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Weak solutions to the compressible Euler equation with an asymptotic γ-law

Dedicated to Professors Takaaki Nishida and Masayasu Mimura
on their sixtieth birthdays

By

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1. Introduction

The one-dimensional motion of a perfect gas is governed by the compressible Euler equation

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) &= 0, \\ \frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + P) &= 0, \end{aligned} \tag{1.1}$$

where unknowns are the density ρ and the velocity u, while the pressure P is supposed to be a given function of ρ. We study the Cauchy problem to the equation under the initial condition

$$\rho|_{t=0} = \rho_0(x), \qquad u|_{t=0} = u_0(x). \tag{1.2}$$

The equation is a prototype of the conservation law

$$U_t + f(U)_x = 0, \tag{1.3}$$

in which

$$U = (\rho, m)^T = (\rho, \rho u)^T, \qquad f(U) = \left(m, \frac{m^2}{\rho} + P\right)^T.$$

A bounded measurable function $U(t,x)$ is a weak solution if

$$\int_0^\infty \int (U\Phi_t + f(U)\Phi_x) dx dt + \int \Phi(0,x) U_0(x) dx = 0$$

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for any test function $\Phi \in C_0^\infty([0,\infty)\times R)$.

Many excellent mathematicians gave existence theorems of global weak solutions to this problem. First we refer T. Nishida, 1968 [5]. He showed the existence of global solutions under the assumption that $P = A\rho$ and

$$T.V.\log\rho_0 < C, \qquad T.V.u_0 < C.$$

The approximate solutions are constructed by the Glimm's scheme and Nishida gave a priori estimates of the growth of the total variations of the approximate solutions by a delicate analysis. On the other hand if we assume $P = A\rho^\gamma, \gamma > 1$, we are interested weak solutions which contains the vacuum. In this case we use the Lax-Friedrichs or Godunov's scheme to construct approximate solutions. A priori L^∞-estimate of the approximate solutions can be obtained comparatively easily. A subsequence therefore converges in the weak star topology. But it is not easy to show that the approximate solutions contain a subsequence which converges almost everywhere. This task was done by the compensated compactness method developed by R. J. DiPerna 1983 [2], [3]. A complete discussion was presented by G.-Q. Chen *et al.* 1985–86 [1]. If we follow their discussions, we find that the Darboux formula

$$\eta = \int_z^w ((w-s)(s-z))^N \phi(s)ds$$

to the Euler-Poisson-Darboux equation

$$\frac{\partial^2\eta}{\partial w\partial z} + \frac{N}{w-z}\left(\frac{\partial\eta}{\partial w} - \frac{\partial\eta}{\partial z}\right) = 0$$

plays a crucial role. The aim of this article is to extend the discussion to the case in which P is proportional to ρ^γ asymptotically.

Thus in this article we assume

(A) $P = P(\rho)$ is a sufficiently smooth function of $\rho > 0$, and

$$0 < P, \qquad 0 < P' = dP/d\rho, \qquad 0 < P'' = d^2P/d\rho^2$$

for $\rho > 0$, and

$$P = A\rho^\gamma(1 + P_1(\epsilon\rho^{\gamma-1}))$$

as $\rho \to 0$. Here A and γ are positive constants,

$$\gamma = 1 + \frac{2}{2N+1},$$

N being a positive integer, ϵ is a positive parameter and $P_1(X)$ is a convergent power series of the form $\sum_{k\geq 1} c_k X^k$.

Our main conclusion is

Theorem 1. *Suppose* (A) *and*

$$0 \leq \rho_0(x) \leq C, \qquad |u_0(x)| \leq C.$$

Then there is a positive number $\epsilon_1 = \epsilon_1(C)$ *such that if* $\epsilon \leq \epsilon_1$ *then* (1.1), (1.2) *has a global weak solution.*

The method of the proof depends upon a generalized Darboux formula to the generalized Euler-Poisson-Darboux equation. The way of discussion is similar to that of C.-H. Hsu, S. S. Lin and T. Makino [4].

As a corollary we have

Theorem 2. *There is a positive number* α *such that if*

$$0 \le \rho_0(x) \le \alpha^{2/(\gamma-1)}, \qquad |u_0(x)| \le \alpha,$$

and if $\epsilon \le 1$, *then* (1.1), (1.2) *admits a global weak solution.*

2. Riemann problem

The Riemann problem is the problem to special initial data of the form

$$\begin{aligned} U_0(x) &= U_L \qquad \text{if} \qquad x < 0, \\ &= U_R \qquad \text{if} \qquad x > 0, \end{aligned}$$

where U_L and U_R are constants. In order to solve Riemann problems we introduce the Riemann invariants

$$w = u + y, \qquad z = u - y,$$

where

$$y = \int_0^\rho \frac{\sqrt{P'}}{\rho} d\rho.$$

Then (1.1) is diagonalized as

$$w_t + \lambda_2 w_x = 0, \qquad z_t + \lambda_1 z_x = 0,$$

where

$$\lambda_1 = u - \sqrt{P'}, \qquad \lambda_2 = u + \sqrt{P'}.$$

The possible states $U = U_R$ connected to U_L on the right by a rarefaction wave are

$$R_1 : \qquad w = w_L, \qquad z > z_L,$$

and

$$R_2 : \qquad w > w_L, \qquad z = z_L.$$

The Rankine-Hugoniot jump condition

$$\sigma[U] = [f(U)],$$

where $[U] = U_R - U_L, [f(U)] = f(U_R) - f(U_L)$, gives the shock curve

$$u_R - u_L = -\sqrt{\frac{[\rho][P]}{\rho_L \rho_R}}.$$

Along this curve we have shocks

$$S_1 : \rho_L < \rho_R, \qquad S_2 : \rho_R < \rho_L.$$

The Riemann problem can be solved uniquely by using these rarefaction waves, shocks and the vacuum state. Moreover if we look at a region of the form

$$\Sigma_B = \{(w, z) : -B \leq z \leq w \leq B\},$$

we have the following

Proposition 2.1. *If the initial data U_L, U_R belong to Σ_B, then the solution of the Riemann problem is confined to Σ_B.*

On the other hand we have

Proposition 2.2. *The region Σ_B is convex in the (ρ, m)-plane.*

Proof. Let us consider the above hedge $m = m(\rho)$ which corresponds to $w = B, -B < z < B$. We have to show $d^2m/d\rho^2 < 0$. Along the hedge we have

$$u = B - \int_0^\rho \frac{\sqrt{P'}}{\rho} d\rho,$$

from which

$$\frac{du}{d\rho} = -\frac{\sqrt{P'}}{\rho}.$$

Therefore

$$\frac{dm}{d\rho} = u - \sqrt{P'}.$$

Differentiate once more, we have

$$\frac{d^2m}{d\rho^2} = -\frac{\sqrt{P'}}{\rho} - \frac{P''}{2\sqrt{P'}} < 0. \qquad \square$$

From Proposition 2.2 we have

Proposition 2.3. *If $U(s) \in \Sigma_B$ for $s \in [a, b]$, then the average*

$$\frac{1}{b-a} \int_a^b U(s) ds$$

belongs to Σ_B.

Let us look at the shock wave which connects the left state U_L to the right state U_R with the shock speed σ. The right state U_R and the shock speed σ are parametrized by $\rho = \rho_R$. Then we have

Proposition 2.4. *Along $S_1(\rho_L < \rho)$, we have $d\sigma/d\rho < 0$, and along $S_2(\rho < \rho_L)$, we have $d\sigma/d\rho > 0$.*

Proof. We can assume $u_L = 0$ and $u = u_R$ is given by

$$u = -\frac{\sqrt{[\rho][P]}}{\rho_L \rho},$$

where $[\rho] = \rho - \rho_L, [P] = P - P_L$. We have

$$\sigma = \frac{[m]}{[\rho]} = \frac{\rho u}{[\rho]}.$$

Therefore

$$\frac{d\sigma}{d\rho} = \frac{[P] - P'[\rho]}{2[\rho]\rho_L\sqrt{[\rho][P]}}.$$

Since $P'' > 0$, we know $[P] < P'[\rho]$. Thus we see $[\rho]d\sigma/d\rho < 0$. □

3. Entropies

A pair of functions η and q of the state U is called an entropy-entropy flux if it satisfies the equation

$$D_U q = D_U \eta . D_U f. \tag{3.1}$$

Using the Riemann invariants, we can write (3.1) as

$$q_w = \lambda_2 \eta_w, \qquad q_z = \lambda_1 \eta_z.$$

By eliminating q, we get the second order equation for η:

$$\frac{\partial^2 \eta}{\partial w \partial z} + \frac{1}{4\sqrt{P'}}\left(1 - \frac{\rho P''}{2P'}\right)\left(\frac{\partial \eta}{\partial w} - \frac{\partial \eta}{\partial z}\right) = 0. \tag{3.2}$$

As $\epsilon = 0$, this equation is reduced to be the Euler-Poisson-Darboux equation

$$\eta_{wz} + \frac{N}{w - z}(\eta_w - \eta_z) = 0. \tag{3.3}$$

Therefore we call (3.2) a generalized Euler-Poisson-Darboux equation.

The kinetic energy

$$\eta^* = \frac{1}{2}\rho u^2 + \Phi(\rho),$$
$$\Phi(\rho) = \rho \int_0^\rho \frac{P'}{\rho} d\rho - P = \rho \int_0^\rho \frac{P}{\rho^2} d\rho,$$

and its flux

$$q^* = \left(\frac{1}{2}\rho u^2 + \Phi_1(\rho)\right) u, \qquad \Phi_1(\rho) = \rho \int_0^\rho \frac{P'}{\rho} d\rho = \Phi(\rho) + P$$

satisfy the generalized Euler-Poisson-Darboux equation. This entropy-entropy flux will be called standard. The important fact is

Proposition 3.1. *The Hessian $D^2_U\eta^*$ is positive definite, i.e., for any fixed B there is a positive constant k such that*

$$(\xi|D^2_U\eta^*.\xi) \geq k|\xi|^2$$

for any $U \in \Sigma_B$ and $\xi = (\xi_0, \xi_1)$ with $|\xi|^2 = (\xi_0)^2 + (\xi_1)^2$.

Proof. By direct computations, we see

$$\eta^*_{\rho\rho} = \frac{u^2}{\rho} + \frac{P'}{\rho},$$
$$\eta^*_{\rho m} = -\frac{u}{\rho},$$
$$\eta^*_{mm} = \frac{1}{\rho}.$$

Hence

$$\begin{aligned}(\xi|D^2\eta^*.\xi) &= \eta_{\rho\rho}\xi_0^2 + 2\eta_{\rho m}\xi_0\xi_1 + \eta_{mm}\xi_1^2 \\ &= \frac{1}{\rho}((u^2+P')\xi_0^2 - 2u\xi_0\xi_1 + \xi_1^2) \\ &\geq \frac{2P'}{\rho(A+C+\sqrt{(A-C)^2+4B^2})},\end{aligned}$$

where

$$A = u^2 + P', \qquad B = -u, \qquad C = 1.$$ □

4. Construction of approximate solutions

Let us construct approximate solutions using the Godunov scheme. The construction is similar if we use the Lax-Friedrichs scheme.

Suppose that the initial data $U_0(x)$ is confined to an invariant region Σ_B. Put $\Lambda_0 = \sup\{|\lambda_j(U)||j = 1, 2, U \in \Sigma_B\}$. Fixing $\Lambda_1 > \Lambda_0$, we take mesh lengths $\Delta x, \Delta t$ such that $\Delta x = \Lambda_1 \Delta t$. We denote $\Delta = \Delta x$.

Let us construct the approxomate solution $U^\Delta(t, x)$. First we put

$$U_0^\Delta(x) = U_0(x)\chi_{[-1/\Delta, 1/\Delta]}.$$

We define

$$U^\Delta(+0, x) = \frac{1}{2\Delta}\int_{2j\Delta}^{(2j+2)\Delta} U_0^\Delta(x)dx$$

for $2j\Delta < x \leq (2j+2)\Delta$. Solving the Riemann problem on each interval $[2(j-1)\Delta, 2(j+1)\Delta]$, we define $U^\Delta(t, x)$ for $0 \leq t < \Delta t$. Since the Courant-Friedrichs-Lewy condition is satisfied, the wave from the center $2j\Delta$ does not intersect. If $U^\Delta(t, x)$ for $0 \leq t < n\Delta t$ has been defined, then we define

$$U^\Delta(n\Delta t, x) = \frac{1}{2\Delta}\int_{2j\Delta}^{(2j+2)\Delta} U^\Delta(n\Delta t - 0, x)dx$$

for $2j\Delta < x \le (2j+2)\Delta$. Solving the Riemann problem, we define $U^\Delta(t,x)$ for $n\Delta t \le t < (n+1)\Delta t$.

By Propositions 2.1 and 2.3, it is inductively guaranteed that U^Δ remains in Σ_B, say,

Proposition 4.1. *The approximate solution $U^\Delta(t,x)$ satisfies $U^\Delta(t,x) \in \Sigma_B$, therefore,*

$$0 \le \rho^\Delta(t,x) \le C, \qquad |u^\Delta(x)| \le C.$$

Moreover we shall prove

Proposition 4.2. *For any test function Φ it holds that*

$$\iint (\Phi_t U^\Delta + \Phi_x f(U^\Delta))dxdt + \int \Phi(0,x)U_0^\Delta(x)dx = O(\Delta^{1/2}).$$

In order to prove Proposition 4.2, we prepare

Proposition 4.3. *For any shock wave from U_L to U_R with the shock speed σ and for any convex entropy η, we have*

$$\sigma[\eta] - [q] \ge 0,$$

where $[\eta] = \eta(U_R) - \eta(U_L), [q] = q(U_R) - q(U_L)$.

Proof. The right state of shocks can be parametrized by $\rho = \rho_R$. Putting

$$Q(\rho) = \sigma[\eta] - [q],$$

we shall see $dQ/d\rho \ge 0$ along $S_1 : [\rho] > 0$ and $dQ/d\rho \le 0$ along $S_2 : [\rho] < 0$. Using the equation (3.1) and the differentiation of the Rankine-Hugoniot condition, we have

$$\begin{aligned}\frac{dQ}{d\rho} &= \frac{d\sigma}{d\rho}([\eta] - D_U\eta(U).[U]) \\ &= -\frac{d\sigma}{d\rho}\int_0^1 \theta(U - U_L|D_U^2\eta(U_L + \theta(U - U_L).(U - U_L))d\theta.\end{aligned}$$

We supposed $D_U^2\eta \ge 0$. By Proposition 2.4, we know $d\sigma/d\rho < 0$ on S_1 and $d\sigma/d\rho > 0$ on S_2. □

Proof of Proposition 4.2. We fix T to consider U^Δ on $0 \le t \le T$. First we shall show

$$\sum_{j,n}\int_{2j\Delta}^{(2j+2)\Delta} |U(n\Delta t - 0, x) - U(n\Delta t + 0, (2j+1)\Delta)|^2 dx \le C. \tag{4.1}$$

Let us consider the standard entropy η^*. Then we have

$$0 = \int \eta^*(U(T,x))dx - \int \eta^*(U(0,x))dx + L + \Sigma,$$

$$L = \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} (\eta^*(U(n\Delta t - 0, x)) - \eta^*(U(n\Delta t + 0, (2j+1)\Delta)))dx,$$

$$\Sigma = \int_0^T \sum_{shocks} (\sigma[\eta^*] - [q^*])dt.$$

We write $U_0 = U(n\Delta t + 0, (2j+1)\Delta), U_1 = U(n\Delta t - 0, x)$. Since

$$U_0 = \frac{1}{2\Delta} \int_{2j\Delta}^{(2j+2)\Delta} U_1 dx,$$

we see

$$L = \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} \int_0^1 (1-\theta)(U_1 - U_0 | D_U^2 \eta^*(U_0 + \theta(U_1 - U_0)).(U_1 - U_0))d\theta dx$$
$$\geq 0.$$

On the other hand we have $\Sigma \geq 0$ from Proposition 4.3. Thus $L \leq C, \Sigma \leq C$. But from Proposition 3.1, we have $D_U^2 \eta^* \geq k$. Therefore

$$C \geq L \geq \frac{k}{2} \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} |U_1 - U_0|^2 dx.$$

Thus we get (4.1).

Now let us consider a test function Φ. Put

$$J = \iint (\Phi_t U^\Delta + \Phi_x f(U^\Delta))dxdt + \int \Phi(0,x) U_0^\Delta dx.$$

Since U^Δ is a weak solution on each time strip $n\Delta t < t < (n+1)\Delta t$, we have

$$J = \sum_n \int \Phi(n\Delta t, x)(U(n\Delta t - 0, x) - U(n\Delta t + 0, x))dx$$
$$= J_1 + J_2,$$
$$J_1 = \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} \Phi(n\Delta t, (2j+1)\Delta)(U(n\Delta t - 0, x) - U(n\Delta t + 0, x))dx,$$
$$J_2 = \sum_{j.n} \int_{2j\Delta}^{(2j+2)\Delta} (\Phi(t,x) - \Phi(n\Delta t, (2j+1)\Delta))(U(n\Delta t - 0, x)$$
$$- U(n\Delta t + 0, x))dx.$$

Since

$$U(n\Delta t + 0, x) = \frac{1}{2\Delta} \int_{2j\Delta}^{(2j+2)\Delta} U(n\Delta t - 0, x)dx$$

for $2j\Delta < x < (2j+2)\Delta$, we see $J_1 = 0$. It follows from (4.1) that

$$|J_2| \le C\Delta^{1/2}||\Phi||_{C^1} \left(\sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} |U(n\Delta t - 0, x) - U(n\Delta t + 0, x)|^2 dx \right)^{1/2}$$
$$\le C'\Delta^{1/2}.$$

Here we have used $T/\Delta t = O(1/\Delta)$. □

Summing up, we have the following theorem.

Theorem 3. *The approximate solution $U^\Delta(t,x)$ satisfies*

$$0 \le \rho^\Delta(t,x) \le C, \qquad |u^\Delta(t,x)| \le C$$

and

$$\iint (\Phi_t U^\Delta + \Phi_x f(U^\Delta)) dxdt + \int \Phi(0,x) U_0^\Delta(x) = O(\Delta^{1/2})$$

for any test function Φ.

We expect that U^Δ tends to a weak solution everywhere. For the case $\epsilon = 0$ this was done by DiPerna [2], [3] and G. Q. Chen *et al.* [1]. In their proof the Darboux formula

$$\eta = \int_z^w ((w-s)(s-z))^N \phi(s) ds$$

which gives solutions of the Euler-Poisson-Darboux equation (3.3), ϕ being arbitrary, plays an important role. Section 5 will be devoted to find such an integral formula for the generalized Euler-Poisson-Darboux equation (3.2).

Remark. We note that

$$\lambda_2 - \lambda_1 = \sqrt{P'} > 0,$$
$$\frac{\partial \lambda_1}{\partial z} = \frac{1}{2}\left(1 + \frac{\rho P''}{2P'}\right) > 0,$$
$$\frac{\partial \lambda_2}{\partial w} = \frac{1}{2}\left(1 + \frac{\rho P''}{2P'}\right) > 0$$

for $\rho > 0$.

This says that the system is strictly hyperbolic and genuinely nonlinear on $\rho > 0$. Therefore the Glimm's theory can be applied if

$$||U_0(x) - U^*||_{L^\infty} + T.V.U_0$$

is sufficiently small, where U^* is a constant state such that $\rho^* > 0$. But the vacuum may not be covered by this application of the general theorem.

5. Generalized Darboux formula

In this section we seek an integration formula for solutions of the generalized Euler-Poisson-Darboux equation (3.2). Using

$$y = \int_0^\rho \frac{\sqrt{P'}}{\rho} d\rho,$$

as an independent variable, we can write (3.2) as

$$\eta_{uu} - \eta_{yy} + \frac{1}{\sqrt{P'}}\left(1 - \frac{\rho P''}{2P'}\right)\eta_y = 0. \tag{5.1}$$

Using the assumption (A), we can write

$$\frac{1}{\sqrt{P'}}\left(1 - \frac{\rho P''}{2P'}\right) = \frac{2N}{y} + a(y,\epsilon), \qquad a = \epsilon y[\epsilon y^2]_0, \tag{5.2}$$

where $[X]_0$ denotes a convergent power series.

Let us introduce the sequence of variables $\eta_0 = \eta, \eta_1, \dots, \eta_N = V$ by

$$\frac{\partial \eta_j}{\partial y} = y\eta_{j+1}$$

and

$$\eta_j(u,y) = I\eta_{j+1}(u,y) = \int_0^y Y\eta_{j+1}(u,Y)dY.$$

The sequence of integro-differential operators L_j is defined by

$$L_j\eta_j = \eta_{j,uu} - \eta_{j,yy} + \left(\frac{2(N-j)}{y} + a\right)\eta_{j,y} + j\tilde{a}\eta_j + \sum_{k=1}^{j-1} c_{jk}\tilde{a}_k I^k \eta_j,$$

where

$$\begin{aligned}
\tilde{a} &= \frac{\partial a}{\partial y} + \frac{a}{y}, \\
\tilde{a}_k &= \left(\frac{1}{y}\frac{d}{dy}\right)^k \tilde{a}, \\
c_{j1} &= \frac{j(j-1)}{2}, \\
c_{j+1,k} &= c_{j,k-1} + c_{jk} \qquad (2 \le k \le j). \\
c_{jj} &= 0
\end{aligned}$$

Clearly $\tilde{a}, \tilde{a}_k$ are of the form $\epsilon[\epsilon y^2]_0$ and are smooth functions of $0 \le y < \infty$. By the definition we have formally

$$\frac{1}{y}\frac{\partial}{\partial y}(L_j\eta_j) = L_{j+1}\eta_{j+1}.$$

Now we consider the equation $L_N V = 0$. The Cauchy problem

$$\text{(Q)} \qquad V_{yy} - V_{uu} = aV_y + N\tilde{a}V + \sum_{k=1}^{N-1} c_k \tilde{a}_k I^k V,$$

$$V = 0, \qquad V_y = 2^{N+1} N! \phi(u) \qquad \text{at} \qquad y = 0$$

is to be considered, where $c_k = c_{Nk}$.

Proposition 5.1. *If $\phi \in C^1(R)$, then the problem* (Q) *admits a unique solution V in $C^2([0,\infty) \times R)$.*

Proof. Let us denote by $H(u,y;V)$ the right hand side of the equation of (Q). Then the problem (Q) is transformed to the integral equation

$$V(u,y) = 2^N N! \int_{u-y}^{u+y} \phi(\xi) d\xi + \frac{1}{2} \int_0^y \int_{u-y+Y}^{u+y-Y} H(X,Y;V) dX dY.$$

We can solve this integral equation by the iteration

$$V^0(u,y) = 2^N N! \int_{u-y}^{u+y} \phi(\xi) d\xi,$$
$$V^{n+1}(u,y) = 2^N N! \int_{u-y}^{u+y} \phi(\xi) d\xi + \frac{1}{2} \int_0^y \int_{u-y+Y}^{u+y-Y} H(X,Y;V^n) dX dY.$$

Then it is easy to get the estimates

$$|V^{n+1}(u,y) - V^n(u,y)| \leq \frac{C^{n+1} y^{n+1}}{(n+1)!}.$$

Thus V^n tends to a limit V uniformly, which solves (Q). □

Now we put

$$\eta_N = V, \qquad \eta_{N-k} = I\eta_{N-k+1}.$$

Since η_{N-k} and its derivatives of order ≤ 2 all vanish on $y=0$ for $k \geq 1$, we see that $L_j \eta_j = 0$ and particularly $\eta = \eta_0$ satisfies the generalized Euler-Poisson-Darboux equation (5.1).

Proposition 5.2. *There is a C^{N+2}-function $G(v,y)$ of $|v| \leq y, 0 \leq y$ such that the solution V of* (Q) *satisfies*

$$\text{(5.3)} \qquad V(u,y) = \int_{u-y}^{u+y} G(\xi - u, y) \phi(\xi) d\xi.$$

Moreover

$$G = 2^N N! + O(\epsilon y^2),$$
$$\partial_v^{p_1} \partial_y^{p_2} G = O(\epsilon) \qquad \mathit{for} \qquad 1 \leq p_1 + p_2 \leq N+2.$$

Proof. We consider the approximate solution $V^n(u, y)$ which appeared in the proof of Proposition 5.1. We write

$$H = (aV)_y + bV + \sum_{k=1}^{N-1} c_k \tilde{a}_k I^k V,$$

where

$$b = N\tilde{a} - \frac{\partial a}{\partial y}.$$

It is easy to see inductively that there is a kernel $G^n(v, y)$ such that

$$V^n(u, y) = \int_{u-y}^{u+y} G^n(\xi - u, y)\phi(\xi)d\xi.$$

In fact $G^0 = 2^N N!$ and

$$G^{n+1} = 2^N N! + \frac{1}{2}\left(T_1 G^n + T_2 G^n + \sum_{k=1}^{N-1} T_{3k} G^n\right),$$

$$T_1 G(v, y) = \int_{(v+y)/2}^{y} a(Y)G(v + y - Y, Y)dY + \int_{(y-v)/2}^{y} a(Y)G(v - y + Y, Y)dY,$$

$$T_2 G(v, y) = \iint_{D(v,y)} b(Y)G(v - Z, Y)dZdY,$$

where

$$D(v, y) = \{(Z, Y) : Z - Y \le v \le Z + Y, -y + Y \le Z \le y - Y\},$$

for $|v| < Y$, by which $0 < Y < y$ on $D(v, y)$,

$$T_{3k} G(v, y) = \iint_{D(v,y)} c_k \tilde{a}_k(Y) J^k G(v - Z, Y)dZdY,$$

where

$$JG(v, y) = \int_{|v|}^{y} Y G(v, Y)dY. \tag{5.4}$$

It is easy to see G^n converges uniformly to a limit G which satisfies (5.3). We can differentiate G^{n+1} $(N + 2)$-times by supposing that $G^n \in C^{N+2}$, and it is easy to see that the derivatives converge uniformly, so $G \in C^{N+2}$. Since the limit G satisfies the integral equation

$$G = 2^N N! + \frac{1}{2}\left(T_1 G + T_2 G + \sum_{k=1}^{N-1} T_{3k} G\right), \tag{5.5}$$

it is easy to observe the stated estimates by keeping in mind that $a = O(\epsilon y), \tilde{a}_k = O(\epsilon)$ and their derivatives are of $O(\epsilon)$. $\square$

By putting

$$K_{N-k} = JK_{N-k+1} = J^k G,$$

we have

$$\eta_{N-k} = \int_{u-y}^{u+y} K_{N-k}(\xi - u, y)\phi(\xi)d\xi.$$

So, if we put

$$K = J^N G,$$

then

$$\eta(u, y) = \int_{u-y}^{u+y} K(\xi - u, y)\phi(\xi)d\xi$$

is the solution of the generalized Euler-Poisson-Darboux equation. By induction we see

$$J^k G(v, y) = \frac{2^N N!}{2^k k!}(y^2 - v^2)^k(1 + O(\epsilon y^2)).$$

Thus we get

Proposition 5.3. *There is a kernal $K(v, y)$ which is of class C^{N+2} in $|v| \leq y, 0 \leq y$ such that*

(5.6) $$\eta(u, y) = \int_{u-y}^{u+y} K(\xi - u, y)\phi(\xi)d\xi$$

gives a solution of the generalized Euler-Poisson-Darboux equation for any smooth ϕ. Moreover

(5.7) $$K(v, y) = (y^2 - v^2)^N(1 + O(\epsilon y^2)).$$

In order to apply this formula (5.6), which will be called the generalized Darboux formula, we need more detailed estimates.

Proposition 5.4. *We have*

(5.8) $$G_v, \qquad G_y = O(\epsilon y).$$

At $(v, y) = ((2s - 1)y, y)$, s being a parameter, we have

(5.9) $$K = 2^{2N}(s - s^2)^N y^{2N} + O(\epsilon y^{2N+2}),$$

(5.10) $$(2s - 1)K_v + K_y = 2^{2N+1}N(s - s^2)^N y^{2N-1} + O(\epsilon y^{2N+1}),$$

(5.11) $$(2s - 1)((2s - 1)K_v + K_y)_v + ((2s - 1)K_v + K_y)_y = 2^{2N+1}N(2N - 1)(s - s^2)^N y^{2N-2} + O(\epsilon y^{2N}).$$

Proof. It is easy to get (5.8) by differentiating the terms of the integral equation (5.5). (5.9) is nothing but (5.7). In order to prove (5.10), it is sufficient to see

$$(2s-1)K_v + K_y = yJ^{N-1}G(v,y) - (2s-1)vG(v,|v|)J^{N-1}1 \\ + (2s-1)J^N G_v.$$

Let us show (5.11). If $N=1$, then

$$\begin{aligned}((2s-1)K_v + K_y)_y &= G + yG_y + (2s-1)yG_v \\ &= 2 + O(\epsilon y^2), \\ ((2s-1)K_v + K_y)_v &= yG_v - (2s-1)G(v,|v|) \\ &\quad - (2s-1)(vG_v + |v|G_y) - (2s-1)vG_v + (2s-1)JG_{vv} \\ &= -2(2s-1) + O(\epsilon y^2),\end{aligned}$$

therefore

$$(2s-1)((2s-1)K_v + K_y)_v + ((2s-1)K_v + K_y)_y = 8(s-s^2) + O(\epsilon y^2).$$

Suppose $N \geq 2$. Then

$$\begin{aligned}((2s-1)K_v + K_y)_y &= J^{N-1}G + y^2 J^{N-2}G \\ &\quad - (2s-1)vG(v,|v|)yJ^{N-2}1 + (2s-1)yJ^{N-1}G_v \\ &= 2^{2N-1}N(s-s^2)^{N-1}y^{2N-2} \\ &\quad + 2^{2N-2}N(N-1)(s-s^2)^{N-2}y^{2N-2} + O(\epsilon y^{2N}), \\ ((2s-1)K_v + K_y)_v &= -yvJ^{N-2}G(v,|v|) + yJ^{N-1}G_v \\ &\quad - (2s-1)G(v,|v|)J^{N-1}1 + (2s-1)(vG_v + |v|G_y)J^{N-1}1 \\ &\quad + (2s-1)v^2G(v,|v|)J^{N-2}1 - (2s-1)vJ^{N-1}G_v \\ &\quad + (2s-1)J^N G_{vv} \\ &= -2^{2N-2}(2s-1)N(N-1)(s-s^2)^{N-2}y^{2N-2} \\ &\quad - 2^{2N-1}(2s-1)N(s-s^2)^{N-1}y^{2N-2} \\ &\quad + 2^{2N-2}(2s-1)^3N(N-1)(s-s^2)^{N-2}y^{2N-2}O(\epsilon y^{2N}).\end{aligned}$$

Thus we get (5.11). □

6. Estimates of the Hessian of entropies

Let us consider the entropy η given by the generalized Darboux formula

$$\eta(u,y) = \int_{u-y}^{u+y} K(\xi - u, y)\phi(\xi)d\xi,$$

where ϕ is a fixed C^2-function. In this section we seek estimates of the derivatives of η with respect to $\rho, m = \rho u$. We introduce the auxiliary variables

$$R = y^{2N+1}, \qquad M = uy^{2N+1}.$$

Proposition 6.1. *We have*

$$\frac{\partial \eta}{\partial M} = 2^{2N+1}\int_0^1 (s-s^2)^N D\phi(u+(2s-1)y)ds + O(\epsilon y^2), \tag{6.1}$$

$$\frac{\partial \eta}{\partial R} = 2^{2N+1}\int_0^1 (s-s^2)^N \phi(u+(2s-1)y)ds + 2^{2N+1}\int_0^1 (s-s^2)^N\left(-u+\frac{2s-1}{2N+1}y\right)D\phi ds + O(\epsilon y^2), \tag{6.2}$$

$$\frac{\partial^2 \eta}{\partial M^2} = 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N D^2\phi(u+(2s-1)y)ds + O(\epsilon y^{-2N+1}), \tag{6.3}$$

$$\frac{\partial^2 \eta}{\partial R\partial M} = 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N\left(-u+\frac{2s-1}{2N+1}y\right)D^2\phi ds + O(\epsilon y^{-2N+1}), \tag{6.4}$$

$$\frac{\partial^2 \eta}{\partial R^2} = 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N \times\left(\left(-u+\frac{2s-1}{2N+1}y\right)^2+\frac{4}{(2N+1)^2}(s-s^2)y^2\right)D^2\phi ds + O(\epsilon y^{-2N+1}). \tag{6.5}$$

Proof. We write

$$\eta = 2y\int_0^1 K((2s-1)y,y)\phi(u+(2s-1)y)ds = 2R^{\frac{1}{2N+1}}\int_0^1 K((2s-1)R^{\frac{1}{2N+1}},R^{\frac{1}{2N+1}})\phi\left(\frac{M}{R}+(2s-1)R^{\frac{1}{2N+1}}\right)ds.$$

Differentiating η with respect to M, we get

$$\frac{\partial \eta}{\partial M} = 2R^{\frac{-2N}{2N+1}}\int_0^1 K((2s-1)y,y)D\phi(u+(2s-1)y)ds.$$

Using Proposition 5.4 (5.9), we see (6.1). Differentiating η with respect to R, we have

$$\frac{\partial \eta}{\partial R} = (1)+(2)+(3),$$

$$(1) = \frac{2}{2N+1}R^{\frac{-2N}{2N+1}}\int_0^1 K\phi ds,$$

$$(2) = \frac{2}{2N+1}R^{\frac{-2N+1}{2N+1}}\int_0^1 ((2s-1)K_v+K_y)\phi ds,$$

$$(3) = 2R^{\frac{-2N}{2N+1}}\int_0^1 K\left(-u+\frac{2s-1}{2N+1}y\right)D\phi ds.$$

Using (5.9), we see

$$(1) = \frac{2^{2N+1}}{2N+1}\int_0^1 (s-s^2)^N \phi ds + O(\epsilon y^2).$$

Using (5.10), we see

$$(2) = \frac{2^{2N+2}N}{2N+1}\int_0^1 (s-s^2)^N \phi ds + O(\epsilon y^2).$$

Using (5.9), we see

$$(3) = 2^{2N+1}\int_0^1 (s-s^2)^N \left(-u + \frac{2s-1}{2N+1}y\right) D\phi ds + O(\epsilon y^2).$$

Summing up, we get (6.2). We have

$$\frac{\partial^2 \eta}{\partial M^2} = 2R^{\frac{-4N-1}{2N+1}}\int_0^1 KD^2\phi ds.$$

Using (5.9), we get (6.3). Next we have

$$\begin{aligned}
\frac{\partial^2 \eta}{\partial R \partial M} &= (4) + (5) + (6), \\
(4) &= -\frac{4N}{2N+1}R^{\frac{-4N-1}{2N+1}}\int_0^1 KD\phi ds, \\
(5) &= \frac{2}{2N+1}R^{\frac{-4N}{2N+1}}\int_0^1 ((2s-1)K_v + K_y)D\phi ds, \\
(6) &= 2R^{\frac{-4N-1}{2N+1}}\int_0^1 K\left(-u + \frac{2s-1}{2N+1}y\right) D^2\phi ds.
\end{aligned}$$

In a similar manner to $\partial\eta/\partial R$, we get (6.4). Finally differentiating $\partial\eta/\partial R$ with respect to R, we have

$$\begin{aligned}
\frac{\partial^2 \eta}{\partial R^2} &= \frac{\partial}{\partial R}(1) + \frac{\partial}{\partial R}(2) + \frac{\partial}{\partial R}(3), \\
\frac{\partial}{\partial R}(1) &= (7) + (8) + (9), \\
(7) &= -\frac{4N}{(2N+1)^2}R^{\frac{-4N-1}{2N+1}}\int_0^1 K\phi ds, \\
(8) &= \frac{2}{(2N+1)^2}R^{\frac{-4N}{2N+1}}\int_0^1 ((2s-1)K_v + K_y)\phi ds, \\
(9) &= \frac{2}{2N+1}R^{\frac{-4N-1}{2N+1}}\int_0^1 K\left(-u + \frac{2s-1}{2N+1}y\right) D\phi ds, \\
\frac{\partial}{\partial R}(2) &= (10) + (11) + (12),
\end{aligned}$$

$$(10) = \frac{2(-2N+1)}{(2N+1)^2} R^{\frac{-4N}{2N+1}} \int_0^1 ((2s-1)K_v + K_y)\phi ds,$$

$$(11) = \frac{2}{(2N+1)^2} R^{\frac{-4N+1}{2N+1}} \int_0^1 ((2s-1)((2s-1)K_v + K_y)_v + ((2s-1)K_v + K_y)_y)\phi ds,$$

$$(12) = \frac{2}{2N+1} R^{\frac{-4N}{2N+1}} \int_0^1 ((2s-1)K_v + K_y)\left(-u + \frac{2s-1}{2N+1}y\right) D\phi ds,$$

$$\frac{\partial}{\partial R}(3) = (13) + (14) + (15) + (16),$$

$$(13) = \frac{-4N}{2N+1} R^{\frac{-4N-1}{2N+1}} \int_0^1 K\left(-u + \frac{2s-1}{2N+1}y\right) D\phi ds,$$

$$(14) = \frac{2}{2N+1} R^{\frac{-4N}{2N+1}} \int_0^1 ((2s-1)K_v + K_y)\left(-u + \frac{2s-1}{2N+1}y\right) D\phi ds,$$

$$(15) = 2R^{\frac{-4N-1}{2N+1}} \int_0^1 K\left(u + \frac{2s-1}{(2N+1)^2}y\right) D\phi ds,$$

$$(16) = 2R^{\frac{-4N-1}{2N+1}} \int_0^1 K\left(-u + \frac{2s-1}{2N+1}y\right)^2 D^2\phi ds.$$

Using (5.12) to estimate (11), we can see (6.5). □

Let us recall the standard entropy η^*, which is generated by

$$\phi^*(u) = \frac{A'}{2}u^2,$$

where

$$A' = (2N+1)^{-2N}(2N-1)!!((2N+1)/(2N+3)A)^{\frac{2N+1}{2}}/2^{N+1}N!.$$

We note that $D^2\phi^*(u) = A'$. We are going to show the Hessian $D_U^2\eta$ is dominated by $D_U^2\eta^*$.

Proposition 6.2. *On each compact suset of* $\{\rho \geq 0\}$ *we have*

$$|(\xi|D_U^2\eta.\xi)| \leq C(\xi|D_U^2\eta^*.\xi),$$

provided that ϵ *is sufficiently small.*

Proof. By the assumption we have

$$R = a\rho(1 + [\epsilon\rho^{\frac{2}{2N+1}}]_1),$$
$$\frac{dR}{d\rho} = a + [\epsilon\rho^{\frac{2}{2N+1}}]_1,$$
$$\frac{d^2R}{d\rho^2} = \epsilon\rho^{\frac{-2N+1}{2N+1}}[\epsilon\rho^{\frac{2}{2N+1}}]_0,$$

where $a = ((2N+3)(2N+1)A)^{((2N+1)/2)}$. Using these, we see

$$\frac{\partial R}{\partial \rho} = a + O(\epsilon y^2), \qquad \frac{\partial R}{\partial m} = 0,$$
$$\frac{\partial M}{\partial \rho} = O(\epsilon y^2)u, \qquad \frac{\partial M}{\partial m} = a + O(\epsilon y^2),$$
$$\frac{\partial^2 R}{\partial \rho^2} = O(\epsilon y^{-2N+1}), \qquad \frac{\partial^2 R}{\partial m \partial \rho} = 0,$$
$$\frac{\partial^2 R}{\partial m^2} = 0, \qquad \frac{\partial^2 M}{\partial \rho^2} = O(\epsilon y^{-2N+1})u,$$
$$\frac{\partial^2 M}{\partial \rho \partial m} = O(\epsilon y^{-2N+1}), \qquad \frac{\partial^2 M}{\partial m^2} = 0.$$

Therefore by the chain rule we get

$$\frac{\partial^2 \eta}{\partial \rho^2} = a^2 \frac{\partial^2 \eta}{\partial R^2} + O(\epsilon y^{-2N+1}),$$
$$\frac{\partial^2 \eta}{\partial \rho \partial m} = a^2 \frac{\partial^2 \eta}{\partial R \partial M} + O(\epsilon y^{-2N+1}),$$
$$\frac{\partial^2 \eta}{\partial m^2} = a^2 \frac{\partial^2 \eta}{\partial M^2} + O(\epsilon y^{-2N+1}).$$

Here we have used $\partial\eta/\partial R, \partial\eta/\partial M = O(1)$. Hence it follows from Proposition 6.1 that

$$\begin{aligned}
(\xi|D^2\eta.\xi) &= \eta_{\rho\rho}\xi_0^2 + 2\eta_{\rho m}\xi_0\xi_1 + \eta_{mm}\xi_1^2 \\
&= \frac{2^{2N+1}a^2}{y^{2N+1}} \int_0^1 (s-s^2)^N Z[\xi] D^2\phi ds + O(\epsilon y^{-2N+1}), \\
Z[\xi] &= Z_{00}\xi_0^2 + 2Z_{01}\xi_0\xi_1 + Z_{11}\xi_1^2, \\
Z_{00} &= \left(-u + \frac{2s-1}{2N+1}y\right)^2 + \frac{4}{(2N+1)^2}(s-s^2)y^2, \\
Z_{01} &= -u + \frac{2s-1}{2N+1}y, \\
Z_{11} &= 1.
\end{aligned}$$

Since

$$Z_{00}Z_{11} - Z_{01}^2 = \frac{4}{(2N+1)^2}(s-s^2)y^2,$$

we see

$$Z[\xi] \geq \kappa(s-s^2)y^2|\xi|^2,$$

κ being a positive constant uniformly taken for $|u| \leq C, 0 \leq y \leq C$. If $|D^2\phi| \leq C_1$, then

$$|(\xi|D^2\eta.\xi)| \leq C_1 \frac{2^{2N+1}a^2}{y^{2N+1}} \int_0^1 (s-s^2)^N Z ds + O(\epsilon y^{-2N+1}).$$

Since $D^2\phi^* = A'$, we have

$$|(\xi|D^2\eta.\xi)| \leq \frac{C_1}{A'}(\xi|D^2\eta^*.\xi) + O(\epsilon y^{-2N+1}).$$

Since

$$(\xi|D^2\eta^*.\xi) \geq \kappa' y^{-2N+1}|\xi|^2,$$

we get the required estimate. □

Remark. All estimates are obtained by assuming that ϕ is C^2, and the smalness of ϵ of Proposition 6.2 does not depend on η.

As for the first derivatives the following is now clear.

Proposition 6.3. *On each compact subset of* $\{\rho \geq 0\}$ *we have*

$$\left|\frac{\partial\eta}{\partial\rho}\right| \leq C, \qquad \left|\frac{\partial\eta}{\partial m}\right| \leq C.$$

7. Compactness of $\eta_t + q_x$

Let us consider an entropy η generated by ϕ through the generalized Darboux formula and its flux q. In this section we will prove

Proposition 7.1. *Let* U^Δ *be the approximate solutions constructed in Section* 4. *Then* $\eta(U^\Delta)_t + q(U^\Delta)_x$ *lies in a compact subset of* $H^{-1}_{loc}(\Omega)$, Ω *being a bounded open subset of* $\{t \geq 0\}$.

Proof. Let Φ be a test function and we consider

$$\begin{aligned} J &= \iint (\eta(U^\Delta)\Phi_t + q(U^\Delta)\Phi_x)dxdt \\ &= N + L + \Sigma, \\ N &= -\int \eta(U^\Delta(+0,x)\Phi(0,x)dx, \\ L &= \sum_n \int [\eta(U^\Delta(t,x)]^{t=n\Delta t-0}_{t=n\Delta t+0}\Phi(n\Delta t,x)dx, \\ \Sigma &= \int \sum_{shock}(\sigma[\eta]-[q])\Phi dt. \end{aligned}$$

Since U^Δ is bounded, we see

$$|N| \leq C||\Phi||_C.$$

Let us look at L. We see

$$\begin{aligned} L &= L_1 + L_2, \\ L_1 &= \sum_{j,n}\Phi(n\Delta t,(2j+1)\Delta)\int_{2j\Delta x}^{(2j+2)\Delta x}[\eta(U^\Delta)]^{t=n\Delta t-0}_{t=n\Delta t+0}dx, \\ L_2 &= \sum_{j,n}\int_{2j\Delta}^{(2j+2)\Delta}(\Phi(n\Delta t,x)-\Phi(n\Delta t,(2j+1)\Delta)[\eta(U^\Delta)]^{t=n\Delta t-0}_{t=n\Delta t+0}dx. \end{aligned}$$

We note

$$[\eta(U^\Delta)]_{t=n\Delta t+0}^{t=n\Delta t-0} = D_U\eta(U^\Delta(n\Delta t+0,x))[U^\Delta] + \int_0^1(1-\theta)([U^\Delta]|D_U^2(U^\Delta(n\Delta t+0)+\theta[U^\Delta]).[U^\Delta])d\theta.$$

and

$$\int_{2j\Delta}^{(2j+2)\Delta}[U^\Delta]dx = 0$$

by the scheme. Therefore

$$|L_1| \leq C||\Phi||_C\sum_{j,n}\iint_0^1(1-\theta)|F(\theta,\eta)|d\theta dx,$$

where

$$F(\theta,\eta) = ([U^\Delta]|D_U^2\eta(U^\Delta(n\Delta t+0)+\theta[U^\Delta]).[U^\Delta]).$$

By Proposition 6.2 we know $|F(\theta,\eta)| \leq CF(\theta,\eta^*)$. But in the proof of Proposition 4.2 we know

$$\sum_{j,n}\iint_0^1(1-\theta)F(\theta,\eta^*)d\theta dx \leq C.$$

Thus we know

$$|L_1| \leq C||\Phi||_C.$$

In the proof of Proposition 4.2 we know

$$\sum_{j,n}\int_{2j\Delta}^{(2j+2)\Delta}|[U^\Delta]|^2dx \leq C.$$

Therefore

$$\begin{aligned}|L_2| &\leq 2^\alpha||\Phi||_{C^\alpha}\sum_n\int(\Delta x)^\alpha|[\eta(U^\Delta)]|dx\\ &\leq 2^{\alpha-1}||\Phi||_{C^\alpha}\sum_n\int((\Delta)^{\alpha+\frac{1}{2}}+(\Delta)^{\alpha-\frac{1}{2}}|[\eta(U^\Delta)]|^2)dx\\ &\leq C||\Phi||_{C^\alpha}((\Delta)^{\alpha-\frac{1}{2}}+(\Delta)^{\alpha-\frac{1}{2}}\sum\int|[U^\Delta]|^2dx\\ &\leq C'(\Delta)^{\alpha-\frac{1}{2}}||\Phi||_{C^\alpha},\end{aligned}$$

where we use the boundedness of $D_U\eta$ and $n = O(1/(\Delta))$. Next we look at Σ. Along the shock we have

$$\sigma[\eta(U)] - [q(U)] = \int_{\rho_L}^{\rho_R}\left(-\frac{d\sigma}{d\rho}\int_0^1\theta(U-U_L|D_U^2\eta(U_L+\theta(U-U_L))(U-U_L))d\theta\right)d\rho.$$

This implies

$$|\sigma[\eta] - [q]| \leq C(\sigma[\eta^*] - [q^*]).$$

But we know

$$\int \sum_{shock} (\sigma[\eta^*] - [q^*])dt \leq C$$

in the proof of Proposition 4.2. Therefore

$$|\Sigma| \leq C||\Phi||_C.$$

Summing up, we know the compactness. □

8. Useful entropies

Let us consider an entropy η generated by ϕ, that is,

$$\eta(u,y) = \int_{u-y}^{u+y} K(\xi - u, y)\phi(\xi)d\xi.$$

The corresponding entropy flux q is given by integrating the equations

$$\frac{\partial q}{\partial w} = \lambda_2 \frac{\partial \eta}{\partial w}, \qquad \frac{\partial q}{\partial z} = \lambda_1 \frac{\partial \eta}{\partial z}.$$

We can solve these equations as

$$\begin{aligned} q &= \lambda_2\eta - \int_z^w \frac{\partial \lambda_2}{\partial w}\eta dw \\ &= \lambda_1\eta + \int_z^w \frac{\partial \lambda_1}{\partial z}\eta dz. \end{aligned}$$

Thus we get the formula

$$q(u,y) = \int_{u-y}^{u+y} L(u,y,\xi)\phi(\xi)d\xi,$$

where

$$\begin{aligned} L(u,y,\xi) &= \lambda_1(u,y)K(\xi-u,y) + L_1(\xi-u,y) \\ &= \lambda_2(u,y)K(\xi-u,y) + L_2(\xi-u,y), \\ L_1(v,y) &= 2\int_{(-v+y)/2}^{y} \mu(Y)K(v-y+Y,Y)dY, \\ L_2(v,y) &= -2\int_{(v+y)/2}^{y} \mu(Y)K(v+y-Y,Y)dY, \\ \mu(y) &= \frac{\partial \lambda_1}{\partial z} = \frac{\partial \lambda_2}{\partial w} \\ &= \frac{1}{2}\left(1 + \frac{\rho P''}{2P'}\right) \end{aligned}$$

$$= \frac{N+1}{2N+1} + O(\epsilon y^2).$$

We are going to construct various useful entropies.

I) Let us put

$$\eta_k^1(u,y) = \int_{u-y}^{u+y} K(\xi - u, y) k^{N+1} e^{k\xi} d\xi,$$
$$\eta_k^2(u,y) = \int_{u-y}^{u+y} K(\xi - u, y) k^{N+1} e^{-k\xi} d\xi,$$

where k is a positive integer.

Proposition 8.1. *We have*

(8.1)
$$\eta_k^1 = 2^N N! y^N (1 + O(\epsilon)) e^{k(u+y)} (1 + O(1/k)),$$
$$\eta_k^2 = 2^N N! y^N (1 + O(\epsilon)) e^{-k(u-y)} (1 + O(1/k))$$

uniformly on each compact subset of $\{y > 0\}$*, and*

(8.2)
$$q_k^1 = \eta_k^1 (\lambda_2 + O(1/k)),$$
$$q_k^2 = \eta_k^2 (\lambda_1 + O(1/k)),$$

(8.3)
$$\eta_k^2 q_k^1 - \eta_k^1 q_k^2 = (2^N N!)^2 y^{2(N-1)} \left(\frac{1}{2N+1} + O(\epsilon) \right) e^{2ky} (y + O(1/k))^3$$

uniformly on each compact subset of $\{y \geq 0\}$*.*

Proof. Since $K = (y^2 - v^2)^N (1 + O(\epsilon))$, we see

$$\eta_k^1 = (1 + O(\epsilon)) 2^{2N+1} y^N e^{ku} f(ky),$$

where

$$f(r) = r^{N+1} e^{-r} \int_0^1 (s - s^2)^N e^{2rs} ds$$
$$= e^{-r} \int_0^r \left(\sigma \left(1 - \frac{\sigma}{r} \right) \right)^N e^{2\sigma} d\sigma.$$

It is easy to see

$$e^{-r} f(r) = 2^{-(N+1)} N! + O(1/r).$$

This implies (7.1). We note

$$\eta_k^1 = (1 + O(\epsilon)) 2^N N! y^{N-1} e^{k(u+y)} (y + O(1/k)),$$
$$\eta_k^2 = (1 + O(\epsilon)) 2^N N! y^{N-1} e^{-k(u-y)} (y + O(1/k))$$

uniformly on $\{y \geq 0\}$. Let us consider the flux. We have

$$\begin{aligned} L_2 &= -2\left(\frac{N+1}{2N+1} + O(\epsilon)\right)\int_{(v+y)/2}^{y} (Y^2 - (v+y-Y)^2)^N dY \\ &= -\left(\frac{1}{2N+1} + O(\epsilon)\right)(y+v)^N(y-v)^{N+1}, \\ q_k^1 - \lambda_2\eta_k^1 &= -\left(\frac{1}{2N+1} + O(\epsilon)\right)\int_{u-y}^{u+y} (y-u+\xi)^N(y+u-\xi)^{N+1} \\ &\quad \times k^{N+1}e^{k\xi}d\xi. \end{aligned}$$

But

$$\begin{aligned} 0 &\leq \int_{u-y}^{u+y} (y-u+\xi)^N(y+u-\xi)^{N+1}k^{N+1}e^{k\xi}d\xi \\ &= (N+1)k^N\int_{u-y}^{u+y} (y^2 - (\xi-u)^2)^N e^{k\xi}d\xi \\ &\quad - Nk^N\int_{u-y}^{u+y} (y-u+\xi)^{N-1}(y+u-\xi)^{N+1}e^{k\xi}d\xi \\ &\leq (N+1)\frac{1}{k}\int_{u-y}^{u+y} (y^2-(\xi-u)^2)^N k^{N+1}e^{k\xi}d\xi. \end{aligned}$$

Thus

$$q_k^1 - \lambda_2\eta_k^1 = O(1/k)\eta_k^1.$$

Since

$$\lambda_2 - \lambda_1 = \sqrt{P'} = \left(\frac{1}{2N+1} + O(\epsilon)\right)y,$$

we get (7.3). □

II) Let ψ be a function in $C_0^\infty(-1,1)$ such that $\psi \geq 0, \psi(x) = \psi(-x)$ and $\int \psi = 1$. We put

$$\begin{aligned} \phi_n^3(u) &= \psi_n(u) = n\psi(n(u-a)), \\ \phi_n^4(u) &= -D\psi_n(u), \\ \eta_n^3(u,y) &= \int_{u-y}^{u+y} K(\xi-u,y)\phi_n^3(\xi)d\xi, \\ \eta_n^4(u,y) &= \int_{u-y}^{u+y} K(\xi-u,y)\phi_n^4(\xi)d\xi, \\ \eta^3(u,y) &= K(a-u,y)X \\ \eta^4(u,y) &= K_v(a-u,y)X \\ q^3(u,y) &= L(a-u,y)X \\ q^4(u,y) &= L_v(a-u,y)X \end{aligned}$$

$$X = 1 \quad \text{for} \quad |u-a| < y$$
$$= \frac{1}{2} \quad \text{for} \quad |u-a| = y$$
$$= 0 \quad \text{for} \quad |u-a| > y$$

Of course $\eta^3_n, \eta^4_n, q^3_n, q^4_n$ tends to η^3, η^4, q^3, q^4 everywhere as $n \to \infty$.

Proposition 8.2. *We have*

(8.4)
$$|\eta^3_n| \leq Cy^{2N}, \qquad |q^3_n| \leq Cy^{2N}(|u|+y),$$
$$|\eta^4_n| \leq Cy^{2N-1}, \qquad |q^4_n| \leq Cy^{2N-1}(|u|+y),$$

(8.5)
$$\eta^3 q^4 - \eta^4 q^3 = \frac{1}{2N+1}(1+O(\epsilon))(y^2-(u-a)^2)^{2N}.$$

Proof. The estimate (7.4) can be easily seen. Let us consider

$$\eta^3 q^4 - \eta^4 q^3 = (KL_v - LK_v)(a-u, y).$$

Suppose $v = a-u \leq 0$. We have

$$\frac{1}{2}(KL_v - LK_v) = K\int_{(-v+y)/2}^{y} \mu K_v(v-y+Y, Y)dY$$
$$- K_v \int_{(-v+y)/2}^{y} \mu K(v-y+Y,Y)dY.$$

Since

$$K_v = -vG(v,|v|)J^{N-1}1 + J^N G_v,$$

we have

$$L_{1,v} = \int_{(-v+y)/2}^{y} \mu K_v(v-y+Y,Y)dY$$
$$= \left(\frac{N+1}{2N+1} + O(\epsilon)\right) 2N \int_{(-v+y)/2}^{y} (-v+y-Y)$$
$$\times (Y^2 - (-v+y-Y)^2)^{N-1} dY$$
$$+ O(\epsilon) \int_{(-v+y)/2}^{y} (Y^2-(-v+y-Y)^2)^N dY$$
$$= \left(\frac{N(N+1)}{2(2N+1)} + O(\epsilon)\right)$$
$$\times (-v+y)^{N+1}(v+y)^N \frac{1}{N(N+1)}(y-(2N-1)v)$$
$$+ O(\epsilon)(v+y)(y^2-v^2)^N$$

Thus

$$K\int_{(-v+y)/2}^{y} \mu K_v dY = \left(\frac{1}{2(2N+1)} + O(\epsilon)\right)(y^2-v^2)^{2N-1}$$
$$\times (v+y)(y-(2N+1)v) + O(\epsilon)(v+y)(y^2-v^2)^{2N}$$

On the other hand we have

$$K_v\int_{(-v+y)/2}^{y}\mu K dY = -\left(\frac{N}{2N+1}+O(\epsilon)\right)v(v+y)(y^2-v^2)^{2N-1} + O(\epsilon)(v+y)(y^2-v^2)^{2N}.$$

Hence

$$\frac{1}{2}(KL_v - LK_v) = \left(\frac{1}{2(2N+1)}+O(\epsilon)\right)(y^2-v^2)^{2N}.$$

Here we have used

$$0 \le -v(y+v) \le y^2-v^2,$$
$$0 \le (y+v)(y-(2N+1)v) \le (2N+1)(y^2-v^2),$$

provided that $-y \le v \le 0$. When $v \ge 0$, we can discuss in a similar manner by using L_2. □

III) Let Φ be a function in $C_0^\infty(-1,1)$ such that $\int\Phi = 0$ and the support $\mathrm{supp}\,\Phi$ is $[-1+\alpha, 1+\alpha]$, where α is a small positive number. We put

$$\begin{aligned}
\psi_n(u) &= n\Phi(n(u-a)),\\
\eta_n^5(u,y) &= \int_{u-y}^{u+y} K(\xi-u,y)D^{N+1}\psi_n(\xi)d\xi,\\
q_n^5(u,y) &= \int_{u-y}^{u+y} L(u,y,\xi)D^{N+1}\psi_n(\xi)d\xi;\\
\hat{\Phi}(u) &= \frac{d}{dx}\left(x\int_{-1}^{x}\Phi\right),\\
\hat{\psi}_n(u) &= n\hat{\Phi}(n(u-a)),\\
\eta_n^6(u,y) &= \int_{u-y}^{u+y} K(\xi-u,y)D^{N+1}\hat{\psi}_n(\xi)d\xi,\\
q_n^6(u,y) &= \int_{u-y}^{u+y} L(u,y,\xi)D^{N+1}\hat{\psi}_n(\xi)d\xi;\\
B_n^3 &= \eta^3 q_n^5 - \eta_n^5 q^3,\\
B_n^4 &= \eta^4 q_n^5 - \eta_n^5 q^4,\\
B_n &= \eta_n^5 q_n^6 - \eta_n^6 q_n^5.
\end{aligned}$$

Let us divide the domain $\Sigma = \{-B \le u-y \le u+y \le B\}$ into the following 5 parts.

$$\begin{aligned}
S_0 &= \left\{-\frac{1}{n} < u+y-a \leq \frac{1}{n}, -\frac{1}{n} \leq u-y-a < \frac{1}{n}\right\} \cap \Sigma, \\
S_1 &= \left\{\frac{1}{n} < u+y-a, u-y-a < -\frac{1}{n}\right\} \cap \Sigma, \\
S_L &= \left\{-\frac{1}{n} < u+y-a \leq \frac{1}{n}, u-y-a < -\frac{1}{n}\right\} \cap \Sigma, \\
S_R &= \left\{\frac{1}{n} < u+y-a, -\frac{1}{n} \leq u-y-a < \frac{1}{n}\right\} \cap \Sigma, \\
S &= \Sigma - (S_0 \cup S_1 \cup S_1 \cup S_L \cup S_R).
\end{aligned}$$

Proposition 8.3. *We have*

$$|B_n^3| \leq C/n, \qquad |B_n^4| \leq C \tag{8.6}$$

on Σ*, and*

$$|B_n| \leq C/n \tag{8.7}$$

on $S_0 \cup S_1 \cup S$*. Moreover, on* S_L*, we have*

$$B_n = ny^{2N}A_1 + y^N A_2 + A_3, \tag{8.8}$$

where

$$\begin{aligned}
A_1 &= \left(\frac{(N+1)(2^N N!)^2}{2N+1} + O(\epsilon)\right)\left(\int_{-1}^{n(u+y-a)} \Phi\right)^2, \\
|A_2| &\leq C\left(\left|\int_{-1}^{n(u+y-a)} \Phi\right| + |\Phi(n(u+y-a))|\right), \\
|A_3| &\leq \frac{C}{n}.
\end{aligned}$$

On S_R*, we have*

$$\begin{aligned}
B_n &= ny^{2N}C_1 + y^N C_2 + C_3, \\
C_1 &= \left(\frac{(N+1)(2^N N!)^2}{2N+1} + O(\epsilon)\right)\left(\int_{-1}^{n(u-y-a)} \Phi\right)^2, \\
|C_2| &\leq C\left(\left|\int_{-1}^{n(u-y-a)} \Phi\right| + |\Phi(n(u-y-a))|\right), \\
|C_3| &\leq \frac{C}{n}.
\end{aligned}$$

Proof. For the simplicity, we write $\eta_n = \eta_n^5, q_n = q_n^5, \hat{\eta}_n = \eta_n^6, \hat{q}_n = q_n^6$. It is easy to see inductively that, for $G_j = J^j G = K_{N-j}$, we have

$$\partial_v^p G_j = J \partial_v^p G_{j-1}$$

for $j \geq p+1$ and

$$\partial_v^p G_p = (-1)^p v^p G(v, |v|) + J \partial_v^p G_{p-1}.$$

Therefore

$$\partial_v^p K = \partial_v^p G_N(v, y) = 0$$

for $p \leq N-1$ and $y = |v|$. Thus by integration by parts we have

$$\begin{aligned} \eta_n &= (-1)^N \partial_v^N K(y, y,) \psi_n(u+y) (-1)^N \partial_v^N K(-y, y) \psi_n(u-y) \\ &\quad + F_n^1(u, y), \\ F_n^1(u, y) &= (-1)^{N+1} \int_{u-y}^{u+y} \partial_v^{N+1} K(\xi - u, y) \psi_n(\xi) d\xi. \end{aligned}$$

We see

$$\partial_v^p L_2(v, y) = -2 \int_{(v+y)/2}^{y} \mu_2 \partial_v^p K(v + y - Y, Y) dY$$

for $p \leq N-1$. Therefore

$$\partial_v^p L_2(y, y) = \partial_v^p L_2(-y, y) = 0$$

for $p \leq N-1$. Moreover we see

$$\partial_v^N L_2(y, y) = 0.$$

Therefore by integration by parts we have

$$\begin{aligned} \sigma_n(u, y) &= q_n(u, y) - \lambda_2 \eta_n(u, y) \\ &= -(-1)^N \partial_v^N L_2(-y, y) \psi_n(u-y) + F_n^2(u, y), \\ F_n^2(u, y) &= (-1)^{N+1} \int_{u-y}^{u+y} \partial_v^{N+1} L_2(\xi - u, y) \psi_n(\xi) d\xi. \end{aligned}$$

Similarly

$$\begin{aligned} \bar{\sigma}_n(u, y) &= q_n(u, y) - \lambda_1 \eta_n(u, y) \\ &= (-1)^N \partial_v^N L_1(y, y) \psi_n(u+y) + \bar{F}_n^2(u, y), \\ \bar{F}_n^2(u, y) &= (-1)^{N+1} \int_{u-y}^{u+y} \partial_v^{N+1} L_1(\xi - u, y) \psi_n(\xi) d\xi. \end{aligned}$$

We note

$$\partial_v^N K(v, y) = (-1)^N v^N G(v, |v|) + J \partial_v^N G_{N-1}.$$

It is easy to see inductively that

$$\partial_v^{p+1}G_p(v,y)=(-1)^p\frac{p(p+1)}{2}v^{p-1}G(v,|v|)+v^pH_p(v)+J\partial_v^{p+1}G_{p-1},$$

where $H_p=O(\epsilon)$. Therefore

$$\partial_v^{N+1}K(v,y)=(-1)^N\frac{N(N+1)}{2}v^{N-1}G(v,|v|)+v^NH_N(v)+J\partial_v^{N+1}G_{N-1}.$$

1) Suppose $(u,y)\in S$. Then it is clear that $\eta^3,\eta^4,q^3,\ q^4,\eta_n,q_n,\ \hat{\eta}_n,\hat{q}_n,$ B_n^3,B_n^4,B_n all vanish.

2) Suppose $(u,y)\in S_0$. Then we see

$$\begin{aligned}
\eta^3&=K(a-u,y)\\
&=O((y^2-(u-a)^2)^N)\\
&=O(n^{-2N}),\\
\eta^4&=K_v(a-u,y)\\
&=O(|u-a|(y^2-(u-a)^2)^{N-1})+O((y^2-(u-a)^2)^N)\\
&=O(n^{-2N+1}),\\
\sigma^3&=L_2(a-u,y)\\
&=-2\int_{(-u+y+a)/2}^{y}\mu K(a-u+y-Y,Y)dY\\
&=O(n^{-2N-1}),\\
\sigma^4&=L_{2,v}(a-u,y)\\
&=-2\int_{(-u+y+a)/2}^{y}\mu K_v(a-u+y-Y,Y)dY\\
&=O(n^{-2N}).
\end{aligned}$$

Since $y=O(1/n)$ and $\psi_n=O(n)$, we see

$$(-1)^N\partial_v^NK(y,y)\psi_n(u+y)-(-1)^N\partial_v^NK(-y,y)\psi_n(u-y)=O(n^{-N+1}).$$

Since $F_n^1=O(1)$, we have $\eta_n=O(1)$. We see

$$\partial_v^NL_2(-y,y)=-2\int_0^y\mu_2\partial_v^NK(-Y,Y)dY=O(n^{-N-1}).$$

Therefore

$$-(-1)^N\partial_v^NL_2(-y,y)\psi_n(u-y)=O(n^{-N}).$$

Since

$$\begin{aligned}
\partial_v^{N+1}L_2(v,y)&=\mu\partial_v^NK((v+y)/2,(v+y)/2)\\
&\qquad-2\int_{(v+y)/2}^{y}\partial_v^{N+1}K(v+y-Y,Y)dY\\
&=O((v+y)^N)+O(-v+y),
\end{aligned}$$

we see

$$F_n^2(x,y) = (-1)^{N+1}\int_{u-y}^{u+y} \partial_v^{N+1} L_2(\xi-u,y)\psi_n(\xi)d\xi$$
$$= O(n^{-1}).$$

Hence $\sigma_n = O(n^{-1})$. Therefore

$$B_n^3 = \eta^3\sigma_n - \eta_n\sigma^3 = O(n^{-2N-1}),$$
$$B_n^4 = \eta^4\sigma_n - \eta_n\sigma^4 = O(n^{-2N}),$$
$$B_n = \eta_n\hat{\sigma}_n - \hat{\eta}_n\sigma_n = O(n^{-1}).$$

3) Suppose $(x,y)\in S_1$, where $u+y > a+(1/n)$ and $u-y < a-(1/n)$. Then $\psi_n(u+y) = \psi_n(u-y) = \hat{\psi}_n(u+y) = \hat{\psi}_n(u-y) = 0$. So, $\eta_n = F_n^1, \sigma_n = F_n^2$, and so on. But

$$F_n^1(u,y) = (-1)^{N+1}\int_{u-y}^{u+y} \partial_v^{N+1}K(\xi-u,y)\psi_n(\xi)d\xi$$
$$= (-1)^{N+1}\int_{-1}^{1}\left(\partial_v^{N+1}K\left(a+\frac{s}{n}-u,y\right) - \partial_v^{N+1}K(a-u,y)\right)\Phi(s)ds$$
$$= O(1/n)$$

since $\int\Phi = 0$ and $\partial_v^{N+1}K$ is Lipschitz continuous. Same estimates hold for $F_n^2, \hat{F}_n^1, \hat{F}_n^2$. Thus

$$B_n^3 = \eta^3F_n^2 - F_n^1\sigma^3 = O(1/n),$$
$$B_n^4 = \eta^4F_n^2 - F_n^1\sigma^4 = O(1/n),$$
$$B_n = F_n^1\hat{F}_n^2 - \hat{F}_n^1F_n^2 = O(1/n^2).$$

4) Suppose $(x,y)\in S_L$, where $|u+y-a|\le 1/n$. It is easy to see $\eta^3 = O(n^{-N}), \eta^4 = O(n^{-N+1}), \sigma^3 = O(n^{-N-1}), \sigma^4 = O(n^{-N})$. Since $n(u-y-a) < -1$, we have $\psi_n(u-y) = 0$. Thus $\eta_n = O(n), \sigma_n = F_n^2 = O(1)$. Therefore

$$B_n^3 = \eta^3\sigma_n - \eta_n\sigma^3 = O(n^{-N}),$$
$$B_n^4 = \eta^4\sigma_n - \eta_n\sigma^4 = O(n^{1-N}).$$

Let us estimate $B_n = \eta_n\hat{\sigma}_n - \hat{\eta}_n\sigma_N$. Since

$$\partial_v^{N+1}K = (-1)^N\frac{N(N+1)}{2}v^{N-1}G(v,|v|) + v^NH_N(v) + J\partial_v^NG_{N-1},$$

we have

$$F_n^1 = (-1)^{N+1}\int_{u-y}^{u+y}\partial_v^{N+1}K(\xi-u,y)\psi_n(\xi)d\xi$$
$$= (-1)^{N+1}\left((-1)^N\frac{N(N+1)}{2}2^NN!(a-u)^{N-1} + F'(a-u)\right)\int_{-1}^{n(u+y-a)}\Phi$$
$$+ O(1/n)$$
$$= -\frac{N(N+1)}{2}2^NN!y^{N-1}(1+F''(a-u,y))\int_{-1}^{n(u+y-a)}\Phi + O(1/n),$$

where $F' = O(\epsilon)|a-u|^N, F'' = O(\epsilon)$. On the other hand

$$\partial_v^N K(y,y) = (-1)^N y^N G(y,y).$$

Hence

$$\eta_n = ny^N G(y,y)\Phi(n(u+y-a)) \\ - \frac{N(N+1)}{2} 2^N N! y^{N-1}(1+F''(a-u,y)) \int_{-1}^{n(u+y-a)} \Phi + O(1/n).$$

Since

$$\begin{aligned}\partial_v^{N+1} L_2(v,y) &= \mu \partial_v^N K((v-y)/2,(v+y)/2) \\ &\quad - 2\int_{(v+y)/2}^{y} \mu\partial_v^{N+1} K(v+y-Y,Y)dY \\ &= \left(\frac{N}{2N+1} + O(\epsilon)\right)(-1)^N \left(\frac{v+y}{2}\right)^N \\ &\quad \times G((v-y)/2,(v+y)/2) + O(-v+y),\end{aligned}$$

we see

$$\begin{aligned}\sigma_n &= F_n^2 \\ &= (-1)^{N+1}\int_{u-y}^{u+y} \partial_v^{N+1} L_2(\xi-u,y)\psi_n(\xi)d\xi \\ &= -\frac{N}{2N+1}2^N N! y^N (1+L'(a-u,y))\int_{-1}^{n(u+y-a)} \Phi + O(1/n),\end{aligned}$$

where $L' = O(\epsilon)$. Here we have used

$$\left(\frac{-u+y+a}{2}\right)^N = \left(y - \frac{u+y-a}{2}\right)^N = y^N + O(1/n).$$

Similar estimates hold for $\hat{\eta}_n, \hat{\sigma}_n$. Thus

$$B_n = ny^{2N}A_1 + y^N A_2 + A_3,$$

where

$$\begin{aligned}A_1 &= -G\frac{N+1}{2N+1}2^N N!(1+L')\Phi(\beta)\int_{-1}^{\beta}\hat{\Phi} + G\frac{N+1}{2N+1}2^N N!(1+L')\hat{\Phi}(\beta)\int_{-1}^{\beta}\Phi \\ &= \frac{N+1}{2N+1}2^N N! G(1+L')\left(\int_{-1}^{\beta}\Phi\right)^2, \\ \beta &= n(u+y-a).\end{aligned}$$

The estimates on S_R can be obtained in a similar manner considering $\bar{\sigma}^3, \bar{\sigma}^4, \bar{\sigma}_n$.
□

If we put

$$\hat{B}^3_n = \eta^3 q^6_n - \eta^6_n q^3,$$
$$\hat{B}^4_n = \eta^4 q^6_n - \eta^6_n q^4,$$

then the same estimates hold.

9. Convergence of approximate solutions

We consider the approximate solutions U^Δ constructed in Section 4. Since U^Δ is bounded, there is a sequence U^{Δ_n} and a family of Young measures $\nu_{t,x}$ such that $\operatorname{supp} \nu_{t,x} \subset \Sigma = \Sigma_B$ and for any continuous function f

$$f(U^{\Delta_n}(t,x)) \to \bar{f} = \langle \nu_{t,x}, f \rangle$$

in L^∞ weak star topology. By Proposition 7.1 we can apply the compensated compactness theory, and we can assume

$$(\eta q' - \eta' q)(U^{\Delta_n}) \to \langle \nu, q \rangle \langle \nu, q' \rangle - \langle \nu, \eta' \rangle \langle \nu, q \rangle$$

in L^∞ weak star. Here $\eta, q; \eta', q'$ are arbitrary Darboux entropy pairs. Thus we have

Proposition 9.1. *For any pairs $(\eta, q), (\eta', q')$ of Darboux entropies-entropy flux, the identity*

$$\langle \nu, \eta q' - \eta' q \rangle = \langle \nu, \eta \rangle \langle \nu, q' \rangle - \langle \nu, \eta' \rangle \langle \nu, q \rangle$$

holds a.e.-(t,x), *where* $\nu = \nu_{t,x}$.

Since entropies we will use are countably many, we can assume that the above identity holds outside a null set which is common to all η. We fix (t,x) at which the identity holds, and we write $\nu = \nu_{t,x}$. Of course $\operatorname{supp.} \nu \subset \Sigma$. Suppose that $\operatorname{supp.} \nu \cap \{\rho > 0\} \neq \phi$. Let Σ_0 be the smallest triangle $\{z_0 \leq z \leq w \leq w_0\}$ such that $\operatorname{supp.} \nu \cap \{\rho > 0\} \subset \Sigma_0$. Let us denote by P_0 the state (w_0, z_0). It will be verified that $\nu = \delta_{P_0}$. (the Dirac measure). First we show

Proposition 9.2.

$$P_0 \in \operatorname{supp.} \nu.$$

Proof. Suppose $P_0 \notin \operatorname{supp.} \nu$. Since Σ_0 is the smallest triangle containing $\operatorname{supp.} \nu \cap \{\rho > 0\}$, $w = w_0$ and $z = z_0$ intersect with $\operatorname{supp.} \nu \cap \{\rho > 0\}$. On neighborhoods of these intersection points we have

$$\eta^1 \geq \frac{1}{C} e^{k(w_0 - \epsilon_1)},$$
$$\eta^2 \geq \frac{1}{C} e^{-k(z_0 + \epsilon_1)}.$$

(See Proposition 8.1). Since ν, η^1, η^2 are nonnegative, we see

$$\langle \nu, \eta^1 \rangle \geq \frac{1}{C} e^{k(w_0 - \epsilon_1)},$$
$$\langle \nu, \eta^2 \rangle \geq \frac{1}{C} e^{-k(z_0 + \epsilon_1)}.$$

Since $P_0 \notin \mathrm{supp.}\, \nu$, we have

$$\langle \nu, \eta^2 q^1 - \eta^1 q^2 \rangle \leq M e^{k(w_0 - z_0 - \delta)}.$$

Taking $2\epsilon_1 < \delta$, we have

$$\begin{aligned} \left| \frac{\langle \nu, q^1 \rangle}{\langle \nu, \eta^1 \rangle} - \frac{\langle \nu, q^2 \rangle}{\langle \nu, \eta^2 \rangle} \right| &= \left| \frac{\langle \nu, \eta^2 q^1 - \eta^1 q^2 \rangle}{\langle \nu, \eta^1 \rangle \langle \nu, \eta^2 \rangle} \right| \\ &\leq C e^{-k(\delta - 2\epsilon_1)} \\ &\to 0 \end{aligned}$$

as $k \to \infty$. Let β be a sufficiently small positive number, and we put

$$\Sigma_2 = \{ z_0 \leq z \leq w < w_0 - \beta \}$$
$$\Sigma_3 = \{ z_0 \leq z \leq w \leq w_0, w_0 - \beta \leq w \}.$$

Then

$$\eta^1 e^{-kw} = (1 + O(\epsilon)) 2^N N! y^{N-1} (y + O(1/k))$$

is bounded on Σ_0 and we have

$$\langle \nu|_{\Sigma_2}, \eta^1 \rangle \leq C e^{k(w_0 - \beta)}.$$

Taking $\epsilon_1 = \beta/2$, we know

$$\frac{\langle \nu|_{\Sigma_2}, \eta^1 \rangle}{\langle \nu, \eta^1 \rangle} \leq C e^{-\beta k/2} \to 0.$$

Since $\partial \lambda_2 / \partial w > 0$, we know

$$\lambda_2(w, z) \geq \lambda_2(w_0 - \beta, z_0)$$

on Σ_3. Therefore we have

$$\begin{aligned} \frac{\langle \nu, q^1 \rangle}{\langle \nu, \eta^1 \rangle} &= \frac{\langle \nu|_{\Sigma_2}, \eta^1 \lambda_2 \rangle}{\langle \nu, \eta^1 \rangle} + \frac{\langle \nu|_{\Sigma_3}, \eta^1 \lambda_2 \rangle}{\langle \nu, \eta^1 \rangle} + O(1/k) \\ &\geq o(1) + \lambda_2(w_0 - \beta, z_0) \end{aligned}$$

Similarly we see

$$\frac{\langle \nu, q^2 \rangle}{\langle \nu, \eta^2 \rangle} \leq o(1) + \lambda_1(w_0, z_0 + \beta).$$

Therefore we have

$$\lambda_2(w_0 - \beta, z_0) - \lambda_1(w_0, z_0 + \beta) \le 0 + o(1).$$

Passing to the limit, we know

$$\lambda_2(w_0, z_0) \le \lambda_1(w_0, z_0).$$

But this means $P_0 \in \{\rho = 0\}$, a contradiction. □

Let us fix a such that $z_0 < a < w_0$. We have

$$\begin{aligned}
\langle \nu, B_n^3 \rangle &= \langle \nu, \eta^3 \rangle \langle \nu, q_n^5 \rangle - \langle \nu, \eta_n^5 \rangle \langle \nu, q^3 \rangle, \\
\langle \nu, B_n^4 \rangle &= \langle \nu, \eta^4 \rangle \langle \nu, q_n^5 \rangle - \langle \nu, \eta_n^5 \rangle \langle \nu, q^4 \rangle, \\
\langle \nu, \eta^3 q^4 - \eta^4 q^3 \rangle &= \langle \nu, \eta^3 \rangle \langle \nu, q^4 \rangle - \langle \nu, \eta^4 \rangle \langle \nu, q^3 \rangle, \\
\langle \nu, B_n \rangle &= \langle \nu, \eta_n^5 \rangle \langle \nu, q_n^6 \rangle - \langle \nu, \eta_n^6 \rangle \langle \nu, q_n^5 \rangle.
\end{aligned}$$

From (8.5) and $P_0 \in \operatorname{supp.} \nu$ we know

$$\langle \nu, \eta^3 q^4 - \eta^4 q^3 \rangle > 0 \qquad \langle \nu, \eta^3 \rangle > 0$$

and from(8.6) we know

$$\langle \nu, B_n^3 \rangle \to 0, \qquad \langle \nu, \hat{B}_n^3 \rangle \to 0$$

Using these we can prove the following propositions. Proofs can be found in Chen *et al.* [1].

Proposition 9.3. *As $n \to \infty$, $\langle \nu, \eta_n^5 \rangle, \langle \nu, q_n^5 \rangle$, $\langle \nu, q_n^6 \rangle$, $\langle \nu, q_n^6 \rangle$ are bounded.*

Proposition 9.4. *As $n \to \infty$, we have $\langle \nu, B_n \rangle \to 0$.*

Now, taking

$$\Phi_0(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| \ge 1 \end{cases}$$

we put

$$\Phi(x) = \frac{1}{\beta} \left(\Phi_0 \left(\frac{x+\beta}{\beta} \right) - \Phi_0 \left(\frac{x-\beta}{\beta} \right) \right)$$

for the generating function of η_n^5. Here $\beta = (1-\alpha)/2$. We put

$$\begin{aligned}
S_+ &= \left\{ z \le w, |w-a| \le \frac{1-3\alpha}{n} \right\}, \\
S_- &= \left\{ z \le w, |z-a| \le \frac{1-3\alpha}{n} \right\}.
\end{aligned}$$

Proposition 9.5. *As $n \to \infty$, we have*

$$\langle \nu|_{S_+}, ny^{2N} \rangle + \langle \nu|_{S_-}, ny^{2N} \rangle \to 0.$$

Proof. Put $S'_L = S_+ \cap S_L, S'_R = S_- \cap S_R$. It is sufficient to prove that

$$\langle \nu|_{S'_L}, ny^{2N}\rangle + \langle \nu|_{S'_R}, ny^{2N}\rangle \to 0.$$

From (8.7) we have

$$\langle \nu|_{S_L}, ny^{2N}A_1 + y^N A_2\rangle + \langle \nu|_{S_R}, ny^{2N}C_1 + y^N C_2\rangle \to 0.$$

Note

$$A_1 = \left(\frac{(N+1)(2^N N!)^2}{2N+1} + O(\varepsilon)\right)\left(\int_{-1}^{n(u+y-a)} \Phi\right)^2 \geq \frac{1}{C_0} > 0$$

on S'_L. Put

$$E_n = \left\{0 \leq y \leq \left(\frac{1}{n}\right)^{\mu}\right\},$$

where μ is a positive parameter. Then $|y^N A_2| \leq C(1/n)^{\mu N} = o(1)$ on $S_L \cap E_n$ and $|y^N A_2| \leq Cny^{2N}(1/n)^{1-\mu N}$ on $S_L - E_n$. Choose $d_n \searrow 0$ such that

$$\int_{-1+\alpha}^{1-\alpha-d_n} \Phi = -\int_{1-\alpha-d_n}^{1-\alpha} \Phi \geq (1/n)^{\mu_0}.$$

Then

$$\left(\int_{-1}^{H} \Phi\right)^2 \geq (1/n)^{2\mu_0}$$

for $|H| \leq 1 - \alpha - d_n$, and

$$|\Phi(H)| + \left|\int_{-1}^{H} \Phi\right| = o(1)$$

for $1 - \alpha - d_n \leq |H| \leq 1$. Put

$$S_+^n = S_L \cap \left\{|w - a| \leq \frac{1-\alpha-d_n}{n}\right\}.$$

Then $S'_L \subset S_+^n \subset S_L$ and

$$|y^N A_2| = o(1)$$

on $S_L - S_+^n$ and

$$\begin{aligned} ny^{2N}A_1 + y^N A_2 &\geq ny^{2N}\left(\frac{1}{C}(1/n)^{2\mu_0} - C(1/n)^{1-\mu N}\right) \\ &\geq 0 \end{aligned}$$

on $S^n_+ - E_n$. Here we take $0 < 2\mu_0 < 1 - \mu N$. Then

$$\begin{aligned}\langle \nu|_{S_L}, ny^{2N}A_1 + y^N A_2\rangle &= \langle \nu|_{S_L\cap E_n}, ny^{2N}A_1\rangle \\ &\quad + \langle \nu|_{S_L - E_n}, ny^{2N}A_1 + y^N A_2\rangle + o(1) \\ &\geq \frac{1}{C_0}\langle \nu|_{S'_L\cap E_n}, ny^{2N}\rangle + \langle \nu|_{S_L - S^n_+\cap E_n}, ny^{2N}A_1\rangle \\ &\quad + \langle \nu|_{S'_L - E_n}, ny^{2N}A_1 + y^N A_2\rangle \\ &\quad + \langle \nu|_{S^n_+ - S'_L - E_n}, ny^{2N}A_1 + y^N A_2\rangle + o(1) \\ &\geq \frac{1}{C_0}\langle \nu|_{S'_L\cap E_n}, ny^{2N}\rangle \\ &\quad + \left\langle \nu|_{S'_L - E_n}, ny^{2N}\left(\frac{1}{C_0} - C(1/n)^{1-\mu N}\right)\right\rangle + o(1) \\ &\geq \frac{1}{2C_0}\langle \nu|_{S'_L}, ny^{2N}\rangle + o(1).\end{aligned}$$

Similarly we know

$$\langle \nu|_{S_R}, ny^{2N}C_1 + y^N C_2\rangle \geq \frac{1}{2C_0}\langle \nu|_{S'_R}, ny^{2N}\rangle + o(1).$$

Thus we see

$$\langle \nu|_{S'_L}, ny^{2N}\rangle + \langle \nu|_{S'_R}, ny^{2N}\rangle \to 0.$$ □

Proposition 9.6. *We have*

$$\nu|_{\{\rho>0\}} = \delta_{P_0}.$$

Proof. Proposition 9.5 says that the projections $P_w\tilde{\nu}, P_z\tilde{\nu}$ of the measure $\tilde{\nu} = y^{2N}\nu$ admits the Lebesgue lower derivatives which vanish at any a. Therefore we can claime that

$$\text{supp.}\,\nu \cap \{\rho > 0\} = \{P_0\}.$$

Since ν is a probability measure, we have

$$\nu|_{\{\rho>0\}} = C\delta_{P_0}.$$

But

$$C(\eta^3 q^4 - \eta^4 q^3) = C^2(\eta^3 q^4 - \eta^4 q^3)$$

at P_0. Hence $C = 1$. □

Summing up we get the proof of Theorem 1.

Let us prove the Theorem 2.

Let α be an arbitrary positive constant. Put

$$\rho = \alpha^{\frac{2}{\gamma-1}}\bar{\rho}, \qquad P = \alpha^{\frac{2\gamma}{\gamma-1}}\bar{P},$$
$$u = \alpha\bar{u}, \qquad x = \alpha\bar{x},$$
$$\epsilon = \alpha^{-2}\bar{\epsilon}.$$

Then the problem for $\bar{\rho}, \bar{u}, \dots$ is the same to the problem (1.1), (1.2) with the same equation of states. Thus Theorem 1 can be applied. Taking $\epsilon_1(1)^{1/2} = \alpha$, we get Theorem 2.

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References

[1] Ding Xiaxi, Guiqiang Chen and Peizhu Luo, Convergence of the Lax-Friedrichs scheme for isentropic gas dynamics, (I), Acta Math. Sci., **5** (1985), 415–432, (II), ibid., **5** (1985), 433–472, (III), ibid., **6** (1986), 75–120.

[2] R. J. DiPerna, Convergence of approximate solutions to the conservation laws, Arch. Rational Mech. Anal., **82** (1983), 27–70.

[3] R. J. DiPerna, Convergence of the viscosity method for isentropic gas dynamics, Commun. Math. Phys., **91** (1983), 1–30.

[4] Cheng-Hsiung Hsu, Song-Sun Lin and T. Makino, On the relativistic Euler equation, preprint.

[5] T. Nishida, Global solutions for an initial boundary value problem of a quasi-linear hyperbolic system, Proc. Japan Acad., **44** (1968), 642–646.