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On commutators of foliation preserving Lipschitz homeomorphisms

By

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Abstract

We consider the group of foliation preserving Lipschitz homeomorphisms of a Lipschitz foliated manifold. First we show that the identity component of the group of leaf preserving Lipschitz homeomorphisms of a Lipschitz foliated manifold is perfect. Next using this result we compute the first homology of the group of foliation preserving Lipschitz homeomorphisms of a codimension one C^2-foliated manifold. Then we have results which are different from those of topological and differentiable cases.

1. Introduction

Let M be an m-dimensional connected Lipschitz manifold. A continuous map $f : M \to M$ is called a Lipschitz map if for any point p in M, there exist a local Lipschitz chart $(U_\alpha, \varphi_\alpha)$ of M around p and a local Lipschitz chart (U_β, φ_β) of M around $f(p)$ such that $f(U_\alpha) \subset U_\beta$ and $\varphi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \varphi_\beta(U_\beta)$ is Lipschitz. We denote by $C_{LIP}(M, M)$ the set of all Lipschitz mappings from M to M. A homeomorphism $f : M \to M$ is called a Lipschitz homeomorphism if both f and f^{-1} are Lipschitz. We denote by $C_{LIP}(M, M)$ the space of all Lipschitz maps from M to M with the compact open Lipschitz topology (see Section 2) and by $\mathcal{H}_{LIP}(M)$ the subspace of $C_{LIP}(M, M)$ which consists of Lipschitz homeomorphisms of M with compact support.

In this note we treat certain subgroups of $\mathcal{H}_{LIP}(M)$. Let $\mathbf{R}^m = \{(x_1, \dots, x_m) \mid x_i \in \mathbf{R}\}$ be an m-dimensional Euclidean space and $\mathcal{F}_0$ the p-dimensional foliation of $\mathbf{R}^m$ whose leaves are defined by $x_{p+1} = \mathit{constant}, \dots, x_m = \mathit{constant}$ $(1 \leq p \leq m)$. A p-dimensional Lipschitz foliation $\mathcal{F}$ of M is defined to be a maximal set of local Lipschitz charts : $\{(U_\alpha, \varphi_\alpha), U_\alpha$ is open in M, $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)(\subset \mathbf{R}^m), \alpha \in A\}$ of M such that $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ preserves the leaves of foliations which are restrictions of $\mathcal{F}_0$ to $\varphi_\beta(U_\alpha \cap U_\beta)$ and $\varphi_\alpha(U_\alpha \cap U_\beta)$.

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A Lipschitz homeomorphism $f : M \to M$ is called *a foliation preserving Lipschitz homeomorphism* (resp. *a leaf preserving Lipschitz homeomorphism*) if for each point x of M, the leaf through x is mapped into the leaf through $f(x)$ (resp. x), that is, $f(L_x) = L_{f(x)}$ (resp. $f(L_x) = L_x$), where L_x is the leaf of $\mathcal{F}$ which contains x. Let $\mathcal{H}_{LIP}(M, \mathcal{F})$ (resp. $\mathcal{H}_{LIP,L}(M, \mathcal{F})$) denote the group of foliation (resp. leaf) preserving homeomorphisms of $(M, \mathcal{F})$ which are isotopic to the identity by foliation (resp. leaf) preserving Lipschitz homeomorphisms fixed outside a compact set.

In Section 2, we consider the homologies of $\mathcal{H}_{LIP,L}(M, \mathcal{F})$, that is, the homology groups of the group $\mathcal{H}_{LIP,L}(M, \mathcal{F})$ and show that the homologies of $\mathcal{H}_{LIP,L}(\mathbf{R}^m, \mathcal{F}_0)$ vanish in all dimension > 0. This is an analogy to Theorem 2.1 of [F-I] which is a generalization of a result of Mather [M].

In Section 3, first we show that any $f \in \mathcal{H}_{LIP,L}(M, \mathcal{F})$ can be expressed as $f = f_k \circ f_{k-1} \circ \cdots \circ f_1$, where each f_i is a leaf preserving Lipschitz homeomorphism with support in a small ball. Next we show using this result and the result in Section 2 that $\mathcal{H}_{LIP,L}(M, \mathcal{F})$ is perfect for a compact Lipschitz foliated manifold $(M, \mathcal{F})$. Furthermore by an argument similar to that in [A-F] we can show that $\mathcal{H}_{LIP,L}(M, \mathcal{F})$ is locally contractible.

In Section 4, we compute the first homology of $\mathcal{H}_{LIP}(M, \mathcal{F})$ for a codimension one C^2-foliated manifold $(M, \mathcal{F})$. For the case that $\mathcal{F}$ has no dense components, we have the same result as that in topological case (Theorem 4.4), which is different from that in differentiable case. For the case that $\mathcal{F}$ has a dense component, we have a phenomenon different from that in topological case (Theorem 4.7).

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2. Compact open Lipschitz topology and homologies of $\mathcal{H}_{LIP,L}(\mathbf{R}^m, \mathcal{F}_0)$

We recall the definitions of a Lipschitz manifold, a Lipschitz map and the compact open Lipschitz topology on $\mathcal{H}_{LIP}(M)$ (cf. [A-F]). Let M be an m-dimensional topological manifold. By a Lipschitz atlas on M we mean a maximal family $\mathcal{S} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ of pairs $(U_\alpha, \varphi_\alpha)$ of open sets U_α in M and homeomorphisms φ_α of U_α to $\varphi_\alpha(U_\alpha)$ in $\mathbf{R}^m$ satisfying the following : (i) $\{U_\alpha\}_{\alpha \in A}$ covers M and (ii) If $U_\alpha \cap U_\beta \neq \emptyset$, the transition function $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ from an open set of $\mathbf{R}^m$ to an open set of $\mathbf{R}^m$ is Lipschitz. We call $(M, \mathcal{S})$ a Lipschitz manifold and simply write M instead of $(M, \mathcal{S})$.

Let M, N be two Lipschitz manifolds. A continuous map $f : M \to N$ is called a Lipschitz map if for any point p in M, there exist a local chart $(U_\alpha, \varphi_\alpha)$ of M around p and a local chart $(V_\lambda, \psi_\lambda)$ of N around $f(p)$ such that $f(U_\alpha) \subset V_\lambda$ and $\psi_\lambda \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \psi_\lambda(V_\lambda)$ is Lipschitz. We denote by $C_{LIP}(M, N)$ the set of all Lipschitz mappings from M to N.

Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ and $\{(V_\lambda, \psi_\lambda)\}_{\lambda \in \Lambda}$ be Lipschitz atlases on M and N respectively. Let K_α be a compact subset of U_α for each $\alpha \in A$ such that the family $\{\text{int}\, K_\alpha\}_{\alpha\in A}$ covers M. Let $f \in C_{LIP}(M,N)$. We take a local chart $(V_\lambda, \psi_\lambda)$ on N such that $f(K_\alpha) \subset V_\lambda$. For $\epsilon_\alpha > 0$, we let $\mathcal{N}^{LIP}(f, (U_\alpha, \varphi_\alpha), (V_\lambda, \psi_\lambda), \epsilon_\alpha, K_\alpha)$ be the set of all $g \in C_{LIP}(M,N)$ such that $g(K_\alpha) \subset V_\lambda$ and $lip(f-g) < \epsilon_\alpha$, where $lip(f-g) < \epsilon_\alpha$ means that

$$||\psi_\lambda \circ f \circ \varphi_\alpha^{-1}(x) - \psi_\lambda \circ g \circ \varphi_\alpha^{-1}(x)|| < \epsilon_\alpha$$

and

$$||(\psi_\lambda\circ f\circ\varphi_\alpha^{-1}(x)-\psi_\lambda\circ g\circ\varphi_\alpha^{-1}(x))-(\psi_\lambda\circ f\circ\varphi_\alpha^{-1}(y)-\psi_\lambda\circ g\circ\varphi_\alpha^{-1}(y))|| < \epsilon_\alpha||x-y||$$

for distinct $x, y \in K_\alpha$. The sets $\mathcal{N}^{LIP}(f, (U_\alpha, \varphi_\alpha), (V_\lambda, \psi_\lambda), \epsilon_\alpha, K_\alpha)$ form a subbasis for a topology on $C_{LIP}(M,N)$. We call this topology the *compact open Lipschitz topology.*

A homeomorphism $f : M \to M$ is called a Lipschitz homeomorphism if f and f^{-1} are Lipschitz. We denote by $\mathcal{H}_{LIP}(M)$ the group of all Lipschitz homeomorphisms of M with compact support (as a subspace of $C_{LIP}(M,M)$ endowed with the compact open Lipschitz topology).

Let $\mathbf{R}^m = \{(x_1, \dots, x_m) \mid x_i \in \mathbf{R}\}$ be an m-dimensional Euclidean space and $\mathcal{F}_0$ the p-dimensional foliation of $\mathbf{R}^m$ whose leaves are defined by $x_{p+1} = \mathit{constant}, \dots, x_m = \mathit{constant}$ $(1 \le p \le m)$. A p-dimensional Lipschitz foliation $\mathcal{F}$ of a Lipschitz manifold M is defined to be a maximal set of Lipschitz charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ of M such that $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ preserves the leaves of foliations which are restrictions of $\mathcal{F}_0$ to $\varphi_\beta(U_\alpha \cap U_\beta)$ and $\varphi_\alpha(U_\alpha \cap U_\beta)$.

Let $(M, \mathcal{F})$ be a Lipschitz foliated manifold. We denote by $\mathcal{H}_{LIP}(M, \mathcal{F})$ (resp. $\mathcal{H}_{LIP,L}(M, \mathcal{F})$) the identity component of the subgroup of $\mathcal{H}_{LIP}(M)$ which consists of foliation (resp. leaf) preserving Lipschitz homeomorphisms of $(M, \mathcal{F})$ (as the subspace of $\mathcal{H}_{LIP}(M)$).

By an argument similar to those in the proofs of Theorem 2.1 of [F-I] and Theorem 2.2 of [A-F], we have the following:

Theorem 2.1. *The homology groups* $H_r(\mathcal{H}_{LIP,L}(\mathbf{R}^m, \mathcal{F}_0)) = 0$ *for* $r > 0$.

Corollary 2.2. $\mathcal{H}_{LIP,L}(\mathbf{R}^m, \mathcal{F}_0)$ *is a perfect group.*

Proof. This is an immediate consequence of Theorem 2.1 because that $H_1(G) \cong G/[G,G]$ for any group G. □

3. Commutators of leaf preserving Lipschitz homeomorphisms

In this section we consider commutators of $\mathcal{H}_{LIP,L}(M, \mathcal{F})$. Let $(M, \mathcal{F})$ be a compact Lipschitz foliated manifold. We take a local foliated chart $(U_\alpha, \varphi_\alpha)$ on M, that is, for coordinate $(x, y) = (x_1, \dots, x_p, y_1, \dots, y_{m-p})$, the set $\{(x, y) \in U_\alpha \mid y_1 = c_1, \dots, y_{m-p} = c_{m-p}\}$ gives a connected component of a leaf of $\mathcal{F}$ and identify U_α with an open subset of $\mathbf{R}^m$ via φ_α, and take relatively compact open subsets W_1, W_2 of U_α such that $\bar{W}_2 \subset W_1$. Then the metric on $\bar{W}_1$ may be considered as the Euclidean metric. Furthermore we take a Lipschitz function $\mu_\alpha : U_\alpha \to [0,1]$ such that $\mu_\alpha = 1$ on $\bar{W}_2$ and $\mu_\alpha = 0$ outside of $\bar{W}_1$. For any $f \in \mathcal{N}^{LIP}(1_M, (U_\alpha,\varphi_\alpha), (U_\alpha,\varphi_\alpha), \epsilon, \bar{W}_1) \cap \mathcal{H}_{LIP,L}(M,\mathcal{F})$, f has the form $f(x,y) = (f_1(x,y), y)$. Then we define a map $f_\alpha : M \to M$ by

$$f_\alpha(x,y) = \begin{cases} (x,y) + (\mu_\alpha(x,y)(f_1(x,y) - x), 0) & \text{for} \quad (x,y) \in U_\alpha \\ (x,y) & \text{for} \quad (x,y) \notin U_\alpha. \end{cases}$$

Then we have the following:

Proposition 3.1. *If $p(1 + lip(\mu_\alpha))\epsilon < 1$, then f_α is a leaf preserving Lipschitz homeomorphism which is isotopic to 1_M through leaf preserving Lipschitz homeomorphisms, that is, $f_\alpha \in \mathcal{H}_{LIP,L}(M,\mathcal{F})$.*

Proof. For distinct $(x,y), (x',y') \in \bar{W}_1$, we have

$$\begin{aligned} &||\mu_\alpha(x,y)(f_1(x,y) - x) - \mu_\alpha(x',y')(f_1(x',y') - x')|| \\ &\quad \le ||\mu_\alpha(x,y)(f_1(x,y) - x) - \mu_\alpha(x,y)(f_1(x',y') - x')|| \\ &\qquad + ||\mu_\alpha(x,y)(f_1(x',y') - x') - \mu_\alpha(x',y')(f_1(x',y') - x')|| \\ &\quad = |\mu_\alpha(x,y)| \cdot ||f_1(x,y) - x - (f_1(x',y') - x')|| \\ &\qquad + |\mu_\alpha(x,y) - \mu_\alpha(x',y')| \cdot ||f_1(x',y') - x'|| \\ &\quad < \epsilon \cdot ||(x,y) - (x',y')|| + lip(\mu_\alpha)\epsilon \cdot ||(x,y) - (x',y')||. \end{aligned}$$

Putting $\mu_\alpha(x,y)(f_1(x,y) - x) = (u_1(x,y), \dots, u_p(x,y))$, we have $lip(u_i) < 1/p$ for each i. We define maps $f_\alpha^i : U_\alpha \to U_\alpha$ $(i = 1, \cdots, p)$ by $f_\alpha^1(x_1, \dots, x_p, y_1, \dots, y_{m-p}) = (x_1 + u_1(x,y), \dots, x_p, y_1, \dots, y_{m-p})$ and $f_\alpha^i(x_1, \dots, x_p, y_1, \dots, y_{m-p}) = (x_1, \dots, x_{i-1}, x_i + u_i((f_\alpha^{i-1} \circ \cdots \circ f_\alpha^1)^{-1}(x,y)), x_{i+1}, \dots, x_p, y_1, \dots, y_{m-p})$ for $i = 2, \dots, p$. By an argument similar to that in the proof of Proposition 4.2 of [A-F], we can prove by induction that each f_α^i is a leaf preserving Lipschitz homeomorphism which is isotopic to 1_M through leaf preserving Lipschitz homeomorphisms. Since $f_\alpha = f_\alpha^p \circ f_\alpha^{p-1} \circ \cdots \circ f_\alpha^1$, we have $f_\alpha \in \mathcal{H}_{LIP,L}(M,\mathcal{F})$. This completes the proof. □

Corollary 3.2 (fragmentation lemma)**.** *For any $f \in \mathcal{H}_{LIP,L}(M,\mathcal{F})$, there are $f_i \in \mathcal{H}_{LIP,L}(M,\mathcal{F})$ $(i = 1, 2, \dots, k)$ such that $f = f_k \circ f_{k-1} \circ \cdots \circ f_1$ and the support of each f_i is contained in a small ball.*

Proof. This follows from Proposition 3.1 because of the compactness of M. □

Corollary 3.3. *$\mathcal{H}_{LIP,L}(M,\mathcal{F})$ is locally contractible.*

Proof. This is an immediate consequence of Proposition 3.1 and Corollary 3.2. □

Theorem 3.4. *$\mathcal{H}_{LIP,L}(M,\mathcal{F})$ is perfect.*

Proof. Let $f \in \mathcal{H}_{LIP,L}(M,\mathcal{F})$. We may assume that f is close to the identity. From Corollary 3.2, we have $f = f_k \circ f_{k-1} \circ \cdots \circ f_1$, where each f_i is a leaf preserving Lipschitz homeomorphism whose support is contained in a small ball. Hence we can assume that $f_i \in \mathcal{H}_{LIP,L}(\mathbf{R}^m,\mathcal{F}_0)$ for each i. From Corollary 2.2, we have that f_i is in the commutator subgroup of $\mathcal{H}_{LIP,L}(\mathbf{R}^m,\mathcal{F}_0)$ and hence f is in the commutator subgroup of $\mathcal{H}_{LIP,L}(M,\mathcal{F})$. Thus $\mathcal{H}_{LIP,L}(M,\mathcal{F})$ is perfect. $\square$

4. $H_1(\mathcal{H}_{LIP}(\mathcal{F}))$ for codimension one foliations

In this section, we consider the first homology of $\mathcal{H}_{LIP}(M,\mathcal{F})$ for a codimension one foliation $\mathcal{F}$. Let M be a compact C^2-manifold without boundary and $\mathcal{F}$ a codimension one C^2-foliation of M. Hereafter we simply write $\mathcal{H}_{LIP}(\mathcal{F})$, $\mathcal{H}_{LIP,L}(\mathcal{F})$ instead of $\mathcal{H}_{LIP}(M,\mathcal{F})$, $\mathcal{H}_{LIP,L}(M,\mathcal{F})$ respectively.

There exists a one dimensional C^2-foliation $\mathcal{T}$ of M transverse to $\mathcal{F}$. Then we have the following:

Lemma 4.1. *Let f be an element of $\mathcal{H}_{LIP}(\mathcal{F})$ sufficiently close to the identity. Then f is uniquely decomposed as $f = g \circ h$, where h (resp. g) is an element of $\mathcal{H}_{LIP}(\mathcal{F}) \cap \mathcal{H}_{LIP,L}(\mathcal{T})$ (resp. $\mathcal{H}_{LIP,L}(\mathcal{F})$) and h and g are also close to the identity.*

Proof. The existence of g and h follows from Lemma 4.1 of [F-I]. For coordinate $(x,y) = (x_1,\dots,x_{m-1},y)$ on a foliated chart U, we may assume that $\mathcal{F}$ is defined by $y = \mathit{constant}$ and $\mathcal{T}$ is defined by $x_1 = \mathit{constant},\dots,x_{m-1} = \mathit{constant}$. Thus f has the form $f(x,y) = (f_1(x,y), f_2(y))$ locally. Then h has the form $h(x,y) = (x, f_2(y))$. Hence h is Lipschitz and is close to the identity and g is also so. $\square$

Lemma 4.2 (Lemma 4.2 of [F-I]). *Let f be an element of $\mathcal{H}_{LIP}(\mathcal{F})$ and L a leaf of $\mathcal{F}$. If $f(L) \neq L$, then the holonomy group of L is trivial.*

We define the subset S_0 of M by

$$S_0 = \{x \in M \mid \textit{there exists an element } f \textit{ of } \mathcal{H}_{LIP}(\mathcal{F}) \textit{ such that } f(L_x) \neq L_x\}.$$

By definition, S_0 is an open $\mathcal{F}$-saturated set and by Lemma 4.2, all leaves in S_0 have trivial holonomy.

Theorem 4.3 (see Theorem 4.3 of [F-I]). *Let S be a connected component of S_0. Then clearly S is invariant under the action of $\mathcal{H}_{LIP}(\mathcal{F})$ and S is one of the following three types*:

Type P : *S is homeomorphic to $L \times (0,1)$ and the foliations $\mathcal{F}|_S$ and $\mathcal{T}|_S$ correspond to the product structure of $L \times (0,1)$.*

Type R : *There exists a closed transversal curve C in S such that C meets each leaf of $\mathcal{F}\mid_S$ at exactly one point and the natural map*

$$p : S \to C, \quad p(x) = L_x \cap C$$

is a fibration and $\mathcal{T}\mid_S$ is a connection of the fibration p.

Type D : *All leaves of $\mathcal{F}$ in S are dense in S and there exists a topological flow $\{\varphi_t\}$ on S which preserves $\mathcal{F}\mid_S$ and whose orbits are leaves of $\mathcal{T}\mid_S$.*

Then following Theorem 4.6 of [F-I], we have:

Theorem 4.4. *Let $\mathcal{F}$ be a codimension one C^2-foliation of a compact C^2-manifold M. Suppose that $\mathcal{F}$ has no components of Type* D *and has only a finite number of components of type* R. *Then $\mathcal{H}_{LIP}(\mathcal{F})$ is perfect.*

Proof. We can suppose that the transverse foliation $\mathcal{T}$ is defined by a C^2 vector field X and on each Type R component S, X has a closed orbit C of period 1 which satisfies the condition of Type R. We make the convention that t is the first component of any local coordinate compatible with the foliation satisfying $\partial/\partial t = fX$, where f is a function which is 1 on C, and differentiations with respect to t will be denoted by $'$ (prime).

For a Type R component S, we can define an flow $\{\varphi_s\}$ on S by the C^1 vector field $(1/p^*dt(X))X$, then $\{\varphi_s\}$ is $\mathcal{F}$-preserving and from the relation $p(\varphi_s(x)) = p(x) + s$, we have the formula ${\varphi_s}'(x) = p'(x)/p'(\varphi_s(x))$. $\square$

Assertion. ${\varphi_s}'$ are bounded on S uniformely for $|s| \leq 1$.

Proof. Since φ_1 (or possibly φ_k) generates the holonomy of a leaf in $\partial\bar{S}$, φ_1 is of class C^2 on $\bar{S}$. In particular $\log({\varphi_1}')$ has bounded variation. Let K be a compact subset of S such that $\{\varphi_s\}$-orbit of K is S, then ${\varphi_s}'$ are bounded on K and $|s| \leq 1$. For $x \notin K$ we have $\varphi_n(x) \in K$ (or possibly $\varphi_{-n}(x) \in K$) for a positive integer n. Since $p = p \circ \varphi_n = p \circ (\varphi_1)^n$, we have

$$|\log {\varphi_s}'(x)|$$
$$\leq |\log p'(\varphi_n(x)) - \log p'(\varphi_{n+s}(x))| + \sum_{k=0}^{n-1} |\log {\varphi_1}'(\varphi_k(x)) - \log {\varphi_1}'(\varphi_{k+s}(x))|.$$

Therefore the assertion holds. $\square$

Proof of Theorem 4.4 *continued.* We have a homomorphism $p_*: \mathcal{H}_{LIP}(\mathcal{F}\mid_S) \to \mathcal{H}_{LIP}(C)$ defined by $p_*(f) = f\mid_C \ (= \bar{f})$ for $f \in \mathcal{H}_{LIP}(\mathcal{F}\mid_S)$. We assert that p_* is surjective. Let $\bar{f}$ be an element of $\mathcal{H}_{LIP}(C)$, then we lift $\bar{f}$ to a foliation preserving homeomorphism f of S by $f(x) = \varphi_{s(x)}(x)$, where $s(x) = \bar{f}(p(x)) - p(x)$. Since $\bar{f}$ is Lipschitz, $\bar{f}$ is almost everywhere differentiable and $\bar{f}'$ is bounded. To prove that f is Lipschitz, it is sufficient to prove that f' exists a.e. and is bounded. From the relation $p(f(x)) = \bar{f}(p(x))$, it is easy to see that $f'(x) = (p'(x)/p'(\varphi_{s(x)}(x)))\bar{f}'(p(x)) = \varphi_{s(x)}{}'(x)\bar{f}'(p(x))$. Since $\varphi_s{}'(x)$ is bounded for $x \in S$ and $|s| \leq 1$, so $f'(x)$ exists a.e. and is bounded. This proves that p_* is surjective.

Since $\mathcal{H}_{LIP}(C)$ is perfect (Theorem 4.6 of [A-F]), we have that for any $f \in \mathcal{H}_{LIP}(\mathcal{F}\mid_{\bar{S}})$, $p_*(f)$ is expressed as the product of commutators $\prod_{i=1}^{v}[\bar{f}_{2i-1}, \bar{f}_{2i}]$, where $\bar{f}_i \in \mathcal{H}_{LIP}(C)$, and we can lift $\bar{f}_i$ to $f_i \in \mathcal{H}_{LIP}(\mathcal{F}\mid_{\bar{S}})$ $(i = 1, \ldots, 2v)$ with $f_i \mid_{\partial\bar{S}} = \mathit{identity}$. Then $f \circ (\prod_{i=1}^{v}[f_{2i-1}, f_{2i}])^{-1}$ is in the kernel of p_* which is $\mathcal{H}_{LIP,L}(\mathcal{F}\mid_{\bar{S}})$.

From Theorem 4.3, for a Type P component S, we have $S \cong L \times (0,1)$, hence $\bar{S} \cong L \times [0,1]$. Then we have the surjective homomorphism $\pi\colon \mathcal{H}_{LIP}(\mathcal{F}\mid_{\bar{S}}) \to \mathcal{H}_{LIP}([0,1])$. T. Tuboi [T] showed that $\mathcal{H}_{LIP}([0,1])$ is uniformly perfect. Thus for any $f \in \mathcal{H}_{LIP}(\mathcal{F}\mid_{\bar{S}})$, $\pi(f)$ is expressed as the product of commutators $\prod_{i=1}^{u}[\bar{f}_{2i-1}, \bar{f}_{2i}]$, where $\bar{f}_i \in \mathcal{H}_{LIP}([0,1])$ and u is the positive integer which does not depend on Type P components. Since π is surjective, we can show by an argument similar to that in the proof of a Type R component that for any $f \in \mathcal{H}_{LIP}(\mathcal{F}\mid_{\bar{S}})$, $f \circ (\prod_{i=1}^{u}[f_{2i-1}, f_{2i}])^{-1}$ is in $\mathcal{H}_{LIP,L}(\mathcal{F}\mid_{\bar{S}})$.

Hence by Theorem 3.4, $\mathcal{H}_{LIP}(\mathcal{F})$ is perfect. This completes the proof. □

The following example shows that the homomorphism p_* is not necessarily surjective for a Type R component of class C^1.

Example 4.5. Let h be a diffeomorphism of $[0,1]$ which is tangent to the identity at $0, 1$ and satisfies the condition $h(t) > t$ for $t \in (0,1)$. Let I be the interval $(1/2, h(1/2))$, Φ a diffeomorphism of I onto $\mathbf{R}$ and let X and Y be vector fields on I defined by $\Phi_* X = x(\partial/\partial x)$ and $\Phi_* Y = \partial/\partial x$. Then the flows $\{f_s\}$ and $\{g_t\}$ on I defined by X and Y respectively satisfies the relation $f_s \circ g_t \circ f_{-s} = g_{te^s}$. We define $f_s(x) = g_t(x) = x$ for $x \in [0,1]\backslash I$, then, by a suitable choice of Φ, we can suppose that f_s and g_t are diffeomorphisms of $[0,1]$. For any sequences s_n and t_n, we define homeomorphisms F and G of $[0,1]$ by

$$F = h \prod_{n=-\infty}^{\infty} h^n \circ f_{s_n} \circ h^{-n}, \quad G = \prod_{n=-\infty}^{\infty} h^n \circ g_{t_n} \circ h^{-n}.$$

We choose the sequence $\{s_n\}$ so that we have $\lim_{n\to\pm\infty} s_n = 0$, $\sum_{n=0}^{\infty} s_n = \infty$, and $\sum_{n=0}^{\infty} s_{-n} = -\infty$, then F is of class C^1. We define t_n by $e^{s_n} t_n = t_{n+1}$ $(t_1 = 1)$, then we have $F \circ G = G \circ F$ and G is not Lipschitz since $\lim_{n\to\pm\infty} t_n = \infty$. We consider $\mathcal{F}$ the foliated $[0,1]$-bundle over S^1 with the holonomy F, then $\mathcal{F}$ is of Type R and of class C^1. G defines an $\mathcal{F}$-preserving homeomorphism g on $[0,1] \times S^1$ and $p_*(g) = \bar{g}$ is smooth but $\bar{g}$ does not lift to an $\mathcal{F}$-preserving Lipschitz homeomorphism of $[0,1] \times S^1$.

Remark 4.6. From Theorem 4.4, we see that $\mathcal{H}_{LIP}(S^3, \mathcal{F}_R)$ is perfect for the Reeb foliation $\mathcal{F}_R$ of S^3. In contrast with Lipschitz case, differentiable case has a different phenomenon as follows. Let $F^r(S^3, \mathcal{F}_R)$ be the group of foliation preserving C^r-diffeomorphisms of $(S^3, \mathcal{F}_R)$ isotopic to the identity through foliation preserving diffeomorphisms. Then Lemma 1 of [F-U] implies that $F^r(S^3, \mathcal{F}_R)$ is not perfect for $r \geq 1$.

For a Type D component S, the flow $\{\varphi_t\}$ is defined as follows (see [I]). Let C be a closed transversal curve of $\mathcal{F}\mid_S$ and we suppose that C is a $\mathcal{T}$-orbit. Then, for a leaf L of $\mathcal{F}\mid_S$, $G = C \cap L$ has a structure of abelian group and G acts on C as the holonomy transformation group. Since all G-orbits are dense, there exists a homeomorphism h of C such that G is included in $h^{-1} \circ SO(2) \circ h$ and we call h, which is unique up to rotations of C, *the linearization map* of the holonomy transformations. We define a flow $\{\varphi_t\}$ on C by $\{h^{-1} \circ R_t \circ h\}$, where R_t is the rotation of C of angle $2\pi t$, and we extend $\{\varphi_t\}$ to a flow on S by using holonomy maps.

We define a submodule $Per(S)$ of $\mathbf{R}$ by

$$Per(S) = \{t \in \mathbf{R} \mid \varphi_t(L) = L \text{ for one and all leaves } L \text{ in } S\}.$$

Then we have the following:

Theorem 4.7. *Let S be a Type* D *component and suppose the linearization map h is not absolutely continuous, then $\mathcal{H}_{LIP}(\mathcal{F}\mid_S)$ coincides with $\mathcal{H}_{LIP,L}(\mathcal{F}\mid_S)$.*

Proof. Suppose that there exists an element f of $\mathcal{H}_{LIP}(\mathcal{F}\mid_S) - \mathcal{H}_{LIP,L}(\mathcal{F}\mid_S)$ and let $\{f_t\}$ be an isotopy in $\mathcal{H}_{LIP}(\mathcal{F}\mid_S)$ and, by restriction, we consider that $\{f_t\}$ is in $\mathcal{H}_{LIP}(C)$. Then we can write $f_t = h^{-1} \circ R_{\alpha(t)} \circ h$, where α is a continuous function of $[0,1]$ onto $[0,\alpha]$. Since f_1 is Lipschitz and h is not absolutely continuous, by Proposition (1.2) of [H](CHAP. XII), we see that h and h^{-1} are almost everywhere differentiable and we have $h'(x) = 0$ (a.e.), $(h^{-1})'(x) = 0$ (a.e.). We choose points x_1, x_2 of C and an angle β so that $h'(x_1) = 0, (h^{-1})'(x_2) = 0$ and $R_\beta(h(x_1)) = x_2$. Moreover we can choose x_2 near to $h(x_1)$ and β sufficiently small. Then for some t_1, we have $\alpha(t_1) = \beta$, $f_{t_1} = h^{-1} \circ R_\beta \circ h$ and $f'_{t_1}(x_1) = 0$. Since $f_{t_1}^{-1}$ is Lipschitz, this is a contradiction. □

Theorem 4.8. *Let S be a Type* D *component and suppose the linearization map h is a C^1-diffeomorphism, then there exists a surjection π of $H_1(\mathcal{H}_{LIP}(\mathcal{F}))$ to $\mathbf{R}/Per(S)$.*

Proof. In this case the flow $\{\varphi_t\}$ is a one parameter subgroup of $\mathcal{H}_{LIP}(\mathcal{F}\mid_S)$ and the proof is the same as that of Theorem 4.3 of [F-I]. □

Theorem 4.9. *Let $\mathcal{F}$ be a foliation of a torus T^m defined by a* 1*-form $\omega = \sum a_i dx_i$. If one of a_i/a_j is irrational, then $H_1(\mathcal{H}_{LIP}(\mathcal{F}))$ is isomorphic to $\mathbf{R}/a_1\mathbf{Z} + \cdots + a_m\mathbf{Z}$.*

Proof. The proof is same as in the proof of Theorem 4.10 of [F-I]. □

Remark 4.10. By Theorem (3.6) of [H](CHAP. XII), there exists a C^∞-foliation $\mathcal{F}'$ which is topologically equivalent to $\mathcal{F}$ of Theorem 4.9 for a suitable $\{a_i\}$ with non absolutely continuous linearization map, therefore by Theorem 3.4 and Theorem 4.7, we have $H_1(\mathcal{H}_{LIP}(\mathcal{F}')) = 0$. We also remark that if $\mathcal{F}$ is $C^r(r \geq 3)$ and $Per(S)$ contains an irrational number which satisfies a Diophantine condition, then the linearization map h is differentiable (see [Y]).

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