)^{-n}$ maps E into $X_n \times \cdots \times X_n \subset E'$. Noting that $(\mu+B)^{-n}$ commutes with $S(t)$, we obtain the following lemma.

Lemma 7.1. *For $x \in E$, if $S(t)x=0$ holds for some $t>0$, then we have $x=0$.*

We return to a proof of the proposition. Using (7.1) and the equation $D_t \mu + B\mu = 0$, we can prove that for each $\tau \geq T$, $D_t(S(\tau-t)u(t))=0$ holds in $\mathcal{D}'(0, T; E)$. This implies that $S(\tau-t)u(t)=0, 0 < t < T$, since $\lim_{t \rightarrow 0} u(t)=0$ in E . By Lemma 7.1 we conclude that $u(t)=0, 0 < t < T$.

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FACULTY OF ENGINEERING,
OSAKA UNIVERSITY

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