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An analogue of Hardy's theorem for the Heckman-Opdam transform

By

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Abstract

A theorem of Hardy asserts that a function on the real line and its Fourier transform cannot both be very small. We generalize Hardy's theorem for the Heckman-Opdam transform associated with hypergeometric functions.

Introduction

Hardy's theorem on Fourier transform [3] asserts that f and its Fourier transform $\hat{f}$ cannot both be very small. More precisely, let p and q be positive constants and assume that f is a function on the real line satisfying $|f(x)| \leq Ce^{-p|x|^2}$ and $|\hat{f}(\lambda)| \leq Ce^{-q|x|^2}$ for some positive constant C. Then (i) $f = 0$ if $pq > 1/4$; (ii) $f = Ae^{-px^2}$ for some constant A if $pq = 1/4$; (iii) there are infinitely many f if $pq < 1/4$.

Theory of Fourier analysis on the real line has been generalized to the setting of harmonic analysis on Lie groups and homogeneous spaces. In [7], [8], [9] and [2], generalizations of part (i) of Hardy's result to Lie groups were studied. Sitaram and Sundari [8] proved an analogue of Hardy's theorem for the Harish-Chandra transform for spherical functions on a Riemannian symmetric space of the non-compact type.

On the other hand, Heckman and Opdam generalized the theory of spherical functions to the theory of hypergeometric functions associated with root systems. Namely, in the case of rank one symmetric spaces, spherical functions can be expressed by the Gauss hypergeometric function $F(a, b, c; z)$ with $a = (\lambda + k_{\alpha/2} + 2k_\alpha)/2$, $b = (-\lambda + k_{\alpha/2} + 2k_\alpha)/2$, $c = k_{\alpha/2} + k_\alpha + 1/2$, where $2k_{\alpha/2} \in \mathbb{Z}_{\geq 0}$ and $2k_\alpha \in \mathbb{Z}_{>0}$ are multiplicities of restricted roots and $\lambda \in \mathbb{C}$ is the spectral parameter, whereas the theory of Heckman and Opdam in the rank one case covers arbitrary values of $k_{\alpha/2}$ and k_α. Opdam [5] generalized the inversion formula and the Plancherel theorem of the Harish-Chandra transform to the case of arbitrary nonnegative multiplicities.

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In this paper we establish an analogue of Hardy's theorem for the Heckman-Opdam transform associated with the multivariate hypergeometric functions. The strategy of Sitaram and Sundari [8] and machinery of Heckman and Opdam [4, Part I] [5], [6] are enough to work out.

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1. The Heckman-Opdam transform

Let $\mathfrak{a}$ be a Euclidean vector space of dimension n with inner product $(\cdot,\cdot)$. We use the same notation for the corresponding inner product on the dual space $\mathfrak{a}^*$. Let $\mathfrak{h} = \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification. Define $A = \exp \mathfrak{a}$ and $e = \exp 0 \in A$.

Let $R \subset \mathfrak{a}^*$ be a root system. For $\alpha \in \mathfrak{a}^*$, let $X_\alpha \in \mathfrak{a}$ be the element determined by $(X_\alpha, X) = \alpha(X)$ for all $X \in \mathfrak{a}$. For $\alpha \in R$, define

$$\alpha^\vee = \frac{2X_\alpha}{(X_\alpha, X_\alpha)}.$$

We put $R^\vee = \{\alpha^\vee\,;\, \alpha \in R\}$. Let $Q = Q(R) = \mathbb{Z}R$ and $Q^\vee = Q(R^\vee)$ be the root lattice and the coroot lattice respectively. Let $P = \mathrm{Hom}_{\mathbb{Z}}(Q^\vee, \mathbb{Z})$ be the weight lattice. Let W be the Weyl group of R. Choose and fix a positive system $R_+ \subset R$. Let $\mathfrak{a}_+ \subset \mathfrak{a}$ be the corresponding positive Weyl chamber and $A_+ = \exp \mathfrak{a}_+$.

A real multiplicity function k is a map $R \to \mathbb{R}$, denoted by $\alpha \mapsto k_\alpha$ and satisfying $k_\alpha = k_\beta$ if α and β are in the same W-orbit. We set

$$\rho(k) = \frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha \in \mathfrak{h}^* \ .$$

Let $\{\xi_1, \ldots, \xi_n\}$ be an orthonormal basis of $\mathfrak{a}$. We define

$$L(k) = \sum_{j=1}^n \partial_{\xi_j}^2 + \sum_{\alpha \in R_+} k_\alpha \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} \partial_\alpha \ .$$

Notice that $L(0) = \sum_{j=1}^n \partial_{\xi_j}^2$ is the Laplacian on $\mathfrak{a}$, which is independent of the choice of orthonormal basis. In a series of papers, Heckman and Opdam proved:

Theorem 1.1 ([4] Part I, [6]). (1) *There is a commutative algebra* $\mathbb{D}(k)$ *of* W*-invariant differential operators such that*

(a) $L(k) \in \mathbb{D}(k)$,

(b) *There is an algebra isomorphism* $\gamma : \mathbb{D}(k) \to S(\mathfrak{h})^W$, *where* $S(\mathfrak{h})^W$ *is the set of* W*-invariant elements in the symmetric algebra* $S(\mathfrak{h})$,

(c) $\gamma(L(k))(\lambda) = (\lambda, \lambda) - (\rho(k), \rho(k))$.

(2) *Assume* $\tilde{c}(\rho(k), k) \neq 0$, *where*

$$\tilde{c}(\lambda, k) = \prod_{\alpha \in R_+} \frac{\Gamma(\lambda(\alpha^\vee))}{\Gamma(\lambda(\alpha^\vee) + k_\alpha)} \ .$$

Then there is a unique function $F(\lambda, k; \cdot)$ on A such that

(a) $DF(\lambda, k; \cdot) = \gamma(D)(\lambda)F(\lambda, k; \cdot)$ *for all* $D \in \mathbb{D}(k)$,

(b) $F(\lambda, k; \cdot)$ *is W-invariant and analytic at e,*

(c) $F(\lambda, k; e) = 1$.

Remark 1.2. (1) $F(\lambda, k; \cdot)$ is called the Heckman-Opdam hypergeometric function.

(2) If $k_\alpha \geq 0$ for any $\alpha \in R$, then $\tilde{c}(\rho(k), k) \neq 0$.

(3) If $2k_\alpha$ ($\alpha \in R$) are root multiplicities of the restricted root system of a Riemannian symmetric space G/K of the non-compact type, then the Heckman-Opdam hypergeometric function is the radial part of the spherical function on G/K.

Let da denote the Lebesgue measure on A normalized by $\mathrm{vol}(A/\exp(Q^\vee)) = 1$ and $d\lambda$ denote the Lebesgue measure on $\sqrt{-1}\mathfrak{a}$ normalized by $\mathrm{vol}(\sqrt{-1}\mathfrak{a}/\sqrt{-1}P) = 1$. For $f \in C_c^\infty(A)^W$ and $\lambda \in \mathfrak{h}^*$, define

$$\mathcal{F}(f)(\lambda) = \int_A f(a)F(-\lambda, k; a)\delta_k(a)da, \tag{1.1}$$

where

$$\delta_k(\exp X) = \prod_{\alpha \in R_+} \left| e^{\alpha(X)/2} - e^{-\alpha(X)/2} \right|^{2k_\alpha}$$

for $X \in \mathfrak{a}$. Notice that δ_k is W-invariant. We call $\mathcal{F}$ the Heckman-Opdam transform. It coincides with the Harish-Chandra transform for spherical functions if $2k_\alpha$ correspond to the root multiplicities of a Riemannian symmetric space. The inversion formula and the Plancherel theorem for $\mathcal{F}$ was proved in this case by Harish-Chandra. For general case, Opdam proved:

Theorem 1.3 ([5], [6]). *Assume $k_\alpha \geq 0$ for any $\alpha \in R$.*

(1) *Let $f \in C_c(A)^W$. Then we have*

$$f(a) = \int_{\sqrt{-1}\mathfrak{a}^*} \mathcal{F}(f)(\lambda)F(\lambda, k; a)\sigma'(\lambda)d\lambda,$$

where

$$\sigma'(\lambda) = \prod_{\alpha \in R_+} \frac{\Gamma(\lambda(\alpha^\vee) + k_\alpha)\Gamma(-\lambda(\alpha^\vee) + k_\alpha)}{\Gamma(\lambda(\alpha^\vee))\Gamma(-\lambda(\alpha^\vee))}.$$

(2) *The Heckman-Opdam transform extends to a unitary isometry*

$$\mathcal{F} \,:\, L^2(A, \delta_k(a)da) \to L^2(\sqrt{-1}\mathfrak{a}^*, \sigma'(\lambda)d\lambda).$$

2. An analogue of Hardy's theorem

We now state and prove an analogue of Hardy's theorem for the Heckman-Opdam transform.

Theorem 2.1. *Assume $k_\alpha \geq 0$ for all $\alpha \in R$. Let p and q be positive constants. Suppose f is a W-invariant measurable function on A satisfying*

$$|f(\exp X)| \leq C \exp(-p(X, X)), \quad X \in \mathfrak{a} \tag{2.1}$$

and

$$|\mathcal{F}(f)(\sqrt{-1}\lambda)| \leq C \exp(-q(\lambda, \lambda)), \quad \lambda \in \mathfrak{a}^*, \tag{2.2}$$

where C is a positive constant. If $pq > 1/4$, then $f = 0$ almost everywhere.

The proof follows closely that of Sitaram and Sundari [8], where an analogue of Hardy's theorem was proved for group case, i.e. the case of the Harish-Chandra transform on a Riemannian symmetric space.

Let $||\cdot||$ denote the Euclidean norm on $\mathfrak{h}$. We claim that

$$|\mathcal{F}(f)(\lambda)| \leq C_0 \exp\left(\frac{1}{4p'}\|\lambda\|^2\right) \quad \text{for all} \quad \lambda \in \mathfrak{h}^*, \tag{2.3}$$

for some positive constants C_0 and p' such that $0 < p' < p$ and $p'q > 1/4$.

To show (2.3), we express $\mathcal{F}(f)$ as the Cherednik-Opdam transform ([1], [5], [6]):

$$\mathcal{F}(f)(\lambda) = \int_A f(a) G(-w_0\lambda, k; w_0 a)\delta_k(a) da, \tag{2.4}$$

where $G(\lambda, k; \cdot)$ is a simultaneous eigenfunction of the Dunkl-Cherednik operators and w_0 is the longest element of W. The function $G(\lambda, k; \cdot)$ is holomorphic in $\lambda \in \mathfrak{h}$ and in $z = \exp X$ on a tubular neighbourhood of A. We have by [5, Proposition 6.1]

$$|G(\lambda, k; \exp X)| \leq |W|^{1/2} e^{\max_w \mathrm{Re}(w\lambda(X))}, \quad X \in \mathfrak{a}, \tag{2.5}$$

where $|W|$ denotes the order of W. Since f is W-invariant, we can rewrite (2.4) as

$$\mathcal{F}(f)(\lambda) = \sum_{w \in W} \int_{\overline{A_+}} f(a) G(-w_0\lambda, k; w_0 w a)\delta_k(a) da\ . \tag{2.6}$$

Moreover we have an estimate of δ_k,

$$\delta_k(\exp X) \leq C_1 e^{C'||X||} \tag{2.7}$$

for some positive constants C_1 and C'. It follows from (2.1), (2.5), (2.7) and holomorphy of $G(\lambda, k; \cdot)$ in $\lambda \in \mathfrak{h}$ that $\mathcal{F}(f)(\lambda)$ defines an entire function on $\lambda \in \mathfrak{h}$. By (2.6), (2.1), (2.5) and (2.7), we have

$$|\mathcal{F}(f)(\lambda)| \leq C_2 \int_{\overline{\mathfrak{a}_+}} \exp(-p(X, X) + (X, X_\mu) + C'||H||) dX$$

for some positive constant C_2, where $\mu \in \operatorname{Re} W\lambda$ such that $X_\mu \in \overline{\mathfrak{a}_+}$ and dX is the Lebesgue measure on $\mathfrak{a}$ corresponding to da on A. Since

$$\exp(-p(X,X) + C'||H||) \leq C_3 \exp(-p'(X,X))$$

for positive constants C_3 and p' such that $0 < p' < p$ and $p'q > 1/4$. Thus we have

$$\begin{aligned} |\mathcal{F}(f)(\lambda)| &\leq C_3 \int_{\overline{\mathfrak{a}_+}} \exp(-p'(X,X) + (X, X_\mu))dX \\ &\leq C_3 \exp\left(\frac{1}{4p'}(X_\mu, X_\mu)\right) \int_{\mathfrak{a}} \exp(-p'(X,X))dX \\ &\leq C_0 \exp\left(\frac{1}{4p'}||\lambda||^2\right) \end{aligned}$$

for some positive constant C_0. This proves (2.3).

On the other hand, since $p'q > 1/4$, it follows from (2.2) that

$$|\mathcal{F}(f)(\sqrt{-1}\lambda)| \leq C \exp\left(-\frac{1}{4p'}||\lambda||^2\right) \quad \text{for all} \quad \lambda \in \mathfrak{a}^*. \tag{2.8}$$

Since $\mathcal{F}(f)$ satisfies the estimates (2.3) and (2.8), it follows from [8, Lemma 2.1] that

$$\mathcal{F}(f)(\lambda) = A \exp\left(-\frac{1}{4p'}(\lambda, \lambda)\right) \tag{2.9}$$

for some constant A. Equations (2.2) and (2.9) imply $A = 0$, since $p'q > 1/4$. Hence $f = 0$ almost everywhere by Theorem 1.3 (2). The proof of Theorem 2.1 is finished.

Remark 2.2. By [4, Part III], Theorem 2.1 gives an analogue of Hardy's theorem for K-invariant functions on certain semisimple symmetric spaces G/H.

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