J. Math. Kyoto Univ. (JMKYAZ)
41-1 (2001), 1–31

Paradifferential Calculus in Gevrey Classes

By

Chen Hua and Luigi Rodino

Abstract

We present a paradifferential calculus adapted to the study of nonlinear partial differential equations in Gevrey classes. We give an application concerning Gevrey microregularity of the solutions of fully nonlinear equations at elliptic points.

1. Introduction

Bony [4], 1981, presented a version of the pseudo-differential calculus, the so-called paradifferential calculus, adapted to the study of the nonlinear equations

$$F[u] = F(x, u, \cdots, \partial^\alpha u, \cdots)_{|\alpha| \leq m} = 0. \tag{1.1}$$

The basic idea was to write

$$F[u] = T_{F'(u)} u + r, \tag{1.2}$$

where $T_{F'(u)}$ is the paradifferential operator having as symbol the linearization F' of F at u, and r is a smooth error. Precisely, if u is assumed of Sobolev class H^{s+m}, $s > n/2$, then $r \in H^{2s-n/2}$. Through (1.2) one is reduced to the study of the paradifferential equation $T_{F'(u)} u \in H^{2s-n/2}$ and obtains linear-type results of existence, regularity and propagation.

Assume now F is analytic in the respective variables, and let u be of Gevrey class G^σ, i.e. locally

$$|\partial^\alpha u(x)| \leq C^{|\alpha|+1} (\alpha!)^\sigma. \tag{1.3}$$

Naively, one could try to reproduce for the Gevrey scale G^σ, $1 < \sigma < \infty$, the results of Bony for the scale H^s, $n/2 < s < \infty$. It is easily seen, however, that the error r in (1.2) turns out to be of class G^σ, with apparent no gain of regularity. This corresponds to the known impossibility of obtaining linear-type results of propagation for (1.1) in the analytic-Gevrey category, but for very

Communicated by Prof. K. Ueno, January 26, 1999
Revised October 3, 2000

special equations, cf. Alinhac and Métivier [1], Godin [8], Chen and Rodino [5], [6] and Sasaki [17].

Here we propose a different approach, developing some preliminary results of Chen and Rodino [5]. Namely, we refer to the Sobolev-Gevrey spaces $H^s_{\tau,\sigma}$, $1 < \sigma < \infty$, $\tau > 0$, $s \in \mathbb{R}$, defined by

$$\| u \|_{H^s_{\tau,\sigma}} = \| \exp[\tau \langle D\rangle^{1/\sigma}] u \|_{H^s} < \infty. \tag{1.4}$$

Spaces of this type have been already studied, for example, in Kajitani and Nishitani [13], Kajitani and Wakabayashi [14] and Taniguchi [18].

Locally we have $G^\sigma = \cup_{\tau,s} H^s_{\tau,\sigma}$. For fixed σ and τ, the scale $H^s_{\tau,\sigma}$, $n/2 < s < \infty$, is indeed appropriate for the paradifferential calculus, the error r in (1.2) belonging to $H^{2s-n/2}_{\tau,\sigma}$, if $u \in H^s_{\tau,\sigma}$.

As an application, we may extend to the fully nonlinear case the result of micro-elliptic Gevrey regularity proved in [5] for the semilinear equations. Namely, from $u \in H^s_{\tau,\sigma}$ we deduce $u \in H^{2s-\lambda}_{\tau,\sigma}$, for some fixed constant λ, microlocally in (1.1) at any elliptic point. Though very weak, this result is nearly the best possible; in fact, starting from a solution $u \in G^\sigma$, there is no hope in general of reaching $G^{\sigma'}$ regularity, where $1 \leq \sigma' < \sigma$, as in the linear case, see the counter-examples in [5] and [6].

Other possible applications in the nonlinear setting, which we leave to the future, concern Gevrey propagation, cf. Bony [4] and Hörmander [11] for the H^s category, existence of Gevrey Riemannian embeddings, cf. Hörmander [10] and Chen and Rodino [7], existence of Gevrey solutions for equations with multiple characteristics, cf. Gramchev and Rodino [9], and weakly hyperbolic Cauchy problems, cf. Mizohata [15] and Kajitani [12].

Finally, we observe that part of the arguments in the following may keep valid for other weighted Sobolev spaces; essentially, one can replace the operator $\exp[\tau\langle D\rangle^{1/\sigma}]$ in (1.4) with a more general operator $\Phi(D)$, provided $\Phi(\xi+\eta) \leq C\Phi(\xi)\Phi(\eta)$.

The paper is organized as follows: in Section 2, we recall the results of [5] about Littlewood-Paley decomposition and $H^s_{\tau,\sigma}$ spaces; in Section 3, we treat some classes of Gevrey pseudo-differential operators; in Sections 4 and 5, we study paraproducts and paradifferential operators, respectively, in the $H^s_{\tau,\sigma}$ frame; in Section 6, we present the above-mentioned application to micro-ellipticity.

2. Gevrey-Sobolev spaces and non-linear operations

Let $\sigma > 1$, $\tau,\ s \in \mathbb{R}$; we introduce Gevrey-Sobolev spaces as follows:

$$H^s_{\tau,\sigma}(\mathbb{R}^n) = \{u \in \mathcal{S}'_{-\tau,\sigma}(\mathbb{R}^n),\ \exp[\tau\langle D\rangle^{1/\sigma}]u \in H^s(\mathbb{R}^n)\},$$

where $\langle D\rangle = (1-\Delta)^{1/2}$, the space $\mathcal{S}'_{\tau,\sigma}$ is defined as the dual of $\mathcal{S}_{\tau,\sigma}$, which in turn for $\tau \geq 0$ is defined by inverse Fourier transform from

$$\widehat{\mathcal{S}}_{\tau,\sigma} = \{v(\xi) \in C^\infty(\mathbb{R}^n) \mid \exp[\tau\langle\xi\rangle^{1/\sigma}]v(\xi) \in \mathcal{S}(\mathbb{R}^n)\};$$

for $\tau < 0$, the space $\mathcal{S}_{\tau,\sigma}$ is defined by transposition of inverse Fourier transform from $\widehat{\mathcal{S}}_{\tau,\sigma}$ (cf. [14]). We shall also write $H^s_{\tau,\sigma}$ for short. The infinite order pseudo-differential operator $\exp[\tau\langle D\rangle^{1/\sigma}]$ is defined by Fourier transform as usual; see for example Rodino [16].

We define the norms in $H^s_{\tau,\sigma}$ by

$$\| u \|_{H^s_{\tau,\sigma}} = \| \exp[\tau\langle D\rangle^{1/\sigma}]u\|_{H^s}.$$

$H^s_{\tau,\sigma}$ is a Hilbert space with inner product

$$\langle u, v\rangle_{H^s_{\tau,\sigma}} = \langle \exp[\tau\langle D\rangle^{1/\sigma}]u, \exp[\tau\langle D\rangle^{1/\sigma}]v\rangle_{H^s}.$$

Taking $K > 1$ a constant, for $p \in \mathbf{Z}_+$, we denote:

$$C_p = \{\xi \in \mathbb{R}^n,\ K^{-1}2^p \le |\xi| \le K2^{p+1}\},$$
$$C_{-1} = B(0, K) = \{\xi \in \mathbb{R}^n,\ |\xi| \le K\}.$$

Thus $\{C_p\}_{p=-1}^{\infty}$ is a circular cover of $\mathbb{R}^n_\xi$. From Bony [4], we have the following result:

Lemma 2.1. *There exists $N_1 \in \mathbf{N}$, depending only on K, such that for any C_p the number of q, such that $C_q \cap C_p \neq \emptyset$, is at most N_1.*

We have the following dyadic partition of unity, see again Bony [4]:

Lemma 2.2. *There exist φ, $\psi \in C_0^\infty(\mathbb{R}^n)$,* $\operatorname{supp}\psi \subset C_{-1}$, $\operatorname{supp}\varphi \subset C_0$, *such that for any $\xi \in \mathbb{R}^n$, and any $l \in \mathbf{N}$, we have*

$$\psi(\xi) + \sum_{p=0}^{\infty} \varphi(2^{-p}\xi) = 1, \tag{2.1}$$

$$\psi(\xi) + \sum_{p=0}^{l-1} \varphi(2^{-p}\xi) = \psi(2^{-l}\xi). \tag{2.2}$$

The Littlewood-Paley decomposition (or say dyadic decomposition) $\{u_p\}_{p=-1}^{\infty}$ for a function $u \in H^s_{\tau,\sigma}$ will be defined as follows:

$$u_{-1}(x) = \psi(D)u(x),\ u_p(x) = \varphi(2^{-p}D)u(x) \ \text{ for } \ p \ge 0. \tag{2.3}$$

It is easy to prove that the series $u = \sum_{p=-1}^{\infty} u_p$ is convergent in the $\mathcal{S}'_{-\tau,\sigma}$ topology. In fact we can use the dyadic decomposition to characterize the Gevrey-Sobolev spaces.

Theorem 2.1. *Let $s > 0$, $\sigma > 1$ and $\tau \in \mathbb{R}$; then the following conditions are equivalent*:

(a) $u \in H^s_{\tau,\sigma}(\mathbb{R}^n)$;

(b) $u = \sum_{p=-1}^{\infty} u_p$, *where* $u_p \in C^\infty$ *and* $\operatorname{supp}\hat{u}_p \subset C_p$, *satisfying* $\|u_p\|_{L^2_{\tau,\sigma}} \le c_p 2^{-ps}$ *with* $\{c_p\} \in \ell^2$, $L^2_{\tau,\sigma} = H^0_{\tau,\sigma}$;

(c) $u = \sum_{p=-1}^{\infty} u_p$, *where* $u_p \in C^\infty$ *and* $\operatorname{supp} \hat{u}_p \subset B(0, K_1 2^p)$ *for some* $K_1 > 0$, *satisfying* $\|u_p\|_{L^2_{\tau,\sigma}} \le c_p 2^{-ps}$, $\{c_p\} \in \ell^2$;

(d) $u = \sum_{p=-1}^{\infty} u_p$, *where* $u_p \in C^\infty$ *and for any* $\alpha \in \mathbf{Z}^n_+$, *we have* $\|D^\alpha u_p\|_{L^2_{\tau,\sigma}} \le c_{p\alpha} 2^{-ps+p|\alpha|}$ *and* $\{c_{p\alpha}\}_p \in \ell^2$.

It is obvious that the proof of Theorem 2.1, for $\tau > 0$, may be deduced directly from the proof of [5, Theorem 1.1], similarly we can prove the case for $\tau \le 0$ as well.

Remark 2.1. In (d) it is actually sufficient to argue for $|\alpha| \le s + 1$.

Remark 2.2. The norm $\|u\|_{H^s_{\tau,\sigma}}$ can be estimated in (b) and (c) by $\|(c_p)\|_{\ell^2}$ and in (d) by $\sum_{|\alpha| \le s+1} \|(c_{p,\alpha})_p\|_{\ell^2}$. The equivalence between (a) and (b) keeps valid for $s \in \mathbb{R}$.

Remark 2.3. Let us recall, see for example Rodino [16], that every $\varphi \in G^\sigma_0(\mathbb{R}^n) = G^\sigma(\mathbb{R}^n) \cap C^\infty_0(\mathbb{R}^n)$, $\sigma > 1$, satisfies for suitable positive constants C and ε:

$$|\hat{\varphi}(\xi)| \le C \exp[-\varepsilon |\xi|^{1/\sigma}]. \tag{2.4}$$

It follows that $G^{\sigma'}_0(\mathbb{R}^n) \subset H^s_{\tau,\sigma}(\mathbb{R}^n)$ for any $\sigma' < \sigma$, $s \in \mathbb{R}$ and $\tau > 0$, with strict inclusion; moreover if $\varphi \in G^\sigma_0(\mathbb{R}^n)$, then $\varphi \in H^s_{\tau,\sigma}(\mathbb{R}^n)$ for all s, if $\tau > 0$ is sufficiently small. In the opposite direction, it is easy to see that $H^s_{\tau,\sigma}(\mathbb{R}^n) \subset G^\sigma(\mathbb{R}^n)$ for all $\tau > 0$, $s \in \mathbb{R}$; cf. the proof of Theorem 1.6.1, (iii) in [16].

Observe if $u \in H^s_{\tau,\sigma}$ and $\varphi \in G^{\sigma'}_0(\mathbb{R}^n)$, $1 < \sigma' < \sigma$, then $\varphi u \in H^s_{\tau,\sigma}$. Thus we can define Gevrey locally Sobolev spaces as follows (cf. [5]):

Definition 2.1. We set $H^s_{\tau,\sigma,loc}$ to be the space of all Gevrey ultra-distributions $u \in \mathcal{D}'_{\sigma'}(\mathbb{R}^n)$ such that for every $\varphi \in G^{\sigma'}_0(\mathbb{R}^n)$ with $1 < \sigma' < \sigma$, we have $\varphi u \in H^s_{\tau,\sigma}$.

We also define, for some open subset $\Omega \subset \mathbb{R}^n$, that

Definition 2.2. We say $u \in H^s_{\tau,\sigma,loc}(\Omega)$, if for all $\varphi \in G^{\sigma'}_0(\Omega)$, $1 < \sigma' < \sigma$, we have $\varphi u \in H^s_{\tau,\sigma}$. We say $u \in H^s_{\tau,\sigma}(x_0)$ for $x_0 \in \mathbb{R}^n$ if there exists a neighborhood V_{x_0} of x_0, such that $u \in H^s_{\tau,\sigma,loc}(V_{x_0})$.

Observe that $\cup_{s \in \mathbb{R}, \tau > 0} H^s_{\tau,\sigma}(x_0) = G^\sigma(x_0)$, the space of all the functions u which are of class G^σ in a neighborhood of x_0; moreover $G^{\sigma'}(x_0) \subset H^s_{\tau,\sigma}(x_0)$ with strict inclusion for all $s \in \mathbb{R}$, $\tau > 0$ and $1 < \sigma' < \sigma$.

Let $(x_0, \xi^0) \in T^*\mathbb{R}^n \backslash \{0\}$; we introduce Gevrey microlocally (i.e. near (x_0, ξ^0)) Sobolev spaces as follows:

Definition 2.3. We write $u \in H^s_{\tau,\sigma}(x_0, \xi^0)$ if there exists V_{x_0} and a conic neighborhood Γ_0 of ξ^0 in $\mathbb{R}^n \backslash \{0\}$, such that for all $\varphi \in G^{\sigma'}_0(V_{x_0})$, $1 < \sigma' < \sigma$, and every $\psi \in C^\infty(\mathbb{R}^n_\xi)$, 0-order homogeneous in ξ for large $|\xi|$ with $\operatorname{supp} \psi \subset \Gamma^0$, we have $\psi(D)(\varphi u) \in H^s_{\tau,\sigma}$.

Next $H^{+\infty}_{\tau,\sigma}$ and $H^{-\infty}_{\tau,\sigma}$ will be defined by $\cap_s H^s_{\tau,\sigma}$ and $\cup_s H^s_{\tau,\sigma}$ respectively. Similarly we can define $H^{+\infty}_{\tau,\sigma,loc}(\Omega)$ and $H^{-\infty}_{\tau,\sigma,loc}(\Omega)$.

To be definite, we recall here the notion of Gevrey wave front set and related remarks used in the sequel.

Definition 2.4. For $u \in \mathcal{D}'_\sigma(\mathbb{R}^n)$ we write $(x_0, \xi^0) \notin WF_\sigma u$ if there exist $\varphi \in G^\sigma_0(\mathbb{R}^n)$, with $\varphi = 1$ in a neighborhood of x_0, and a conic neighborhood Γ_0 of ξ^0 such that for positive constants C, ε:

$$|(\varphi u)\hat{}(\xi)| \le C \exp[-\varepsilon |\xi|^{1/\sigma}], \quad \xi \in \Gamma_0. \tag{2.5}$$

Remark 2.4. Equivalently (cf. Rodino [16, Lemma 1.7.3]), we may say that $(x_0, \xi^0) \notin WF_\sigma u$ if there exist V_{x_0} and Γ_0 such that for all $\varphi \in G^\sigma_0(V_{x_0})$ and every $\psi \in C^\infty(\mathbb{R}^n_\xi)$, 0-order homogeneous in ξ for large $|\xi|$ with $\operatorname{supp}\psi \subset \Gamma_0$, we have for some C, $\varepsilon > 0$:

$$|(\psi(\xi)(\varphi u))\hat{}(\xi)| \le C \exp[-\varepsilon |\xi|^{1/\sigma}]. \tag{2.6}$$

It is then clear that $(x_0, \xi^0) \notin WF_{\sigma'} u$ for $1 < \sigma' < \sigma$ implies $u \in H^s_{\tau,\sigma}(x_0, \xi^0)$ for all $\tau > 0$, $s \in \mathbb{R}$.

We have the following Hausdorff-Young inequality:

Theorem 2.2. *Let $u \in L^1$, $v \in L^2_{\tau,\sigma}$, $\tau \in \mathbb{R}$, $\sigma > 1$, then*

$$\|u * v\|_{L^2_{\tau,\sigma}} \le \|u\|_{L^1} \cdot \|v\|_{L^2_{\tau,\sigma}}.$$

Moreover we can easily extend the results in [5, Theorems 2.1 through 2.3] to the case of $\tau \le 0$, i.e. we have the following results:

Theorem 2.3. *We have $u \in H^s_{\tau,\sigma}(x_0, \xi^0)$ $(\tau,\ s \in \mathbb{R},\ \sigma > 1)$ if and only if there exist V_{x_0} and a decomposition*

$$u = u_1 + u_2, \quad \text{for } x \in V_{x_0},$$

where $u_1 \in H^s_{\tau,\sigma}(\mathbb{R}^n)$ and $(x_0, \xi^0) \notin WF_{\sigma'}(u_2)$ for $1 < \sigma' < \sigma$.

Theorem 2.4. *Let $s' > s$, $\tau \in \mathbb{R}$ and $\sigma > 1$, the following two conditions are equivalent*:

(a) $u \in H^s_{\tau,\sigma}(x_0) \cap H^{s'}_{\tau,\sigma}(x_0, \xi^0)$.

(b) *There exists $\varphi_1 \in G^{\sigma'}_0(\mathbb{R}^n)$, $1 < \sigma' < \sigma$, with $\varphi_1 = 1$ in a neighborhood V_{x_0} of x_0, and there exists a conic neighborhood Γ_0 of ξ^0 such that $\varphi_1 u = u_{-1} + \sum_{p=0}^\infty (u'_p + u''_p)$, where $u_{-1} \in G^{\sigma'}(\mathbb{R}^n)$, $\|u'_p\|_{L^2_{\tau,\sigma}} \le c'_p 2^{-ps}$, $\operatorname{supp}\hat{u}'_p \subset C_p \cap \Gamma^C_0$ (Γ^C_0 is complement of Γ_0), $\|u''_p\|_{L^2_{\tau,\sigma}} \le c''_p 2^{-ps'}$, $\operatorname{supp}\hat{u}''_p \subset C_p$, $\{c'_p\}$, $\{c''_p\} \in \ell^2$.*

Theorem 2.5. *Let $u \in L^2_{\tau,\sigma}$ (or $u \in L^2_{|\tau|,\sigma}$), $\tau \in \mathbb{R}$, $\sigma > 1$, and $v \in H^s_{|\tau|,\sigma}$ (or $v \in H^s_{\tau,\sigma}$), with $s > n/2$. Then $uv \in L^2_{\tau,\sigma}$ and for some $C > 0$:*

$$\|uv\|_{L^2_{\tau,\sigma}} \le C\|v\|_{H^s_{|\tau|,\sigma}} \, \|u\|_{L^2_{\tau,\sigma}} \quad (\textit{or} \quad \|uv\|_{L^2_{\tau,\sigma}} \le C\|v\|_{H^s_{\tau,\sigma}} \, \|u\|_{L^2_{|\tau|,\sigma}}).$$

It is important for us to consider when a function space would become an algebra, which in particular is useful to study nonlinear partial differential equations. Here for Gevrey-Sobolev spaces we have (similar to [5]):

Theorem 2.6. *Let $s > n/2$ and $\tau \geq 0$, then*

(a) *$H^s_{\tau,\sigma}$ is an algebra, and there exists $C > 0$ such that for all u, $v \in H^s_{\tau,\sigma}$, $s > n/2$:*

$$\|uv\|_{H^s_{\tau,\sigma}} \leq C\|u\|_{H^s_{\tau,\sigma}} \ \|v\|_{H^s_{\tau,\sigma}}.$$

(b) *Furthermore, if $s' < 2s - n/2$, then $H^s_{\tau,\sigma}(x_0) \cap H^{s'}_{\tau,\sigma}(x_0, \xi^0)$ is an algebra.*

3. Some classes of Gevrey pseudo-differential operators

We first define some classes of Gevrey symbols.

Definition 3.1. Let $m \in \mathbb{R}$, $\tau \geq 0$, $\sigma > 1$ and $\varepsilon > 0$. We denote by $S^{m,\varepsilon}_{\tau,\sigma}$, the class of all the symbols $p(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\|D^\beta_\xi p(\cdot, \xi)\|_{H^{n/2+\varepsilon}_{\tau,\sigma}} \leq c_\beta \langle\xi\rangle^{m-|\beta|}, \tag{3.1}$$

for constants c_β independent of $\xi \in \mathbb{R}^n$.

Denoting by $\hat{p}(\eta, \xi)$ the partial Fourier transform of $p(x, \xi)$ with respect to the first variables, we can re-write (3.1) as

$$\begin{aligned} &\| \exp[\tau\langle D_x\rangle^{1/\sigma}]D^\beta_\xi p(x, \xi)\|_{H^{n/2+\varepsilon}(\mathbb{R}^n_x)} \\ &\quad = \| \exp[\tau\langle\eta\rangle^{1/\sigma}]\langle\eta\rangle^{n/2+\varepsilon} D^\beta_\xi \hat{p}(\eta, \xi)\|_{L^2(\mathbb{R}^n_\eta)} \leq c_\beta\langle\xi\rangle^{m-|\beta|}. \end{aligned} \tag{3.2}$$

In our setting the classes $S^{m,\varepsilon}_{\tau,\sigma}$ play the role of symbols with "limited smoothness"; they will contain the subclasses $l^{m,\varepsilon}_{\tau,\sigma}$ and $\Sigma^{m,\varepsilon}_{\tau,\sigma}$ of the Gevrey paradifferential symbols, see the next sections for precise definitions. Here let us fix attention on the corresponding class of "smooth" symbols.

Definition 3.2. Let $m \in \mathbb{R}$, $\tau \geq 0$, $\sigma > 1$. We denote by $S^m_{\tau,\sigma}$ the class given by $\cap_{\varepsilon>0} S^{m,\varepsilon}_{\tau,\sigma}$; i.e. $p(x, \xi) \in S^m_{\tau,\sigma}$ if (3.1), or (3.2), is valid for every $\varepsilon > 0$.

Equivalent definitions for $S^m_{\tau,\sigma}$ are obtained by imposing for every α, β:

$$\|D^\alpha_x D^\beta_\xi p(\cdot, \xi)\|_{H^s_{\tau,\sigma}} \leq c_{\alpha\beta}\langle\xi\rangle^{m-|\beta|}, \tag{3.3}$$

for some fixed $s \in \mathbb{R}$; in particular for $s = 0$

$$\|D^\alpha_x D^\beta_\xi p(\cdot, \xi)\|_{L^2_{\tau,\sigma}} \leq c_{\alpha\beta}\langle\xi\rangle^{m-|\beta|}, \tag{3.4}$$

with suitable constants $c_{\alpha\beta}$. Let us write $S^{-\infty}_{\tau,\sigma} = \cap_m S^m_{\tau,\sigma}$.

Example 3.1. Let $\phi \in G^{\sigma'}_0(\mathbb{R}^n)$, $1 < \sigma' < \sigma$. Then $\phi(x)$ belongs to $H^s_{\tau,\sigma}$ for every $\tau > 0$, $s \in \mathbb{R}$, and it can be regarded as symbol in $S^0_{\tau,\sigma}$.

Example 3.2. Let $a(\xi) \in C^\infty(\mathbb{R}^n)$ satisfy the estimates $|D_\xi^\beta a(\xi)| \leq c_\beta \langle\xi\rangle^{m-|\beta|}$. Then $\phi(x)a(\xi)$, with $\phi(x)$ as in Example 3.1, belongs to $S^m_{\tau,\sigma}$.

If $p \in S^m_{\tau,\sigma}$, $q \in S^{m'}_{\tau,\sigma}$, then $pq \in S^{m+m'}_{\tau,\sigma}$, as we have from (3.1), the Leibniz rule and the algebra property of $H^{n/2+\varepsilon}_{\tau,\sigma}$. If $p \in S^m_{\tau,\sigma}$, then $D_x^\alpha D_\xi^\beta p \in S^{m-|\beta|}_{\tau,\sigma}$.

Let $p_j \in S^{m_j}_{\tau,\sigma}$, $j = 1, 2, \cdots$, $m_j \to -\infty$ with $m_{j+1} \leq m_j$ for all j, and let $p \in S^{m_1}_{\tau,\sigma}$; we write $p \sim \sum_{j=1}^\infty p_j$ if for all integers $N \geq 2$ we have

$$p - \sum_{1 \leq j < N} p_j \in S^{m_N}_{\tau,\sigma}.$$

Given an asymptotic sum $\sum_{j=1}^\infty p_j$ as before, we can actually construct as standard $p \in S^{m_1}_{\tau,\sigma}$ with $p \sim \sum_{j=1}^\infty p_j$.

We then define symbols corresponding to the classical smooth case.

Definition 3.3. Let $m \in \mathbb{R}$, $\tau \geq 0$, $\sigma > 1$. We denote by $S^m_{\tau,\sigma,cl}$ the subclass of all $p(x,\xi) \in S^m_{\tau,\sigma}$ such that $p(x,\xi) \sim \sum_{j=0}^\infty p_{m-j}(x,\xi)$ where $p_{m-j}(x,\xi) \in S^{m-j}_{\tau,\sigma}$ is positively homogeneous with respect to ξ of degree $m-j$, for large $|\xi|$.

For later reference we also introduce more general classes of ρ,δ-type.

Definition 3.4. Let $m \in \mathbb{R}$, $\tau \geq 0$, $\sigma > 1$, $0 \leq \rho \leq 1$, $0 \leq \delta \leq 1$. We denote by $S^m_{\tau,\sigma,\rho,\delta}$ the class of all $p(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying for every α, β

$$\|D_x^\alpha D_\xi^\beta p(\cdot,\xi)\|_{L^2_{\tau,\sigma}} \leq c_{\alpha\beta} \langle\xi\rangle^{m-\rho|\beta|+\delta|\alpha|}. \tag{3.5}$$

In particular we have $S^m_{\tau,\sigma,1,0} = S^m_{\tau,\sigma}$.

Consider now pseudo-differential operators

$$Pu(x) = p(x,D)u(x) = (2\pi)^{-n} \int e^{ix\xi} p(x,\xi)\hat{u}(\xi)d\xi,$$

with symbol in the preceding classes. For simplicity we shall limit ourselves to the main properties of the operators with symbols in $S^m_{\tau,\sigma}$; variants and generalizations to $S^m_{\tau,\sigma,\rho,\delta}$ are left to the reader. Proofs will follow closely the calculus for pseudo-differential operators with limited Sobolev smoothness, see for example Beals [2], Beals and Reed [3] and Taylor [19]. In particular the following lemma, taken from Beals and Reed [3], will be very useful in our context.

Lemma 3.1. *Suppose that*

$$C^2 = \sup_{\xi \in \mathbb{R}^n} \int |g(\lambda,\xi)|^2 d\lambda < \infty \quad \textit{and} \quad K^2 = \sup_{\eta \in \mathbb{R}^n} \int |G(\xi,\eta)|^2 d\xi < \infty.$$

For $h \in L^2$ *define*

$$Ah(\eta) = \int G(\xi, \eta) g(\eta - \xi, \xi) h(\xi) d\xi.$$

Then $\|Ah\|_{L^2} \le CK \|h\|_{L^2}$.

The proof of Lemma 3.1 is elementary, by writing $\|Ah\|_{L^2} = \sup_{\|f\| \le 1} |\int f(\eta) Ah(\eta) d\eta|$, interchanging integrals and using Schwarz inequality.

Theorem 3.1. *If* $p(x, \xi) \in S^m_{\tau,\sigma}(m \in \mathbb{R},\ \tau \ge 0,\ \sigma > 1)$, *then* $P : H^s_{\tau,\sigma} \to H^{s-m}_{\tau,\sigma}$, $P : H^s_{-\tau,\sigma} \to H^{s-m}_{-\tau,\sigma}$ *continuously for every* $s \in \mathbb{R}$.

It follows $P : H^{+\infty}_{\tau,\sigma} \to H^{+\infty}_{\tau,\sigma}$, $P : H^{-\infty}_{\tau,\sigma} \to H^{-\infty}_{\tau,\sigma}$. *In particular we have that if* $p(x, \xi) \sim 0$, *i.e.* $p(x, \xi) \in S^{-\infty}_{\tau,\sigma}$, *then* P *is regularizing, in the sense that* $P : H^{-\infty}_{\tau,\sigma} \to H^{+\infty}_{\tau,\sigma}$.

Proof. Let us prove $P : H^s_{\tau,\sigma} \to H^{s-m}_{\tau,\sigma}$ continuously. We first write

$$\hat{Pu}(\eta) = (2\pi)^{-n} \int \hat{p}(\eta - \xi, \xi) \hat{u}(\xi) d\xi,$$

where as before $\hat{p}$ is the partial Fourier transform of p with respect to the x-variables. We then have to estimate the L^2-norm of

$$(3.6)\ \exp[\tau \langle \eta \rangle^{1/\sigma}] \langle \eta \rangle^{s-m} (\hat{Pu})(\eta)$$
$$= (2\pi)^{-n} \int H(\xi, \eta) \langle \eta \rangle^{s-m} \langle \xi \rangle^{-s} \exp[\tau \langle \eta - \xi \rangle^{1/\sigma}] \hat{p}(\eta - \xi, \xi) \hat{v}(\xi) d\xi,$$

where

$$v = \exp[\tau \langle D \rangle^{1/\sigma}] \langle D \rangle^s u,$$

so that $\|u\|_{H^s_{\tau,\sigma}} = \|v\|_{L^2}$, and we have set

$$H(\xi, \eta) = \exp[\tau \langle \eta \rangle^{1/\sigma} - \tau \langle \xi \rangle^{1/\sigma} - \tau \langle \eta - \xi \rangle^{1/\sigma}].$$

Note that $H(\xi, \eta) \le 1$. We apply Lemma 3.1 by taking there

$$g(\lambda, \xi) = \exp[\tau \langle \lambda \rangle^{1/\sigma}] \langle \lambda \rangle^N \hat{p}(\lambda, \xi) \langle \xi \rangle^{-m},$$

which satisfies for every fixed N

$$C^2 = \sup_{\xi \in \mathbb{R}^n} \int |g(\lambda, \xi)|^2 d\lambda < \infty,$$

in view of Definition 3.2. We set also

$$G(\xi, \eta) = H(\xi, \eta) \langle \eta \rangle^{s-m} \langle \xi \rangle^{m-s} \langle \eta - \xi \rangle^{-N},$$

for which

$$K^2 = \sup_{\eta \in \mathbb{R}^n} \int |G(\xi, \eta)|^2 d\xi < \infty$$

if N has been chosen sufficiently large. Therefore by Lemma 3.1, the L^2-norm of (3.6) is estimated by $CK \|v\|_{L^2}$, and this gives the conclusion. Similarly we prove $P : H^s_{-\tau,\sigma} \to H^{s-m}_{-\tau,\sigma}$. $\square$

Remark 3.1. Concerning symbols with limited Gevrey-Sobolev smoothness, $p(x,\xi) \in S^{m,\varepsilon}_{\tau,\sigma}$, the corresponding operators act with continuity from $H^s_{\tau,\sigma}$ to $H^{s'}_{\tau,\sigma}$, $s' \le s-m$, provided s is sufficiently large and s' sufficiently small, depending on ε and m.

Theorem 3.2. (1) *Let $p(x,\xi) \in S^m_{\tau,\sigma}$ and consider the corresponding pseudo-differential operator P. Define the L^2-adjoint P^* by*

$$\langle P^* u, v\rangle_{L^2} = \langle u, Pv\rangle_{L^2}, \quad u \in H^{s+m}_{\tau,\sigma}, \quad v \in H^{-s}_{-\tau,\sigma}.$$

Then P^* is a pseudo-differential operator with symbol $p^*(x,\xi) \in S^m_{\tau,\sigma}$, having asymptotic expansion

$$p^*(x,\xi) \sim \sum_\alpha (\alpha!)^{-1} \partial_\xi^\alpha D_x^\alpha \bar{p}(x,\xi). \tag{3.7}$$

(2) *Let $p_j \in S^{m_j}_{\tau,\sigma}$, $j = 1,2$, and consider the corresponding pseudo-differential operators P_j, $j = 1,2$. Then P_1P_2 is a pseudo-differential operator with symbol $p(x,\xi) \in S^{m_1+m_2}_{\tau,\sigma}$, having asymptotic expansion*

$$p \sim p_1 \# p_2 = \sum_\alpha (\alpha!)^{-1} \partial_\xi^\alpha p_1(x,\xi) D_x^\alpha p_2(x,\xi). \tag{3.8}$$

Proof. We prove (2) and leave (1) to the reader. By standard computations we may express the symbol of P_1P_2 in the form

$$\begin{aligned} p(x,\zeta) &= (2\pi)^{-n} \int \exp[-i(y-x)(\xi-\zeta)] p_1(x,\xi) p_2(y,\zeta)\, dy d\xi \\ &= (2\pi)^{-n} \int \exp[ix\xi] p_1(x,\zeta+\xi) \hat{p}_2(\xi,\zeta)\, d\xi, \end{aligned} \tag{3.9}$$

where $\hat{p}_2$ is the Fourier transform of p_2 with respect to the x-variables.

Let us first show that $p(x,\zeta) \in S^{m_1+m_2}_{\tau,\sigma}$. To this end, we compute $\hat{p}(\eta,\zeta)$, Fourier transform of $p(x,\zeta)$ with respect to the x-variables, and obtain

$$\hat{p}(\eta,\zeta) = (2\pi)^{-n} \int \hat{p}_1(\eta-\xi, \zeta+\xi) \hat{p}_2(\xi,\zeta)\, d\xi.$$

We have to estimate for any $s \in \mathbb{R}$ the L^2-norm with respect to η of

$$\langle\zeta\rangle^{-m_1-m_2+|\beta|} \exp[\langle\eta\rangle^{1/\sigma}] \langle\eta\rangle^s D_\zeta^\beta \hat{p}(\eta,\zeta),$$

uniformly in the parameter ζ. Let us limit ourselves to treat the case $\beta = 0$, the generalization to arbitrary β being trivial by Leibniz rule. We have

$$\begin{aligned} &\langle\zeta\rangle^{-m_1-m_2} \exp[\tau\langle\eta\rangle^{1/\sigma}] \langle\eta\rangle^s \hat{p}(\eta,\zeta) \\ &\quad = (2\pi)^{-n} \int H(\xi,\eta) \langle\eta\rangle^s \langle\zeta\rangle^{-m_1-m_2} \widetilde{p}_1(\eta-\xi, \zeta+\xi) \widetilde{p}_2(\xi,\zeta)\, d\xi, \end{aligned} \tag{3.10}$$

where $H(\xi,\eta)=\exp[\tau\langle\eta\rangle^{1/\sigma}-\tau\langle\xi\rangle^{1/\sigma}-\tau\langle\eta-\xi\rangle^{1/\sigma}]\le 1$ and

$$\widetilde{p}_1(\eta-\xi,\zeta+\xi)=\exp[\tau\langle\eta-\xi\rangle^{1/\sigma}]\hat{p}_1(\eta-\xi,\zeta+\xi),$$
$$\widetilde{p}_2(\xi,\zeta)=\exp[\tau\langle\xi\rangle^{1/\sigma}]\hat{p}_2(\xi,\zeta).$$

We shall apply again Lemma 3.1, all the terms there depending on the parameter ζ with uniformly bounded norms. Namely, we set

$$h(\xi,\zeta)=\widetilde{p}_2(\xi,\zeta)\langle\zeta\rangle^{-m_2}\langle\xi\rangle^{L}$$

with L^2-norm with respect to ξ bounded uniformly with respect to ζ, for any L; moreover

$$g(\lambda,\xi,\zeta)=\widetilde{p}_1(\lambda,\zeta+\xi)\langle\zeta+\xi\rangle^{-m_1}\lambda^{M}$$

with L^2-norm with respect to λ bounded uniformly with respect to ζ and ξ, for any M. Finally, we take

$$G(\xi,\eta,\zeta)=H(\xi,\eta)\langle\eta\rangle^{s}\langle\zeta\rangle^{-m_1}\langle\xi\rangle^{-L}\langle\zeta+\xi\rangle^{m_1}\langle\eta-\xi\rangle^{-M},$$

for which

$$\sup_{\eta,\zeta}\int|G(\xi,\eta,\zeta)|^2d\xi<\infty$$

if L and M have been chosen sufficiently large. From Lemma 3.1 we therefore deduce that the L^2-norm with respect to η of (3.10) is bounded, uniformly with respect to ζ. We pass now to prove the asymptotic formula in (2). As standard in the pseudo-differential calculus, after Taylor expanding $p_1(x,\zeta+\xi)$ in (3.9) with respect to ξ, we are reduced to consider the remainder

$$r_N(x,\zeta)=\sum_{|\gamma|=N}\frac{N}{\gamma!}\int_0^1 r_{N\gamma}(x,\zeta,t)(1-t)^{N-1}dt,$$

where

$$r_{N\gamma}(x,\zeta,t)=(2\pi)^{-n}\int\exp[ix\xi]\partial_\xi^\gamma p_1(x,\zeta+t\xi)\xi^\gamma\hat{p}_2(\xi,\zeta)d\xi.$$

We have to prove that $r_{N\gamma}\in S^{m_1+m_2-N}_{\tau,\sigma}$, $N=|\gamma|$, with uniform bounds with respect to the parameter t, $0\le t\le 1$.

Arguing as before, we are led to consider

$$\hat{r}_{N\gamma}(\eta,\zeta,t)=(2\pi)^{-n}\int\partial_\xi^\gamma\hat{p}_1(\eta-\xi,\zeta+t\xi)\xi^\gamma\hat{p}_2(\xi,\zeta)d\xi.$$

Repeating the preceding arguments, and in particular applying Lemma 3.1 with ζ and t as parameters, we get easily the conclusion. □

Corollary 3.1. *If $u\in H^{-\infty}_{\tau,\sigma}$, $u\in H^s_{\tau,\sigma}(x_0)$ and P has symbol in $S^m_{\tau,\sigma}$, then $Pu\in H^{s-m}_{\tau,\sigma}(x_0)$.*

Proof. Take $\phi \in G_0^{\sigma'}(\mathbb{R}^n),\ 1 < \sigma' < \sigma$, with $\phi(x) = 1$ in a neighborhood V of x_0, such that $\phi u \in H^s_{\tau,\sigma}$. Take then any $\phi' \in G_0^{\sigma'}(\mathbb{R}^n)$ with $\operatorname{supp}\phi' \subset V$. Writing

$$\phi' P u = \phi' P \phi u + \phi' P(1-\phi)u,$$

applying Theorem 3.1 to the first term in the right hand side and (3.8) in Theorem 3.2 to the second term, we get $\phi' P u \in H^{s-m}_{\tau,\sigma}$, hence $Pu \in H^{s-m}_{\tau,\sigma}(x_0)$. □

Corollary 3.2. *If $u \in H^{-\infty}_{\tau,\sigma}$, $u \in H^s_{\tau,\sigma}(x_0,\xi^0)$ and P has symbol in $S^m_{\tau,\sigma}$, then $Pu \in H^{s-m}_{\tau,\sigma}(x_0,\xi^0)$.*

Proof. Let $\psi(\xi),\ \psi'(\xi) \in C^\infty(\mathbb{R}^n)$ be 0-order homogeneous for large $|\xi|$ with $\psi(\xi) = 1$ in a conic neighborhood Γ of ξ^0 and $\operatorname{supp}\ \psi' \subset \Gamma$. Let $\phi,\ \phi'$ be as in the preceding proof, such that $\psi(D)(\phi u) \in H^s_{\tau,\sigma}$. The conclusion is easily obtained by writing

$$\psi'(D)(\phi' P u) = \psi'(D)\phi' P(\psi(D)\phi u) + \psi'(D)\phi' P(1-\psi(D)\phi)u$$

and applying (3.8) to the second term in the right hand side. □

Given Ω open subset of $\mathbb{R}^n$, the class of symbols $S^m_{\tau,\sigma}(\Omega)$ is the set of all $p(x,\xi) \in C^\infty(\Omega \times \mathbb{R}^n)$ such that $\phi(x)p(x,\xi) \in S^m_{\tau,\sigma}$ for every $\phi \in G_0^{\sigma'}(\Omega),\ 1 < \sigma' < \sigma$. Similarly we define $S^{m,\varepsilon}_{\tau,\sigma}(\Omega)$, $S^m_{\tau,\sigma,cl}(\Omega)$ and $S^m_{\tau,\sigma,\rho,\delta}(\Omega)$. The preceding Theorems 3.1 and 3.2 have obvious variants for the corresponding pseudo-differential operators.

Let $p(x,\xi) \sim \sum_{j=0}^\infty p_{m-j}(x,\xi)$ in $S^m_{\tau,\sigma,cl}(\Omega)$ be elliptic, i.e. for every $K \subset\subset \Omega$ we have

$$|p_m(x,\xi)| \geq c_K|\xi|^m, \quad x \in K, \quad \xi \in \mathbb{R}^n,$$

for a suitable positive constant c_K. Then $q_{-m}(x,\xi) = (p_m(x,\xi))^{-1} \in S^{-m}_{\tau,\sigma}(\Omega)$ for large ξ and we may recursively construct as standard $q(x,\xi) \sim \sum_{j=0}^\infty q_{-m-j}(x,\xi)$ in $S^{-m}_{\tau,\sigma,cl}(\Omega)$, such that $q\#p = 1,\ p\#q = 1$. From (2) in Theorem 3.2 we therefore obtain:

Theorem 3.3. *Let $p(x,\xi) \in S^m_{\tau,\sigma,cl}(\Omega)$ be elliptic in Ω. Assume $P = p(x,D)$ is properly supported, i.e. it is well defined as a map $P : H^{+\infty}_{\tau,\sigma,loc}(\Omega) \to H^{+\infty}_{\tau,\sigma,loc}(\Omega)$, $H^{-\infty}_{\tau,\sigma,loc}(\Omega) \to H^{-\infty}_{\tau,\sigma,loc}(\Omega)$, preserving compactness of supports. Then for P there exists a properly supported parametrix $Q = q(x,D)$, $q(x,\xi) \in S^{-m}_{\tau,\sigma,cl}(\Omega)$; namely $QP = I + R_1,\ PQ = I + R_2$, where R_1 and R_2 have symbols in $S^{-\infty}_{\tau,\sigma}(\Omega)$.*

Corollary 3.3. *Let $p(x,\xi) \in S^m_{\tau,\sigma,cl}$ be elliptic in a neighborhood of x_0. Then $u \in H^{-\infty}_{\tau,\sigma},\ Pu \in H^s_{\tau,\sigma}(x_0)$ imply $u \in H^{s+m}_{\tau,\sigma}(x_0)$.*

The proof is by Theorem 3.3, Corollary 3.1 and Theorem 3.1.

Using Corollary 3.2 and constructing microlocal parametrices, we deduce similarly the following micro-regularity result.

Corollary 3.4. *Let $p(x,\xi) \in S^m_{\tau,\sigma,cl}$ satisfy $p_m(x_0,\xi^0) \neq 0$ for some $x_0 \in \Omega$, $\xi^0 \neq 0$. Then $u \in H^{-\infty}_{\tau,\sigma}$, $Pu \in H^s_{\tau,\sigma}(x_0,\xi^0)$ imply $u \in H^{s+m}_{\tau,\sigma}(x_0,\xi^0)$.*

Comments 3.1. In the classes presented here we require only C^∞ regularity with respect to ξ. The corresponding symbols are comparable, for example, with those in Taniguchi [18] defined by the estimates

$$|D^\alpha_x D^\beta_\xi p(x,\xi)| \leq c_\beta M^{-|\alpha|}\alpha!^\sigma \langle\xi\rangle^{m-|\beta|}. \tag{3.11}$$

Namely, if (3.11) is satisfied we have $p(x,\xi) \in S^m_{\tau,\sigma}$ for a small $\tau > 0$.

We point out that, with respect to the standard calculus requiring also Gevrey estimates in ξ, cf. Rodino [16] and the references there, our present regularizing operators R are such only in the $H^{-\infty}_{\tau,\sigma}$ frame. More precisely, if $R = r(x,D)$ with $r \in S^{-\infty}_{\tau,\sigma}$, we have $R : H^{-\infty}_{\tau,\sigma} \to H^\infty_{\tau,\sigma}$, but for $f \in \mathcal{E}'(\mathbb{R}^n)$ or even $f \in C^\infty_0(\mathbb{R}^n)$, in general $Rf \in C^\infty$ is not of Gevrey class, neither the possible Gevrey local regularities and micro-regularities of f are preserved under applications of R or $P = p(x,D)$ with symbol $p \in S^m_{\tau,\sigma}$.

4. Gevrey paraproduct calculus

Let $a \in H^{n/2+\varepsilon}_{|\tau|,\sigma}$, $\varepsilon > 0$, $\tau \in \mathbb{R}$, $\sigma > 1$. We can define the paraproduct operator T_a as follows:

$$T_a u = \sum_q (S_q a)u_q, \qquad u \in H^s_{\tau,\sigma}, \tag{4.1}$$

where $\{u_q\}^\infty_{q=-1}$ denotes the dyadic decomposition of u, $S_q a = \sum_{-1\leq p\leq q-N_1} a_p$, $\{a_p\}$ the dyadic decomposition of a. Let N_1 be sufficiently large, cf. [4],[5], then we have

Theorem 4.1. *$T_a : H^s_{\tau,\sigma} \to H^s_{\tau,\sigma}$ is a continuous mapping for every $s \in \mathbb{R}$. Moreover, $u \in H^{-\infty}_{\tau,\sigma}$ and $u \in H^s_{\tau,\sigma}(x_0)$ imply $T_a u \in H^s_{\tau,\sigma}(x_0)$ for any $s \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. We have $\|T_a\|_{\mathcal{L}(H^s_{\tau,\sigma},H^s_{\tau,\sigma})} \leq C_s\|a\|_{H^{n/2+\varepsilon}_{|\tau|,\sigma}}$. Fix further $\xi^0 \neq 0$. If $u \in H^s_{\tau,\sigma}(x_0)$ with $s > 0$, then $u \in H^t_{\tau,\sigma}(x_0,\xi^0)$ implies $T_a u \in H^t_{\tau,\sigma}(x_0,\xi^0)$ for $s < t < s+\varepsilon$.*

Proof. The same statement was already proved in [5, Theorem 3.1]. We think however it is worth to give in the following a precise argument for the pseudo-local property, i.e. $u \in H^{-\infty}_{\tau,\sigma}$ and $u \in H^s_{\tau,\sigma}(x_0)$ imply $T_a u \in H^s_{\tau,\sigma}(x_0)$ for every $s \in \mathbb{R}$, since details in this connection are missing in [5]. Our present proof will be based on the pseudo-differential calculus of the preceding Section 3.

Let us assume for simplicity $\tau > 0$. We may take $\phi \in G^{\sigma'}_0(\mathbb{R}^n)$, $1 < \sigma' < \sigma$, with $\phi(x) = 1$ in a neighborhood V_{x_0} of x_0, such that $\phi u \in H^s_{\tau,\sigma}$. Let us then show that $\phi_1 T_a u \in H^s_{\tau,\sigma}$ for every $\phi_1 \in G^{\sigma'}_0(V_{x_0})$. In fact

$$\phi_1 T_a u = \phi_1 T_a \phi u + \phi_1 T_a(1-\phi)u$$

where $\phi_1 T_a \phi u \in H^s_{\tau,\sigma}$, granted the boundedness of T_a on $H^s_{\tau,\sigma}$.

Let us prove that $\phi_1 T_a(1-\phi)u \in H^h_{\tau,\sigma}$ for all $h \in \mathbb{R}$. Basing on (4.1), we write

$$\phi_1 T_a(1-\phi)u = \phi_1 \sum_{p\geq 0}(S_p a)\varphi(2^{-p}D)(1-\phi)u + f, \tag{4.2}$$

where $f \in H^{+\infty}_{\tau,\sigma}$ and φ is defined as in Lemma 2.2. Since $\phi_1\varphi(2^{-p}\xi)$ and $\phi(x)$ are symbols in $S^0_{\tau,\sigma}$, we may apply Theorem 3.2 and write, with N to be fixed later:

$$\phi_1\varphi(2^{-p}D)(1-\phi) = \phi_1 \sum_{|\alpha|<N} (\alpha!)^{-1} D^\alpha_x(1-\phi)2^{-p|\alpha|}(\partial^\alpha_\xi\varphi)(2^{-p}D) + \phi_1 r_{pN}(x,D)$$

where $\phi_1 r_{p,N}(x,\xi) \in S^{-N}_{\tau,\sigma}$. Inserting in (4.2) and observing that $\phi_1 D^\alpha_x(1-\phi) \equiv 0$, we are reduced to study the boundedness of the operator

$$R_N = \sum_{p\geq 0}(S_p a) r_{pN}(x,D).$$

Namely, we shall prove that $u \in H^{h'}_{\tau,\sigma}$ implies $R_N u \in H^h_{\tau,\sigma}$ for every $h, h' \in \mathbb{R}$. To this end, assuming without loss of generality $h > 0$ and applying (d) in Theorem 2.1, we may limit ourselves to check that

$$\|D^\alpha(S_p a) r_{pN}(x,D)u\|_{L^2_{\tau,\sigma}} \leq c_{p\alpha} 2^{-ph+p|\alpha|}, \qquad \{c_{p\alpha}\}_p \in l^2. \tag{4.3}$$

By Leibniz formula and Theorem 2.5, we are further reduced to prove the same estimates for the terms

$$\|D^{\alpha_1}(S_p a)\|_{H^{n/2+\varepsilon'}_{\tau,\sigma}} \|D^{\alpha_2} r_{pN}(x,D)u\|_{L^2_{\tau,\sigma}}, \qquad \alpha_1+\alpha_2=\alpha, \tag{4.4}$$

with $0 < \varepsilon' < \varepsilon$. Now from the definition of S_p we have easily

$$\|D^{\alpha_1}(S_p a)\|_{H^{n/2+\varepsilon'}_{\tau,\sigma}} \leq c'_{p,\alpha_1} 2^{p|\alpha_1|} \|a\|_{H^{n/2+\varepsilon}_{\tau,\sigma}}, \qquad \{c'_{p\alpha_1}\}_p \in l^2. \tag{4.5}$$

It will be then convenient to write the explicit expression of $r_{pN\alpha_2}(x,\xi)$, the symbol of $D^{\alpha_2} r_{pN}(x,D)$. Namely, according to the last part of the proof of Theorem 3.2:

$$r_{pN\alpha_2}(x,\xi) = \sum_{|\gamma|=N} \frac{N}{\gamma!} \int_0^1 r_{pN\alpha_2\gamma}(x,\xi,t)(1-t)^{N-1}dt, \tag{4.6}$$

where $r_{pN\alpha_2\gamma}$ is a linear combination of terms of the form

$$e(x,\xi,t) = (2\pi)^{-n} \int e^{ix\zeta} 2^{-pN}(\partial^\gamma_\xi\varphi)(2^{-p}(\xi+t\zeta))\xi^{\beta_1}\zeta^{\gamma+\beta_2}\hat{\phi}(\zeta)d\zeta. \tag{4.7}$$

with $\beta_1+\beta_2=\alpha_2$.

We want to estimate the norm of the corresponding operator as a map from $H^{h'}_{\tau,\sigma}$ to $L^2_{\tau,\sigma}$. Going back to Lemma 3.1 and to the proof of Theorem 3.1, we have then to consider

$$\hat{e}(\lambda,\xi,t) = 2^{-pN}(\partial_\xi^\gamma \varphi)(2^{-p}(\xi+t\lambda))\xi^{\beta_1}\lambda^{\gamma+\beta_2}\hat{\phi}(\lambda)$$

and evaluate the L^2-norm with respect to λ of

$$g(\lambda,\xi,t) = e^{\tau\lambda^{1/\sigma}}\langle\lambda\rangle^M \hat{e}(\lambda,\xi,t)\langle\xi\rangle^{-h'},$$

where M is determined as in the proof of Theorem 3.1, depending on h'. Assuming without loss of generality $h'<0$, we have

$$\begin{aligned}|(\partial_\xi^\gamma\varphi)(2^{-p}(\xi+t\lambda))| &\le c_\gamma\langle 2^{-p}(\xi+t\lambda)\rangle^{h'-|\alpha_2|}\\ &\le c_\gamma 2^{-ph'+p|\alpha_2|}\langle\xi+t\lambda\rangle^{h'-|\alpha_2|}\\ &\le c'_\gamma 2^{-ph'+p|\alpha_2|}\langle\xi\rangle^{h'-|\alpha_2|}\langle t\lambda\rangle^{-h'+|\alpha_2|}\end{aligned}$$

and moreover for some $\delta>0$

$$|\hat{\phi}(\lambda)| \le ce^{-\delta\lambda^{1/\sigma'}},$$

so we obtain

$$\sup_{0\le t\le 1}\sup_{\xi\in\mathbb{R}^n}\|g(\lambda,\xi,t)\|_{L^2(\mathbb{R}^n_\lambda)} \le c2^{-pN-ph'+p|\alpha_2|}$$

for a constant c independent of p. In view of (4.6), (4.7) and Lemma 3.1, we deduce that

$$\|D^{\alpha_2}r_{pN}(x,D)u\|_{L^2_{\tau,\sigma}} \le c'_{\alpha_2}2^{-pN-ph'+p|\alpha_2|}\|u\|_{H^{h'}_{\tau,\sigma}}$$

and therefore from (4.4) and (4.5)

$$\|D^\alpha((S_pa)r_{pN}(x,D)u)\|_{L^2_{\tau,\sigma}} \le c''_{p\alpha}2^{-pN-ph'+p|\alpha|}\|u\|_{H^{h'}_{\tau,\sigma}}$$

where $\{c''_{p\alpha}\}_p\in l^2$. To obtain (4.3) it will be then sufficient to fix $N>h-h'$.

This concludes the proof of the pseudo-local property. For the other statements in Theorem 4.1 we refer to the proof in [5]. □

Remark 4.1. If $a\in H^{n/2+\varepsilon}_{\tau,\sigma}$ $(\tau<0)$, then in the the same way we can define the paraproduct operator T_a, which is a continuous mapping from $H^s_{|\tau|,\sigma}$ to $H^s_{\tau,\sigma}$.

Remark 4.2. Observe in Theorem 4.1 that $u\in H^{-\infty}_{\tau,\sigma}, u\in H^s_{\tau,\sigma}(x_0)$ imply $T_au\in H^s_{\tau,\sigma}(x_0)$ without any restriction on $s\in\mathbb{R}$, whereas the microlocal statement depends on the local regularity of u. In fact when $\tau=0$ the paraproduct T_a belongs to the Hörmander's class $L^0_{1,1}$, cf. [4], and it is well known that the corresponding pseudo-differential operators are pseudo-local but not micro-local in general.

From (4.1), the definition of T_a seems dependent on the dyadic decomposition of Gevrey-Sobolev space $H^s_{\tau,\sigma}$ (i.e. depending on the choice of $\{K,\varphi,N_1\}$). We suppose there exist two dyadic decompositions which depend on $\{K,\varphi,N_1\}$ and $\{K',\varphi',N_1'\}$ respectively, and denote by T_a and T_a' as two paraproducts corresponding to $\{K,\varphi,N_1\}$ and $\{K',\varphi',N_1'\}$ respectively, then we have

Theorem 4.2. *If $a\in H^{n/2+\varepsilon}_{|\tau|,\sigma}$, then $T_a-T_a'\in\mathcal{L}(H^s_{\tau,\sigma},H^{s+\varepsilon_1}_{\tau,\sigma})$, for any $0<\varepsilon_1<\varepsilon$, and*

$$\|T_a-T_a'\|_{\mathcal{L}(H^s_{\tau,\sigma},H^{s+\varepsilon_1}_{\tau,\sigma})}\le C_s\|a\|_{H^{n/2+\varepsilon}_{|\tau|,\sigma}},\tag{4.8}$$

Proof. Let function a (resp. $u\in H^s_{\tau,\sigma}$) have two decompositions $\sum a_p$ and $\sum a_p'$ (resp. $\sum u_p$ and $\sum v_p$), and $S_qa=\sum_{p\le q-N_1}a_p$, $S_q'a=\sum_{p\le q-N_1'}a_p'$, then

$$\begin{aligned}T_au-T_a'u&=\sum_q\sum_{p\le q-N_1}a_pu_q-\sum_q\sum_{p\le q-N_1}a_pv_q\\&\quad+\sum_q\sum_{p\le q-N_1}a_pv_q-\sum_q(S_q'a)v_q\\&=\sum_p a_p\left[\sum_{q\ge p+N_1}(u_q-v_q)\right]+\sum_q\left[\sum_{p\le q-N_1}a_p-S_q'a\right]v_q\\&=\sum_p a_p\omega_p+\sum_q\tilde{\omega}_q.\end{aligned}\tag{4.9}$$

Without loss of generality, we let $K'>K$, then

$$\operatorname{supp}\hat{\omega}_p\subset C'_{p+N_1},\quad \|\omega_p\|_{L^2_{\tau,\sigma}}\le c_p2^{-ps}.$$

So if we choose N_1 large enough, we have $\operatorname{supp}\{\widehat{a_p\omega_p}\}\subset C''_{p+N_1}$, and

$$\begin{aligned}\|a_p\omega_p\|_{L^2_{\tau,\sigma}}&\le C\|a_p\|_{H^{n/2+\varepsilon'}_{|\tau|,\sigma}}\|\omega_p\|_{L^2_{\tau,\sigma}}\\&\le C2^{p(\varepsilon'-\varepsilon)}\|a_p\|_{H^{n/2+\varepsilon}_{|\tau|,\sigma}}c_p2^{-ps},\end{aligned}$$

where $\varepsilon'\in(0,\varepsilon)$, then

$$\|a_p\omega_p\|_{L^2_{\tau,\sigma}}\le\tilde{c}_p\|a_p\|_{H^{n/2+\varepsilon}_{|\tau|,\sigma}}2^{-p(s+\varepsilon_1)},\tag{4.10}$$

where $\varepsilon_1=\varepsilon-\varepsilon'\in(0,\varepsilon)$, $\{\tilde{c}_p\}\in l^2$.

Next we have, for N_1 large enough, that $\operatorname{supp}\widehat{\tilde{\omega}_q}\subset C_q'$, and for any $\varepsilon'\in(0,\varepsilon)$, we have

$$
\begin{aligned}
(4.11)\ \|\tilde{\omega_q}\|_{L^2_{\tau,\sigma}} &\le \left[\left\| a - \sum_{p\le q-N_1} a_p \right\|_{H^{n/2+\varepsilon'}_{|\tau|,\sigma}} + \|a - S'_q a\|_{H^{n/2+\varepsilon'}_{|\tau|,\sigma}} \right] \|v_q\|_{L^2_{\tau,\sigma}} \\
&\le C\|a\|_{H^{n/2+\varepsilon}_{|\tau|,\sigma}} 2^{-q(\varepsilon-\varepsilon')} c'_q 2^{-qs} \\
&= \tilde{c}'_q \|a\|_{H^{n/2+\varepsilon}_{|\tau|,\sigma}} 2^{-q(s+\varepsilon_1)}, \quad \varepsilon_1 = \varepsilon - \varepsilon' \in (0,\varepsilon),\ \{\tilde{c}'_q\} \in l^2.
\end{aligned}
$$

This implies that $T_a u - T'_a u \in H^{s+\varepsilon_1}_{\tau,\sigma}$, for any $0 < \varepsilon_1 < \varepsilon$, and the estimate (4.8) is obvious from the process above. Theorem 4.2 is proved. □

From Theorem 4.2, we know $T_a \equiv T'_a (\mathrm{mod}\mathcal{L}(H^s_{\tau,\sigma}, H^{s+\varepsilon_1}_{\tau,\sigma}))$. If we denote $\mathcal{L}^{-\varepsilon}_{\tau,\sigma}$ as the Gevrey ε-regular operator class, i.e. $A \in \mathcal{L}^{-\varepsilon}_{\tau,\sigma}$ means $A \in \mathcal{L}(H^s_{\tau,\sigma}, H^{s+\varepsilon}_{\tau,\sigma})$ for any $s \in \mathbb{R}$, then we also have $T_a \equiv T'_a (\mathrm{mod}\mathcal{L}^{-\varepsilon_1}_{\tau,\sigma})$, or $T_a - T'_a \in \mathcal{L}^{-\varepsilon_1}_{\tau,\sigma}$.

We have the following composition result for the paraproduct operators:

Theorem 4.3. *Let* $a,\ b \in H^{\varepsilon+n/2}_{\tau,\sigma}$, $\tau \ge 0$, $\sigma > 1$, $\varepsilon > 0$, *thus (see Theorem* 2.6 *above)* $ab \in H^{\varepsilon+n/2}_{\tau,\sigma}$. *Then for any* $0 < \varepsilon_1 < \varepsilon$, $T_a \circ T_b - T_{ab} \in \mathcal{L}^{-\varepsilon_1}_{\tau,\sigma}$, *and we have* $\|T_a \circ T_b - T_{ab}\|_{\mathcal{L}(H^s_{\tau,\sigma}, H^{s+\varepsilon_1}_{\tau,\sigma})} \le C\|a\|_{H^{n/2+\varepsilon}_{\tau,\sigma}} \|b\|_{H^{n/2+\varepsilon}_{\tau,\sigma}}$.

Proof. Let $u \in H^s_{\tau,\sigma}$, $\{a_p\}$, $\{b_p\}$ and $\{u_p\}$ the L-P decompostions with respect to a, b and u. Then we know from Theorem 4.1 that $v = T_b u \in H^s_{\tau,\sigma}$, and $v = \sum v_q$, $\mathrm{supp}\,\hat{v}_q \subset C'_q$; $S_q a = \sum_{p_2 \le q-N_1} a_{p_2}$. Then $T_a \circ T_b u = T_a v = \sum_q (S_q a) v_q + Rv$, $v_q = \sum_{p_1 \le q-N_1} b_{p_1} u_q$, i.e.

$$
(4.12) \qquad T_a \circ T_b u = \sum_q \sum_{p_1 \le q-N_1} \sum_{p_2 \le q-N_1} a_{p_2} b_{p_1} u_q + R(T_b u).
$$

Since $\mathrm{supp}\,\widehat{S_q a} \subset B(0, C2^q)$, $\mathrm{supp}\,\hat{v}_q \subset C'_q$ and $C_q \subset C'_q$, then it is similar to the proof of (4.9), we have easily

$$
(4.13) \qquad R(T_b u) \in H^{s+\varepsilon_1}_{\tau,\sigma}, \quad 0 < \varepsilon_1 < \varepsilon.
$$

Now we let

$$
(4.14) \qquad d_q = \sum_{p_1 \le q-N_1} \sum_{p_2 \le q-N_1} a_{p_2} b_{p_1}.
$$

Observe $\mathrm{supp}\,\hat{d}_q \subset B(0, C2^q)$, and

$$
ab - d_q = \sum_{p_1 > q-N_1 \ or \ p_2 > q-N_1} a_{p_2} b_{p_1},
$$

we have, by Schauder-Gevrey estimate and assuming as before $p_1 > q - N_1$ or $p_2 > q - N_1$ in the sums, that

$$
\begin{aligned}
(4.15)\ \|(ab - d_q)u_q\|_{L^2_{\tau,\sigma}} &\le \sum \|a_{p_2} b_{p_1}\|_{H^{n/2+\varepsilon'}_{\tau,\sigma}} \|u_q\|_{L^2_{\tau,\sigma}} \\
&\le C \sum \|a_{p_2}\|_{H^{n/2+\varepsilon'}_{\tau,\sigma}} \|b_{p_1}\|_{H^{n/2+\varepsilon'}_{\tau,\sigma}} \|u_q\|_{L^2_{\tau,\sigma'}}, \quad \varepsilon' \in (0, \varepsilon) \\
&\le C\|a\|_{H^{\varepsilon+n/2}_{\tau,\sigma}} \|b\|_{H^{\varepsilon+n/2}_{\tau,\sigma}} \left(\sum 2^{-(p_1+p_2)\varepsilon_1}\right) c_q 2^{-qs}, \quad \varepsilon_1 = \varepsilon - \varepsilon' \\
&\le \tilde{c_q} \|a\|_{H^{n/2+\varepsilon}_{\tau,\sigma}} \|b\|_{H^{n/2+\varepsilon}_{\tau,\sigma}} 2^{-q(s+\varepsilon_1)}.
\end{aligned}
$$

Thus it is similar to the proof of (4.9), we have, for $u \in H^s_{\tau,\sigma}$, that

$$
T_{ab} u - \sum_q d_q u_q \in H^{s+\varepsilon_1}_{\tau,\sigma},
$$

this means that $T_a \circ T_b u - T_{ab} u = R(T_b u) + [\sum_q d_q u_q - T_{ab} u] \in H^{s+\varepsilon_1}_{\tau,\sigma}$.

The result on norm-estimate of composition may be easily checked from the proof process above. Theorem 4.3 is proved. □

With respect to the L^2-scalar product we can define the conjugation operator T_a^* for paraproduct $T_a : H^s_{\tau,\sigma} \to H^s_{\tau,\sigma}$ by

$$
\langle T_a^* u, v\rangle = \langle u, T_a v\rangle, \quad u \in H^{-s}_{-\tau,\sigma}, \quad v \in H^s_{\tau,\sigma}. \tag{4.16}
$$

So $T_a^* : H^{-s}_{-\tau,\sigma} \to H^{-s}_{-\tau,\sigma} = (H^s_{\tau,\sigma})'$ (the dual space of $H^s_{\tau,\sigma}$). More precisely we have

Theorem 4.4. *Let $a \in H^{n/2+\varepsilon}_{|\tau|,\sigma}$ ($\varepsilon > 0$, $\tau \in \mathbb{R}$ and $\sigma > 1$), then T_a^* is also a paraproduct operator and $T_a^* - T_{\bar{a}} \in \mathcal{L}^{-\varepsilon_1}_{\tau,\sigma}$, for any $\varepsilon_1 \in (0, \varepsilon)$, and*

$$
\|T_a^* - T_{\bar{a}}\|_{\mathcal{L}(H^s_{\tau,\sigma}, H^{s+\varepsilon_1}_{\tau,\sigma})} \le C\|a\|_{H^{n/2+\varepsilon}_{|\tau|,\sigma}}.
$$

Proof. Let $u \in H^s_{\tau,\sigma}$, $v \in H^{-(s+\varepsilon_1)}_{-\tau,\sigma}$, then we have

$$
\begin{aligned}
\langle (T_a^* - T_{\bar{a}})u, v\rangle &= \langle T_a^* u, v\rangle - \langle T_{\bar{a}} u, v\rangle \\
&= \langle u, T_a v\rangle - \langle T_{\bar{a}} u, v\rangle,
\end{aligned}
$$

where

$$
\begin{aligned}
\langle u, T_a v\rangle &= \sum_{q,r} \sum_{p \le r - N_1} \int u_q \bar{a}_p \bar{v}_r dx, \\
\langle T_{\bar{a}} u, v\rangle &= \sum_{r,q} \sum_{p \le q - N_1} \int \bar{a}_p u_q \bar{v}_r dx.
\end{aligned}
$$

Observe $\operatorname{supp}(\widehat{\sum_{p \le r - N_1} a_p v_r}) \subset C'_r$, $\operatorname{supp} \hat{u}_q \subset C_q$, and there exists $N_2 > 0$, large enough, such that $C_q \bigcap C'_r = \emptyset$ if $|q - r| > N_2$. Hence if $|q - r| > N_2$, we have

$$
\int u_q(x)(\overline{a_p v_r}) dx = \int \hat{u}_q(-\eta) \widehat{\overline{a_p v_r}}(\eta) d\eta = 0.
$$

This implies

$$\langle T_a^* u, v\rangle = \sum_q \sum_{q-N_2\le r\le q+N_2} \sum_{p\le r-N_1} \int u_q \bar a_p \bar v_r dx,$$

$$\langle T_{\bar a} u, v\rangle = \sum_q \sum_{q-N_2\le r\le q+N_2} \sum_{p\le q-N_1} \int \bar a_p u_q \bar v_r dx.$$

Thus there exists a large integer N_3, such that

$$|\langle T_a^* u, v\rangle - \langle T_{\bar a} u, v\rangle| \le \sum_q \sum_{q-N_2\le r\le q+N_2} \sum_{q-N_3\le p\le q+N_3} \|a_p u_q v_r\|_{L^1}.$$

Let $p = q+j_1, r = q+j_2$, then by Cauchy-Schwarz inequality and Theorem 2.5, we have for $\varepsilon' = \varepsilon - \varepsilon_1 \in (0,\varepsilon)$

$$\begin{aligned}|\langle T_a^* u, v\rangle - \langle T_{\bar a} u, v\rangle| \\ \le \sum_q \sum_{|j_2|\le N_2} \sum_{|j_3|\le N_3} \|a_{q+j_1}\|_{H^{n/2+\varepsilon'}_{|\tau|,\sigma}} \|u_q\|_{L^2_{\tau,\sigma}} \|v_{q+j_2}\|_{L^2_{-\tau,\sigma}}.\end{aligned}$$

Because $N_2,\ N_3$ are finite and fixed, then we can further estimate by

$$C\|a_q\|_{H^{n/2+\varepsilon}_{|\tau|,\sigma}} 2^{-q\varepsilon_1} c_q 2^{-qs} c'_q 2^{q(s+\varepsilon_1)} \le C\|a\|_{H^{n/2+\varepsilon}_{|\tau|,\sigma}} c_q c'_q,$$

where $\varepsilon_1 = \varepsilon-\varepsilon' \in (0,\varepsilon)$, and $\{c_q\},\ \{c'_q\} \in l^2$, $\|\{c_q\}\|_{l^2} \le C\|u\|_{H^s_{\tau,\sigma}}$, $\|\{c'_q\}\|_{l^2} \le C\|v\|_{H^{-(s+\varepsilon_1)}_{-\tau,\sigma}}$. Thus we obtain

$$|\langle T_a^* u, v\rangle - \langle T_{\bar a} u, v\rangle| \le C\|a\|_{H^{n/2+\varepsilon}_{|\tau|,\sigma}} \|u\|_{H^s_{\tau,\sigma}} \|v\|_{H^{-(s+\varepsilon_1)}_{-\tau,\sigma}}. \tag{4.17}$$

Theorem 4.4 is proved. □

From [5, Section 3], we also have the following paralinearization results:

Theorem 4.5. *Let $F : \mathbf{C} \to \mathbf{C}$ be an entire analytic function, and satisfy $F(0) = 0$. Let f be in $H^s_{\tau,\sigma},\ s > n/2,\ \tau > 0,\ \sigma > 1$. Then $F(f) \in H^s_{\tau,\sigma}$ and $F(f) = T_{F'(f)} f + g$, where $g \in H^t_{\tau,\sigma}$ for all $t < 2s - n/2$.*

Theorem 4.5 has the following obvious corollaries:

Corollary 4.1. *Let $F : \mathbf{C} \to \mathbf{C}$ be an entire analytic function, and let f be in $H^s_{\tau,\sigma}(x_0),\ s > n/2,\ \tau > 0,\ \sigma > 1,\ x_0 \in \mathbb{R}^n$. Then $F(f)$, which is well defined in a neighborhood of x_0, belongs to $H^s_{\tau,\sigma}(x_0)$. Fix further $\xi^0 \ne 0$. If $f \in H^t_{\tau,\sigma}(x_0,\xi^0)$ for $s < t < 2s - n/2$, then also $F(f) \in H^t_{\tau,\sigma}(x_0,\xi^0)$.*

Corollary 4.2. *Let $F(x,z) = \sum_\beta c_\beta(x) z^\beta$, entire with respect to z, for $c_\beta \in G^{\sigma'}(\Omega)$ $(1 < \sigma' < \sigma)$, $z \in \mathbf{C}^N$ and $x_0 \in \Omega \subset \mathbb{R}^n$, and let the components of $f = (f_1,\cdots,f_N)$ be in $H^s_{\tau,\sigma}(x_0),\ s > n/2,\ \tau > 0$; then $F(x,f) \in H^s_{\tau,\sigma}(x_0)$. After cutting off F and f by a function $\varphi \in G^{\sigma'}_0(\Omega)$ with $\varphi(x) = 1$ in a neighborhood of x_0, we have $F(x,f) = \sum_{j=1}^N T_{\partial F/\partial z_j(x,f)} f_j + g$, where $g \in H^t_{\tau,\sigma}(x_0)$ for all $t < 2s - n/2$. If all the components of f are in $H^t_{\tau,\sigma}(x_0,\xi^0)$ for $s < t < 2s - n/2,\ \xi^0 \ne 0$, then also $F(x,f) \in H^t_{\tau,\sigma}(x_0,\xi^0)$.*

5. Paradifferential operators in Gevrey classes

In this section, $m \in \mathbb{R}$, $\sigma > 1$ as usual, but we shall assume $\tau > 0$.

Definition 5.1. For $\varepsilon > 0$, let

$$l^{m,\varepsilon}_{\tau,\sigma} = \{l(x,\xi) \mid l \text{ is } m \text{ order homogeneous } C^\infty(\mathbb{R}^n \setminus 0) \text{ function in } \xi, \text{ and } H^{n/2+\varepsilon}_{\tau,\sigma} \text{ function in } x \text{ for } \xi \text{ uniformly}\}.$$

The functions $l \in l^{m,\varepsilon}_{\tau,\sigma}$ can be regarded as symbols in the classes $S^{m,\varepsilon}_{\tau,\sigma}$ from Definition 3.1. Observe however that the corresponding pseudo-differential operators $l(x,D)$ are not $\mathcal{L}^m_{\tau,\sigma}$ class operators; in fact continuity from $H^s_{\tau,\sigma}$ to $H^{s-m}_{\tau,\sigma}$ fails for large s, because of the limited Gevrey smoothness of $l(x,\xi)$ with respect to x. Following Bony [4] and using the Gevrey paraproduct calculus of the preceding section, we shall then consider paradifferential operators associated to $l(x,\xi)$, which will turn out to be of class $\mathcal{L}^m_{\tau,\sigma}$.

Definition 5.2. For $l \in l^{m,\varepsilon}_{\tau,\sigma}$, we can define an operator T_l as

$$(T_l u)(x) = \sum_q S_q(l(x,D))u_q(x), \quad u = \sum u_q \in H^s_{\tau,\sigma},$$

where $S_q(l(x,D))$ is the pseudo-differential operator with symbol $S_q(l(x,\xi))$, defined by letting S_q act on the x variables, cf. (4.1).

If $l(x,\xi) = \sum_j l_j(x,\xi)$ is a finite sum, then we denote $T_l = \sum_j T_{l_j}$.

If $l(x,\xi) = a(x)h(\xi)$, $a(x) \in H^{n/2+\varepsilon}_{\tau,\sigma}$, $h(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$ and m order homogeneous, then

$$S_q(a(x)h(\xi)) = (S_q a)h(\xi). \tag{5.1}$$

Hence we have

$$(T_l u)(x) = \sum_q S_q(a)(h(D)u)_q,$$

$$(h(D)u)_q = (2\pi)^{-n} \int e^{ix\xi}\phi(2^{-q}\xi)h(\xi)\hat{u}(\xi)d\xi = h(D)u_q,$$

i.e.

$$(T_l u)(x) = T_a \circ h(D)u, \quad \text{if } \; l = a(x)h(\xi). \tag{5.2}$$

For general $l \in l^{m,\varepsilon}_{\tau,\sigma}$, we can rewrite

$$l(x,\xi) = |\xi|^m l(x,\omega), \quad \omega = \frac{\xi}{|\xi|} \in S^{n-1}. \tag{5.3}$$

Let Δ be Laplace-Beltrami operator on S^{n-1}, $\{\lambda_j\}$ and $\{\tilde{h}_j(\omega)\}$ are corresponding eigenvalues and eigenfuctions (i.e. $\Delta\tilde{h}_j = \lambda_j\tilde{h}_j$), we know $\{\tilde{h}_j\}$ is a complete orthonormal basis in $L^2(S^{n-1})$, and $\lim_{j\to\infty}\lambda_j j^{-2/n} \in (0,+\infty)$. Since $l(x,\omega)\in L^2(S^{n-1})$, $\omega\in S^{n-1}$, we have, by using Fourier expansion, that

$$l(x,\omega)=\sum_j a_j(x)\tilde{h}_j(\omega), \tag{5.4}$$

where $a_j(x)=\int_{S^{n-1}} l(x,\omega)\overline{\tilde{h}_j}(\omega)d\omega$.

Since Δ is self-adjoint, we have

$$\begin{aligned}\lambda_j^k a_j(x) &= \int_{S^{n-1}} l(x,\omega)\overline{\Delta^k\tilde{h}_j(\omega)}d\omega\\ &= \int_{S^{n-1}} \Delta^k l(x,\omega)\overline{\tilde{h}_j(\omega)}d\omega.\end{aligned}$$

Thus by Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\lambda_j|^k\|a_j(x)\|_{H^{n/2+\varepsilon}_{\tau,\sigma}} &\le \left(\int_{S^{n-1}}\|\Delta^k l(x,\omega)\|^2_{H^{n/2+\varepsilon}_{\tau,\sigma}}d\omega\right)^{\frac12}\|\tilde{h}_j(\omega)\|_{L^2(S^{n-1})}\\ &= \left(\int_{S^{n-1}}\|\Delta^k l(x,\omega)\|^2_{H^{n/2+\varepsilon}_{\tau,\sigma}}d\omega\right)^{\frac12}.\end{aligned}$$

Since $\Delta^k l(x,\omega)\in H^{n/2+\varepsilon}_{\tau,\sigma}$ in x, hence we have obtained $|\lambda_j|^k\|a_j(x)\|_{H^{n/2+\varepsilon}_{\tau,\sigma}}$ $\le C_k$, $\{C_k\}$ is a bounded constant set. This implies $a_j(x)\in H^{n/2+\varepsilon}_{\tau,\sigma}$, and

$$\|a_j\|_{H^{n/2+\varepsilon}_{\tau,\sigma}}\le C_k j^{-\frac{2}{n}k},\quad \forall k, \tag{5.5}$$

is rapidly decreasing in j.

On the other hand from Sobolev lemma, we have for an even integer s_1, satisfying $s_1>n/2+M$

$$\begin{aligned}\|\tilde{h}_j(\omega)\|_{C^M(S^{n-1})} &\le C\|\tilde{h}_j(\omega)\|_{H^{s_1}(S^{n-1})}\\ &\le C\sum_{k=0}^{s_1/2}\|\Delta^k\tilde{h}_j(\omega)\|_{L^2(S^{n-1})}\\ &\le C\sum_{k=0}^{s_1/2}|\lambda_j|^k.\end{aligned}$$

That is

$$\|\tilde{h}_j(\omega)\|_{C^M(S^{n-1})}\le C_M j^{\frac{(M+n/2+1)}{n}}, \tag{5.6}$$

is temperedly increasing in j. Actually we have proved the following result:

Lemma 5.1. *Let $l\in l^{m,\varepsilon}_{\tau,\sigma}$, then l has the following spherical harmonic decomposition*

$$l(x,\xi)=\sum_j a_j(x)h_j(\xi), \tag{5.7}$$

where $a_j(x) \in H^{n/2+\varepsilon}_{\tau,\sigma}$ *and* $\|a_j\|_{H^{n/2+\varepsilon}_{\tau,\sigma}}$ *is rapidly decreasing in* j, $h_j(\xi) = |\xi|^m \tilde{h}_j(\xi/|\xi|)$ *and* $\|\tilde{h}_j(\omega)\|_{C^M(S^{n-1})}$ *is temperedly increasing in* j *for any* M *fixed.*

Since $(h_j(D)u)_q = h_j(D)u_q$, now we can define the operator T_l as follows:

$$T_l u = \sum_j T_{a_j} \circ h_j(D)u, \quad u \in H^s_{\tau,\sigma}, \tag{5.8}$$

where $\|T_{a_j}\|_{\mathcal{L}(H^s_{\tau,\sigma}, H^s_{\tau,\sigma})} \leq C\|a_j\|_{H^{n/2+\varepsilon}_{\tau,\sigma}}$ is rapidly decreasing in j, and the norm of $h_j(D)u$ is temperedly increasing, then the series (5.8) is convergent.

We can prove T_l, as defined by (5.8), is $\mathcal{L}^m_{\tau,\sigma}$ class operator, i.e.

Theorem 5.1. *For* $l \in l^{m,\varepsilon}_{\tau,\sigma}$, $T_l : H^s_{\tau,\sigma} \to H^{s-m}_{\tau,\sigma}$ $(\forall s \in \mathbb{R})$ *is a bounded linear operator.*

Proof. We may write

$$T_l u = \sum_j \sum_q S_q(a_j) h_j(D) u_q, \quad u \in H^s_{\tau,\sigma}, \tag{5.9}$$

where $\operatorname{supp} \widehat{h_j(D)}u_q \subset C_q$, $\operatorname{supp} \widehat{S_q(a_j)} \subset B(0, K2^{q-N_1})$. So for N_1 large enough we have

$$\operatorname{supp} \widehat{S_q(a_j) h_j}(D)u_q = \operatorname{supp} \widehat{S_q(a_j)} * \widehat{h_j(D)}u_q \subset C_q + B(0, K2^{q-N_1}) \subset C'_q,$$

i.e. $\operatorname{supp} S_q(l\widehat{(x,D)})u_q \subset C'_q$. Thus

$$\begin{aligned} S_q(l(x,D))u_q(x) &= (2\pi)^{-n} \int e^{ix\xi} S_q(l(x,\xi))\hat{u}_q(\xi)d\xi \\ &= \sum_j S_q(a_j)h_j(D)u_q, \end{aligned}$$

and

$$\begin{aligned} \|S_q(l(x,D))u_q(x)\|_{L^2_{\tau,\sigma}} &\leq \sum_j \|S_q(a_j)h_j(D)u_q\|_{L^2_{\tau,\sigma}} \\ &\leq \sum_j \|S_q(a_j)\|_{H^{n/2+\varepsilon}_{\tau,\sigma}} \|h_j(D)u_q\|_{L^2_{\tau,\sigma}} \\ &\leq \sum_j \|a_j\|_{H^{n/2+\varepsilon}_{\tau,\sigma}} \|h_j(D)u_q\|_{L^2_{\tau,\sigma}}, \end{aligned}$$

where

$$\begin{aligned} \|h_j(D)u_q\|_{L^2_{\tau,\sigma}} &= \|\exp(\tau\langle\xi\rangle^{\frac{1}{\sigma}})h_j(\xi)\hat{u}_q\|_{L^2} \\ &= \|\exp(\tau\langle\xi\rangle^{\frac{1}{\sigma}})\tilde{h}_j(\xi/|\xi|)|\xi|^m\hat{u}_q\|_{L^2} \\ &\leq \|\tilde{h}_j(\omega)\|_{C(S^{n-1})}(K2^{(q+1)})^m\|\hat{u}_q\|_{L^2_{\tau,\sigma}} \\ &\leq \|\tilde{h}_j(\omega)\|_{C(S^{n-1})}c_q 2^{-q(s-m)}. \end{aligned}$$

Since $\|a_j\|_{H^{n/2+\varepsilon}_{\tau,\sigma}}$ is rapidly decreasing in j and $\|\tilde{h}_j(\omega)\|_{C(S^{n-1})}$ is temperedly increasing in j, we have

$$\|S_q(l(x,D))u_q\|_{L^2_{\tau,\sigma}} \le Cc_q 2^{-q(s-m)}, \quad \{c_q\} \in l^2. \tag{5.10}$$

Since $T_l u = \sum_q S_q(l(x,D))u_q$, and $\|\{c_q\}\|_{l^2} \le C_1 \|u\|_{H^s_{\tau,\sigma}}$, then we have proved $T_l u \in H^{s-m}_{\tau,\sigma}$, and $\|T_l\|_{\mathcal{L}(H^s_{\tau,\sigma}, H^{s-m}_{\tau,\sigma})} \le CC_1$. Theorem 5.1 is proved. □

It seems, from Definition 5.2, the operator T_l depends on the dyadic decomposition. However if $\{K', \phi_1, N_1'\}$ is another dyadic decomposition, and T_l' is the corresponding operator, then

$$T_l - T_l' = \sum_j (T_{a_j} - T'_{a_j}) \circ h_j(D).$$

We have proved in Theorem 4.2 that

$$T_{a_j} - T'_{a_j} \in \mathcal{L}^{-\varepsilon_1}_{\tau,\sigma}, \quad \varepsilon_1 \in (0,\varepsilon), \quad \text{and} \quad \|T_{a_j} - T'_{a_j}\|_{\mathcal{L}(H^s_{\tau,\sigma}, H^{s+\varepsilon_1}_{\tau,\sigma})} \le C\|a_j\|_{H^{n/2+\varepsilon}_{\tau,\sigma}},$$

and $h_j(D) : H^s_{\tau,\sigma} \to H^{s-m}_{\tau,\sigma}$. Thus it is easy to prove that

$$T_l - T_l' \in \mathcal{L}^{m-\varepsilon_1}_{\tau,\sigma}, \forall \varepsilon_1 \in (0,\varepsilon). \tag{5.11}$$

Let us consider the composition of two operators.

Theorem 5.2. *Let $l_k(x,\xi) \in l^{m_k,\varepsilon}_{\tau,\sigma}$ $(k=1,2)$, $\varepsilon \notin \mathbb{N}$, and*

$$l(x,\xi) = \sum_{|\alpha|<[\varepsilon]} \frac{1}{\alpha!} \partial^\alpha_\xi l_1(x,\xi) D^\alpha_x l_2(x,\xi) = (l_1 \# l_2)(x,\xi)$$

then

$$T_{l_1} \circ T_{l_2} - T_l \in \mathcal{L}^{m_1+m_2-[\varepsilon]}_{\tau,\sigma}.$$

The proof of Theorem 5.2 depends on the following lemma:

Lemma 5.2. *Let $h(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$, m order homogeneous in ξ; $a \in H^{n/2+\varepsilon}_{\tau,\sigma}$, $\varepsilon > 0$ and $\varepsilon \notin \mathbb{N}$. Then*

$$R = h(D) \circ T_a - \sum_{|\alpha|<[\varepsilon]} \frac{1}{\alpha!} T_{D^\alpha a} \circ h^\alpha(D) \in \mathcal{L}^{m-[\varepsilon]}_{\tau,\sigma},$$

where $h^\alpha(\xi) = \partial^\alpha_\xi h(\xi)$, and for suitable M, we have

$$\|R\|_{\mathcal{L}(H^s_{\tau,\sigma}, H^{s-m+[\varepsilon]}_{\tau,\sigma})} \le C\|a\|_{H^{n/2+\varepsilon}_{\tau,\sigma}} \|\tilde{h}\|_{C^{2M}(S^{n-1})}, \quad \tilde{h}(\omega) = h\left(\frac{\xi}{|\xi|}\right).$$

Proof. We have $\operatorname{supp}\widehat{S_q(a)u_q} \subset C'_q$, $u = \sum u_q \in H^s_{\tau,\sigma}$. Take $C'_q \subset C''_q$, and a function $\phi_0 \in C_0^\infty(C''_0)$, satisfying $\phi_0 = 1$ on C'_0 and $\phi_0 = 0$ near $\xi = 0$. Let $h_1(\xi) = h(\xi)\phi_0(\xi)$, then $h_1(\xi) \in \mathcal{S}$ and we know if $\xi \in C'_q$

$$h(\xi) = 2^{mq} h(2^{-q}\xi) = 2^{mq} h_1(2^{-q}\xi).$$

Taking a function $r(x)$, satisfying $\hat{r}(\xi) = h_1(\xi)$, then $r \in \mathcal{S}$, and for $M > n + \varepsilon/2$, we obtain easily

$$\|(1 + |x|^\varepsilon) r(x)\|_{L^1(\mathbb{R}^n)} \leq C \|\tilde{h}\|_{C^{2M}(S^{n-1})}. \tag{5.12}$$

For $u \in H^s_{\tau,\sigma}$, we have

$$Ru = \sum_q 2^{mq} \left[h_1(2^{-q}D) S_q(a) - \sum_{|\alpha|<[\varepsilon]} \frac{1}{\alpha!} S_q(D^\alpha a) h_1^\alpha(2^{-q}D) \right] u_q,$$

where $h_1^\alpha(\xi) = \partial_\xi^\alpha h_1(\xi)$ is Fourier transformation of $(-ix)^\alpha r(x)$. Thus we obtain, by using convolution formula, that

$$\begin{aligned} Ru = \sum_q 2^{mq} \int r(t) \Bigg[& S_q(a)(x - 2^{-q}t) \\ & - \sum_{|\alpha|<[\varepsilon]} \frac{1}{\alpha!} S_q(D^\alpha a)(x)(-i2^{-q}t)^\alpha \Bigg] u_q(x - 2^{-q}t) dt \\ = \sum_q f_q. \end{aligned}$$

Observe $\operatorname{supp} \hat{f}_q \subset C'_q$, and apply Taylor formula to $S_q(a)$, with remainder expressed in terms of $D^\alpha S_q(a) \in H^{n/2+\varepsilon_0}_{\tau,\sigma}$, for $|\alpha| = [\varepsilon]$, $\varepsilon = [\varepsilon] + \varepsilon_0$. We have by using Hausdorff-Young inequality and Theorem 2.5

$$\|f_q\|_{L^2_{\tau,\sigma}} \leq C 2^{mq} \|\langle D\rangle^{[\varepsilon]} S_q(a)\|_{H^{n/2+\varepsilon_0}_{\tau,\sigma}} 2^{-q[\varepsilon]} \||t|^{[\varepsilon]} r(t)\|_{L^1} \|u_q\|_{L^2_{\tau,\sigma}},$$

i.e. from the estimate (5.12)

$$\begin{aligned} \|f_q\|_{L^2_{\tau,\sigma}} &\leq C 2^{mq-[\varepsilon]q} \|\langle D\rangle^{[\varepsilon]} S_q(a)\|_{H^{n/2+\varepsilon_0}_{\tau,\sigma}} \|\tilde{h}\|_{C^{2M}(S^{n-1})} \|u_q\|_{L^2_{\tau,\sigma}} \\ &\leq C c_q \|a\|_{H^{n/2+\varepsilon}_{\tau,\sigma}} \|\tilde{h}\|_{C^{2M}(S^{n-1})} 2^{q[m-([\varepsilon]+s)]}, \end{aligned}$$

where $\{c_q\} \in l^2$, $\|\{c_q\}\|_{l^2} \leq C\|u\|_{H^s_{\tau,\sigma}}$. Thus Lemma 5.2 is proved. □

The proof of Theorem 5.2 is as follows:

Let $l_k(x,\xi) = \sum_j a_{kj}(x) h_{kj}(\xi)$, $k = 1, 2$, the spherical harmonic decomposition of l_k, then

$$T_{l_1} \circ T_{l_2} = \sum_{j,i} T_{a_{1j}} \circ h_{1j}(D) \circ T_{a_{2i}} \circ h_{2i}(D) = \sum_{j,i} A_{j,i}.$$

From Lemma 5.2, we know

$$A_{j,i} = T_{a_{1j}} \left(\sum_{|\alpha|<[\varepsilon]} \frac{1}{\alpha!} T_{D^\alpha a_{2j}} h_{1j}^\alpha(D) h_{2i}(D) \right) + T_{a_{1j}} R_{ji} h_{2i}(D),$$

where $T_{a_{1j}} R_{ji} h_{2i} \in \mathcal{L}_{\tau,\sigma}^{m_1+m_2-[\varepsilon]}$, and we can easily see that

$$\begin{aligned} &\|T_{a_{1j}} R_{ji} h_{2i}\|_{\mathcal{L}(H^s_{\tau,\sigma}, H^{s-(m_1+m_2-[\varepsilon])}_{\tau,\sigma})} \\ &\quad \le C\|a_{1j}\|_{H^{n/2+\varepsilon}_{\tau,\sigma}} \|a_{2i}\|_{H^{n/2+\varepsilon}_{\tau,\sigma}} \|\tilde{h}_{1j}\|_{C^{2M}(S^{n-1})} \|\tilde{h}_{2i}\|_{C^{2M}(S^{n-1})}. \end{aligned}$$

Also from Theorem 4.3, we have

$$T_{a_{1j}} T_{D^\alpha a_{2i}} h_{1j}^\alpha(D) h_{2i}(D) = T_{a_{1j} D^\alpha a_{2i}} h_{1j}^\alpha(D) h_{2i}(D) + R_{ji}^\alpha h_{1j}^\alpha(D) h_{2i}(D),$$

where $R^\alpha_{ji} \in \mathcal{L}^{-\varepsilon_2}_{\tau,\sigma}$ $(\forall \varepsilon_2 \in (0, \varepsilon - |\alpha|))$, and

$$\|R^\alpha_{ji}\|_{\mathcal{L}(H^s_{\tau,\sigma}, H^{s+\varepsilon_2}_{\tau,\sigma})} \le C\|a_{1j}\|_{H^{n/2+\varepsilon}_{\tau,\sigma}} \|D^\alpha a_{2i}\|_{H^{n/2+\varepsilon-|\alpha|}_{\tau,\sigma}},$$

i.e. $R^\alpha_{ji} h^\alpha_{1j}(D) h_{2i}(D) \in \mathcal{L}^{m_1+m_2-\varepsilon_1}_{\tau,\sigma}$, $\forall \varepsilon_1 \in (0,\varepsilon)$, and

$$\|R^\alpha_{ji} h^\alpha_{1j}(D) h_{2i}(D)\| \le C\|a_{1j}\|_{H^{n/2+\varepsilon}_{\tau,\sigma}} \|a_{2i}\|_{H^{n/2+\varepsilon}_{\tau,\sigma}} \|\tilde{h}_{1j}\|_{C^{2M}(S^{n-1})} \|\tilde{h}_{2i}\|_{C^{2M}(S^{n-1})}.$$

Hence Theorem 5.2 is proved, and we have obtained, taking $\varepsilon_1 = [\varepsilon] < \varepsilon$, that

$$\begin{aligned} &\|T_{l_1} \circ T_{l_2} - T_{l_1 \# l_2}\|_{\mathcal{L}(H^s_{\tau,\sigma}, H^{s-(m_1+m_2-[\varepsilon])}_{\tau,\sigma})} \\ &\quad \le C \sum_{j,i} \|a_{1j}\|_{H^{n/2+\varepsilon}_{\tau,\sigma}} \|a_{2i}\|_{H^{n/2+\varepsilon}_{\tau,\sigma}} \|\tilde{h}_{1j}\|_{C^{2M}(S^{n-1})} \|\tilde{h}_{2i}\|_{C^{2M}(S^{n-1})}. \end{aligned}$$

Theorem 5.3. *Let $l(x,\xi) \in l^{m,\varepsilon}_{\tau,\sigma}$, $\varepsilon > 0$, and $\varepsilon \notin \mathbb{N}$. Denote*

$$l^*(x,\xi) = \sum_{|\alpha|<[\varepsilon]} \frac{1}{\alpha!} \partial^\alpha_\xi D^\alpha_x \bar{l}(x,\xi).$$

Then $T^*_l - T_{l^*} \in \mathcal{L}^{m-[\varepsilon]}_{\tau,\sigma}$, and $T^*_l : H^s_{\tau,\sigma} \to H^{s-m}_{\tau,\sigma}$.

Proof. Let $l(x,\xi) = \sum_j a_j(x) h_j(\xi)$, $u \in H^s_{\tau,\sigma}$, $v \in H^{m-s-[\varepsilon]}_{-\tau,\sigma}$,

$$\begin{aligned} \langle (T^*_l - T_{l^*})u, v\rangle &= \langle T^*_l u, v\rangle - \langle T_{l^*} u, v\rangle \\ &= \langle u, T_l v\rangle - \langle T_{l^*} u, u\rangle. \end{aligned}$$

Secondly $T_l v = \sum_j T_{a_j} \circ h_j(D) v$, $\langle u, T_l v\rangle = \sum_j \langle T^*_{a_j} u, h_j(D) v\rangle$.

From Theorem 4.4, we have

$$\langle u, T_l v\rangle = \sum_j [\langle T_{\bar{a}_j} u, h_j(D) v\rangle + \langle R_j u, h_j(D) v\rangle],$$

where $R_j \in \mathcal{L}^{-[\varepsilon]}_{\tau,\sigma}$, and

$$\|R_j u\|_{H^{s+[\varepsilon]}_{\tau,\sigma}} \le C\|a_j\|_{H^{n/2+\varepsilon}_{\tau,\sigma}}\|u\|_{H^s_{\tau,\sigma}}.$$

Then we have

$$\begin{aligned}|\langle R_j u, h_j(D)v\rangle| &\le C\|a_j\|_{H^{n/2+\varepsilon}_{\tau,\sigma}}\|u\|_{H^s_{\tau,\sigma}}\|h_j(D)v\|_{H^{-s-[\varepsilon]}_{-\tau,\sigma}}\\ &\le C\|a_j\|_{H^{n/2+\varepsilon}_{\tau,\sigma}}\|\tilde{h}_j\|_{C^{2M}(S^{n-1})}\|u\|_{H^s_{\tau,\sigma}}\|v\|_{H^{m-s-[\varepsilon]}_{-\tau,\sigma}}.\end{aligned}$$

Next from Lemma 5.2 we have

$$\begin{aligned}\langle T_{\bar{a}_j}u, h_j(D)v\rangle &= \langle \bar{h}_j(D)T_{\bar{a}_j}u, v\rangle\\ &= \sum_{|\alpha|<[\varepsilon]}\frac{1}{\alpha!}\langle T_{D^\alpha \bar{a}_j}\bar{h}^{\alpha}_j(D)u, v\rangle + \langle R'_j u, v\rangle,\end{aligned}$$

where $R'_j \in \mathcal{L}^{m-[\varepsilon]}_{\tau,\sigma}$, and

$$|\langle R'_j u, v\rangle| \le C\|a_j\|_{H^{n/2+\varepsilon}_{\tau,\sigma}}\|\tilde{h}_j\|_{C^{2M}(S^{n-1})}\|u\|_{H^s_{\tau,\sigma}}\|v\|_{H^{m-s-[\varepsilon]}_{-\tau,\sigma}}.$$

Thus we obtain

$$\langle T^*_l u, v\rangle = \langle T_{l^*}u, v\rangle + \sum_j[\langle R_j u, h_j(D)v\rangle + \langle R'_j u, v\rangle],$$

and

$$\begin{aligned}&\sum_j|\langle R_j u, h_j(D)v\rangle + \langle R'_j u, v\rangle|\\ &\quad\le \sum_j C\|a_j\|_{H^{n/2+\varepsilon}_{\tau,\sigma}}\|\tilde{h}_j\|_{C^{2M}(S^{n-1})}\|u\|_{H^s_{\tau,\sigma}}\|v\|_{H^{m-s-[\varepsilon]}_{-\tau,\sigma}}.\end{aligned}$$

Since $\|a_j\|_{H^{n/2+\varepsilon}_{\tau,\sigma}}$ is rapidly decreasing in j, $\|\tilde{h}_j\|_{C^{2M}(S^{n-1})}$ is temperedly increasing in j, we have actually proved that $T^*_l - T_{l^*} \in \mathcal{L}^{m-[\varepsilon]}_{\tau,\sigma}$ and $T^*_l : H^s_{\tau,\sigma} \to H^{s-m}_{\tau,\sigma}$. □

For $l \in l^{m,\varepsilon}_{\tau,\sigma}$, $\varepsilon > m$, we may use standard way to define a pseudo-differential operator $l(x, D)$, cf. Remark 3.1. Concerning what is the relation between $l(x, D)$ and T_l, we have the following result:

Theorem 5.4. *If $l \in l^{m,\varepsilon}_{\tau,\sigma}$, $\varepsilon > m$, then for all $s > m - \varepsilon$ we have*

$$l(x, D) - T_l \in \mathcal{L}(H^s_{\tau,\sigma}, H^{s'}_{\tau,\sigma}), \tag{5.13}$$

where $s' < \min\{\varepsilon, s + \varepsilon - m\}$.

Proof. Without loss of generality, let $l(x,\xi)=a(x)h(\xi)$. For $u\in H^s_{\tau,\sigma}$, we have $v=h(D)u\in H^{s-m}_{\tau,\sigma}$, and $T_l=T_a\circ h(D)u=T_av$. Thus

$$T_lu-l(x,D)u=T_av-av=-T_va-R(a,v),$$

where $av=T_av+T_va+R(a,v)$, $a\in H^{n/2+\varepsilon}_{\tau,\sigma}$.

Since $s+\varepsilon-m>0$, we know $R(a,v)\in H^{s-m+\varepsilon_1}_{\tau,\sigma}$, for any $\varepsilon_1\in(m,\varepsilon)$. Also $T_va=\sum_q S_q(v)a_q=\sum f_q$, where $\operatorname{supp}\hat f_q\subset\operatorname{supp}\hat a_q+\operatorname{supp}(\widehat{S_q(v)})\subset C'_q$, and

$$\begin{aligned}\|f_q\|_{L^2_{\tau,\sigma}}&\le\|a_q\|_{H^{n/2+\varepsilon'}_{\tau,\sigma}}\|S_q(v)\|_{L^2_{\tau,\sigma}}\quad(\text{for }\forall\varepsilon'\in(0,\varepsilon))\\&\le\|a_q\|_{H^{n/2+\varepsilon'}_{\tau,\sigma}}\sum_{p\le q-N_1}\|v_p\|_{L^2_{\tau,\sigma}}\\&\le\|a_q\|_{L^2_{\tau,\sigma}}2^{q(n/2+\varepsilon')}\sum_{p\le q-N_1}c_p2^{-p(s-m)}\\&\le c'_q2^{-q(\varepsilon-\varepsilon')}\sum_{p\le q-N_1}c_p2^{-p(s-m)}.\end{aligned}$$

If $s>m$, then $\sum_p c_p2^{-p(s-m)}\le C<\infty$, which implies $T_va\in H^{\varepsilon-\varepsilon'}_{\tau,\sigma}$ for any $\varepsilon'\in(0,\varepsilon)$. If $s<m$, then $\|f_q\|_{L^2_{\tau,\sigma}}\le c'_q2^{-q(\varepsilon-\varepsilon')}C2^{-q(s-m)}=c''_q2^{-q(s+\varepsilon-\varepsilon'-m)}$, i.e. $T_va\in H^{s+\varepsilon-\varepsilon'-m}_{\tau,\sigma}$ for any $\varepsilon'\in(0,\varepsilon)$. If $s=m$, then we have $\|f_q\|_{L^2_{\tau,\sigma}}\le Cc'_q2^{-q(\varepsilon-\varepsilon')}\|v\|_{L^2_{\tau,\sigma}}$, i.e. $T_va\in H^{\varepsilon-\varepsilon'}_{\tau,\sigma}$ for any $\varepsilon'\in(0,\varepsilon)$. Therefore we have proved $T_va\in H^{s'}_{\tau,\sigma}$ for $s'<\min\{\varepsilon,s+\varepsilon-m\}$. Theorem 5.4 is proved. □

Since $\bigcap_\varepsilon l^{m,\varepsilon}_{\tau,\sigma}\subset S^m_{\tau,\sigma}$, from Theorem 5.4, Corollaries 3.1 and 3.2 we get:

Corollary 5.1. *If $l\in l^{m,\varepsilon}_{\tau,\sigma}$ for all $\varepsilon>0$, then $l(x,D)-T_l$ is "regularizing" operator, i.e. $l(x,D)-T_l\in\mathcal{L}(H^s_{\tau,\sigma},H^{s'}_{\tau,\sigma})$ for any s and s'. Thus $u\in H^{s'}_{\tau,\sigma}(x_0)$ implies $T_lu\in H^{s'-m}_{\tau,\sigma}(x_0)$, and $u\in H^{s'}_{\tau,\sigma}(x_0,\xi^0)$ implies $T_lu\in H^{s'-m}_{\tau,\sigma}(x_0,\xi^0)$.*

Applying further Corollary 3.4, we deduce

Corollary 5.2. *Let $l\in l^{m,\varepsilon}_{\tau,\sigma}$, $\varepsilon>m$, $\varepsilon\notin\mathbb{N}$. If $u\in H^s_{\tau,\sigma}$, then $u\in H^{s'}_{\tau,\sigma}(x_0,\xi^0)$ implies $T_lu\in H^t_{\tau,\sigma}(x_0,\xi^0)$ for $t=\min\{s+[\varepsilon]-m,s'-m\}$.*

Theorem 5.5. *Let $l\in l^{m,\varepsilon}_{\tau,\sigma}$, $\varepsilon>0$, then for the symbol $\sigma(T_l)$ of T_l, we have*

$$\|\partial^\alpha_\xi\partial^\beta_x\sigma(T_l)\|_{L^2_{\tau,\sigma}}\le C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|+|\beta|-n/2-\varepsilon},\tag{5.14}$$

that is $\sigma(T_l)\in S^{m-n/2-\varepsilon}_{\tau,\sigma,1,1}$, cf. Definition 3.4.

Proof. Without loss of generality, we take $l(x,\xi)=a(x)h(\xi)$, $a\in H^{n/2+\varepsilon}_{\tau,\sigma}$ and order of h is m. Then

$$\begin{aligned}\sigma(T_l)(x,\xi)&=\sum_q S_q(l(x,\xi))\varphi(2^{-q}\xi)\\&=\sum_q S_q(a)(x)h(\xi)\varphi(2^{-q}\xi).\end{aligned}$$

Thus

$$\partial_\xi^\alpha\partial_x^\beta\sigma(T_l)(x,\xi)=\sum_q \partial_x^\beta S_q(a)(x)\partial_\xi^\alpha(h(\xi)\varphi(2^{-q}\xi)),$$

$$\|\partial_\xi^\alpha\partial_x^\beta\sigma(T_l)(\cdot,\xi)\|_{L^2_{\tau,\sigma}}\le\sum_q\|\partial_x^\beta S_q(a)\|_{L^2_{\tau,\sigma}}|\partial_\xi^\alpha h(\xi)\varphi(2^{-q}\xi)|$$

$$\le\sum_q\sum_{p\le q-N_1}\|\partial_x^\beta a_p\|_{L^2_{\tau,\sigma}}C_\alpha\sum_{\alpha_1+\alpha_2=\alpha}|h^{(\alpha_1)}\left(\frac{\xi}{|\xi|}\right)||\xi|^{m-|\alpha_1|}2^{-q|\alpha_2|}|\varphi^{(\alpha_2)}(2^{-q}\xi)|.$$

Then, from Theorem 2.1,

$$\begin{aligned}&\|\partial_\xi^\alpha\partial_x^\beta\sigma(T_l)(\cdot,\xi)\|_{L^2_{\tau,\sigma}}\\&\quad\le\tilde C_\alpha\sum_q c_{q\beta}2^{q[|\beta|-(n/2+\varepsilon)]}\|\tilde h\|_{C^\alpha(S^{n-1})}2^{q(m-|\alpha|)}|\varphi^{(\alpha_2)}(2^{-q}\xi)|\\&\quad\le\tilde C'_\alpha\sum_q c_{q\beta}2^{q(m-|\alpha|+|\beta|-n/2-\varepsilon)}|\varphi(2^{-q}\xi)|,\end{aligned}$$

where $\{c_{q\beta}\}_q\in l^2$. Since on $\operatorname{supp}\varphi(2^{-q}\xi)$, $|\xi|\approx 2^q$, then the estimate above implies that (5.14) holds. □

Next, let $\Omega\subset\mathbb{R}^n$ be an open subset, $m\in\mathbb{R}$, $\varepsilon>0$, and $\varepsilon\notin\mathbb{N}$, we define Gevrey paradifferential symbol class $\Sigma^{m,\varepsilon}_{\tau,\sigma}(\Omega)$ as follows:

Definition 5.3. We call $\Gamma(x,\xi)\in\Sigma^{m,\varepsilon}_{\tau,\sigma}(\Omega)$, if

$$\Gamma(x,\xi)=\Gamma_m(x,\xi)+\Gamma_{m-1}(x,\xi)+...+\Gamma_{m-[\varepsilon]}(x,\xi), \tag{5.15}$$

where $\Gamma_{m-k}(x,\xi)$ is $C^\infty(\mathbb{R}^n\setminus 0)$ and $m-k$ order homogeneous in ξ, and is $H^{n/2+\varepsilon-k}_{\tau,\sigma,loc}(\Omega)$ in x for ξ uniformly.

If $\Gamma^k\in\Sigma^{m_k,\varepsilon}_{\tau,\sigma}(\Omega)$, $(k=1,2)$, we define $\Gamma^1\#\Gamma^2\in\Sigma^{m_1+m_2,\varepsilon}_{\tau,\sigma}(\Omega)$ as

$$\Gamma^1\#\Gamma^2=\sum_{|\alpha|+k_1+k_2<[\varepsilon]}\frac{1}{\alpha!}\partial_\xi^\alpha(\Gamma^1_{m_1-k_1})D_x^\alpha(\Gamma^2_{m_2-k_2}). \tag{5.16}$$

If $\Gamma\in\Sigma^{m,\varepsilon}_{\tau,\sigma}(\Omega)$, we define $\Gamma^*\in\Sigma^{m,\varepsilon}_{\tau,\sigma}(\Omega)$ as

$$\Gamma^*=\sum_{|\alpha|+k<[\varepsilon]}\frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha\bar\Gamma_{m-k}. \tag{5.17}$$

For a compact set $K \subset\subset \Omega$ we take $\chi \in G_0^{\sigma'}(\Omega),\ 1 < \sigma' < \sigma$, and $\chi \equiv 1$ near K. We know for $\Gamma \in \Sigma_{\tau,\sigma}^{m,\varepsilon}(\Omega)$, $\chi\Gamma = \chi\Gamma_m + \chi\Gamma_{m-1} + ... + \chi\Gamma_{m-[\varepsilon]}$, $\chi\Gamma_{m-k} \in l_{\tau,\sigma}^{m-k,\varepsilon-k}$, for $0 \le k \le [\varepsilon]$. Thus we could define $T_{\chi\Gamma}$ as $T_{\chi\Gamma} = \sum_{k=0}^{[\varepsilon]} T_{\chi\Gamma_{m-k}}$.

We denote $H_{\tau,\sigma,K}^s = \{u \mid u \in H_{\tau,\sigma}^s, \operatorname{supp} u \subset K\}$, where $K \subset\subset \Omega$ is compact; $H_{\tau,\sigma,comp.}^s(\Omega) = \cup_K H_{\tau,\sigma,K}^s = H_{\tau,\sigma,loc}^s(\Omega) \cap \mathcal{E}'_\sigma$. Then we have

Definition 5.4. We say $L : H_{\tau,\sigma,loc}^{-\infty}(\Omega) \to H_{\tau,\sigma,loc}^{-\infty}(\Omega)$, with proper support, is a m-order ε-class Gevrey paradifferential operator defined on Ω, if there exists $\Gamma \in \Sigma_{\tau,\sigma}^{m,\varepsilon}(\Omega)$ such that for any compact $K \subset\subset \Omega$ and a cut-off function $\chi \in G_0^{\sigma'}(\Omega),\ 1 < \sigma' < \sigma$, $\chi \equiv 1$ near K, we have the mapping

$$L - \chi T_{\chi\Gamma} : H_{\tau,\sigma,K}^s \to H_{\tau,\sigma,comp.}^{s-m+[\varepsilon]}(\Omega)$$

is continuous for any $s \in \mathbb{R}$. We shall use $Op\{\Sigma_{\tau,\sigma}^{m,\varepsilon}(\Omega)\}$ as the notation of such operator class, and say Γ is the symbol of L, denoted also by $\sigma(L)$.

Observe if $L \in Op\{\Sigma_{\tau,\sigma}^{m,\varepsilon}(\Omega)\}$, then for any $s \in \mathbb{R}$, we have

$$L : H_{\tau,\sigma,loc}^s(\Omega) \to H_{\tau,\sigma,loc}^{s-m}(\Omega). \tag{5.18}$$

We also have the following results, the proof being similar to that in Bony [4], we shall leave it to readers.

Theorem 5.6. (a) *$L \in Op\{\Sigma_{\tau,\sigma}^{m,\varepsilon}(\Omega)\}$ has an unique symbol $\sigma(L) \in \Sigma_{\tau,\sigma}^{m,\varepsilon}(\Omega)$, and mapping $\sigma : Op\{\Sigma_{\tau,\sigma}^{m,\varepsilon}(\Omega)\} \to \Sigma_{\tau,\sigma}^{m,\varepsilon}(\Omega)$ is surjective*; $\operatorname{Ker}(\sigma) = \{L \mid L : H_{\tau,\sigma,loc}^s(\Omega) \to H_{\tau,\sigma,loc}^{s-m+[\varepsilon]}(\Omega)\}$.
(b) *If $L_j \in Op\{\Sigma_{\tau,\sigma}^{m_j,\varepsilon}(\Omega)\}$ $(j = 1, 2)$, then $L_1 \circ L_2 \in Op\{\Sigma_{\tau,\sigma}^{m_1+m_2,\varepsilon}(\Omega)\}$, $\sigma(L_1 \circ L_2) = \sigma(L_1)\#\sigma(L_2)$.*
(c) *If $L \in Op\{\Sigma_{\tau,\sigma}^{m,\varepsilon}(\Omega)\}$, then $L^* \in Op\{\Sigma_{\tau,\sigma}^{m,\varepsilon}(\Omega)\}$, and $\sigma(L^*) = \sigma(L)^*$.*
(d) *If $l(x,\xi) \sim \sum_{j=0}^{\infty} l_{m-j}(x,\xi) \in S_{\tau,\sigma,cl}^m(\Omega)$, then for any fixed $h \in \mathbf{Z}_+$, $l^h(x,\xi) = \sum_{j=0}^{h} l_{m-j}(x,\xi) \in \Sigma_{\tau,\sigma}^{m,\varepsilon}(\Omega)$ for $[\varepsilon] = h$. The pseudo-differential operator $l(x,D)$ can be regarded as m-order ε-class Gevrey paradifferential operator in Ω, with symbol $l^h(x,\xi)$ in the sense of Definition* 5.4.

We may be also able to construct parametrix of a Gevrey paradifferential operator, i.e.

Theorem 5.7. *Let $\Gamma \in \Sigma_{\tau,\sigma}^{m,\varepsilon}(\Omega)$, $\Gamma' \in \Sigma_{\tau,\sigma}^{m',\varepsilon}(\Omega)$, and $\Gamma_m(x,\xi) \neq 0$ on a neighborhood of* $\operatorname{supp}\Gamma'$. *Then there exist $\Gamma^k \in \Sigma_{\tau,\sigma}^{m'-m,\varepsilon}(\Omega),\ k = 1, 2$, such that*

$$\Gamma\#\Gamma^1 = \Gamma^2\#\Gamma = \Gamma'.$$

6. Application

Let us consider the following nonlinear equation

$$F[u] = F(x, u, \cdots, \partial^\beta u, \cdots)_{|\beta| \le m} = 0, \tag{6.1}$$

where F is of class $G^{\sigma'}$, $\sigma' < \sigma$, in x near $x_0 \in \Omega$, entire function with respect to other variables, cf. the hypotheses of Corollary 4.2.

Let $u \in H^s_{\tau,\sigma}(x_0)$, $s > m + n/2$, be a local solution of (6.1). Then we are able to introduce the symbol of F, near x_0, as

(6.2)

$$p(x,\xi) = \sum_{|\beta|\le m} F_\beta (i\xi)^\beta, \quad F_\beta = \frac{\partial F}{\partial u_\beta}(x, u, \cdots, \partial^\alpha u, \cdots)_{|\alpha|\le m}, \quad u_\beta = \partial^\beta u,$$

and the principal symbol of F is defined as

$$p_m(x,\xi) = \sum_{|\beta|=m} F_\beta (i\xi)^\beta. \tag{6.3}$$

Theorem 6.1. *Under the preceding assumptions, we have* $u \in H^t_{\tau,\sigma}(x_0,\xi^0)$ *for all* $\xi^0 \neq 0$ *satisfying* $p_m(x_0,\xi^0) \neq 0$, *and* $s < t < 2s - \lambda$ *with* $\lambda = m + n/2$.

Proof. Observe first that $F_\beta \in H^{s-m}_{\tau,\sigma}(x_0)$ in view of Corollary 4.2. Therefore $p(x,\xi) \in \Sigma^{m,\varepsilon}_{\tau,\sigma}(V)$ for a neighborhood V of x_0, with $\varepsilon = s - m - n/2$. Applying the paralinearization result in Corollary 4.2, we may write

$$F[u] = \sum_{|\beta|\le m} T_{F_\beta}\partial^\beta u + v,$$

that is,

$$F[u] = T_p u + v,$$

where $v \in H^r_{\tau,\sigma}(x_0)$ for $r < 2s - 2m - n/2$. At this moment we are reduced to treat the paradifferential equation

$$T_p u = -v. \tag{6.4}$$

Assume $p_m(x,\xi) \neq 0$ in the conic neighborhood Λ of (x_0,ξ^0). To prove $u \in H^t_{\tau,\sigma}(x_0,\xi^0)$ we fix $l(x,\xi) \in S^0_{\tau,\sigma,cl}(V)$ with $l(x,\xi) \sim l_0(x,\xi)$ homogeneous of order 0 in ξ and supported in Λ, and $l_0(x,\xi) = 1$ in a smaller conic neighborhood Λ' of (x_0,ξ^0). In view of Corollary 3.4, it will be sufficient to check $l(x,D)u \in H^t_{\tau,\sigma}(x_0,\xi^0)$.

Applying Theorem 5.6 (d), we may regard $l(x,D)$ as paradifferential operator with symbol $l_0 \in \Sigma^{0,\varepsilon}_{\tau,\sigma}(V)$. We then apply Theorem 5.7 and find $q \in \Sigma^{-m,\varepsilon}_{\tau,\sigma}(V)$ such that $q\#p = l_0$. We have from Theorem 5.6 (b) $T_q \circ T_p \in Op\{\Sigma^{0,\varepsilon}_{\tau,\sigma}(V)\}$ with symbol l_0 and then, from Theorem 5.6 (a)

$$T_q \circ T_p u = l(x,D)u + Ru,$$

where $R \in \mathcal{L}^{-[\varepsilon]}_{\tau,\sigma}$. Therefore from (6.4)

$$l(x,D)u = -Ru - T_q v,$$

which gives the conclusion. □

Acknowledgement. The research was supported by the NNSF of China and the MURST of Italy.

The final version of the paper was finished while the first author was a visiting professor at the Department of Mathematics, University of Torino in 1998, he would like to thank the Institution for its hospitality.

In conclusion, the authors would like also to thank the referee of the paper for some useful comments.

Department of Mathematics
Wuhan University
Wuhan, 430072, China
e-mail: chenhua@whu.edu.cn

Dipartimento di Matematica
Università di Torino
Via Carlo Alberto 10, 10123, Torino, Italy
e-mail: rodino@dm.unito.it

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