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A coupling of infinite particle systems II

By

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This paper follows an earlier work by the second author [M]. Its purpose is to point out some simple consequences of the first work, both general and specific.

In the first section we adopt a general setting for interacting particle systems (I.P.S.), we suppose that the state space for our system is $D^{\mathbf{Z}}$ where D is a finite set. We suppose that the process is of finite range and that all flip rates are uniformly bounded over configurations. The following was shown in [M]:

Theorem 0.1. *Let ξ_t be an I.P.S. satisfying the above conditions. Then, any weakly covergent subsequence of ξ_t has an invariant measure as its limit.*

Theorem 0.2. *Suppose all the hypothesis of Theorem* 0.1. *hold. Then, $\forall f \in C(D^{\mathbf{Z}})$, T, $\delta > 0$, $\exists t_0$ s.t. for any initial condition ξ*

$$|E^{\xi}(f(\xi_t)) - E^{\xi}(f(\xi_{t+T}))| < \delta \quad \forall t \geq t_0 .$$

Although Theorem 0.2 is not explicitely stated in [M], it follows from the arguments given there. More precisely from the following observation: Lemma 1.2 and Proposition 2.1 in that paper say that certain probabilities converge to 1, but the proofs given there show in fact that both these convergences are uniform in the initial configuration.

A simple consequence is that to show a measure ν is an invariant distribution it is only necessary to show that $P_t^* \nu = \nu$ for a single strictly positive t, rather than for all t, as the definition of invariant distribution a priori requires. In Section 1 we show two more results, which are to our best knowledge, new. In the sequel we denote by S the set of translation invariant probability measures on $D^{\mathbf{Z}}$ and by I the set of stationary measures for the process. The extreme points of a set A of probabilities measures will be called A_e.

Theorem 1.1. *Suppose the above hypotheses are true and in addition that our process is translation invariant. Then,*

a) *Any element of $(I \cap S)_e$ is ergodic with respect to translations on $\mathbf{Z}$ and*

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b) *any element of* $(I)_e \cap S$ *is mixing with respect to translations on* $\mathbf{Z}$.

This Theorem partially answers some questions raised in [A] where a counter example to this result when processes on $T^{\mathbf{Z}}$ are allowed is given.

It is well known and follows from the general theory (see Theorem 3.3.2 in [AG]) that if a measure μ is extremal invariant, then under P^{μ} $\{\xi_t\ 0 \leq t < \infty\}$ is ergodic with respect to shifts in time. This does not imply that our process is mixing. We show

Theorem 1.2. *Under the above hypothesis, if* μ *is an extremal invariant distribution for* (ξ_t) *then* $E^{\mu} f(\xi_0) g(\xi_t) - \langle \mu, f \rangle \langle \mu, g \rangle \to 0$ *as* $t \to \infty$ $\forall f, g \in L^2$.

The second section is given over to the treatment of 1-dimensional voter model like processes.

It is prompted by a conjecture by [D] and by results of [ALM]. [D] considered spin systems ξ_t where the flip rate $C(x, \xi) = f(D_x)$ for a function f with $f(0) = 0$, where $D_x = I_{\xi(x) \neq \xi(x-1)} + I_{\xi(x) \neq \xi(x+1)}$ is the number of nearest neighbours in disagreement. For a translation invariant initial distribution it was shown that $\forall x \neq y$ $P(\xi_t(x) \neq \xi_t(y)) \to 0$ as $t \to \infty$. It was conjectured that for any initial ξ_0, and $x \neq y$

$$P^{\xi_0}(\xi_t(x) \neq \xi_t(y)) \to 0 \quad \text{as} \quad t \to \infty.$$

If $f(2) \geq f(1)$, then the process is attractive and arguments given in [ALM] can easily be adapted to show this result.

In fact we are able to solve a considerable generalization of the conjecture:

Theorem 2.1. *Let the spin system* ξ_t *have flip function* $C(x, \xi)$ *which is*

(i) *Translation invariant.*

(ii) $C(x, \xi) = 0$ *iff* $D_x = 0$.

(iii) *Is finite range.*

Then $\forall x \neq y$, ξ_0 $P^{\xi_0}(\xi_t(x) \neq \xi_t(y)) \to 0$ *as* $t \to \infty$.

It should be noted that unlike the voter model, the flip rate may make distinctions between 1's and 0' and between right and left neighbors. Given that the process is finite range, there exists a constant $\delta > 0$ so that $C(x, \xi) \geq \delta$ if $D_x \geq 1$.

We can also generalise the conjecture in another direction. Say a process ξ_t is voter model like if

(i)' The flip function $C(x, \xi)$ has a finite range m and is translation invariant.

(ii)' a) $C(x, \xi) = 0$ *iff* $\Sigma_{0 < |x-y| \leq m} I_{\xi(x) = \xi(y)} > m$.

b) there exists $c > 0$ such that $C(x, \xi) = c$ if either

$$\xi(x+1) = \xi(x+2) = \cdots = \xi(x+m) = 1 - \xi(x) \text{ or}$$

$$\xi(x-1) = \xi(x-2) = \cdots = \xi(x-m) = 1 - \xi(x)$$

Theorem 2.2. *Under hypotheses* (i)' *and* (ii)' *for any configuration* ξ_0 *and any* $x \neq y$ $P^{\xi_0}(\xi_t(x) \neq \xi_t(y)) \to 0$ *as* $t \to \infty$.

This was proven in [ALM] for the threshold voter model, that is when

$$C(x, \xi) = 1 \quad \text{if} \quad \sum_{0<|x-y|\leq m} I_{\xi(x)=\xi(y)} \leq m$$
$$= 0 \quad \text{otherwise.}$$

Section 1. Under the usual discrete product topology $D^{\mathbf{Z}}$ is a compact metric space. Therefore $P(D^{\mathbf{Z}})$ under the weak topology is also a compact metric space, whose distance we denote by d, and I is a closed (therefore compact) subset of $P(D^{\mathbf{Z}})$.

Lemma 1.1. $\forall \delta > 0$ $\exists n < \infty$, $f_i \in C(D^{\mathbf{Z}})$, $\varepsilon_i, > 0$, t_i, $i = 1, \cdots, n$ *so that for any* $\mu \in P(D^{\mathbf{Z}})$

$$|\langle \mu, f_i \rangle - \langle \mu, P_{t_i} f_i \rangle| \leq \varepsilon_i \quad \forall i = 1, \cdots, n.$$
$$\Rightarrow d(\mu, I) < \delta.$$

Proof. The set of probability measures which are at least distance at least δ from I is a closed and therefore compact subset of $P(D^{\mathbf{Z}})$. If μ satisfies $d(\mu, I) \geq \delta$, then by definition of I, $\exists t$, $f, \varepsilon > 0$, so that

$$|\langle \mu, f \rangle - \langle \mu, P_t f \rangle| > \varepsilon.$$

By compactness we can find a finite collection of open sets

$$U_i = \{\mu \,|\, |\langle \mu, f_i \rangle - \langle \mu, P_{t_i} f_i \rangle| > \varepsilon_i\} \text{ s.t.}$$
$$\{\mu : d(\mu, I) \geq \delta\} \subseteq \cup U_i$$

Lemma 1.2. *If* μ *is an extremal invariant distribution for* ξ_t *and* $g \in L^2(\mu)$, *then* $P_t g \to \langle \mu, g \rangle$ *in* μ *probability as* $t \to \infty$.

Proof. If not then there exists $\varepsilon > 0$ and $t_n \uparrow \infty$ so that

$$(*) \qquad |P_{t_n} g(\xi) - \langle \mu, g \rangle| > \varepsilon \quad \text{with } \mu \text{ probability at least } \varepsilon.$$

But $\mu = \int \mu(d\xi) P_{t_n}^* \delta_\xi$ by definition of invariance, and by Theorem 0.1 $d(P_{t_n}^* \delta_\xi, I)$ tends to zero. If we define a measure λ_{t_n} on $P(D^{\mathbf{Z}})$ by $\lambda_{t_n}(A) = \mu(\{\xi : P_{t_n}^* \delta_\xi \in A\})$, then any limit probability measure of λ_{t_n} is a probability measure on I. Since $\forall n$, $\mu = \int \lambda_{t_n}(dv) v$, λ_{t_n} must tend to δ_μ, by extremality of μ, but this violates $(*)$.

Proof of Theorem 1.1. a) Let μ be in $(I \cap S)_e$. As μ is translation invariant it can be written as a mixture of ergodic translation invariant measures

$$\mu = \int \gamma d\lambda(\gamma).$$

Now $P_t^*\mu = \mu = \int P_t^*\gamma d\lambda(\gamma) \quad \forall t$.
Let λ_t be the image of λ under P_t^*:

$$\lambda_t(A) = \lambda(\gamma : P_t^*(\gamma) \in A)$$

then

$$P_t^*\mu = \mu = \int P_t^*\gamma d\lambda(\gamma) = \int \gamma d\lambda_t(\gamma), \qquad \forall t.$$

By Theorem 0.2 and Lemma 1.1, $\forall \delta > 0$, $\exists t_0$ s.t. $\forall t \geq t_0$, λ_t concentrates on $\{\gamma : d(\gamma, I) \leq \delta\}$. So any limit point of λ_t (as $t \to \infty$) must be concentrated on $I \cap S$. Since μ is extremal on that set, any limit point of λ_t must be δ_μ. Hence, $P_t^*: S \to S$ converges in λ measure, as t goes to infinity, to the point mass at measure μ. Therefore, for some subsequence t_n, $P_{t_n}^*(\gamma)$ converges λ *a.e.* to μ and the result follows from part i) of Theorem 1.4 in [A] (condition 1.5 in that theorem is satisfied for particle systems, as explained in Section 3 of that paper).

b) is an immediate consequence of Lemma 1.2 and part ii) of Theorem 1.4 in [A].

Proof of Theorem 1.2. As $E^\mu f(\xi_0)g(\xi_t) = E^\mu(f(\xi_0)P_t g(\xi_0))$ the desired result follows from Lemma 1.2.

Section 2. In this section D is always equal to $\{0, 1\}$. We first consider spin systems satisfying

(a) $C(x, \xi) = 0 \Leftrightarrow \xi(x-1) = \xi(x) = \xi(x+1)$.

(b) If $C(x, \xi) \neq 0 \Rightarrow C(x, \xi) \geq C_0 > 0$.

(c) $C(x, \xi)$ is translation invariant and bounded.

Note that these systems include all the systems considered by Theorem 2.1. In view of this and of Theorem 0.1, Theorem 2.1 is a consequence of the following proposition:

Proposition 2.1. *A system satisfying the above conditions admits no nontrivial stationary distributions.*

We do this in two stages; first we show there are no nontrivial translation invariant stationary distributions, then we show there are no nontrivial stationary distributions at all.

Lemma 2.2 is essentially that on page 48 of [D] and so is not proven here.

Lemma 2.2. *The above system admits no nontrivial translation invariant* stationary distributions.

Proof of Proposition 2.1. Let γ be an invariant measure and consider

$$\gamma^n = \frac{1}{n}\sum_{i=1}^{n} \theta_i\gamma \qquad \gamma^{-n} = \frac{1}{n}\sum_{i=-n}^{-1} \theta_i\gamma.$$

Since any weak limit of these sequences is stationary and translation invariant, it follows from Lemma 2.2 that for all $\epsilon > 0$ there exists $m = m(\epsilon)$ such that:

$$\gamma\{(\xi(m) \neq \xi(m+1))\} < \epsilon$$

and

$$\gamma\{(\xi(-m) \neq \xi(-m-1))\} < \epsilon$$

Let $f_m(\xi) = \Sigma_{x=-m}^{m} I_{\xi(x)\neq\xi(x+1)}$ and $g(\xi) = I_{\xi(-1)\neq\xi(0)\neq\xi(1)}$ and denote by L the generator of the process, then

$$\int Lf_m(\xi)d\gamma = 0$$

Since $f_m(\xi)$ can only increase when either the $-m$ coordinate flips and $\xi(-m) \neq \xi(-m-1)$ or the m coordinate flips and $\xi(m) \neq \xi(m+1)$ we get that the integral of the positive terms of $Lf_m(\xi)$ with respect to γ is bounded above by $2\epsilon K$ where K is an upper bound for $C(x, \xi)$. But the negative terms of $Lf_m(\xi)$ are in absolute value bounded below by $2g(\xi)C_0$, therefore

$$\int 2g(\xi)d\gamma \leq 2\epsilon K,$$

since ϵ is arbitrary, we have

$$\int g(\xi)d\gamma = 0.$$

Similarly $\int I_{\xi(k-1)\neq\xi(k)\neq\xi(k+1)}d\gamma = 0$ and a simple inductive argument shows that $\int I_{\xi(k)\neq\xi(m)\neq\xi(n)}d\gamma = 0$ for any $k < n < m$. Therefore γ concentrates on configurations which are either constant or change only once. However if $\xi(x)$ changes its value only once, the site at which this change takes places evolves as a random walk on Z, therefore any weak limit of ξ_t must be a linear combination of δ_0 and δ_1, hence γ must also be a linear combination of δ_0 and δ_1.

Next we consider a different generalization of non-linear voter models: we call a spin system ξ_t on $\{0, 1\}^{\mathbf{Z}}$ a voter model like process if it satisfies conditions (i)' and (ii)'. Let $c_0 = \inf c(0, \xi)$ where the infimum is over all ξ s.t. $\Sigma_{0<|i|\leq m} I_{\xi(i)=\xi(0)} \leq m$, let $C_0 = \sup C(0, \xi)$ where the supremum is over all ξ, and let c be as in part b) of (ii)'.

Properties (i)' and (ii)' imply that $c_0>0$ and that C_0 is finite.

Lemma 2.5. *Let* $\sigma=\inf\{t:\xi_t$ *is not constant on* $[-m,m]\}$, *then there exists a constant* $k>0$ *so that* $\forall\xi_0$ *constant on* $[-m,m]$ *we have*

$$P^{\xi_0}(\sigma>t)>\frac{k}{t} \qquad \forall t\geq 1.$$

Proof. Define

$$r_t=\sup\{n:\xi_t(i)=\xi_t(0) \quad \forall \quad 0\leq i\leq n\}$$

$$\text{and} \qquad l_t=\inf\{l:\xi_t(i)=\xi_t(0) \quad \forall -l\leq i\leq -1\}.$$

Then on $[0,\sigma]$, r_t jumps up at least 1 at rate c, and jumps down 1 at rate c if

$$\xi_t(r_t+1)=\xi_t(r_t+2)\cdots\xi_t(r_t+m)=1-\xi_t(r_t)$$

and at rate 0 otherwise. Similarly with l_t.
Thus on $[0,\sigma]$,

$$r_t=S_1(A_t)+V_t$$

$$-l_t=S_2(B_t)+U_t$$

where

$$A_t=\int_0^t I_{\xi_s(r_s)=1-\xi_s(r_s+i)i=1,\cdots,m}ds,$$

$$B_t=\int_0^t I_{\xi_s(l_s)=1-\xi_s(l_s-i)i=1,\cdots,m}ds,$$

U and V are increasing jump processes such that $U_0=V_0=0$, S_1 and S_2 are independant random walks jumping up or down 1 at rate c. So

$$\begin{aligned}P(\sigma>t)&\geq P\left(\inf_{s\leq t}S_1(s)\geq S_1(0),\ \inf_{s\leq t}S_2(s)\geq S_2(0)\right)\\&=P\left(\inf_{s\leq t}S_1(s)\geq S_1(0)\right)P\left(\inf_{s\leq t}S_2(s)\geq S_2(0)\right)\\&\geq\frac{k}{t}\quad\text{for some } k \text{ and all } t>1.\end{aligned}$$

Lemma 2.6. *Let* $\tau=\inf\{t>0:\xi_t(i)=\xi_t(0)\ \forall i\in[-m,m]\}$. *Then there exists a r.v.* X *with some exponential moments so that* $\forall t$, ξ_0, $P^{\xi_0}(\tau>t)\leq P(X>t)$.

Proof. It follows from the lemma in Section 2 of [ALM] that $\exists$ integer $k(m)$ so that $\forall \xi_0$, $\exists \xi_1, \xi_2, \cdots, \xi_r$, $r \leq k(m)$ with the following properties

a) $\xi_i = \xi_{i+1}$ except for one site in $[-m, m]$
b) ξ_r is constant on $[-m, m]$
c) ξ_i jumps to ξ_{i+1} at a strictly positive rate.

From this it is clear that if Z is a Markov chain on $\{0, k(m)\}$ with jump rates

$$q_{x,x+1} = c_0, \; q_{x,0} = 4mC \qquad x = \{0, 1, 2, \cdots\}$$

and if $\rho = \inf\{t : Z_t = k(m)\}$. Then $\forall \xi_0$, $P^{\xi_0}(\tau > t) \leq P^0(\rho > t)$.

Lemma 2.7. *If* $X_1, X_2, \cdots$ *are iid r.v. with distribution equal to that of r.v.* X *in Lemma* 2.6 *and if* Y_1, Y_2 *are iid, positive r.v. with* $P(Y_i > t) \geq \frac{k}{t}$ *for* $t \geq 1$ *and some* $k > 0$.

Then

$$\frac{\Sigma_{i=1}^{n} X_i}{\Sigma_{i=1}^{n-1} Y_i} \to 0. \qquad a.s.$$

Proof. Simply the S.L.L.N.

Proof of Theorem 2.2. Define the sequence of stopping times σ_i, τ_i by

(1) $\tau_1 = \inf\{t : \xi_t$ is constant on $[-m, m]\}$.
(2) $\sigma_i = \inf\{t > \tau_i : \xi_t$ is not constant on $[-m, m]\}$ for $i \geq 1$
(3) $\tau_{i+1} = \inf\{t > \sigma_i : \xi_t$ is constant on $[-m, m]\}$ for $i \geq 1$.

Then by Lemma 2.6. $\tau_1, \tau_2 - \sigma_1, \tau_3 - \sigma_2 \cdots$ is stochastically dominated by a sequence $X_1, X_2 \cdots$ of iid r.v. of distribution X described in the proof of Lemma 2.6. By Lemma 2.5 $(\sigma_i - \tau_i)$ stochastically dominates sequence of Y_1, Y_2 of iid r.v. satisfying $P(Y_i > t) \geq \frac{k}{t}, t > 1$ (it should be understood that if for some i, $\sigma_i - \tau_i = \infty$ then the r.v. $\sigma_j - \tau_j$, $\tau_j - \sigma_{j-1}$, $j > i$ are taken to be iid r.v. of appropriate distributions independent of ξ_t).

By Lemma 2.7 we have (uniformly in ξ_0)

$$\frac{1}{t}\int_0^t P^{\xi_0}(\xi_s \text{ is not constant on } [-m, m])ds \to 0.$$

Thus for any measure μ on $\{0, 1\}^{\mathbf{Z}}$

$$\frac{1}{t}\int_0^t P^{\mu}(\xi_s \text{ is not constant on } [-m, m])ds \to 0.$$

Hence the only invariant distribution for our process μ satisfies

$$\mu(\{\xi : \xi \text{ is nonconsant on } [-m, m]\}) = 0.$$

This clearly implies that any invariant measure is trivial and therefore, by Theorem one of [M], that $\forall \xi_0$, $x<y$

$$P^{\xi_0}(\xi_t(x) \neq \xi_t(y)) \to 0.$$

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References

[A] E. Andjel, Ergodic and mixing properties of equilibrium measures for Markov processes, Trans. Amer. Math. Soc., **318** (1990), 601–614.

[ALM] E. Andjel, T. Liggett and T. Mountford, Clustering in one-dimensional threshold voter models, Stoch. Proc. and their Applications, **42** (1992), 73–90.

[A-G] R. Ash and M. Gardner, Topics in Stochastic Processes, Academic Press, 1975.

[D] R. Durrett, Ten lectures on particle systems, St. Flour Lectures, Springer, 1995.

[M] T. S. Mountford, A coupling of infinite particle systems, Journal of Mathematics of Kyoto University, **35**–1 (1995), 43–52.