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Classification of equivariant complex vector bundles over a circle

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Abstract

In this paper we characterize the fiber representations of equivariant complex vector bundles over a circle and classify these bundles. We also treat the triviality of them by investigating the extensions of the fiber representations. As a corollary of our results, we calculate the reduced equivariant K-group of a circle for any compact Lie group, which extends a result of Y. Yang [Yan95].

1. Introduction

The classification of vector bundles over a topological space is one of the fundamental problems in topology, and many theories have been developed to solve the problem. However the problem becomes more complex and difficult when one considers it in the equivariant category. For instance, every complex vector bundle over a circle is trivial, but equivariant ones are abundant and not necessarily trivial. In this paper, we classify equivariant complex vector bundles over a circle. The real case is treated in another paper [CKMS00].

In order to state our main results, let us fix some notation and terminology. Let G be a compact Lie group and let $\rho\colon G \to O(2)$ be an orthogonal representation of G. The unit circle of the corresponding G-module is denoted by $S(\rho)$. We set $H = \rho^{-1}(1)$, so that H acts trivially on $S(\rho)$ and the fiber H-module of a complex G-vector bundle over $S(\rho)$ is determined uniquely up to isomorphism. On the other hand, for a character χ of H and $g \in G$, a new character ${}^g\chi$ of H is defined by ${}^g\chi(h) = \chi(g^{-1}hg)$ for $h \in H$. We say that the character χ is *G-invariant* if ${}^g\chi = \chi$ for all $g \in G$. Our first main theorem characterizes the fiber H-module of a complex G-vector bundle over $S(\rho)$.

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Theorem A. *A complex H-module is the fiber H-module of a complex G-vector bundle over $S(\rho)$ if and only if its character is G-invariant.*

We need more notation to state our second main theorem. Let $\mathrm{Irr}(H)$ be the set of characters of irreducible H-modules. It has a G-action defined above. Since a character is a class function, the isotropy subgroup G_χ of G at $\chi \in \mathrm{Irr}(H)$ contains H. We choose and fix a representative from each G-orbit in $\mathrm{Irr}(H)$ and denote the set of those representatives by $\mathrm{Irr}(H)/G$. Denote by $\mathrm{Vect}_G(X)$ the set of isomorphism classes of complex G-vector bundles over a connected G-space X and by $\mathrm{Vect}_{G_\chi}(X, \chi)$ the subset of $\mathrm{Vect}_{G_\chi}(X)$ with a multiple of χ as the character of fiber H-modules. They are semi-groups under Whitney sum. The decomposition of a G-vector bundle into the χ-isotypical components induces an isomorphism

$$\mathrm{Vect}_G(X) \cong \bigoplus_{\chi \in \mathrm{Irr}(H)/G} \mathrm{Vect}_{G_\chi}(X, \chi).$$

This reduces the study of $\mathrm{Vect}_G(X)$ to that of $\mathrm{Vect}_{G_\chi}(X, \chi)$ (see Section 2).

Theorem B. *The semi-group* $\mathrm{Vect}_{G_\chi}(S(\rho), \chi)$ *is generated by*

(1) *one element L_χ if $\rho(G_\chi) \subset SO(2)$,*

(2) *two elements $L_\chi^\pm$ if $\rho(G_\chi) = O(2)$, and*

(3) *four elements $L_\chi^{\pm\pm}$ with a relation $L_\chi^{++} + L_\chi^{--} = L_\chi^{+-} + L_\chi^{-+}$ otherwise.*

Using this theorem, one can easily enumerate complex G-vector bundles over $S(\rho)$ with a fixed H-module as the fiber H-module (see Corollary 5.2).

Our last main theorem is about the triviality of G-vector bundles. Here a G-vector bundle is said to be *trivial* if it is isomorphic to a product bundle with a G-module as its fiber.

Theorem C. *The triviality of the generators appeared in Theorem* B *is as follows.*

(1) *L_χ is trivial.*

(2) *$L_\chi^\pm$ are both trivial or both nontrivial.*

(3) *Two of $L_\chi^{\pm\pm}$ are trivial and the other two are nontrivial if $|\rho(G_\chi)|/2$ is odd, and $L_\chi^{\pm\pm}$ are all trivial or all nontrivial if $|\rho(G_\chi)|/2$ is even, where $|\rho(G_\chi)|$ denotes the order of the dihedral group $\rho(G_\chi)$.*

Since $\rho(G_\chi) \subset SO(2)$ for any χ if $\rho(G) \subset SO(2)$, it follows that

Corollary D. *Every G-vector bundle over $S(\rho)$ is trivial if $\rho(G) \subset SO(2)$.*

The reader will find that there are many nontrivial G-vector bundles as well as trivial ones unless $\rho(G) \subset SO(2)$.

This paper is organized as follows. Sections 2 and 3 deal with results on G-vector bundles which hold for an arbitrary base space. We also recall some results from representation theory, which turn out to be closely related to the semi-group structure on $\mathrm{Vect}_G(S(\rho))$ and the triviality of G-vector bundles over $S(\rho)$. Theorems A and B are proved in Sections 4 and 5. In Section 6 we present another approach to study $\mathrm{Vect}_G(S(\rho))$. The triviality of G-vector bundles over $S(\rho)$ is discussed and the proof of Theorem C is given in Section 7. We describe G-line bundles explicitly in Section 8. In Section 9 we apply the general results obtained in the previous sections to the case when G is abelian. In Section 10 we determine the reduced equivariant K-group of $S(\rho)$, which extends a result of Y. Yang [Yan95, Theorem A] for a finite cyclic group to any compact Lie group G.

The subject treated in this paper is classical and the reader may wonder why we were led to study this subject. In fact, we were concerned with what is called the manifold realization problem. It asks whether a closed smooth G-manifold is equivariantly diffeomorphic to a non-singular real affine G-variety. This problem was originally considered in the non-equivariant category by J. Nash [Nas52], and affirmatively solved by A. Tognoli [Tog73]. Then R. Palais [Pal81] considered the equivariant case above, and some partial affirmative solutions are obtained, see [DM95] for instance. It is even considered to realize smooth G-vector bundles over closed smooth G-manifolds by algebraic ones, which is called the bundle realization problem, and some partial affirmative solutions are obtained as well, see [DMS94]. Apparently the latter problem is more general than the former, but they are linked. Namely we encounter the bundle realization problem to solve the manifold realization problem. For instance, we were faced with realizing real or complex G-line bundles over a circle by algebraic ones to solve the manifold realization problem for two- or three-dimensional manifolds [KM94, CS97]. This motivated us to investigate G-vector bundles over a circle. At the beginning of this research we suspected that the problem might already be solved, but there is no literature as far as we know. We hope that it is worth while publishing the results obtained in this paper in print.

2. Decomposition of G-vector bundles

Hereafter we omit the adjective “complex” for complex vector bundles and complex modules since we work in the complex category. Let G be a compact Lie group and let H be a closed normal subgroup of G. Given a character χ of H and $g \in G$, a new character ${}^g\chi$ of H is defined by ${}^g\chi(h) = \chi(g^{-1}hg)$ for $h \in H$. This defines an action of G on the set $\mathrm{Irr}(H)$ of characters of irreducible H-modules. Since a character is a class function, H acts on $\mathrm{Irr}(H)$ trivially. Therefore, the isotropy subgroup of G at $\chi \in \mathrm{Irr}(H)$, denoted by G_χ, contains H. We choose a representative from each G-orbit in $\mathrm{Irr}(H)$ and denote by $\mathrm{Irr}(H)/G$ the set of those representatives.

Let X be a connected G-space on which H acts trivially. Then all the fibers of a G-vector bundle E over X are isomorphic as H-modules. We call the unique (up to isomorphism) H-module the *fiber H-module* of E. As is well-known, E decomposes according to irreducible H-modules. For $\chi \in \mathrm{Irr}(H)$, we denote by $E(\chi)$ the χ-isotypical component of E, that is, the largest H-subbundle of E with a multiple of χ as the character of the fiber H-module. Note that $gE(\chi)$, that is $E(\chi)$ mapped by $g \in G$, is ${}^g\chi$-isotypical component of E. This means that $E(\chi)$ is actually a G_χ-vector bundle and that $\bigoplus_{\lambda\in G(\chi)} E(\lambda)$, where $G(\chi)$ denotes the G-orbit of χ, is a G-subbundle of E. Here, $G(\chi) \cong G/G_\chi$ is always finite since χ is invariant for any connected component of G, and G has only finitely many connected components. Then $\bigoplus_{\lambda\in G(\chi)} E(\lambda)$ is nothing but the induced G-vector bundle $\operatorname{ind}_{G_\chi}^G E(\chi)$ so that we have the following decomposition

$$E = \bigoplus_{\chi\in\operatorname{Irr}(H)/G} \operatorname{ind}_{G_\chi}^G E(\chi)$$

as G-vector bundles. For the definition of the induced bundle $\operatorname{ind}_{G_\chi}^G E(\chi)$ we refer the reader to [CS85], [MP85], and to [CM00, Section 3] for the complete description.

Lemma 2.1. *Two G-vector bundles E and E' over X are isomorphic if and only if $E(\chi)$ and $E'(\chi)$ are isomorphic as G_χ-vector bundles for each $\chi \in \operatorname{Irr}(H)/G$. In particular, E is trivial if and only if $E(\chi)$ is trivial for each $\chi \in \operatorname{Irr}(H)/G$.*

Proof. The necessity is obvious since a G-vector bundle isomorphism $E \to E'$ restricts to a G_χ-vector bundle isomorphism $E(\chi) \to E'(\chi)$, and the sufficiency follows from the fact that $\operatorname{ind}_{G_\chi}^G$ is functorial. □

The observation above can be restated as follows. Denote by $\operatorname{Vect}_G(X)$ the set of isomorphism classes of G-vector bundles over X, and by $\operatorname{Vect}_{G_\chi}(X,\chi)$ the subset of $\operatorname{Vect}_{G_\chi}(X)$ with a multiple of χ as the character of fiber H-modules. They are semi-groups under Whitney sum. Then the map sending E to $\oplus_{\chi\in\operatorname{Irr}(H)/G} E(\chi)$ gives a semi-group isomorphism

$$\Phi\colon \operatorname{Vect}_G(X) \to \bigoplus_{\chi\in\operatorname{Irr}(H)/G} \operatorname{Vect}_{G_\chi}(X,\chi).$$

This reduces the study of $\operatorname{Vect}_G(X)$ to that of $\operatorname{Vect}_{G_\chi}(X,\chi)$.

Lemma 2.2. *If there is a G_χ-vector bundle over X with χ as the character of the fiber H-module, then $\operatorname{Vect}_{G_\chi}(X,\chi)$ is isomorphic to $\operatorname{Vect}_{G_\chi/H}(X)$ as semi-groups. In fact, if L is such a G_χ-vector bundle, then the map*

$$\operatorname{Vect}_{G_\chi}(X,\chi) \to \operatorname{Vect}_{G_\chi/H}(X)$$

sending E to $\operatorname{Hom}_H(L,E)$ gives an isomorphism.

Proof. It is easy to check that the map

$$\operatorname{Vect}_{G_\chi/H}(X) \to \operatorname{Vect}_{G_\chi}(X,\chi)$$

sending $F \in \mathrm{Vect}_{G_\chi/H}(X)$ to $L \otimes F$ gives the inverse of the map in the lemma. $\square$

Remark. The lemma above does not hold in the real category in general, but it does if χ is of real type, i.e., if χ is the character of a real irreducible H-module with the endomorphism algebra isomorphic to $\mathbb{R}$.

We conclude this section with the following well-known fact.

Proposition 2.3. *If the G-action on X is transitive, i.e., X is homeomorphic to G/K for a closed subgroup K, then any G-vector bundle over X is of the form*

$$G \times_K W \to G/K = X$$

for some K-module W. In fact, W is the fiber over a point of X with K as the isotropy subgroup.

The proposition above implies that there is an isomorphism

$$\mathrm{Vect}_G(G/K) \cong \mathrm{Vect}_K(*),$$

where $*$ denotes the one-point K-space.

3. Fiber H-modules and extension

We say that a character χ of H is *G-invariant* if ${}^g\chi = \chi$ for all $g \in G$. The following proposition gives a necessary condition for an H-module to be the fiber H-module of a G-vector bundle E over a connected G-space X.

Proposition 3.1. *The character of the fiber H-module of E is G-invariant.*

Proof. Let $x \in X$ and $g \in G$. We know that the fibers E_x and E_{gx} of E at x and gx are isomorphic as H-modules. On the other hand, since $g(g^{-1}hg)v = hgv$ for $v \in E_x$, the map $g\colon E_x \to E_{gx}$ becomes an H-equivariant isomorphism if we consider an H-action on E_x given through an automorphism of H given by $h \to g^{-1}hg$. This implies the proposition. $\square$

For a group K containing H, we say that an H-module V *extends* to a K-module W (or W is a *K-extension* of V) if the restriction $\mathrm{res}_H W$ of W to H is isomorphic to V.

There are two reasons why we are concerned with the extension of an H-module. One is that if V is the fiber H-module of E and there is a point in the base space with the isotropy subgroup K larger than H, then the fiber over the point gives a K-extension of V. The other is that if E is trivial, i.e, isomorphic to a product bundle with a G-module as its fiber, then the fiber H-module of E must extend to the G-module.

If H is a normal subgroup of K and an H-module has a K-extension, then its character must be K-invariant because a character is a class function. But the converse does not hold in general.

Example 3.2. Let $K = O(2)$ and let H be the central subgroup of K, that is, the order two subgroup generated by minus the identity matrix. Since H is central, any character of H is K-invariant. As is easily seen, there are only two one-dimensional K-modules, on which $SO(2)$ acts trivially. Therefore, a nontrivial one-dimensional H-module has no K-extension.

A similar example can be found for $K = D_{4m}$ the dihedral group of order $4m$ and for H the central subgroup of K which is also an order two subgroup. Suppose that $K = D_{4m}$ is generated by

$$\{r, s \mid r^{4m} = s^2 = (rs)^2 = 1\}.$$

Then H is generated by r^{2m}. If $\varphi\colon K \to \mathrm{GL}(1, \mathbb{C})$ is a one-dimensional representation of K, then $\varphi(r^2) = 1$ because $s^2 = (rs)^2 = 1$ and $\mathrm{GL}(1, \mathbb{C})$ is abelian. In particular, $\varphi(r^{2m}) = 1$. Therefore, a nontrivial one-dimensional H-module has no K-extension.

The following proposition gives an answer to the extension problem when K/H is isomorphic to a subgroup of $O(2)$.

Proposition 3.3. *Let H be a normal subgroup of K and let V be an irreducible H-module with K-invariant character.*

(1) *If K/H is finite cyclic or isomorphic to $SO(2)$, then V has a K-extension.*

(2) *If K/H is isomorphic to $O(2)$, then V has two K-extensions or none.*

(3) *If K/H is a dihedral group D_n of order $2n$, then*

  (i) *in case n is odd, V has two K-extensions,*

  (ii) *in case n is even, V has either four K-extensions or none.*

In any case, if W is a K-extension of V, then any K-extension of V is of the form $W \otimes U$, where U is a one-dimensional K/H-module viewed as a K-module through the projection $K \to K/H$, and different U's produce different K-extensions. Therefore, if V has a K-extension, then the number of K-extensions of V agrees with the number of one-dimensional K/H-modules.

Remark. (1) The character of an H-module is K-invariant whenever K/H is isomorphic to $SO(2)$ because $SO(2)$ is connected. So the K-invariance of the character of V in the proposition above is unnecessary in this case, i.e., any H-module extends to a K-module whenever K/H is isomorphic to $SO(2)$.

(2) The proposition above does not hold in the real category but it does when K/H is cyclic and of odd order.

Proof. We shall prove the latter statement in the proposition. In this proof, we do not need the assumption that K/H is isomorphic to a subgroup of $O(2)$. Suppose that V has K-extensions W and W'. Then $\mathrm{Hom}_H(W, W')$ is a K/H-module and one-dimensional by Schur's lemma because $\mathrm{res}_H W = \mathrm{res}_H W' = V$ is irreducible. One can easily check that the map

$$W \otimes \mathrm{Hom}_H(W, W') \to W'$$

sending $w \otimes f$ to $f(w)$ is a K-linear isomorphism. Therefore, any K-extension of V is the tensor product of W and a one-dimensional K/H-module viewed as a K-module. Moreover, since $\mathrm{Hom}_H(W, W \otimes U) \cong U$ as K/H-modules, different one-dimensional K/H-modules produce different K-extensions of V. This proves the latter statement.

It is well-known and easy to see that there are two $O(2)$- or D_n-modules of dimension one for n odd, and four D_n-modules of dimension one for n even. This proves the statements on the number of K-extensions in (2) and (3).

It remains to see that V has a K-extension in the cases (1) and (3–i). The case (1) is well-known if K/H is finite cyclic, see for instance [Isa76, Corollary 11.22]. The book [Isa76] treats only finite groups (so that H is finite), but some direct proofs work even if H is infinite. The reader can find one of the direct proofs in [CKS99, Proposition 4.2]. We suspect that the case (1) is known even when K/H is isomorphic to $SO(2)$, but there is no literature as far as we know. Since the proof we found is rather long and has independent interest, we gave it in [CKS99, Proposition 2.3]. The case (3–i) is also well-known, see [Isa76, Corollary 11.31] or [CKS99, Proposition 4.3] for an elementary proof. $\square$

It turns out that the above facts on representation theory greatly influence the semi-group structure on $\mathrm{Vect}_G(X)$ when X is a circle with G-action.

4. Fiber H-modules of G-vector bundles over a circle

Henceforth, we restrict our concern to G-vector bundles over a circle with G-action. For an orthogonal representation $\rho\colon G \to O(2)$ of G, we denote by $S(\rho)$ the unit circle of the corresponding representation space. It is well-known that a circle with continuous (resp. smooth) G-action is equivariantly homeomorphic (resp. diffeomorphic) to $S(\rho)$ for some representation ρ [Sch84, Theorem 2.0]. We identify $S(\rho)$ with the unit circle of the complex line $\mathbb{C}$, and denote a point in $S(\rho)$ by $z \in \mathbb{C}$ with absolute value 1. Set $H = \rho^{-1}(1)$.

Let us observe the subgroup $\rho(G)$ of $O(2)$. If $\rho(G)$ is infinite, then it is either $SO(2)$ or $O(2)$ itself. Otherwise $\rho(G)$ is a finite cyclic or dihedral group. Suppose $\rho(G)$ is a finite cyclic subgroup Z_n of $SO(2)$ of order $n \geq 1$. Choose and fix an element $a \in G$ such that $\rho(a)$ is the rotation through an angle $2\pi/n$. Then G is generated by H and a under the relation $a^n \in H$, and all the isotropy subgroups G_z at $z \in S(\rho)$ are equal to H. If $\rho(G)$ is a proper subgroup of $O(2)$ not contained in $SO(2)$, then we may assume that $\rho(G)$ is a dihedral subgroup D_n of $O(2)$ generated by the reflection matrix about the x-axis and the rotation matrix through an angle $2\pi/n$. Choose and fix one more element $b \in G$ such that $\rho(b)$ is the reflection matrix about the x-axis. Then G is generated by H, a and b under the relations, a^n, b^2, and $(ab)^2 \in H$. The isotropy subgroup G_1 at $1 \in S(\rho)$ is generated by H and b, and G_μ at $\mu = e^{\pi i/n} \in S(\rho)$ is generated by H and ab.

Here is a characterization of the fiber H-modules of G-vector bundles over $S(\rho)$.

Theorem A. *An H-module is the fiber H-module of a G-vector bundle over $S(\rho)$ if and only if its character is G-invariant.*

Proof. The necessity follows from Proposition 3.1, so we prove the sufficiency. Let V be an H-module with G-invariant character. We distinguish three cases according to $\rho(G)$.

Case 1. The case where $\rho(G)$ is infinite, i.e., $\rho(G) = SO(2)$ or $O(2)$. In this case the G-action on $S(\rho)$ is transitive, and the isotropy subgroup K of a point in $S(\rho)$ is H when $\rho(G) = SO(2)$, and contains H as an index two subgroup when $\rho(G) = O(2)$. Therefore V has a K-extension by Proposition 3.3 since the character of V is G-invariant (in particular, K-invariant) and K/H is trivial or of order two. This together with Proposition 2.3 implies the existence of a G-vector bundle over $S(\rho)$ with V as the fiber H-module.

Case 2. The case where $\rho(G) = Z_n$. In this case $H = G_1 = G_\mu$. Set $W = \operatorname{ind}_H^G V$ and consider the product G-vector bundle $\underline{W} = S(\rho) \times W$. Choose an H-submodule isomorphic to V in the fiber of $\underline{W}$ at 1, and identify it with V. The G-invariance of the character of V implies that $\operatorname{res}_H V \cong \operatorname{res}_H(aV)$. Viewing V and aV as H-invariant subspaces of W, one can connect V and aV through a continuous family of H-invariant subspaces along the arc of $S(\rho)$ joining 1 and $e^{2\pi i/n}$, in other words, for each z in the arc one can find an H-invariant subspace in the fiber of $\underline{W}$ at z so that the family of H-invariant subspaces varies continuously on the points z. This is always possible because the set of such H-invariant subspaces of W is homeomorphic to a product of Grassmann manifolds which is arcwise connected. Translating the family of H-invariant subspaces by the action of a repeatedly yields the desired G-subbundle of $\underline{W}$.

Case 3. The case where $\rho(G) = D_n$. By Proposition 3.3, there exist G_1- and G_μ-extensions V_1 and V_μ of V respectively. Set $W = \operatorname{ind}_{G_1}^G V_1 \oplus \operatorname{ind}_{G_\mu}^G V_\mu$ and consider the product G-vector bundle $\underline{W}$. Then V_1 and V_μ are contained as G_1- and G_μ-submodules in the fibers of $\underline{W}$ at 1 and μ, respectively. Since $\operatorname{res}_H V_1 \cong \operatorname{res}_H V_\mu$, it is possible to connect V_1 and V_μ through a continuous family of H-invariant subspaces along the arc of $S(\rho)$ joining 1 and μ as we did in Case 2 above. We translate it using the action of b and then using the action of a repeatedly to obtain the desired G-subbundle of $\underline{W}$. □

Remark. (1) The proof above shows that any G_1-extension (and G_μ-extension when $\rho(G) = D_n$) of V can be realized as the fiber at 1 (and at μ when $\rho(G) = D_n$) of a G-vector bundle over $S(\rho)$.

(2) The proof above also works in the real category except the extension problem of V. Namely, since Proposition 3.3 does not hold in the real category, we need to assume that, in addition to the G-invariance of the character of V, V has a G_1-extension when $\rho(G) = O(2)$, and both G_1- and G_μ-extensions when $\rho(G) = D_n$.

5. The semi-group structure on $\mathrm{Vect}_G(S(\rho))$

In this section we determine the semi-group structure on $\mathrm{Vect}_G(S(\rho))$. We begin with a simple case.

Lemma 5.1. *Suppose the G-action on $S(\rho)$ is effective, in other words, $\rho\colon G \to O(2)$ is injective. Then the semi-group $\mathrm{Vect}_G(S(\rho))$ is generated by*

(1) *one trivial G-line bundle if $\rho(G) \subset SO(2)$,*

(2) *two trivial G-line bundles $L^{\pm}$ if $\rho(G) = O(2)$, and*

(3) *four G-line bundles $L^{\pm\pm}$ with a relation $L^{++} + L^{--} = L^{+-} + L^{-+}$ otherwise.*

Proof. Since ρ is injective, $P = \rho^{-1}(SO(2))$ acts freely on $S(\rho)$; so taking orbit spaces by P gives an isomorphism

$$\mathrm{Vect}_G(S(\rho)) \cong \mathrm{Vect}_{G/P}(S(\rho)/P).$$

In fact, the inverse is given by pulling back elements in $\mathrm{Vect}_{G/P}(S(\rho)/P)$ by the quotient map from $S(\rho)$ to $S(\rho)/P$. Because of this isomorphism, it suffices to study the semi-group structure on $\mathrm{Vect}_{G/P}(S(\rho)/P)$. Note that $S(\rho)/P$ is again a circle or a point, and that the pullback of a trivial bundle is again trivial.

(1) The case where $\rho(G) \subset SO(2)$. In this case $P = G$, i.e., G/P is the trivial group, so the semi-group $\mathrm{Vect}_{G/P}(S(\rho)/P)$ is generated by one element, that is the trivial line bundle, as is well known. This implies (1) in the lemma.

(2) The case where $\rho(G) = O(2)$. In this case G/P is of order two and $S(\rho)/P$ is a point. Therefore, $\mathrm{Vect}_{G/P}(S(\rho)/P)$ is generated by two elements of dimension one. This implies (2) in the lemma.

(3) The case where $\rho(G) = D_n$ for some n. In this case $S(\rho)/P$ is again a circle, G/P is of order two, and the action of G/P on $S(\rho)/P$ is a reflection. In the sequel it suffices to treat the case where $n = 1$. But this case is already studied in [Kim94]. (Kim treats real bundles but the same argument works for complex bundles.) The result in [Kim94] says that D_1-vector bundles over $S(\rho)$ are distinguished by the fiber D_1-modules over the fixed points $\pm 1 \in S(\rho)$, and that any pair of D_1-modules of the same dimension is realized as the fiber D_1-modules at ± 1 of a D_1-vector bundle over $S(\rho)$. Since D_1 is of order two, there are two one-dimensional D_1-modules (one is the trivial one $\mathbb{C}_+$ and the other is the nontrivial one $\mathbb{C}_-$) and that any D_1-module is a direct sum of them. Therefore, there are four inequivalent D_1-line bundles $L^{\pm\pm}$, where $L^{\epsilon\delta}$ (ϵ and δ stand for $+$ or $-$) denotes the D_1-line bundle with the fiber D_1-modules $\mathbb{C}_\epsilon$ at 1 and $\mathbb{C}_\delta$ at -1, and the structure of $\mathrm{Vect}_{D_1}(S(\rho))$ is as stated in (3).

For the reader's convenience, we shall give the argument in [Kim94] briefly. First we observe that any pair of one-dimensional D_1-modules can be realized as the fiber D_1-modules at ± 1 of a D_1-line bundle over $S(\rho)$. In fact, the trivial (real) line bundle and the (real) Hopf line bundle have respectively two different D_1-liftings to the total space, and each D_1-lifting of the trivial (real) line bundle has the same fiber D_1-modules at ± 1 while that of the (real) Hopf line bundle has different fiber D_1-modules at ± 1. We consider complexification of them. Then any pair of D_1-modules of dimension m can be realized as the fiber D_1-modules at ± 1 by taking the Whitney sum of suitable m number of those complexified D_1-line bundles. On the other hand, the same technique used in the proof of Theorem A shows that any D_1-vector bundle over $S(\rho)$ decomposes into the Whitney sum of the above D_1-line bundles. $\square$

Theorem A applied with $G = G_\chi$ and the irreducible H-module with character χ says that the assumption in Lemma 2.2 is satisfied when $X = S(\rho)$, so $\mathrm{Vect}_{G_\chi}(S(\rho), \chi)$ has the same semi-group structure as $\mathrm{Vect}_{G_\chi/H}(S(\rho))$. Here the action of G_χ/H on $S(\rho)$ is effective, so the lemma above can be applied to $\mathrm{Vect}_{G_\chi/H}(S(\rho))$. In the sequel the semi-group structure on $\mathrm{Vect}_{G_\chi}(S(\rho), \chi)$ is divided into three types depending on $\rho(G_\chi)$. We denote the generators of $\mathrm{Vect}_{G_\chi}(S(\rho), \chi)$ corresponding to the generators in Lemma 5.1 by

$$\begin{cases} L_\chi, & \text{if } \rho(G_\chi) \subset SO(2), \\ L_\chi^\pm, & \text{if } \rho(G_\chi) = O(2), \\ L_\chi^{\pm\pm}, & \text{otherwise.} \end{cases}$$

The following theorem follows immediately from Lemma 5.1.

Theorem B. *The semi-group* $\mathrm{Vect}_{G_\chi}(S(\rho), \chi)$ *is generated by*
(1) *one element* L_χ *if* $\rho(G_\chi) \subset SO(2)$,
(2) *two elements* $L_\chi^\pm$ *if* $\rho(G_\chi) = O(2)$, *and*
(3) *four elements* $L_\chi^{\pm\pm}$ *with the relation* $L_\chi^{++} + L_\chi^{--} = L_\chi^{+-} + L_\chi^{-+}$ *otherwise.* $\square$

Remark. If $\rho(G) \subset SO(2)$, then $\rho(G_\chi)$ is of type (1) above for any χ. If $\rho(G) = O(2)$, then $\rho(G_\chi)$ is of type (1) or (2) above; more precisely $\rho(G_\chi) = SO(2)$ or $O(2)$ because the G-action on $\mathrm{Irr}(H)$ reduces to an action of $G/H = \rho(G) = O(2)$ and the action of $SO(2)$ on $\mathrm{Irr}(H)$ is trivial since $SO(2)$ is connected. Moreover, if $\rho(G) = D_n$, then $\rho(G_\chi)$ is of type (1) or (3) above. Therefore, the semi-group structure on $\mathrm{Vect}_G(S(\rho))$ can be read from the theorem above and the isomorphism Φ in Section 2.

Using the theorem above, one can easily enumerate G-vector bundles over $S(\rho)$ with a fixed H-module V as the fiber H-module. Since V must have a G-invariant character by Theorem A, one can express the character of V as

$$\sum_{\chi \in \mathrm{Irr}(H)/G} m_\chi \Big(\sum_{\lambda \in G(\chi)} \lambda \Big)$$

with non-negative integers m_χ, where m_χ's are zero for all but finitely many χ's in $\mathrm{Irr}(H)/G$ because V is of finite dimension. Set

$$e(\chi) = \begin{cases} 0, & \text{if } \rho(G_\chi) \subset SO(2) \\ 1, & \text{if } \rho(G_\chi) = O(2) \\ 2, & \text{otherwise.} \end{cases}$$

With this understood

Corollary 5.2. *The number of isomorphism classes of G-vector bundles over $S(\rho)$ with V as the fiber H-modules is given by $\prod_{\chi\in\mathrm{Irr}(H)/G}(m_\chi+1)^{e(\chi)}$.* □

6. Isomorphism theorem

In this section we present another approach to study the semi-group structure on $\mathrm{Vect}_G(S(\rho))$. The following theorem, which we call an isomorphism theorem, reduces the study of $\mathrm{Vect}_G(S(\rho))$ to representation theory.

Theorem 6.1. *Two G-vector bundles E and E' over $S(\rho)$ are isomorphic if and only if the fiber G_z-modules E_z and E'_z at $z \in S(\rho)$ are isomorphic for $z = 1$ (and for $z = \mu$ when $\rho(G) = D_n$).*

Proof. The necessity part is obvious, so we prove the sufficiency. We note that if there exists an equivariant isomorphism $\Psi\colon E \to E'$, then it must satisfy the equivariance condition

$$\Psi_{\rho(g)z} = g\Psi_z g^{-1}$$

for any $g \in G$ where $\Psi_z = \Psi|_{E_z}$. By the assumption we have a G_1-linear isomorphism Ψ_1 (and a G_μ-linear isomorphism Ψ_μ when $\rho(G) = D_n$). In the following we will define Ψ_z for all $z \in S(\rho)$ using the above equivariance condition to get an equivariant isomorphism Ψ. We consider three cases according to the images of G by ρ.

Case 1. The case where $\rho(G) = SO(2)$ or $O(2)$. In this case the G-action on $S(\rho)$ is transitive, so for any $z \in S(\rho)$ we define $\Psi_z = g\Psi_1 g^{-1}$ with $g \in G$ such that $z = \rho(g)1$. The well-definedness follows from the G_1-equivariance of Ψ_1. This gives the desired equivariant isomorphism Ψ.

Case 2. The case where $\rho(G) = Z_n$. Let $\zeta = e^{2\pi i/n}$ and define Ψ_ζ by $a\Psi_1 a^{-1}$. The map Ψ_ζ is also an H-equivariant isomorphism. We connect Ψ_1 and Ψ_ζ along the arc of $S(\rho)$ joining 1 and ζ, in other words, we find an H-equivariant linear isomorphism Ψ_z for each z in the arc of $S(\rho)$ so that Ψ_z is continuous at those z. (This is always possible because the set of H-linear isomorphisms between E_z and E'_z is arcwise connected, in fact, homeomorphic to a product of $\mathrm{GL}(N,\mathbb{C})$'s.) Now we define Ψ_z for any $z \in S(\rho)$ using the equivariance condition $\Psi_{\zeta z} = a\Psi_z a^{-1}$. This gives the desired isomorphism Ψ.

Case 3. The case where $\rho(G) = D_n$. Note that the equivariance condition of Ψ is

$$\Psi_z = h\Psi_z h^{-1} \text{ for any } h \in H, \quad \Psi_{\zeta z} = a\Psi_z a^{-1}, \quad \Psi_{\bar z} = b\Psi_z b^{-1}.$$

We connect Ψ_1 and Ψ_μ along the arc joining 1 and μ to obtain Ψ_z for z in the arc. Then using the equivariance condition $\Psi_{\bar z} = b\Psi_z b^{-1}$, we define Ψ_z

for z in the arc joining 1 and μ^{-1}. Thus we have defined Ψ_z for z in the arc joining μ^{-1} and μ. We then define Ψ_z for all z using the equivariance condition $\Psi_{\zeta z} = a\Psi_z a^{-1}$. This gives the desired isomorphism. $\square$

For a group K we denote by $\mathrm{Rep}(K)$ the set of isomorphism classes of K-modules, and by $\mathrm{Rep}(G_1, G_\mu)$ the set of elements $(V, W) \in \mathrm{Rep}(G_1) \times \mathrm{Rep}(G_\mu)$ with $\mathrm{res}_H V = \mathrm{res}_H W$. Restriction of a G-vector bundle over $S(\rho)$ to fibers at 1 (and μ when $\rho(G) = D_n$) yields a map

$$\Gamma\colon \mathrm{Vect}_G(S(\rho)) \to \begin{cases} \mathrm{Rep}(H)^G & \text{if } \rho(G) \subset SO(2), \\ \mathrm{Rep}(G_1) & \text{if } \rho(G) = O(2), \\ \mathrm{Rep}(G_1, G_\mu) & \text{if } \rho(G) = D_n \end{cases}$$

where $\mathrm{Rep}(H)^G$ denotes the subset of $\mathrm{Rep}(H)$ with G-invariant character. The target of the map Γ is a semi-group under direct sum. With this understood

Proposition 6.2. *The map Γ is an isomorphism.*

Proof. It is obvious that Γ is a homomorphism and that the characters of all representations in $\mathrm{Rep}(G_1)$ (when $\rho(G) = O(2)$) and $\mathrm{Rep}(G_1, G_\mu)$ (when $\rho(G) = D_n$) are G-invariant. The surjectivity follows from Theorem A (when $\rho(G) \subset SO(2)$) and the first remark following it in Section 4 (when $\rho(G) = O(2)$ or D_n), and the injectivity follows from Theorem 6.1. $\square$

As a matter of fact, the source and target of the map Γ have more structures, that is, they have products given by tensor product and $R(G)$ acts on them naturally through the tensor product. In fact, $R(G)$ acts on the target through the restriction to H, G_1, or G_μ. Clearly the map Γ preserves these structures.

7. Triviality of G-vector bundles over a circle

In this section we investigate when a G-vector bundle over $S(\rho)$ is trivial. Here is the criterion of triviality of a G-vector bundle over $S(\rho)$.

Lemma 7.1. (1) *A G-vector bundle over $S(\rho)$ is trivial if and only if the fiber G_z-module at $z = 1$ (and at $z = \mu$ when $\rho(G) = D_n$) extends to a (same when $\rho(G) = D_n$) G-module.*

(2) *Unless $\rho(G) \subset SO(2)$, the number of the isomorphism classes of trivial G-vector bundles over $S(\rho)$ with an irreducible fiber H-module V agrees with the number of G-extensions of V.*

Proof. (1) The necessity is trivial and the sufficiency follows from Theorem 6.1.

(2) Let W and W' be two G-extensions of V and suppose that the product bundles $\underline{W}$ and $\underline{W'}$ are isomorphic. Then $\mathrm{res}_{G_1} W \cong \mathrm{res}_{G_1} W'$ (and $\mathrm{res}_{G_\mu} W \cong \mathrm{res}_{G_\mu} W'$ if $\rho(G) = D_n$). It is easy to see from Proposition 3.3 that each G-extension of V is distinguished by its restriction to G_1 (and G_μ if $\rho(G) = D_n$). Therefore, W and W' are isomorphic as G-modules, proving (2). $\square$

Theorem C. *The triviality of the generators appeared in Theorem* B *is as follows.*

(1) L_χ *is trivial.*

(2) $L_\chi^\pm$ *are both trivial or both nontrivial.*

(3) *Two of* $L_\chi^{\pm\pm}$ *are trivial and the other two are nontrivial if* $|\rho(G_\chi)|/2$ *is odd, and* $L_\chi^{\pm\pm}$ *are all trivial or all nontrivial if* $|\rho(G_\chi)|/2$ *is even.*

Proof. This follows from Proposition 3.3 and Lemma 7.1. □

In fact, $L_\chi^\pm$ are related through the tensor product with a one-dimensional nontrivial representation $G_\chi \to O(2)/SO(2) = \{\pm 1\}$ (which is ρ composed with the projection $O(2) \to O(2)/SO(2)$), and $L_\chi^{\pm\pm}$ are related through the tensor product with four G_χ-line bundles $L^{\pm\pm}$ which are pullback of the four D_1-line bundles by the quotient map from $S(\rho)$ to $S(\rho)/P$ where $P = \rho^{-1}(SO(2))$ as before. Note that the fiber H-modules of $L_\chi^{\pm\pm}$ are trivial. The G_χ-line bundles $L^{\pm\pm}$ are well understood and one (actually two) of $L_\chi^{\pm\pm}$ is trivial in case n is odd, so we completely understand $L_\chi^{\pm\pm}$ in this case. But we do not know $L_\chi^{\pm\pm}$ explicitly when n is even and they are all nontrivial.

Corollary 7.2. *If* $\rho(G_\chi) \subset SO(2)$, *then every element in* $\mathrm{Vect}_{G_\chi}(S(\rho), \chi)$ *is trivial.*

Proof. The corollary follows from Theorems B (1) and C (1). □

Corollary D. *Every* G*-vector bundle over* $S(\rho)$ *is trivial if* $\rho(G) \subset SO(2)$.

Proof. If $\rho(G) \subset SO(2)$, then $\rho(G_\chi) \subset SO(2)$ for any χ. Therefore the corollary follows from Lemma 2.1 and Corollary 7.2. □

8. Description of G-line bundles over a circle

In this section, we describe $L_\chi^{\pm\pm}$ explicitly when χ is the character of a one-dimensional H-module. In the following, φ denotes a one-dimensional H-representation with G-invariant character χ. Since φ is one-dimensional, χ agrees with φ.

When $\rho(G) = D_n$, let a and b be as before, i.e., they denote elements of G whose images by ρ are respectively the rotation through an angle $2\pi/n$ and the reflection by x-axis. Note that G is generated by H, a and b under the relations that a^n, b^2 and $(ab)^2$ belong to H.

Lemma 8.1. *Suppose* $\rho(G) = D_n$. *Then*

(1) $\varphi(a^n)^2 = \varphi(abab^{-1})^n$,

(2) *when* n *is even,* φ *has a* G*-extension if and only if* $\varphi(a^n) = \varphi(abab^{-1})^{n/2}$.

Remark. When n is odd, we know that φ has a G-extension by Proposition 3.3 (3). It can also be seen from the proof of (2) below.

Proof. Let $\widetilde{\varphi}\colon G_1 \to \mathrm{GL}(1,\mathbb{C})$ be an extension of φ to G_1. Since (the character of) φ is G-invariant (in particular, G_1-invariant) and H is an index two subgroup of G_1, such an extension exists by Proposition 3.3 (1).

(1) It is elementary to see that $a^iba^i \in G_1$ and $a^iba^iaba \in H$ for $1 \le i \le n-1$. Since φ is G-invariant we have

$$\widetilde{\varphi}(a^iba^i)\widetilde{\varphi}(aba) = \varphi(a^iba^iaba) = \varphi(a(a^iba^iaba)a^{-1}) = \widetilde{\varphi}(a^{i+1}ba^{i+1})\widetilde{\varphi}(b)$$

for $1 \le i \le n-1$. By an inductive application of the above identity, we have

$$\widetilde{\varphi}(aba)^n = \widetilde{\varphi}(a^nba^n)\widetilde{\varphi}(b)^{n-1}.$$

It follows that

$$\varphi(abab^{-1})^n = \widetilde{\varphi}(aba)^n\widetilde{\varphi}(b^{-1})^n = \widetilde{\varphi}(a^nba^n)\widetilde{\varphi}(b)^{n-1}\widetilde{\varphi}(b^{-1})^n = \varphi(a^n)^2.$$

(2) The necessity is obvious, so we shall prove the sufficiency. Let A be an n-th root of $\varphi(a^n)$. Then $(A^{-2}\varphi(abab^{-1}))^n = 1$ by the identity in (1) above, so there is an integer k (determined modulo n) such that $A^{-2}\varphi(abab^{-1}) = \zeta^k$ where $\zeta = e^{2\pi i/n}$. The equality $\varphi(a^n) = \varphi(abab^{-1})^{n/2}$ is equivalent to k being even. We define $\widetilde{\varphi}(a) = A\zeta^{k/2}$. Then $\widetilde{\varphi}(a)^n = (A\zeta^{k/2})^n = A^n = \varphi(a^n)$. Therefore, to see that the extended $\widetilde{\varphi}$ is a G-extension of φ, it only remains to check that $\widetilde{\varphi}(a)^2\widetilde{\varphi}(b)^2 = \varphi((ab)^2)$. (Remember that $(ab)^2 \in H$.) The left hand side at the identity is equal to $A^2\zeta^k\widetilde{\varphi}(b)^2$ while the right hand side is equal to $\varphi(abab^{-1})\varphi(b^2)$ which agrees with $A^2\zeta^k\widetilde{\varphi}(b)^2$ because $A^{-2}\varphi(abab^{-1}) = \zeta^k$ by the choice of k above. □

Let $\varphi\colon H \to U(1)$ be a unitary representation with G-invariant character. If $\rho(G) = D_n$, then there are exactly four G-line bundles with φ as the fiber H-representation by Theorem B (3). They are described explicitly in the following example.

Example 8.2. Assume that $\rho(G) = D_n$. Let $\varphi\colon H \to U(1)$ be a unitary representation with G-invariant character, and let $\widetilde{\varphi}\colon G_1 \to U(1)$ be a G_1-extension of φ. Let A be an n-th root of $\varphi(a^n)$. As observed in the proof of Lemma 8.1 (2), there is an integer k such that $A^{-2}\widetilde{\varphi}(abab^{-1}) = \zeta^k$. One can check that

$$h(z,v) = (z, \varphi(h)v) \text{ for } h \in H,\ a(z,v) = (\zeta z, Av), \text{ and } b(z,v) = (\bar{z}, \widetilde{\varphi}(b)z^kv)$$

define an action of G on $S^1 \times \mathbb{C}$. In fact, it defines a G-line bundle over $S(\rho)$ such that the fiber representation at 1 is $\widetilde{\varphi}$ and that at μ is given by

$$h \mapsto \varphi(h), \quad ab \mapsto A\mu^k\widetilde{\varphi}(b).$$

Since the integer k is only determined modulo n (once A is chosen) and $\mu^n = -1$, this construction gives two G-line bundles with $\widetilde{\varphi}$ as the fiber representation at 1. (If we take $k+n$ instead of k, then the fiber representation at μ evaluated on ab changes the sign.) Since there are two G_1-extensions of φ by Proposition 3.3, the above construction describes all the four G-line bundles over $S(\rho)$ with φ as the fiber H-module.

We now have a classification result for G-line bundles over $S(\rho)$.

Theorem 8.3. *Let $\varphi\colon H \to U(1)$ be an H-representation with G-invariant character. Let N be the number of G-line bundles over $S(\rho)$ with φ as the fiber H-module.*

(1) *If $\rho(G) \subset SO(2)$, then $N = 1$ and the bundle is trivial.*

(2) *If $\rho(G) = O(2)$, then $N = 2$ and both bundles are trivial or both are nontrivial.*

(3) *If $\rho(G) = D_n$, then $N = 4$ and all the four bundles are given in Example 8.2, and*

(i) *if n is odd, then two of them are trivial and the other two are nontrivial*;

(ii) *if n is even, then all the four bundles are trivial if $\varphi(a^n) = \varphi(abab^{-1})^{n/2}$, and all are nontrivial otherwise.*

Proof. This follows from Theorems B, C, and the observation done in Example 8.2. ☐

9. The case when G is abelian

When G is abelian (and hence so is the subgroup H), any irreducible H-module is one-dimensional and $G_\chi = G$ for any character χ of H. Therefore $\mathrm{Vect}_G(S(\rho))$ is generated by G-line bundles. Moreover, since $\rho(G)$ is an abelian subgroup of $O(2)$, it is contained in $SO(2)$ or isomorphic to D_1 or D_2. When $\rho(G) \subset SO(2)$, any G-line bundle over $S(\rho)$ is trivial as we know. When $\rho(G) = D_2$, the condition $\varphi(a^n) = \varphi(abab^{-1})^{n/2}$ in Theorem 8.3 (3–ii) for $n = 2$ holds because G is abelian; so any G-line bundle over $S(\rho)$ is trivial in this case, too. But there are two nontrivial G-line bundles when $\rho(G) = D_1$ as claimed in Theorem 8.3 (3–i). The following example is simply an interpretation of Example 8.2 to the special case when $\rho(G) = D_1$.

Example 9.1. Suppose G is abelian and $\rho(G) = D_1$. Then $G = G_1$. Let $\varphi\colon H \to U(1)$ be a unitary representation of H. (Any φ is G-invariant because G is abelian.) As in Example 8.2, choosing a G-extension $\widetilde{\varphi}$ of φ induces a G-action on $S(\rho) \times \mathbb{C}$ defined by

$$h(z, v) = (z, \varphi(h)v) \quad \text{for } h \in H \quad \text{and} \quad b(z, v) = (\bar{z}, \widetilde{\varphi}(b)zv).$$

It gives a nontrivial G-line bundle over $S(\rho)$. Since there are two G-extensions of φ, this produces two nontrivial G-line bundles over $S(\rho)$ with φ as the fiber H-module.

Summing up, we have

Proposition 9.2. *Suppose G is abelian and let E be a G-vector bundle over $S(\rho)$.*

(1) *If $\rho(G) \neq D_1$, then E is trivial.*

(2) *If $\rho(G) = D_1$, then E is the Whitney sum of trivial bundles and the nontrivial line bundles in Example* 9.1.

10. Equivariant K-groups of a circle

In this section we apply the results discussed in the previous sections to the calculation of the reduced equivariant K-group of $S(\rho)$. For a compact G-space X, the equivariant K-group $K_G(X)$ of X is defined to be the Grothendieck group of finite dimensional G-vector bundles over X. If X has a base point $*$ fixed by the G-action, then the reduced equivariant K-group $\widetilde{K}_G(X)$ is defined to be the kernel of the restriction homomorphism $K_G(X) \to K_G(*)$ induced from the inclusion map. In fact, $K_G(X)$ and $\widetilde{K}_G(X)$ are algebras over $R(G)$ (although there is no identity element in $\widetilde{K}_G(X)$).

The additive structure on $K_G(S(\rho))$ can be determined completely by Lemma 2.1 and Theorem B. One can also describe the $R(G)$-algebra structure in terms of the representation ring $R(G)$ through the map Γ in Section 6. In the following, we shall compute $\widetilde{K}_G(S(\rho))$. Note that $\widetilde{K}_G(S(\rho))$ is defined only when $S(\rho)$ has a fixed point, i.e., $G = H$ or $\rho(G) = D_1$, and that $\widetilde{K}_G(S(\rho))$ is trivial if $G = H$. Suppose $\rho(G) = D_1$. Then the G-fixed point set $S(\rho)^G$ consists of the two points $\{\pm 1\}$ and we take -1 to be a base point. It follows from Theorem 6.1 that the restriction homomorphism

$$\widetilde{K}_G(S(\rho)) \to \widetilde{K}_G(S(\rho)^G) \cong R(G)$$

to fibers at 1 is injective. The following theorem determines the image of the homomorphism as an ideal of $R(G)$, which extends Y. Yang's result for G finite cyclic [Yan95, Theorem A] to any compact Lie group G. Denote by $\mathbb{C}_+$ and $\mathbb{C}_-$ the G-modules of dimension one induced from the trivial and the nontrivial D_1-modules of dimension one, respectively, by the homomorphism $G \to G/H \cong D_1$. Note that the one-point compactification of $\mathbb{C}_-$ is nothing but $S(\rho)$.

Theorem 10.1. *If $\rho(G) = D_1$, then $\widetilde{K}_G(S(\rho))$ is isomorphic to the ideal $R(G)(\mathbb{C}_+ - \mathbb{C}_-)$ in $R(G)$ generated by $\mathbb{C}_+ - \mathbb{C}_-$. In particular, $\widetilde{K}_G(S(\rho))$ is torsion-free for any compact Lie group G.*

Proof. The remark (1) at the end of Section 4 implies that $R(G)(\mathbb{C}_+ - \mathbb{C}_-)$ is contained in the image of $\widetilde{K}_G(S(\rho)) \to \widetilde{K}_G(S(\rho)^G) \cong R(G)$, so we prove the converse.

Choose an element $E - F$ in $\widetilde{K}_G(S(\rho))$. Then the fibers of E and F at the base point -1 are isomorphic as G-modules. In particular, E and F have the same fiber H-module and thus $\mathrm{res}_H\, E_1$ is isomorphic to $\mathrm{res}_H\, F_1$. Hence, one can express the image of $E-F$, i.e., E_1-F_1 in $R(G)$ as $E_1-F_1 \cong \bigoplus(E_i-F_i)$ where E_i and F_i are irreducible G-submodules of E_1 and F_1, respectively, such that $\mathrm{res}_H\, E_i \cong \mathrm{res}_H\, F_i$. Here, we note that an irreducible G-module W is uniquely determined by $\mathrm{res}_H\, W$ if it is reducible, because $2W \cong \mathrm{ind}_H^G \mathrm{res}_H\, W$ in this case, see [BtD85, Theorem 7.3 (ii), Chapter VI]. Therefore, if $\mathrm{res}_H\, E_i \cong \mathrm{res}_H\, F_i$ is reducible, then $E_i - F_i = 0$ in $R(G)$. On the other hand, if $\mathrm{res}_H\, E_i \cong \mathrm{res}_H\, F_i$ is irreducible, then both E_i and F_i are G-extensions of $\mathrm{res}_H\, E_i$. Thus F_i is isomorphic to E_i or $E_i \otimes \mathbb{C}_-$ by the last statement of Proposition 3.3. It follows that $E_i - F_i$ is either zero or $E_i \otimes (\mathbb{C}_+ - \mathbb{C}_-)$ in $R(G)$. Therefore, the image of $E - F$ is contained in the ideal $R(G)(\mathbb{C}_+ - \mathbb{C}_-)$. □

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