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The torsion group of a certain numerical Godeaux surface

By

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Abstract

We compute the torsion part of the Picard group of a numerical Godeaux surface Y which was constructed by Stagnaro [7] as a double cover of $\mathbb{P}^2$ branching along a curve of degree 10. We show that this surface is a classical Godeaux surface whose universal cover is the Fermat quintic in $\mathbb{P}^3$ (cf. [1, p.170]).

Introduction

E. Stagnaro [7] constructed a numerical Godeaux surface Y as a double plane by giving a degree-ten plane curve C_{10} having certain singularities. Here by a numerical Godeaux surface we mean a minimal surface Y of general type having invariants $p_g(Y) = 0$, $(K_Y^2) = 1$, where $p_g(Y)$, K_Y are the geometric genus and a canonical divisor of Y, respectively. The curve C_{10} is irreducible, and has five $[3, 3]$ points (for the definition see Notation and Terminology below) and a quadruple point. In this paper, we show that the torsion group $\mathrm{Tor}(\mathrm{Pic}(Y))$ is $\mathbb{Z}/5\mathbb{Z}$, and show that the universal cover $\widetilde{Y}$ of Y corresponding to $\mathrm{Tor}(\mathrm{Pic}(Y))$ is a degree-five surface in $\mathbb{P}^3$ of Fermat type.

Numerical Godeaux surfaces are classified with respect to their torsion parts of the Picard groups. We have $\sharp\mathrm{Tor}(\mathrm{Pic}(Y)) \leq 5$ and $\mathrm{Tor}(\mathrm{Pic}(Y)) \not\simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ for a numerical Godeaux surface Y ([2, p.155–160]). Any cyclic group G of order ≤ 5 indeed appears as the torsion group of a numerical Godeaux surface (see for example [1, p.237]). Complete description of numerical Godeaux surfaces with $\mathrm{Tor}(\mathrm{Pic}(Y)) \simeq \mathbb{Z}/5\mathbb{Z}$ are given by Miyaoka in [3].

One of effective methods to obtain a surface of general type with $p_g = 0$ is to take a double cover of $\mathbb{P}^2$ branched along a plane curve of degree ten. Several examples of surfaces are obtained by this method (see [1, p.237]). In most cases, the branch curves are reducible. We deal with an irreducible branch curve in this paper. It is difficult in this case to find torsion divisors, or to find good curves in the pluri-canonical systems.

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In Ssection 1, we recall the construction of the curve C_{10}, and study geometry of desingularization Y of the double cover of $\mathbb{P}^2$ branched along C_{10}. In Section 2, we show that the torsion group of Y is isomorphic to $\mathbb{Z}/5\mathbb{Z}$. Finally in Section 3, we show that the unramified cover $\widetilde{Y}$ of Y corresponding to $\mathrm{Tor}(\mathrm{Pic}(Y))$ is a quintic surface in $\mathbb{P}^3$ of Fermat type. Throughout this paper we work over the complex number field $\mathbb{C}$.

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Notation and Terminology. Let S be a compact complex manifold of dimension two. We set $\mathrm{Tors}(S) = \mathrm{Tor}(\mathrm{Pic}(S))$ to be the torsion part of the Picard group of S. We use the symbols $q(S)$, $p_g(S)$ and K_S for the irregularity, the geometric genus and a canonical divisor of S respectively. Let C be a curve on a surface S, and P a triple point of the curve C. The triple point P is called a $[3,3]$ point, if C has three distinct smooth branches at P which intersect one another with multiplicity exactly two.

1. Construction of the numerical Godeaux surface

The curve C_{10} of Stagnaro [7] is the plane curve defined by the following equation:

$$\begin{aligned}(1)\qquad & -8\,x^5y^5 + 2\,y^{10} + 10\left(\sqrt{5}-1\right)x^3y^6 + \left(11\sqrt{5}-25\right)x^6y^2 \\ & - 4\sqrt{5}xy^7 + 10\left(7\sqrt{5}-15\right)x^4y^3 + 10\left(2\sqrt{5}-5\right)x^2y^4 \\ & + 8\left(17\sqrt{5}-38\right)x^5 + 2\left(3\sqrt{5}-5\right)y^5 + 20\left(4\sqrt{5}-9\right)x^3y \\ & + 2\left(11\sqrt{5}-25\right)xy^2 + \left(-3\sqrt{5}+7\right) = 0.\end{aligned}$$

We denote by $f_{10}(x,y)$ the degree-ten polynomial of the left hand side of the equation (1). Let $\varepsilon = \exp(2\pi\sqrt{-1}/5)$ be a primitive 5th root of unity, and φ_0 an automorphism of $\mathbb{P}^2$ given by

$$\varphi_0 : (x,y) \mapsto (\varepsilon x, \varepsilon^2 y).$$

Then, the cyclic group $I_5 = \langle\varphi_0\rangle \simeq \mathbb{Z}/5\mathbb{Z}$ generated by φ_0 acts on the complex projective plane $\mathbb{P}^2$. Since we have $f_{10}(\varepsilon x, \varepsilon^2 y) = f_{10}(x,y)$, the degree-ten plane curve C_{10} is invariant under the action of I_5. We set $q = (\sqrt{5}-1)/2$. We denote by $(X : Y : Z)$ the homogeneous coordinates of the projective plane satisfying $x = X/Z$, $y = Y/Z$.

Proposition 1 (Stagnaro)**.** (i) *The curve C_{10} has a quadruple point at $P_6 = (1:0:0)$. This quadruple point is a union of two cusps, and the cuspidal lines* (*i.e. tangent cones*) *are $Y = 0$ and $Z = 0$ respectively.*

(ii) *C_{10} has a $[3,3]$ point at $P_i = (\varepsilon^{i-1}, \varepsilon^{2(i-1)}q)$ for $1 \leq i \leq 5$. The (tangent) singular line t_i at P_i is given by $y - \varepsilon^{2(i-1)}q + \varepsilon^{(i-1)}(x - \varepsilon^{(i-1)}) = 0$ for $1 \leq i \leq 5$.*

(iii) *The above six singularities $P_1, \dots P_6$ do not lie on a conic.*

Note that the set of five points $\{P_1, \dots P_5\}$ is an orbit of the action of I_5.

Proposition 2 (Stagnaro)**.** *The curve C_{10} is irreducible, and it has no singularities other than those mentioned above. The desingularization of C_{10} is a non-singular rational curve.*

For a proof of these facts, see [7].

Remark 1. The defining equation in [7] of the curve $C_{10} = C_{10}^{(A,B)}$ contains apparent two parameters A and $B \in \mathbb{C}^*$. Setting f to be an automorphism of $\mathbb{P}^2$ given by $f(x,y) = (Ax/B, y/B)$, we have $f(C_{10}^{(A,B)}) = C_{10}^{(1,1)}$. Thus any two curves $C_{10}^{(A,B)}$ and $C_{10}^{(A',B')}$ are biholomorphically mapped onto each other by a plane projective transformation.

Now let us consider the numerical Godeaux surface due to Stagnaro. We obtain this surface as a desingularization of the double cover of $\mathbb{P}^2$ branched along C_{10}.

Proposition 3. *Let V be the double cover of $\mathbb{P}^2$ branched along C_{10}. Then the minimal desingularization of V is a numerical Godeaux surface.*

This proposition is a direct consequence of [4, Proposition 2.1]. We shall give an explicit construction. Let $p_1 : X_1 \to \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2$ at the six points $P_1, \cdots, P_6$, and s_i the exceptional curve appearing by the blowing up at P_i. The strict transform $p_1^{-1}[C_{10}]$ has an ordinary triple point on the curve s_i for each $1 \leq i \leq 5$. Blowing up at these five ordinary triple points, we get a surface X with a holomorphic map $p : X \to \mathbb{P}^2$.

We also use the same symbol s_i for the strict transform on X of the curve s_i, and denote by s_i' the exceptional curve appearing by the blowing up at the ordinary triple point of $p_1^{-1}[C_{10}]$ on the curve s_i. Let $p^{-1}[C_{10}]$ be the strict transform on X of the curve C_{10}. The reduced curve

$$\Gamma = p^{-1}[C_{10}] + \sum_{i=1}^{5} s_i \tag{2}$$

has no singularity. Moreover we have

$$\Gamma \sim 2B, \qquad B = p^*(5H) - \sum_{i=1}^{5}(s_i + 3s_i') - 2s_6, \tag{3}$$

where H is a line on $\mathbb{P}^2$, and the symbol $\sim$ means the linear equivalence. Thus we have the double cover $r : Z \to X$ branched along Γ with smooth Z. We have

$$r^*(p^{-1}[C_{10}]) = 2\overline{C_{10}}, \quad r^*(s_i) = 2\overline{s_i}, \quad 1 \leq i \leq 5$$

for certain curves $\overline{C_{10}}$ and $\overline{s_1}, \ldots, \overline{s_5}$ on Z. The curves $\overline{s_1}, \ldots, \overline{s_5}$ are exceptional curves of the first kind. Blowing down $\overline{s_i}$ for $1 \leq i \leq 5$, we obtain a smooth surface Y and the following commutative diagram:

$$\begin{array}{ccc} Z & \xrightarrow{\sigma} & Y \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{p} & \mathbb{P}^2. \end{array} \tag{4}$$

This surface Y is the minimal desingularization of the double cover V.

Proposition 4. *The surface Y is a numerical Godeaux surface, that is, a minimal surface of general type with $p_g(Y) = 0$, $(K_Y^2) = 1$, where $p_g(Y)$, K_Y are the geometric genus and a canonical divisor of Y respectively.*

Corollary 1.1. *The n-th pluri- genus $P_n(Y)$ is given by $P_n(Y) = (n^2 - n + 2)/2$ for $n \geq 2$. In particular we have $P_2(Y) = 2$ and $P_3(Y) = 4$.*

2. The torsion group

In this section, we compute the torsion group $\mathrm{Tors}(Y)$ of the numerical Godeaux surface Y constructed in Section 1. We use the following Theorem. For a proof of this theorem, see [3, Theorem 2'] and [2, p.159 Proposition 3].

Theorem 2.1 (Miyaoka). *Let Y be a numerical Godeaux surface. Then $|3K_Y|$ has no fixed components, and the number b of the base points is given as follows*:

$$b = \begin{cases} 0 & \textit{if} \quad \mathrm{Tors}(Y) = 0 \textit{ or } \mathbb{Z}/2\mathbb{Z}; \\ 1 & \textit{if} \quad \mathrm{Tors}(Y) = \mathbb{Z}/3\mathbb{Z} \textit{ or } \mathbb{Z}/4\mathbb{Z}; \\ 2 & \textit{if} \quad \mathrm{Tors}(Y) = \mathbb{Z}/5\mathbb{Z}. \end{cases}$$

We first give two explicit bicanonical curves of our surface Y. Since we have

$$K_Z \sim r^*(K_X + B) \sim r^* \left(p^*(2H) - \sum_{i=1}^{5} s'_i - s_6 \right), \tag{5}$$

this is done by finding suitable plane quartic curves. Consider two polynomials of degree four depending on two coefficients respectively:

$$f_4(x,y) = axy^3 + x^2 + by, \quad g_4(x,y) = x^2y^2 + cy^3 + dx.$$

The polynomials f_4 and g_4 define plane quartics C_4 and D_4, respectively. Since we have equalities

$$f_4(\varepsilon x, \varepsilon^2 y) = \varepsilon^2 f_4(x,y), \quad g_4(\varepsilon x, \varepsilon^2 y) = \varepsilon g_4(x,y),$$

two curves C_4 and D_4 are both stable under the action of I_5. It is easy to show the following:

Lemma 2.1. *Let C_4 and D_4 be the quartic curves as above, and $(X : Y : Z)$ the homogeneous coordinates of $\mathbb{P}^2$ s.t. $x = X/Z, y = Y/Z$. If $a \neq 0$, then the curve C_4 has a cusp at P_6, and its cuspidal line at P_6 is $Z = 0$. If $d \neq 0$, then the curve D_4 has a cusp at P_6, and its cuspidal line at P_6 is $Y = 0$.*

We choose four coefficients $a, \cdots, d$ in such a way that both C_4 and D_4 pass through P_1 and are tangent to t_1 at P_1. Then we get

$$C_4 : -\left(2+\sqrt{5}\right) y + x^2 + \frac{(7+3\sqrt{5})\, y^3 x}{2} = 0,$$
$$D_4 : x^2 y^2 - \frac{(\sqrt{5}-1)\, y^3}{2} - \left(\sqrt{5}-2\right) x = 0.$$

Since C_4 and D_4 are invariant under I_5, they are tangent to t_i at P_i for $1 \leq i \leq 5$.

Lemma 2.2. *The quartic curves C_4 and D_4 have no singularities other than P_6. The strict transforms $p^{-1}[C_4]$ and $p^{-1}[D_4]$ are both non-singular curves of genus 2.*

Proof. Let us first show the irreducibility of C_4. Since C_4 has a cusp at P_6, we only need to show that C_4 does not split into a line C_1 and an irreducible cubic C_3. If this happens, then C_1 must be invariant under I_5, hence one of the following three lines: $L_X : X = 0$, $L_Y : Y = 0$ or $L_Z : Z = 0$, which is clearly imposed in view of the equation. Since the virtual genus of a plane quartic is 3, the irreducibility of C_4 implies that C_4 has at most three singularities. Thus C_4 has no singularities outside $P_6 = (1 : 0 : 0)$, $P_7 = (0 : 1 : 0)$ and $P_8 = (0 : 0 : 1)$. This proves the statement for C_4. In the same way, we can show the statement for D_4. □

Put $\overline{C_4} = r^*(p^{-1}[C_4])$ and $\overline{D_4} = r^*(p^{-1}[D_4])$. Then by (5) we have

$$2K_Y \sim \widetilde{C_4} \sim \widetilde{D_4}, \tag{6}$$

where we set $\sigma(\overline{C_4}) = \widetilde{C_4}$ and $\sigma(\overline{D_4}) = \widetilde{D_4}$.

Let us survey the geometry of two bicanonical curves $\widetilde{C_4}$ and $\widetilde{D_4}$. Since σ maps $\overline{C_4}$ and $\overline{D_4}$ biholomorphically onto $\widetilde{C_4}$ and $\widetilde{D_4}$, it is enough to survey the geometry of two curves $\overline{C_4}$ and $\overline{D_4}$.

Computation of the intersection multiplicity gives

$$\begin{aligned} &\mathit{\Gamma} \cdot p^{-1}[C_4] = 2Q_1, \qquad \mathit{\Gamma} \cdot p^{-1}[D_4] = 2Q_2, \\ &p^{-1}[C_4] \cdot p^{-1}[D_4] = P_7 + P_8, \end{aligned} \tag{7}$$

where Q_1 and Q_2 are points on the exceptional curve s_6. Thus if we put $\overline{Q_i} = r^{-1}(Q_i)$ for $i = 1, 2$, the curve $\overline{C_4}$ has a node at $\overline{Q_1}$ and that $\overline{Q_1}$ is the only singularity of $\overline{C_4}$. Also $\overline{D_4}$ has a node at $\overline{Q_2}$, which is the only singularity of $\overline{D_4}$.

Lemma 2.3. *Both $\overline{C_4}$ and $\overline{D_4}$ are reducible.*

Proof. Let $r_1' : C_4{}' \to \overline{C_4}$ and $r_2' : D_4{}' \to \overline{D_4}$ be the desingularization of $\overline{C_4}$ and $\overline{D_4}$ respectively. The mapping $r \circ r_1'$ of $C_4{}'$ onto $p^{-1}[C_4]$ and the mapping $r \circ r_2'$ of $D_4{}'$ onto $p^{-1}[D_4]$ are both unramified mappings of degree two. We denote by $[B]$ the line bundle associated with the divisor B, and denote by $[B]|_{p^{-1}[C_4]}$ its restriction to $p^{-1}[C_4]$. The surface Z is a complex submanifold in the line bundle $[B]$ over X. Then $\overline{C_4}$ is a subspace of $[B]|_{p^{-1}[C_4]}$. Denoting by S_0 the zero-section of the line bundle $[B]|_{p^{-1}[C_4]}$, we easily infer from (7) that S_0 meets each of the two branches of $\overline{C_4}$ transversally at $\overline{Q_1}$. $\overline{Q_1}$ is a unique common points of S_0 and $\overline{C_4}$. Let L_X, L_Y and L_Z be the lines on $\mathbb{P}^2$ given by $L_X : X = 0$, $L_Y : Y = 0$, $L_Z : Z = 0$. Using the equivalence

$$5H \sim D_4 + C_2 + L_Z - L_X - L_Y, \tag{8}$$

we infer from (3) the following:

$$B \sim p^{-1}[D_4] + p^{-1}[C_2] + p^{-1}[L_Z] - p^{-1}[L_X] - p^{-1}[L_Y] + \sum_{i=1}^{5} s_i. \tag{9}$$

On the other hand we have

$$\begin{aligned} &p^{-1}[C_4] \cdot p^{-1}[D_4] = P_7 + P_8, &\quad &p^{-1}[C_4] \cdot p^{-1}[C_2] = P_7 + 2P_8,\\ &p^{-1}[C_4] \cdot p^{-1}[L_Z] = P_7 + Q_1, &\quad &p^{-1}[C_4] \cdot p^{-1}[L_X] = 3P_7 + P_8,\\ &p^{-1}[C_4] \cdot p^{-1}[L_Y] = 2P_8, &\quad &p^{-1}[C_4] \cdot s_i = 0 \quad 1 \le i \le 5. \end{aligned} \tag{10}$$

Thus by (9) and (10), the point Q_1 on $p^{-1}[C_4]$ gives an effective divisor corresponding to the line bundle $[B]|_{p^{-1}[C_4]}$. This implies the existence of a non-zero holomorphic section of $[B]|_{p^{-1}[C_4]}$ which vanishes simply at Q_1 and does not vanish at any other points on $p^{-1}[C_4]$. We take one of such non-zero sections and denote it by θ. Take any point p on $C_4{}' \setminus r_1^{-1}(\overline{Q_1})$. Since $r_1(p)$ and $\theta(r \circ r_1(p))$ lie over the same point $r \circ r_1(p)$ on $p^{-1}[C_4]$, and since θ vanishes at no points other than Q_1, a constant $u(p)$ satisfying the following condition is uniquely determined:

$$r_1(p) = u(p) \cdot \theta(r \circ r_1(p)).$$

$u(p)$ extends to a holomorphic function on the whole curve $p^{-1}[C_4]$, because the multiplicity of θ at Q_1 is equal to one. Thus u is constant on each component of $C_4{}'$. Now for a pair (p, q) of distinct points on $C_4{}'$ with $r \circ r_1(p) = r \circ r_1(q)$, we have $r_1(p) = -r_1(q)$. Thus the equality $u(p) = -u(q)$ holds for such p, q. This proves that $\overline{C_4}$ has at least two components.

In the same way, we can prove that $\overline{D_4}$ is reducible. In this case we use

$$5H \sim C_4 + C_2 + L_Y - L_X - L_Z$$

in place of the equivalence (8). $\square$

We set $\widetilde{Q_1} = \sigma(\overline{Q_1})$ and $\widetilde{Q_2} = \sigma(\overline{Q_2})$. By the above lemma, the bicanonical curve $\widetilde{C_4}$ is a union of two non-singular curves of genus 2 meeting transversally at $\widetilde{Q_1}$. Also the curve $\widetilde{D_4}$ is a union of two non-singular curves of genus two meeting transversally at $\widetilde{Q_2}$. The point $\widetilde{Q_1}$ is the unique singularity of $\widetilde{C_4}$, and $\widetilde{Q_2}$ is the only singularity of $\widetilde{D_4}$. We are now ready to compute the torsion group of our surface Y.

Proposition 5. *Let Y be the numerical Godeaux surface constructed in Section* 1*. Then the torsion group* $\mathrm{Tors}(Y)$ *is isomorphic to* $\mathbb{Z}/5\mathbb{Z}$*.*

Proof. Let $\pi' : Y' \to Y$ be the blowing up of Y at $\widetilde{Q_1}$, and E the exceptional curve over $\widetilde{Q_1}$. By Lemma 2.3, the strict transform $C_4' = {\pi'}^{-1}[\widetilde{C_4}]$ is a disjoint union of two non-singular curves of genus 2. Consider the exact sequence

$$0 \to \mathcal{O}_{Y'}(-C_4') \to \mathcal{O}_{Y'} \to \mathcal{O}_{C_4'} \to 0. \tag{11}$$

Then from the associated exact cohomology sequence, we infer equality $H^1(C_4', \mathcal{O}_{C_4'}) \simeq H^2(Y', \mathcal{O}_{Y'}(-C_4'))$. On the other hand, the curve C_4' is a disjoint union of two non-singular curves of genus 2. Thus we have $\dim H^2(Y', \mathcal{O}_{Y'}(-C_4')) = 4$. The duality theorem and equivalence (6) assert that

$$\begin{aligned} H^2(Y', \mathcal{O}_{Y'}(-C_4'))^* &\simeq H^0(Y', \mathcal{O}_{Y'}(K_{Y'} + C_4')) \\ &\simeq H^0(Y', \mathcal{O}_{Y'}({\pi'}^*(3K_Y) - E)). \end{aligned}$$

Thus $H^2(Y', \mathcal{O}_{Y'}(-C_4'))^*$ is the space of global sections of $3K_Y$ which vanish at $\widetilde{Q_1}$. Comparing $\dim H^2(Y', \mathcal{O}_{Y'}(-C_4')) = 4$ with $P_3(Y) = 4$, we see that $\widetilde{Q_1}$ is a base point of $|3K_Y|$.

Replacing the curve $\widetilde{C_4}$ by $\widetilde{D_4}$, we see in the same way that $\widetilde{Q_2}$ is also a base point of $|3K_Y|$. Thus $|3K_Y|$ has at least two base points. Then by virtue of Theorem (2.1), we see that $\mathrm{Tors}(Y)$ is isomorphic to $\mathbb{Z}/5\mathbb{Z}$. $\square$

3. The universal cover

In this section, we shall show that the universal cover of the surface Y constructed in Section 1 is a quintic surface in $\mathbb{P}^3$ of Fermat type. We start with a preparation.

Definition 3.1. Let X and Z be connected complex manifolds, r a surjective holomorphic mapping of Z to X, and G a group acting on X. We say that an action of G on Z is a lifting of the action on X, if the action on Z is compatible with the projection r, that is, $r(\varphi \cdot z) = \varphi \cdot r(z)$ for every element φ in G and every point z on Z.

Assume that X is a compact connected complex manifold, and that Γ is a non-singular reduced curve on X which is linearly equivalent to nB for a non-trivial divisor B on X with an integer $n \geq 2$. We have $\Gamma - nB = (f)$ for an meromorphic function f on X. Let $r : Z \to X$ be an n-ple cover of X branched along Γ. Furthermore, we assume that a finite group G acts on X.

Lemma 3.1. *Assume that both Γ and B are invariant under G as divisors. Then $\varphi \mapsto c_\varphi = \varphi^* f/f$ gives a charactor c of G. Let* $\mathrm{Char}(G)$ *be the character group of G and Ψ be an endomorphism of* $\mathrm{Char}(G)$ *given by $\Psi : \psi \mapsto n\psi$. Then the action of G on X lifts to a one on Z, if and only if $c \in \mathrm{Im}\Psi$. If a lifting of the action of G to Z exists, there exist exactly* $\sharp(\ker \Psi)$ *liftings.*

Lemma 3.2. *If $\Gamma = \emptyset$, the cover $Z \to X$ corresponding to B is unramified. Assume that $G = \langle \iota \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ and that $\iota^* B = -B$ as divisors. Then there exist exactly n liftings of the action of $\langle \iota \rangle$.*

A proof of the two lemmas above is by direct computation with writing down defining equations of X in the total space of the line bundle $[B]$. We do not show it here. For a proof of the following lemma, see for example [2, p.158, Lemma].

Lemma 3.3 (Reid)**.** *Let Y be a numerical Godeaux surface.*

(1) *For any $g \neq 0 \in$* $\mathrm{Tors}(Y)$*, we have* $\dim |K_Y + g| = 0$.

(2) *We denote by D_g the only curve in $|K_Y + g|$. Then, D_g and $D_{g'}$ have no common components, if g and g' are distinct.*

The following Theorem is Miyaoka's result reformulated by Reid. For a proof of this theorem, see [5].

Theorem 3.1 (Miyaoka, Reid)**.** *Let Y be a numerical Godeaux surface with* $\mathrm{Tors\,Pic}\,(Y) \simeq \mathbb{Z}/5\mathbb{Z}$*, and g a generator of* $\mathrm{Tors}(Y)$*. Let $\phi : \widetilde{Y} \to Y$ be the unramified cover of degree five corresponding to* $\mathrm{Tors}(Y)$*. We denote by ξ_i a non-zero holomorphic 1-form of $\widetilde{Y}$ coming from $H^0(Y, \mathcal{O}_Y(K_Y - ig))$ for each $1 \leq i \leq 4$.*

(i) *The holomorphic 1-forms $\xi_1, \cdots, \xi_4$ form a base of $H^0(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}(K_{\widetilde{Y}}))$, and the canonical system $|K_{\widetilde{Y}}|$ has no base points. The mapping $\Phi : p \mapsto (\xi_1(p) : \cdots \xi_4(p))$ of $\widetilde{Y} \to \mathbb{P}^3$ gives the canonical model of $\widetilde{Y}$.*

(ii) *Let $(X_1 : \cdots : X_4)$ be a homogeneous coordinate of $\mathbb{P}^3$. We can set $\xi_1, \cdots, \xi_4$ in such a way that the defining equation of $\Phi(\widetilde{Y})$ is of the following form:*

$$
\begin{aligned}
F(X_1, \cdots, X_4) &= X_1^5 + X_2^5 + X_3^5 + X_4^5 \\
&+ a_1 X_1^3 X_3 X_4 + a_2 X_2^3 X_1 X_3 + a_3 X_3^3 X_2 X_4 + a_4 X_4^3 X_1 X_2 \\
&+ b_1 X_3^2 X_4^2 X_1 + b_2 X_1^2 X_3^2 X_2 + b_3 X_2^2 X_4^2 X_3 + b_4 X_1^2 X_2^2 X_4 = 0.
\end{aligned} \tag{12}
$$

Remark 2. For a properly chosen generator ϕ_0 of the Galois group $\mathrm{Gal}(\widetilde{Y}/Y)$ we have $\phi_0^*(\xi_i) = \varepsilon^i \xi_i$, where we set $\varepsilon = \exp(2\pi\sqrt{-1}/5)$. Thus the automorphism of $\Phi(\widetilde{Y})$ induced by ϕ_0 is given by $(X_1 : \cdots : X_4) \mapsto (\varepsilon X_1 : \varepsilon^2 X_2 : \varepsilon^3 X_3 : \varepsilon^4 X_4)$. We use the same symbol ϕ_0 for this induced automorphism of $\Phi(\widetilde{Y})$.

In what follows, We let Y denote the numerical Godeaux surface constructed in Section 1, and show that the unramified cover $\widetilde{Y}$ is of Fermat type. Let $\pi : Y \to \mathbb{P}^2$ be the morphism given in (4), and ι the involution determined by π.

Lemma 3.4. *The action of $I_2 = \langle \iota \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ on Y lifts to a one on $\widetilde{Y}$, where $\widetilde{Y}$ is as in Theorem* 3.1.

Proof. First note that the action of I_2 on $\mathrm{Tors}(Y)$ is non-trivial. Indeed if this action is trivial, the action of I_2 on Y lifts to a one on $\widetilde{Y}$ by Lemma 3.1. In this case the induced involution of $\mathbb{P}^2$ is given by $(X_1, \cdots, X_4) \mapsto (\alpha_1 X_1, \cdots, \alpha_4 X_4)$, where $\alpha_1, \cdots, \alpha_4 \in \mathbb{C}^*$. However this is impossible since by (12) we would have $\alpha_1 = \cdots = \alpha_4$. Thus we have $\iota^* D_g = D_{4g}$ and $\iota^* D_{2g} = D_{3g}$. By $D_{3g} - D_{2g} \sim g$ and Lemma 3.2, the action lifts to a one on $\widetilde{Y}$. □

We fix an action of I_2 on $\widetilde{Y}$. By (12) and the proof of the lemma above, the automorphism of $\mathbb{P}^2$ induced by ι is given by

$$(X_1 : X_2 : X_3 : X_4) \mapsto (X_4 : X_3 : X_2 : X_1). \tag{13}$$

for suitable $\xi_1, \cdots, \xi_4$. Then we have

$$a_1 = a_4, \quad a_2 = a_3, \quad b_1 = b_4, \quad b_2 = b_3, \tag{14}$$

where $a_1, \cdots, a_4, b_1 \cdots b_4$ are the coefficients defined in (12). We set $x_i = X_i/X_4$ for each $1 \leq i \leq 3$. Then the defining equation of $\Phi(\widetilde{Y})$ is $F(x_1, x_2, x_3, 1) = 0$. We denote by $f(x_1, x_2, x_3)$ the left hand side of this equation. Let ϕ_0 be the induced automorphism of $\Phi(\widetilde{Y})$ defined in Remark 2. Then ϕ_0 is given by

$$(x_1, x_2, x_3) \mapsto (\varepsilon^2 x_1, \varepsilon^3 x_2, \varepsilon^4 x_3). \tag{15}$$

Let X and Z be as in Section 1. By Lemma 3.1 together with (2) and (3), the action of $I_5 = \langle \varphi_0 \rangle$ on $\mathbb{P}^2$ uniquely lifts to a one Z, hence to a one on Y. The group I_5 acts trivially on $\mathrm{Tors}(Y)$. Thus the divisor $D_{3g} - D_{2g}$ is invariant under I_5. Computing the multiplicity of torsion divisors at a fixed point of action of I_5, we see by Lemma 3.1 that there exist exactly five liftings to the action of I_5 on the surface $\widetilde{Y}$. We fix one of these five liftings, and denote by φ both a generator of the group of induced automorphisms of $\widetilde{Y}$ and that of the group of induced automorphisms of $\Phi(\widetilde{Y})$. Since we have $\varphi^*(\phi^* D_{ig}) = \phi^* D_{ig}$ for each $1 \leq i \leq 5$, the 1-forms $\xi_1, \cdots, \xi_4$ are eigenvectors of the automorphism φ^*, where we set φ^* to be the automorphism of $H^0(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}(K_{\widetilde{Y}}))$ induced by the analytic automorphism φ. Thus the induced automorphism φ of $\Phi(\widetilde{Y})$ is given by $(x_1, x_2, x_3) \mapsto (\varepsilon^{k_1} x_1, \varepsilon^{k_2} x_2, \varepsilon^{k_3} x_3)$, where k_1, k_2, k_3 are certain integers.

Lemma 3.5. *Let l_1, l_2, l_3 be integers and ψ an automorphism of* $\Phi(\widetilde{Y})$ *given by* $(x_1, x_2, x_3) \mapsto (\varepsilon^{l_1} x_1, \varepsilon^{l_2} x_2, \varepsilon^{l_3} x_3)$. *Assume that at least one of the four constants $a_1 a_4$, $a_2 a_3$, $b_1 b_4$, $b_2 b_3$ does not vanish. Then the automorphism ψ is induced by an element in* $\mathrm{Gal}(\widetilde{Y}/Y)$.

Proof. Assume $a_1 a_4$ does not vanish. Then comparing the coefficients of $\psi^* f(x_1, x_2, x_3)$ with those of $f(x_1, x_2, x_3)$, we have $3k_1 + k_2 \equiv 0$, and $k_1 + k_2 \equiv 0$ mod 5. This implies $\psi = \phi_0^3$, where the automorphism ϕ_0 is given in (15). Thus ψ is induced by an element in $\mathrm{Gal}(\widetilde{Y}/Y)$. For the remaining cases, we can give a proof in the same way. $\square$

Our automorphism φ of $\Phi(\widetilde{Y})$ is not induced by an element in $\mathrm{Gal}(\widetilde{Y}/Y)$. By the equalities (14) and the Lemma above, we see that all of the eight coefficients $a_1, \cdots, a_4, b_1, \cdots, b_4$ are equal to zero. Thus we have the following:

Proposition 6. *Let Y be the numerical Godeaux surface constructed in Section* 1, *and Φ the mapping as in Theorem* 3.1. *Then the mapping $\Phi : \widetilde{Y} \to \mathbb{P}^3$ gives an embedding of $\widetilde{Y}$. The surface $\widetilde{Y}$ is given by $X_1^5 + X_2^5 + X_3^5 + X_4^5 = 0$ in $\mathbb{P}^3$. Thus the surface Y is nothing but the Godeaux surface given in* [1, p.170].

Remark 3. The curve $\widetilde{C_{10}} = \sigma(\overline{C_{10}})$ is a one-dimensional component of the fixed locus of the involution ι. Thus by (13), the Stagnaro's curve C_{10} is corresponding to the curve in $\mathbb{P}^2$ given by

$$\widetilde{C} = \{(X_1 : X_2 : X_3 : X_4) \in \mathbb{P}^3 : X_1 + X_4 = 0,\ X_2 + X_3 = 0\}.$$

See also [6].

Remark 4. Actually, we have $\{\widetilde{C_4}, \widetilde{D_4}\} = \{D_g + D_{4g}, D_{2g} + D_{3g}\}$, where $\widetilde{C_4}$ and $\widetilde{D_4}$ are the curves as in Section 2. Indeed, two points $D_g \cap D_{4g}$ and $D_{2g} \cap D_{3g}$ are the base points of $|3K_Y|$ (see [3]). Thus, counting the intersection numbers, we have the assertion.

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