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Abhyankar-Sathaye embedding problem in dimension three

By

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Abstract

Abhyankar-Moh and Suzuki proved that if an irreducible polynomial $f \in \mathbf{C}[x_1, x_2]$ in two complex variables x_1 and x_2 defines the affine plane curve $C = (f = 0) \subset \mathbf{A}^2$, which is isomorphic to the affine line: $C \cong \mathbf{A}^1$, then f is a variable of $\mathbf{C}[x_1, x_2]$, i.e., there exists a polynomial $g \in \mathbf{C}[x_1, x_2]$ such that $\mathbf{C}[f, g] = \mathbf{C}[x_1, x_2]$ (cf. [A-M75], [Su74]). In this article, we prove under some additional assumptions that the similar result holds in the three-dimensional case, namely, if an irreducible polynomial $f \in \mathbf{C}[x_1, x_2, x_3]$ in three complex variables x_1, x_2 and x_3 defines the hypersurface $S = (f = 0) \subset \mathbf{A}^3$, which is isomorphic to the affine plane: $S \cong \mathbf{A}^2$, then f is a variable of $\mathbf{C}[x_1, x_2, x_3]$, i.e., there are polynomials $g, h \in \mathbf{C}[x_1, x_2, x_3]$ such that $\mathbf{C}[f, g, h] = \mathbf{C}[x_1, x_2, x_3]$. Moreover, we shall determine the detailed form of such a polynomial $f \in \mathbf{C}[x_1, x_2, x_3]$ for the special case.

1. Introduction

Throughout the present article we work over the field of complex numbers $\mathbf{C}$. Let $f \in \mathbf{C}[x_1, \dots, x_n]$ be an irreducible polynomial in n complex variables $x_1, \dots, x_n$. Suppose that the hypersurface $S := (f = 0) \subset \mathbf{A}^n := \mathrm{Spec}(\mathbf{C}[x_1, \dots, x_n])$ defined by f is isomorphic to the affine $(n-1)$-space: $S \cong \mathbf{A}^{n-1}$. Then *Abhyankar-Sathaye Embedding Problem in dimension n* (abbreviated (A-S;n)) asks whether or not the polynomial f is a *variable* of $\mathbf{C}[x_1, \dots, x_n]$, i.e., there exist the polynomials $f_1 := f, f_2, \dots, f_n \in \mathbf{C}[x_1, \dots, x_n]$ such that $\mathbf{C}[f_1, \dots, f_n] = \mathbf{C}[x_1, \dots, x_n]$. Another relevant important problems are to determine the form of such a polynomial $f \in \mathbf{C}[x_1, \dots, x_n]$ defining a hypersurface which is isomorphic to $\mathbf{A}^{n-1}$ (we call this problem (Poly;n)) and to settle the structure of the group $\mathrm{Aut}_{\mathbf{C}}(\mathbf{C}[x_1, \dots, x_n])$ of polynomial automorphisms of the polynomial ring $\mathbf{C}[x_1, \dots, x_n]$ (we call this problem (Aut;n)).

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Note that if (A-S;n) is true, then (Poly;n) and (Aut;n) are essentially the same problem.

In the case $n = 1$ the problems (A-S;1), (Aut;1) and (Poly;1) are easy. In the case $n = 2$, Abhyankar-Moh and Suzuki showed independently that (A-S;2) holds true (cf. [A-M75], [Su74]). Furthermore, Jung and Van der Kulk settled the problem (Aut;2) (cf. [Ju42], [Ku53]). Hence we can solved (Poly;2) by combining (A-S;2) with (Aut;2). Meanwhile, in the case where $n \geq 3$ the problems (A-S;n), (Poly;n) and (Aut;n) are still open. We consider the problems (A-S;3) and (Poly;3) mainly in this article.

Suppose that an irreducible polynomial $f \in \mathbf{C}[x_1, x_2, x_3]$ in the polynomial ring in three complex variables x_1, x_2 and x_3 defines the hypersurface $S := (f = 0) \subset \mathbf{A}^3 := \mathrm{Spec}(\mathbf{C}[x_1, x_2, x_3])$ which is isomorphic to the affine plane: $S \cong \mathbf{A}^2$. In these notations and the setting, we shall summarize the partial affirmative results concerning (A-S;3):

When the above polynomial f is of the form $f = gx_3 - h$, where $g, h \in \mathbf{C}[x_1, x_2]$, then (A-S;3) holds true (cf. Sathaye [Sa76], Russell [Ru76], Miyanishi [Miy78a]). More generally, Wright proved that (A-S;3) holds true for the case where f is of the form $f = gx_3^m - h$, where $g, h \in \mathbf{C}[x_1, x_2]$ and m is a positive integer (cf. [Wr78]). In the case where the polynomial $f \in \mathbf{C}[x_1, x_2, x_3]$ is of degree $\deg(f) = 3$, Ohta (cf. [Oh99]) determined all the standard forms of such a polynomial f (which consists of nine different types). On the other hand, in the case where $\deg(f) = 4$ and the closure in $\mathbf{P}^3$ of $S \cong \mathbf{A}^2$ is normal and has a triple point, Ohta (cf. [Oh01]) also determined all the standard forms of such a polynomial f (which consists of fourteen types). In both cases, Ohta constructed explicit automorphisms of $\mathbf{C}[x_1, x_2, x_3]$ sending f to the standard variable x_1. Consequently he showed that (A-S;3) holds true for such cases.

With the notations $f \in \mathbf{C}[x_1, x_2, x_3]$ and $\mathbf{A}^2 \cong S = (f = 0) \subset \mathbf{A}^3 = \mathrm{Spec}(\mathbf{C}[x_1, x_2, x_3])$ as above, since the problem (A-S;3) is obvious in the case where the degree $\deg(f)$ of f is one, we may and shall assume that $d := \deg(f) \geq 2$ in the subsequent argument. We embed the affine three space $\mathbf{A}^3$ into the projective three space $\mathbf{P}^3$ canonically as the complement of the hyperplane $H_0 := (x_0 = 0)$, where $(x_0 : x_1 : x_2 : x_3)$ is the homogeneous coordinate of $\mathbf{P}^3$. We denote by X the closure in $\mathbf{P}^3$ of the hypersurface $S \subset \mathbf{A}^3$, and denote by $L := X \cap H_0$ the (set-theoretic) intersection of X and H_0. In this article we consider the hypersurface X satisfying the following condition (†):

(†) L is a line in $\mathbf{P}^3$ and the hypersurface $X \subset \mathbf{P}^3$ has the multiplicity $d - 1$ along the line L; $\mathrm{mult}\,_L X = d - 1$, where $d \geq 2$ is the degree of X.

For the simplicity we say that a polynomial $f \in \mathbf{C}[x_1, x_2, x_3]$ satisfies (†) if the closure X in $\mathbf{P}^3$ of the hypersurface $(f = 0) \subset \mathbf{A}^3$ satisfies the above condition (†). Then our main result in the present article is the following:

Theorem 1.1. *Let $f \in \mathbf{C}[x_1, x_2, x_3]$ be an irreducible polynomial of degree $d := \deg(f) \geq 2$ in three complex variables x_1, x_2 and x_3. Suppose that* *the hypersurface* $S := (f = 0) \subset \mathbf{A}^3 := \mathrm{Spec}(\mathbf{C}[x_1, x_2, x_3])$ *defined by* f *is isomorphic to the affine plane*: $S \cong \mathbf{A}^2$, *and that* f *satisfies* (†). *Then* f *is a variable of* $\mathbf{C}[x_1, x_2, x_3]$.

Remark 1. Three dimensional Abhyankar-Sathaye Embedding Problem (A-S;3) may not be solved in the full generality. It is, however, important to consider the criteria for the polynomial $f \in \mathbf{C}[x_1, x_2, x_3]$ to be a variable. In this point of view, Theorem 1.1 is significant.

Our argument for the proof of Theorem 1.1 is, roughly speaking, stated as follows: With the notations and the assumptions as in Theorem 1.1, we consider the irreducible linear pencil $\Lambda(f) := \{X_\alpha | \alpha \in \mathbf{C}\} \cup \{X_\infty := dH_0\}$ on $\mathbf{P}^3$ spanned by $X_0 = X$ and X_∞, where X_α is the closure in $\mathbf{P}^3$ of the hypersurface $S_\alpha := (f = \alpha) \subset \mathbf{A}^3$ defined by $f - \alpha$ for $\alpha \in \mathbf{C}$ and $d = \deg(f) \geq 2$. In order to prove Theorem 1.1, we are devoted to showing that the affine surfaces S_α are isomorphic to the affine plane $\mathbf{A}^2$ for all $\alpha \in \mathbf{C}$. To see this, we blow up $\mathbf{P}^3$ along the line $L = X \cap H_0$, say $\sigma : V \to \mathbf{P}^3$, and denote by $\overline{E} := \sigma^{-1}(L) \cong \mathbf{P}^1 \times \mathbf{P}^1$ the exceptional divisor. For any irreducible Cartier divisor $Z \subset \mathbf{P}^3$, we denote by $\overline{Z} \subset V$ the proper transform of Z on V. Then the proper transform $\overline{\Lambda(f)}$ by σ of $\Lambda(f)$ is the linear pencil on V spanned by $\overline{X}$ and $d\overline{H_0} + \overline{E}$. It is then an important step to investigate the scheme-theoretic intersection $\overline{X} \cdot \overline{E}$ of $\overline{X}$ and the exceptional divisor $\overline{E}$. As seen in Section 3, there are $(d-1)$ different types, say $\mathbf{TYPE}(d, \lambda)$ $(0 \leq \lambda \leq d-2)$, concerning the configuration of the intersection $\overline{X} \cdot \overline{E}$ (cf. Lemma 3.3). Nevertheless $\overline{\Lambda(f)}$ still has the base locus, we can see that $\overline{X}$ is smooth and, moreover, that the members $\overline{X_\alpha}$ of $\overline{\Lambda(f)}$ are normal and smooth along $\overline{X_\alpha} - S_\alpha$ for all $\alpha \in \mathbf{C}$. For any $\alpha \in \mathbf{C}$, the normal affine surface S_α has a compactification $S_\alpha \hookrightarrow \overline{X_\alpha}$ such that the boundary $\overline{X_\alpha} - S_\alpha$ has the same weighted dual graph as that of the boundary $\overline{X} - S$ (cf. Lemma 3.5). Consequently, we can see that S_α are isomorphic to the affine plane for all $\alpha \in \mathbf{C}$; $S_\alpha \cong \mathbf{A}^2$ (cf. Lemma 3.7). Thus all the closed fibers of the polynomial map $f : \mathbf{A}^3 = \mathrm{Spec}(\mathbf{C}[x_1, x_2, x_3]) \to \mathbf{A}^1 := \mathrm{Spec}(\mathbf{C}[f])$ associated to the canonical inclusion $\mathbf{C}[f] \hookrightarrow \mathbf{C}[x_1, x_2, x_3]$ are isomorphic to the affine plane and, furthermore, the generic fiber of it is isomorphic to the affine plane defined over the function field $\mathbf{C}(f)$ of the base curve $\mathbf{A}^1$, i.e., $\mathbf{C}[x_1, x_2, x_3] \otimes_{\mathbf{C}[f]} \mathbf{C}(f) \cong \mathbf{C}(f)^{[2]}$. As a consequence, it follows that this polynomial map is a trivial $\mathbf{A}^2$-bundle structure over the base curve $\mathbf{A}^1 = \mathrm{Spec}(\mathbf{C}[f])$, hence there are two indeterminates u and v over $\mathbf{C}[f]$ such that $\mathbf{C}[f] \otimes \mathbf{C}[u, v] = \mathbf{C}[f, u, v] \cong \mathbf{C}[x_1, x_2, x_3]$. Thus f is a variable of $\mathbf{C}[x_1, x_2, x_3]$.

As remarked above there are $(d-1)$ cases $\mathbf{TYPE}(d, \lambda)$ $(0 \leq \lambda \leq d-2)$ with respect to the scheme-theoretic intersection $\overline{X} \cdot \overline{E}$. Applying the arguments in Section 3 to prove Theorem 1.1 and the general theory of the generically rational polynomials (cf. Miyanishi-Sugie [Mi-Su80]), we can determine the concrete form of such a polynomial $f \in \mathbf{C}[x_1, x_2, x_3]$ of $\mathbf{TYPE}(d, \lambda)$ with $d - 2\lambda \geq 0$ (cf. Theorem 4.1).

This article is organized as follows: In Section 2, we have some preliminary results about the (minimal) normal compactifications of the affine plane $\mathbf{A}^2$.

We shall give the characterization of the affine plane $\mathbf{A}^2$, which asserts that the affine plane can be determined by the weighted dual graph of the boundary divisor. This characterization is needed in Section 3 in order to consider the intersection $\overline{X} \cdot \overline{E}$ and deduce that the affine surfaces S_α are isomorphic to the affine plane; $S_\alpha \cong \mathbf{A}^2$ for all $\alpha \in \mathbf{C}$. Furthermore, we recall the recent result of Kaliman and Zaidenberg (cf. [Ka-Za01]) concerning the generic fiber of a given $\mathbf{A}^2$-fibration. In Section 3, we shall prove Theorem 1.1. In Section 4, we consider the problem (Poly;3) and solved it for $\mathbf{TYPE}(d, \lambda)$ with $d - 2\lambda \geq 0$ (see Theorem 4.1).

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2. Preliminaries

In this section we shall study the compactifications and the characterization of the affine plane $\mathbf{A}^2$. Furthermore, we recall the result of Kaliman and Zaidenberg concerning the generic fiber of an $\mathbf{A}^2$-fibration. These results are used in Section 3 to prove Theorem 1.1.

Definition 2.1. A pair (Y, Δ) of a smooth projective surface Y and a (reduced) divisor $\Delta := \sum_i \Delta_i$ on Y is called a *compactification of* $\mathbf{A}^2$ if the complement $Y - \Delta$ is isomorphic to the affine plane $\mathbf{A}^2$. We say a pair (Y, Δ) to be a *normal compactification of* $\mathbf{A}^2$ if it is a compactification of $\mathbf{A}^2$ and Δ is a simple normal crossing divisor on Y. If, furthermore, the divisor $\Delta = \sum_i \Delta_i$ satisfies the following condition $(*)$, then a pair (Y, Δ) is called a *minimal normal compactification of* $\mathbf{A}^2$:

$(*)$ if a component Δ_i of Δ is a (-1)-curve on Y, then $\Delta_i \cap \overline{\Delta - \Delta_i}$ consists of at least three points.

The classification of the weighted dual graphs of the boundaries of the minimal normal compactifications of the affine planes $\mathbf{A}^2$ is summarized as follows:

Lemma 2.1 ([Mo73], [Ki02]). *Let (Y, Δ) be a minimal normal compactification of the affine plane $\mathbf{A}^2$. Then the weighted dual graph of Δ is a linear tree of smooth rational curves which is one of the following types* (i)–(iii):

(i) $\overset{1}{\circ}$

(ii) $\overset{0}{\circ}$——$\overset{m}{\circ}$ $(m \neq -1)$

(iii) $\Delta^{(1)}$——$\overset{-(m+1)}{\circ}$——$\overset{0}{\circ}$——$\overset{m}{\circ}$——$\Delta^{(2)}$ $(m>0)$, *where $\Delta^{(i)}$ $(i = 1, 2)$ is either an empty or a linear tree of smooth rational curves each of which has self-intersection number less than or equal to -2.*

Remark 2. In fact, Morrow [Mo73] (see also [Ki02]), furthermore, gives the complete list of the minimal normal compactifications of the affine plane $\mathbf{A}^2$. But, for our purpose to prove Theorem 1.1, we have only to obtain Lemma 2.1.

The following lemma means that the affine plane $\mathbf{A}^2$ can be characterized by the weighted dual graph of the boundary divisor.

Lemma 2.2. *Let T be a normal affine surface. Suppose that T is embedded into a normal projective surface Y in such a way that Y is smooth along the boundary $\Delta := Y - T$, each irreducible component of Δ is isomorphic to the smooth rational curve and the weighted dual graph of Δ coincides with that of a boundary divisor with respect to a suitable normal compactification of the affine plane $\mathbf{A}^2$. Then T is isomorphic to $\mathbf{A}^2$.*

Proof. Our proof consists of five steps:

Step 1. By our assumption we can construct a birational map $\phi : Y \cdots \to Y'$ which is composed of the blowing-ups of the points on the boundary Δ (including infinitely near points) and the contractions of the (-1)-curves each of which is either a proper transform of a component in Δ or a proper transform of a component produced in the previous blowing-up process in such a way that the new boundary divisor $\Delta' := Y' - T$ is a smooth rational curve with self-intersection number 1. (Since ϕ has no affection on T, we may and shall assume that Y' contains T as a Zariski open subset.) Let $\mathrm{Sing}(Y') = \{p_1, \ldots, p_s\}$ be the singular points on Y' (if there exist at all). Note that these points are located on T by the assumption. We denote by $\mu : \overline{Y} \to Y'$ the minimal resolution of Y' and denote by $\overline{E_j} := \mu^{-1}(p_j)$ the exceptional set over the point p_j for $1 \leq j \leq s$.

Step 2. Let $\overline{\Delta}$ denote the proper transform of the curve Δ' on $\overline{Y}$. Since the minimal resolution μ has no affection on Δ', we have $(\overline{\Delta}^2) = (\Delta'^2) = 1$. Then we claim the following:

Claim. $\overline{Y}$ is a rational surface.

Indeed, since $(\overline{\Delta}^2) > 0$ and $(K_{\overline{Y}} \cdot \overline{\Delta}) < 0$, $\overline{Y}$ is either a rational surface or an irrational ruled surface. Suppose that $\overline{Y}$ is an irrational ruled surface. Then there exists a $\mathbf{P}^1$-fibration $g : \overline{Y} \to B$, where B is a smooth projective curve of genus > 0. Since $(\overline{\Delta}^2) > 0$, $\overline{\Delta}$ is not contained in any fiber of g, hence it is a quasi-section of g. Since $\overline{\Delta} \cong \mathbf{P}^1$ and $\overline{\Delta}$ is a quasi-section of g, it follows that $B \cong \mathbf{P}^1$ by Lüroth theorem, which is a contradiction. Hence $\overline{Y}$ is a rational surface as desired.

We take an arbitrary point on $\overline{\Delta}$, say p, and denote by $\pi : \widehat{Y} \to \overline{Y}$ the blowing-up at p, and denote by $\widehat{\Delta}$ (resp. $\widehat{E} := \pi^{-1}(p)$) the proper transform of $\overline{\Delta}$ on $\widehat{Y}$ (resp. the exceptional curve of π). Since $\widehat{\Delta} \cong \mathbf{P}^1$ and $(\widehat{\Delta}^2) = 0$, we have the following exact sequence:

$$0 \to \mathcal{O}_{\widehat{Y}} \to \mathcal{O}_{\widehat{Y}}(\widehat{\Delta}) \to \mathcal{O}_{\mathbf{P}^1} \to 0.$$

From the induced cohomology exact sequence and the fact that $H^1(\widehat{Y}, \mathcal{O}_{\widehat{Y}}) = 0$ (see Claim), we have $h^0(\widehat{\Delta}, \mathcal{O}_{\widehat{Y}}(\widehat{\Delta})) = 2$. The linear pencil $|\widehat{\Delta}|$ on $\widehat{Y}$ is base point free and defines the $\mathbf{P}^1$-fibration $\widehat{h} := \Phi_{|\widehat{\Delta}|} : \widehat{Y} \to \mathbf{P}^1$.

Step 3. Let $\rho : \widetilde{Y} \to Y'$ denote the blowing-up at the point $\mu(p) \in \Delta'$ with the exceptional curve $\widetilde{E}$ and let $\widetilde{\Delta}$ denote the proper transform of Δ' on $\widetilde{Y}$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \widehat{Y} & \xrightarrow{\nu} & \widetilde{Y} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\rho} \\ \overline{Y} & \xrightarrow{\mu} & Y' \end{array}$$

where ν is the contraction of $\widehat{E_j} := \pi^{-1}(\overline{E_j})$ to the point p_j for $1 \le j \le s$. (Note that since ρ induces an isomorphism between $\rho^{-1}(T)$ and T, we may and shall identify the point $\rho^{-1}(p_j)$ with p_j for $1 \le j \le s$.) Since $\widehat{E_j} \cap \widehat{\Delta} = \emptyset$, $\widehat{E_j}$ is contained in a fiber of $\widehat{h}$ for $1 \le j \le s$. Hence $\widehat{h}$ induces the $\mathbf{P}^1$-fibration $\widetilde{h} : \widetilde{Y} \to \mathbf{P}^1$ satisfying $\widehat{h} = \widetilde{h} \circ \nu$. The restriction of $\widetilde{h}$ to the complement $\widetilde{Y} - (\widetilde{\Delta} \cup \widetilde{E}) \cong T$ gives rise to a fibration $h : T \to B$, where $B := \mathbf{P}^1 - \widetilde{h}(\widetilde{\Delta}) \cong \mathbf{A}^1$. The general fibers of h are isomorphic to the affine line $\mathbf{A}^1$ because, for any fiber $l \in |\widetilde{\Delta}|$ of $\widetilde{h}$, we have $(l \cdot \widetilde{E}) = (\widetilde{\Delta} \cdot \widetilde{E}) = (\widehat{\Delta} \cdot \widehat{E}) = 1$, i.e., $\widetilde{E}$ is a cross-section of $\widetilde{h}$. Note that since $T \cong \widetilde{Y} - (\widetilde{\Delta} \cup \widetilde{E})$ is affine and $\widetilde{E}$ is a cross-section of $\widetilde{h}$, all the fibers of $\widetilde{h}$ are irreducible and reduced.

Step 4. We shall show that T is, in fact, smooth. Assume to the contrary that T is not smooth, i.e., $\mathrm{Sing}(T) = \{p_1, \ldots, p_s\} \neq \emptyset$. Let $\widetilde{l_1}$ denote the fiber of $\widetilde{h}$ passing through the sigular point p_1. As remarked above, $\widetilde{l_1}$ is irreducible and reduced. Let $p_1, \ldots, p_t$ exhaust all the singular points of T contained in $\widetilde{l_1}$. Then the fiber $\nu^*(\widetilde{l_1})$ of $\widehat{h}$ consists of the proper transform $\widehat{l_1}$ of $\widetilde{l_1}$ on $\widehat{Y}$ and the components of the exceptional sets $\widehat{E_1}, \ldots, \widehat{E_t}$. Note that each component of $\widehat{E_j}$ is not a (-1)-curve because ν is the minimal resolution of $\widetilde{Y}$. Hence $\widehat{l_1}$ is the unique (-1)-curve contained in $\nu^*(\widetilde{l_1})$. Note that the multiplicity of $\widehat{l_1}$ in the fiber $\nu^*(\widetilde{l_1})$ is one, so there exists a (-1)-curve in $\nu^*(\widetilde{l_1})$ other than $\widehat{l_1}$ (cf. [Miy01, Chapter 3, Lemma 1.4.1]). This is a contradiction. Thus T is smooth.

Step 5. We consider the $\mathbf{A}^1$-fibration $h : T \to B$ on the smooth affine surface T, all the fibers of which are isomorphic to the affine line $\mathbf{A}^1$. Thus h is an $\mathbf{A}^1$-bundle structure over the base curve $B \cong \mathbf{A}^1$ (cf. [Kam-Miy78]). Since $\mathrm{Pic}(B) = 0$, h is, in fact, a trivial $\mathbf{A}^1$-bundle structure over $B \cong \mathbf{A}^1$, i.e., $T \cong B \times \mathbf{A}^1 \cong \mathbf{A}^1 \times \mathbf{A}^1 \cong \mathbf{A}^2$. This completes the proof of Lemma 2.2. □

It is well-known that the generic fiber of a given $\mathbf{A}^1$-fibration between the smooth quasi-projective varieties is isomorphic to the affine line defined over the function field of the base variety (cf. [Kam-Miy78]). The following result due to Kaliman and Zaidenberg is the two-dimensional analogy, and we shall use it to prove Theorem 1.1 at the final step.

Theorem 2.1 ([Ka-Za01, Theorem 0.1]). *Let $f : Z \to B$ be the morphism between smooth, quasi-projective varieties. Suppose that the general fibers of f are isomorphic to the complex affine plane $\mathbf{A}^2$. Then the generic fiber of f is isomorphic to the affine plane $\mathbf{A}^2_{\mathbf{C}(B)}$ defined over the function field $\mathbf{C}(B)$ of the base variety B. Equivalently, there exists a nonempty Zariski open subset $U \subseteq B$ such that the restriction $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a trivial $\mathbf{A}^2$-bundle structure.*

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Let $f \in \mathbf{C}[x_1, x_2, x_3]$ be an irreducible polynomial of degree $d := \deg(f) \geq 2$ in three complex variables x_1, x_2 and x_3 such that the hypersurface $S := (f = 0) \subset \mathbf{A}^3 := \mathrm{Spec}(\mathbf{C}[x_1, x_2, x_3])$ defined by f is isomorphic to the affine plane $S \cong \mathbf{A}^2$. We embed the affine three space $\mathbf{A}^3$ into the projective three space $\mathbf{P}^3$ as the complement of the hyperplane $H_0 := (x_0 = 0) \subset \mathbf{P}^3 : (x_0 : x_1 : x_2 : x_3)$. We denote by $X \subset \mathbf{P}^3$ the closure of S in $\mathbf{P}^3$ and denote by $L := X \cap H_0$ the set-theoretic intersection of X and H_0. Suppose that X satisfies (†) (see Section 1), namely, L is a line in $\mathbf{P}^3$ and X has the multiplicity $d-1$ along L; $\mathrm{mult}_L X = d - 1$. In order to show that f is a variable of $\mathbf{C}[x_1, x_2, x_3]$, we have to see that the affine surfaces $S_\alpha := (f - \alpha = 0) \subset \mathbf{A}^3$ defined by $f - \alpha$ are isomorphic to the affine plane; $S_\alpha \cong \mathbf{A}^2$ for all $\alpha \in \mathbf{C}$. The almost all parts of this section are devoted to proving $S_\alpha \cong \mathbf{A}^2$ for all $\alpha \in \mathbf{C}$.

Let $\Lambda(f)$ be the irreducible linear pencil on $\mathbf{P}^3$ spanned by X and dH_0. It is clear that $\Lambda(f)$ consists of X_α ($\alpha \in \mathbf{C}$) and the non-reduced member dH_0, and it has the base locus along the line L, i.e., $\mathrm{Bs}\,\Lambda(f) = L$, where X_α is the closure in $\mathbf{P}^3$ of the affine surface S_α ($\alpha \in \mathbf{C}$). We denote by $\sigma : V \to \mathbf{P}^3$ the blowing-up of $\mathbf{P}^3$ along the line L and denote by $\overline{E} := \sigma^{-1}(L) \cong \mathbf{P}^1 \times \mathbf{P}^1$ the exceptional divisor of σ. Then we have:

Lemma 3.1. *With the notations as above, we have $\overline{E}|_{\overline{E}} \sim -M_0 + f_0$, where M_0 (resp. f_0) is the class of the fiber on $\overline{E} \cong \mathbf{P}^1 \times \mathbf{P}^1$ such that $\sigma(M_0) = L$ (resp. $\sigma(f_0)$ is a point).*

Proof. We can write $K_V = \sigma^*(K_{\mathbf{P}^3}) + \overline{E} \sim -4\sigma^*(H_0) + \overline{E} = -4\overline{H_0} - 3\overline{E}$, where $\overline{H_0}$ is the proper transform of H_0 on V. On the other hand, by the adjunction formula, we have $K_{\overline{E}} = (K_V + \overline{E})|_{\overline{E}} \sim (-4\overline{H_0} - 2\overline{E})|_{\overline{E}}$. Since $K_{\overline{E}} \sim -2M_0 - 2f_0$ and $\overline{H_0}|_{\overline{E}}$ is linear equivalent to M_0 on $\overline{E}$, we have $\overline{E}|_{\overline{E}} \sim -M_0 + f_0$. □

We denote by $\overline{\Lambda(f)}$ the proper transform by σ of $\Lambda(f)$ and denote by $\overline{Z}$ the proper transform on V of a Cartier divisor Z on $\mathbf{P}^3$. It is then clear that

$\overline{\Lambda(f)}$ is spanned by $\overline{X}$ and $d\overline{H_0}+\overline{E}$. Indeed, since $\mathrm{mult}_L X = d-1$ by the assumption, we have $\sigma^*(X) = \overline{X} + (d-1)\overline{E} \sim d\sigma^*(H_0) = d(\overline{H_0}+\overline{E})$, hence $\overline{X} \sim d\overline{H_0}+\overline{E}$. We calculate the intersection of $\overline{X}$ and $\overline{E}$:

Lemma 3.2. *With the notations as above, we have* $\overline{X}|_{\overline{E}} \sim (d-1)M_0 + f_0$.

Proof. Since $\overline{X} \sim d\overline{H_0}+\overline{E}$, $\overline{H_0}|_{\overline{E}} \sim M_0$ and $\overline{E}|_{\overline{E}} \sim -M_0+f_0$ (cf. Lemma 3.1), we have $\overline{X}|_{\overline{E}} \sim d\overline{H_0}|_{\overline{E}} + \overline{E}|_{\overline{E}} \sim dM_0 + (-M_0+f_0) = (d-1)M_0+f_0$. $\square$

We put $\overline{L} := \overline{H_0}\cap\overline{E}$, which is a line on $\overline{H_0} \cong \mathbf{P}^2$ and is linear equivalent to M_0 on $\overline{E} \cong \mathbf{P}^1\times\mathbf{P}^1$. Now we have to observe not only the relation $\overline{X}|_{\overline{E}} \sim (d-1)M_0+f_0$ (cf. Lemma 3.2) but also the concrete configuration of the (scheme-theoretic) intersection $\overline{X}|_{\overline{E}}$. Namely, we have the following result, which is an important step to prove Theorem 1.1:

Lemma 3.3. *With the notations as above, the configuration of the scheme-theoretic intersection* $\overline{X}|_{\overline{E}}$ ($\sim (d-1)M_0+f_0$; *see Lemma* 3.2) *coincides with one of the following* $(d-1)$*-types*; $\mathbf{TYPE}(d,\lambda)$ $(0\le\lambda\le d-2)$:

$\mathbf{TYPE}(d,\lambda)$: $\overline{X}|_{\overline{E}} = \overline{C_\lambda} + (d-\lambda-1)\overline{L}$, *where* $\overline{C_\lambda}$ *is an irreducible curve such that* $\overline{C_\lambda} \sim \lambda M_0 + f_0$ *on* $\overline{E}\cong\mathbf{P}^1\times\mathbf{P}^1$ $(0\le\lambda\le d-2)$.

Proof. Our proof consists of four steps.

Step 1. We can write the intersection $\overline{X}|_{\overline{E}} (\sim (d-1)M_0+f_0$; see Lemma 3.2) of $\overline{X}$ and the exceptional divisor $\overline{E}$ as $\overline{X}|_{\overline{E}} = \overline{C_\lambda} + \sum_{i=1}^r m_i\overline{L_i}$, where $\overline{C_\lambda}$ is an irreducible curve such that $\overline{C_\lambda}\sim\lambda M_0+f_0$ $(\lambda\ge 0)$ on $\overline{E}$, and each $\overline{L_i}$ is linear equivalent to M_0 on $\overline{E}$ such that $\overline{L_i}\ne\overline{L_j}$ $(i\ne j)$ and $\sum_{i=1}^r m_i = d-\lambda-1$. In Steps 1 through 3 below, we shall prove that the situation $r\ge 2$ can not occur and, in fact, $r=1$. Assume to the contrary that $r\ge 2$ and denote by $H^{(i)}\subset\mathbf{P}^3$ the hyperplane passing through the line L such that its proper transform $\overline{H^{(i)}}$ on V meets the exceptional divisor $\overline{E}$ along $\overline{L_i}$ for $1\le i\le r$. Note that $\overline{L_i}$ is a line on $\overline{H^{(i)}}\cong\mathbf{P}^2$. We then have the following:

Claim 1. $\overline{H^{(i)}}$ meets $\overline{X}$ along the curve $\overline{L_i}$ scheme-theoretically, i.e., $\overline{X}\cdot\overline{H^{(i)}} = \overline{L_i}$ for $1\le i\le r$.

Note that $\overline{X}|_{\overline{H_0}} \sim (d\overline{H_0}+\overline{E})|_{\overline{H_0}} \sim \overline{L}$, hence $\overline{X}$ meets $\overline{H_0}$ along the curve $\overline{L}$ scheme-theoretically, i.e., $\overline{X}\cdot\overline{H_0} = \overline{L}$. Since $\overline{H^{(i)}}$ is linear equivalent to $\overline{H_0}$, it follows that $\overline{X}\cdot\overline{H_0}=\overline{L}$ is homologous to $\overline{X}\cdot\overline{H^{(i)}}$. Thus we have $\overline{X}\cdot\overline{H^{(i)}} = \overline{L_i}$ as desired.

As seen in the proof of Claim 1, $\overline{X}$ meets $\overline{H_0}$ along the curve $\overline{L}(=\overline{H_0}\cap\overline{E})$, hence $\overline{L}$ is contained in the intersection $\overline{X}\cap\overline{E}$. Thus we may assume that $\overline{L_1}=\overline{L}$.

Step 2. Next we prove the following:

Claim 2. $\overline{X}$ is smooth.

Indeed, since $\overline{X} - (\overline{X} \cap (\overline{H_0} \cup \overline{E})) = \overline{X} - (\overline{C_\lambda} \cup \overline{L_1} \cup \cdots \cup \overline{L_r}) \cong S \cong \mathbf{A}^2$, it follows that $\mathrm{Sing}(\overline{X}) \subseteq \overline{C_\lambda} \cup \overline{L_1} \cup \cdots \cup \overline{L_r}$. We take an arbitrary point $p \in \overline{C_\lambda} \cup \overline{L_1} \cup \cdots \cup \overline{L_r}$. When $p \in \overline{L_i}$ for some i, we take the line $l_p (\neq \overline{L_i})$ on $\overline{H^{(i)}} \cong \mathbf{P}^2$ intersecting $\overline{L_i}$ at p. Then we have the following inequality (cf. [Oh99, Lemma 1]):

$$\mathrm{mult}_{\,p}\overline{X} \leq i(\overline{X}|_{\overline{H^{(i)}}} \cdot l_p; p).$$

Note that $\overline{X}|_{\overline{H^{(i)}}} = \overline{L_i}$ by Claim 1, hence the right hand side of the above inequality is 1 by the choice of the curve l_p. Thus we have $\mathrm{mult}_{\,p}\overline{X} = 1$, i.e., p is a smooth point of $\overline{X}$. On the other hand, when $p \in \overline{C_\lambda}$ and $p \notin \overline{L_i}$ for all i, we take the curve l_p on $\overline{E}$ intersecting $\overline{C_\lambda}$ at p transversally. (Note that since $\overline{C_\lambda} \cong \mathbf{P}^1$ by the genus formula, in particular $\overline{C_\lambda}$ is smooth, we can take such a curve l_p.) Then, as above, we have the following inequality (cf. [Oh99, Lemma 1]):

$$\mathrm{mult}_{\,p}\overline{X} \leq i(\overline{X}|_{\overline{E}} \cdot l_p; p).$$

Note that $\overline{X}|_{\overline{E}} = \overline{C_\lambda} + \sum_{i=1}^{r} m_i \overline{L_i}$, the curve l_p meets $\overline{C_\lambda}$ at p transversally and that l_p does not meet $\overline{L_i}$ at p for all i. Hence the right hand side of the above inequality is $i(\overline{C_\lambda} \cdot l_p; p)$, which is 1 by the choice of the curve l_p. Thus $\mathrm{mult}_{\,p}\overline{X} = 1$, i.e., $\overline{X}$ is smooth at p. As a consequence, $\overline{X}$ is smooth.

Step 3. Now we need the numerical data of the intersection numbers on the smooth projective surface $\overline{X}$. Namely, we shall prove the following:

Claim 3. $\overline{X}$ is rational, $(\overline{L_i} \cdot \overline{L_j})_{\overline{X}} = 0$ and $(\overline{C_\lambda} \cdot \overline{L_i})_{\overline{X}} = 1$ for $1 \leq i, j \leq r$.

The first assertion is clear. Note that $\overline{H^{(i)}} \cdot \overline{X} = \overline{L_i}$ (cf. Claim 1) and $\overline{H^{(i)}} \cdot \overline{H^{(j)}} = 0$ for any $1 \leq i, j \leq r$ in $H^4(V; \mathbf{Z})$. Hence we have $(\overline{L_i} \cdot \overline{L_j})_{\overline{X}} = \overline{H^{(i)}} \cdot \overline{H^{(j)}} \cdot \overline{X} = 0$, which proves the second assertion. For the last assertion, we note that $\overline{C_\lambda} = \overline{E}|_{\overline{X}} - \sum_{i=1}^{r} m_i \overline{L_i} = \overline{E}|_{\overline{X}} - \sum_{i=1}^{r} m_i \overline{H^{(i)}}|_{\overline{X}}$. Hence we have $(\overline{C_\lambda} \cdot \overline{L_i})_{\overline{X}} = (\overline{E} - \sum_{j=1}^{r} m_j \overline{H^{(j)}} \cdot \overline{H^{(i)}} \cdot \overline{X}) = (\overline{E} \cdot \overline{H^{(i)}} \cdot \overline{X})$. On the other hand, since $\overline{X} \sim d\overline{H^{(i)}} + \overline{E}$, the above equality is $(\overline{E} \cdot \overline{H^{(i)}} \cdot d\overline{H^{(i)}} + \overline{E}) = (\overline{E} \cdot \overline{H^{(i)}} \cdot \overline{E}) = (\overline{L_i}^2)_{\overline{H^{(i)}}} = 1$ as desired.

By Claims 2 and 3 above, it follows that $S \hookrightarrow \overline{X}$ is a normal compactification of the affine plane $S \cong \mathbf{A}^2$ (see Definition 2.1) with the boundary divisor $\overline{X} - S = \overline{C_\lambda} \cup \overline{L_1} \cup \cdots \cup \overline{L_r}$. We can obtain the minimal normal compactification of the affine plane $S \cong \mathbf{A}^2$ (see Definition 2.1) by the successive contractions of the (-1)-curves in $\overline{C_\lambda} \cup \overline{L_1} \cup \cdots \cup \overline{L_r}$. By Claim 3, only the component $\overline{C_\lambda}$ can be a (-1)-curve in the boundary components $\overline{C_\lambda} \cup \overline{L_1} \cup \cdots \cup \overline{L_r}$, and if $\overline{C_\lambda}$ is a (-1)-curve then we denote by $\tau : \overline{X} \to X'$ the contraction of $\overline{C_\lambda}$ and denote by L'_i the image on X' of $\overline{L_i}$. Note that $({L'_i}^2) = 1$. Thus either $S \hookrightarrow \overline{X}$ or $S \hookrightarrow X'$ is the minimal normal compactification of the affine plane $S \cong \mathbf{A}^2$. But in any case, if $r \geq 2$, this is a contradiction to Lemma 2.1. Hence we have $r \leq 1$.

Step 4. As remarked in Step 1, since the intersection $\overline{X}\cap\overline{E}$ contains the curve $\overline{L}=\overline{L_1}$, we have $r=1$ and, so, we can write $\overline{X}|_{\overline{E}}=\overline{C_\lambda}+(d-\lambda-1)\overline{L}$, where $\overline{C_\lambda}$ is an irreducible curve such that it is linear equivalent to λM_0+f_0 on $\overline{E}\cong\mathbf{P}^1\times\mathbf{P}^1$ for some $0\le\lambda\le d-2$. This completes the proof of Lemma 3.3. □

Furthermore, we have the following:

Lemma 3.4. *With the notations as above, $\overline{X}$ is isomorphic to a Hirzebruch surface of a suitable degree. The curve $\overline{C_\lambda}$ (resp. $\overline{L}$) is a cross-section (resp. a fiber) on $\overline{X}$.*

Proof. By the same arguments as in Claims 2 and 3 of Lemma 3.3, it follows that $\overline{X}$ is smooth and that $\overline{C_\lambda}\cong\overline{L}\cong\mathbf{P}^1$, $(\overline{L}^2)_{\overline{X}}=0$ and $(\overline{C_\lambda}\cdot\overline{L})_{\overline{X}}=1$. Hence $S\hookrightarrow\overline{X}$ is a normal compactification of the affine plane $S\cong\mathbf{A}^2$ (see Definition 2.1) such that the boundary $\overline{X}-S=\overline{C_\lambda}\cup\overline{L}$ consists of two smooth rational curves $\overline{C_\lambda}$ and $\overline{L}$. We consider the linear system $|L|$ on $\overline{X}$, which is, in fact, a base point free linear pencil on $\overline{X}$ (see the argument of Lemma 2.2, Step 2). It is then easy to see that this pencil defines a $\mathbf{P}^1$-bundle structure $\Phi_{|L|}:\overline{X}\to\mathbf{P}^1$ and that $\overline{C_\lambda}$ is a cross-section of $\Phi_{|L|}$ by noting that $(\overline{C_\lambda}\cdot\overline{L})_{\overline{X}}=1$ and $\overline{X}-(\overline{C_\lambda}\cup\overline{L})\cong S\cong\mathbf{A}^2$ is affine. Hence $\overline{X}$ is a Hirzebruch surface of a suitable degree and $\overline{C_\lambda}$ (resp. $\overline{L}$) is a cross-section (resp. a fiber) on $\overline{X}$. □

Now we shall observe the general members of the linear pencil $\overline{\Lambda(f)}$ on V; see the argument before Lemma 3.2. It is clear that $\overline{\Lambda(f)}$ is composed of the proper transforms $\overline{X_\alpha}$ on V of X_α ($\alpha\in\mathbf{C}$) and the reducible member $d\overline{H_0}+\overline{E}$. Since $\overline{X}\cap\overline{H_0}=\overline{L}$ and $\overline{X}\cap\overline{E}=\overline{C_\lambda}\cup\overline{L}$ (cf. Lemma 3.3), we have $\mathrm{Bs}\overline{\Lambda(f)}=\overline{C_\lambda}\cup\overline{L}$. We have the following result concerning the general members $\overline{X_\alpha}$ of $\overline{\Lambda(f)}$:

Lemma 3.5. *With the notations as above, we have*:

(1) $\overline{X_\alpha}|_{\overline{H_0}}=\overline{L}$ *and* $\overline{X_\alpha}|_{\overline{E}}=\overline{C_\lambda}+(d-\lambda-1)\overline{L}$ *for all* $\alpha\in\mathbf{C}$,

(2) $\overline{X_\alpha}-S_\alpha=\overline{C_\lambda}\cup\overline{L}$ *and* $\overline{X_\alpha}$ *is smooth along the curve* $\overline{C_\lambda}\cup\overline{L}$ *for all* $\alpha\in\mathbf{C}$,

(3) $(\overline{C_\lambda}^2)_{\overline{X_\alpha}}=-d+2\lambda$, $(\overline{L}^2)_{\overline{X_\alpha}}=0$ *and* $(\overline{C_\lambda}\cdot\overline{L})_{\overline{X_\alpha}}=1$ *for all* $\alpha\in\mathbf{C}$.

Proof. (1) Since $\overline{X_\alpha}$ is linear equivalent to $\overline{X}$, it follows that $\overline{X_\alpha}|_{\overline{H_0}}$ is numerically equivalent to $\overline{X}|_{\overline{H_0}}=\overline{L}$ on $\overline{H_0}\cong\mathbf{P}^2$. Note that $\overline{X_\alpha}$ meets $\overline{H_0}$ along $\overline{L}$ because of $\mathrm{Bs}\,\overline{\Lambda(f)}=\overline{C_\lambda}\cup\overline{L}$. Hence we have $\overline{X_\alpha}|_{\overline{H_0}}=\overline{L}$. Similarly, $\overline{X_\alpha}|_{\overline{E}}$ is numerically equivalent to $\overline{X}|_{\overline{E}}=\overline{C_\lambda}+(d-\lambda-1)\overline{L}$ (cf. Lemma 3.3) on $\overline{E}\cong\mathbf{P}^1\times\mathbf{P}^1$. Note that the (set-theoretic) intersection $\overline{X_\alpha}\cap\overline{E}$ is $\overline{C_\lambda}\cup\overline{L}$ because of $\mathrm{Bs}\,\overline{\Lambda(f)}=\overline{C_\lambda}\cup\overline{L}$. Hence we have $\overline{X_\alpha}|_{\overline{E}}=\overline{C_\lambda}+(d-\lambda-1)\overline{L}$.

(2) Note that the affine three space $\mathbf{A}^3=\mathrm{Spec}(\mathbf{C}[x_1,x_2,x_3])$ is contained in V as the complement of $\overline{H_0}\cup\overline{E}$. Since $\overline{X_\alpha}|_{\overline{H_0}}=\overline{L}$ and $\overline{X_\alpha}|_{\overline{E}}=\overline{C_\lambda}+(d-\lambda-1)\overline{L}$

by (1), we have $\overline{X_\alpha} - S_\alpha = \overline{X_\alpha} \cap (\overline{H_0} \cup \overline{E}) = \overline{C_\lambda} \cup \overline{L}$. Moreover, by the same argument as in Claim 2 of Lemma 3.3, $\overline{X_\alpha}$ is smooth along the curve $\overline{C_\lambda} \cup \overline{L}$.

(3) Note that since $\overline{X_\alpha}$ is smooth along the curve $\overline{C_\lambda} \cup \overline{L}$, it makes sense to take the intersections of curves $\overline{C_\lambda}$ and $\overline{L}$ on $\overline{X_\alpha}$. Since $\overline{X_\alpha} \cdot \overline{H_0} = \overline{L}$ and $\overline{X_\alpha} \cdot \overline{E} = \overline{C_\lambda} + (d - \lambda - 1)\overline{L}$ (see the assertion (1)), we have:

$$(\overline{C_\lambda})^2_{\overline{X_\alpha}} = (\overline{E}|_{\overline{X_\alpha}} - (d - \lambda - 1)\overline{H_0}|_{\overline{X_\alpha}})^2_{\overline{X_\alpha}} = (\overline{E} - (d - \lambda - 1)\overline{H_0})^2 \cdot \overline{X_\alpha}.$$

On the other hand, we note that $\overline{X_\alpha} \sim d\overline{H_0} + \overline{E}$ and $\overline{H_0}^2 = 0$ in $H^4(V; \mathbf{Z})$. Hence the right hand side of the above equality is

$$(-d + 2\lambda + 2)\overline{H_0} \cdot \overline{E}^2 + \overline{E}^3 = (-d + 2\lambda + 2)(\overline{L})^2_{\overline{H_0}} + (-M_0 + f_0)^2_{\overline{E}} = -d + 2\lambda,$$

as desired (cf. Lemma 3.1). Similarly, we have:

$$(\overline{L})^2_{\overline{X_\alpha}} = (\overline{H_0}|_{\overline{X_\alpha}})^2_{\overline{X_\alpha}} = \overline{H_0}^2 \cdot \overline{X_\alpha} = 0,$$

and

$$\begin{aligned}(\overline{C_\lambda} \cdot \overline{L})_{\overline{X_\alpha}} &= (\overline{E}|_{\overline{X_\alpha}} - (d - \lambda - 1)\overline{H_0}|_{\overline{X_\alpha}} \cdot \overline{H_0}|_{\overline{X_\alpha}})_{\overline{X_\alpha}} \\ &= (\overline{E} - (d - \lambda - 1)\overline{H_0} \cdot \overline{H_0} \cdot \overline{X_\alpha}) \\ &= (\overline{E} - (d - \lambda - 1)\overline{H_0} \cdot \overline{H_0} \cdot d\overline{H_0} + \overline{E}) \\ &= (\overline{E} \cdot \overline{H_0} \cdot \overline{E}) = (\overline{L})^2_{\overline{H_0}} = 1\end{aligned}$$

as desired. □

Although we have that $\overline{X_\alpha}$ is smooth along the complement $\overline{X_\alpha} - S_\alpha = \overline{C_\lambda} \cup \overline{L}$ (cf. Lemma 3.5), we have, moreover, the following:

Lemma 3.6. *With the notations as above, the affine surfaces S_α are normal for all $\alpha \in \mathbf{C}$.*

Proof. Suppose that S_α is non-normal for some $\alpha \in \mathbf{C}$, then $\overline{X_\alpha}$ is a non-normal projective surface. We denote by $\Gamma := \mathrm{Sing}(\overline{X_\alpha})$ the singular locus of $\overline{X_\alpha}$, which is a one-dimensional closed subscheme of $\overline{X_\alpha}$. Since $S_\alpha \cong \overline{X_\alpha} - (\overline{C_\lambda} \cup \overline{L})$ is affine, the curve Γ is not contained in S_α, i.e., $\Gamma \cap (\overline{C_\lambda} \cup \overline{L}) \neq \emptyset$. But since $\overline{X_\alpha}$ is smooth along $\overline{C_\lambda} \cup \overline{L}$ (cf. Lemma 3.5 (2)), this is a contradiction. □

By combining Lemmas 3.5 and 3.6 with Lemma 2.2, we have the following:

Lemma 3.7. *With the notations as above, the affine surfaces S_α are isomorphic to the affine plane*; $S_\alpha \cong \mathbf{A}^2$ *for all* $\alpha \in \mathbf{C}$.

Proof. By Lemmas 3.5 and 3.6, for any $\alpha \in \mathbf{C}$, a normal affine surface S_α has a compactification $S_\alpha \hookrightarrow \overline{X_\alpha}$ such that $\overline{X_\alpha}$ is smooth along the complement $\overline{X_\alpha} - S_\alpha = \overline{C_\lambda} \cup \overline{L}$ and that the weighted dual graph of $\overline{C_\lambda} \cup \overline{L}$ on $\overline{X_\alpha}$ is the same as that of $\overline{C_\lambda} \cup \overline{L}$ on $\overline{X}$. Hence, by Lemma 2.2, S_α is isomorphic to the affine plane; $S_\alpha \cong \mathbf{A}^2$. $\square$

Now we have completed the preparation to prove Theorem 1.1. By Lemma 3.7, all the fibers S_α of the polynomial map $f : \mathbf{A}^3 = \mathrm{Spec}(\mathbf{C}[x_1, x_2, x_3]) \to \mathbf{A}^1 = \mathrm{Spec}(\mathbf{C}[f])$ associated to the canonical inclusion $\mathbf{C}[f] \hookrightarrow \mathbf{C}[x_1, x_2, x_3]$ are isomorphic to the affine plane $\mathbf{A}^2$. Then Theorem 2.1 and the fact that $\mathrm{Pic}(\mathbf{A}^1) = 0$ implies that this polynomial map is a trivial $\mathbf{A}^2$-bundle structure over the base curve $\mathbf{A}^1 = \mathrm{Spec}(\mathbf{C}[f])$. Hence there are two indeterminants over $\mathbf{C}[f]$, say $u, v \in \mathbf{C}[x_1, x_2, x_3]$, such that $\mathbf{C}[f] \otimes \mathbf{C}[u, v] = \mathbf{C}[f, u, v] \cong \mathbf{C}[x_1, x_2, x_3]$, i.e., f is a variable of $\mathbf{C}[x_1, x_2, x_3]$. This completes the proof of Theorem 1.1.

Remark 3. Though we have seen that S_α are isomorphic to the affine plane; $S_\alpha \cong \mathbf{A}^2$ for *all* $\alpha \in \mathbf{C}$ in Lemma 3.7, we have only to show $S_\alpha \cong \mathbf{A}^2$ for a *general* $\alpha \in \mathbf{C}$ in order to prove f to be a variable of $\mathbf{C}[x_1, x_2, x_3]$ (cf. [Ka02], [Ka-Za01]).

4. The figures of the polynomials in $\mathbf{C}[x_1, x_2, x_3]$ defining the hypersurfaces isomorphic to $\mathbf{A}^2$

Let $f \in \mathbf{C}[x_1, x_2, x_3]$ be an irreducible polynomial of degree $d := \deg(f) \geq 2$ defining the hypersurface $S := (f = 0) \subset \mathbf{A}^3 := \mathrm{Spec}(\mathbf{C}[x_1, x_2, x_3])$, which is isomorphic to the affine plane; $S \cong \mathbf{A}^2$. We embed the affine three space $\mathbf{A}^3$ into the projective three space $\mathbf{P}^3$ as the complement of the hyperplane $H_0 := (x_0 = 0) \subset \mathbf{P}^3$, where $(x_0 : x_1 : x_2 : x_3)$ is the homogeneous coordinate of $\mathbf{P}^3$. Suppose that the closure X in $\mathbf{P}^3$ of the hypersurface S satisfies the condition (†) (cf. Section 1), i.e., the set-theoretic intersection $L := X \cap H_0$ is a line in $\mathbf{P}^3$ and X has the multiplicity $d-1$ along the line L; $\mathrm{mult}_L X = d-1$. As shown in Section 3, f is then a variable of $\mathbf{C}[x_1, x_2, x_3]$ (cf. Theorem 1.1), but we do not know the concrete figure of such a polynomial f. In this section, we consider the problem (Poly;3), namely, we shall determine the concrete figure of such a polynomial f satisfying (†) (cf. Theorem 4.1).

Let $\sigma : V \to \mathbf{P}^3$ denote the blowing-up of $\mathbf{P}^3$ along the line L and let $\overline{E} := \sigma^{-1}(L) \cong \mathbf{P}^1 \times \mathbf{P}^1$ denote the exceptional divisor of σ. We denote by $\overline{H_j} \subset V$ the proper transforms by σ of the hyperplanes $H_j := (x_j = 0) \subset \mathbf{P}^3$ for $0 \leq j \leq 3$. We may and shall assume that the line L, which is the center of $\sigma : V \to \mathbf{P}^3$, is the intersection $L = H_0 \cap H_1$ of the hyperplanes H_0 and H_1. We consider the proper transform $\overline{X} \subset V$ by σ of the hypersurface $X \subset \mathbf{P}^3$. With the notations as in Section 3, $\overline{X}$ is isomorphic to a Hirzebruch surface of a suitable degree, say $\overline{X} \cong \mathbf{F}_N$ ($N \in \mathbf{Z}_{\geq 0}$), and the boundary $\overline{X} - S$ consists of two smooth rational curves $\overline{C_\lambda}$ and $\overline{L}$ such that $\overline{C_\lambda}$ (resp. $\overline{L}$) is a cross-section (resp. a fiber) on $\overline{X}$ (cf. Lemma 3.4). Then the following lemma is easy to see, so we shall omit its proof:

Lemma 4.1. *With the notations as above, there exists a birational map* $\tau : \overline{X} \cong \mathbf{F}_N \cdots \to \mathbf{P}^2$, *which induces an isomorphism* $\tau : \overline{X} - (\overline{C_\lambda} \cup \overline{L}) \xrightarrow{\sim} \mathbf{P}^2 - \overline{L}^*$

when restricted to $\overline{X} - (\overline{C_\lambda} \cup \overline{L}) \cong S$, *where* $\overline{L}^*$ *is a line on* $\mathbf{P}^2$. *More precisely,* τ *is written as a composite* $\tau = \tau_1 \circ \cdots \circ \tau_N$, *where* $\tau_j : \mathbf{F}_j \cdots \to \mathbf{F}_{j-1}$ *for* $j \geq 2$ *and* $\tau_0 : \mathbf{F}_0 \cdots \to \mathbf{F}_1$ *are elementary transformations, and* $\tau_1 : \mathbf{F}_1 \to \mathbf{P}^2$ *is the contraction of the minimal section on* $\mathbf{F}_1$, *such that each* τ_j *is performed outside* S.

By Lemma 4.1, we have the following diagram (cf. Figure 1):

$$\begin{tikzcd}
V \arrow[d, "\sigma"'] & \supset & \overline{X} \arrow[d, "\sigma|_{\overline{X}}"'] \arrow[r, dashed, "\tau"] & \mathbf{P}^2 \arrow[dl, dotted, "\phi"] \\
\mathbf{P}^3 & \supset & X &
\end{tikzcd}$$

Figure 1

where we put $\phi := (\sigma|_{\overline{X}}) \circ \tau^{-1}$. Let $|\mathcal{O}_{\mathbf{P}^3}(1)|$ be the complete linear system on $\mathbf{P}^3$ of degree 1, which is of dimension $\dim |\mathcal{O}_{\mathbf{P}^3}(1)| = 3$ and the hyperplanes H_j $(0 \leq j \leq 3)$ are the basis as elements of the $\mathbf{C}$-vector space $H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))$. We denote by $\mathbf{L} := \mathrm{Tr}_X |\mathcal{O}_{\mathbf{P}^3}(1)|$ the trace of $|\mathcal{O}_{\mathbf{P}^3}(1)|$ on X. It is then not difficult to see that $\mathbf{L}$ is the complete linear system on X of dimension $\dim \mathbf{L} = 3$ and that the hyperplane sections $D_j := H_j|_X$ $(0 \leq j \leq 3)$ are the basis as elements of the $\mathbf{C}$-vector space $\mathbf{M} := H^0(X, \mathcal{O}_X(1))$. Let $\overline{\mathbf{L}} := (\sigma|_{\overline{X}})^*(\mathbf{L})$ be the pull-back by $\sigma|_{\overline{X}}$ of the linear system $\mathbf{L}$, and let $\overline{\mathbf{L}}^*$ be the proper transform of $\overline{\mathbf{L}}$ via the birational map $\tau : \overline{X} \cdots \to \mathbf{P}^2$. Then the following lemma holds:

Lemma 4.2. *With the notations as above, we have*:

(1) $\dim \overline{\mathbf{L}} = 3$ *and the* $\overline{D_j} := (\sigma|_{\overline{X}})^*(D_j)$ $(0 \leq j \leq 3)$ *are the basis as elements of the associated* $\mathbf{C}$*-vector space* $\overline{\mathbf{M}} := H^0(\overline{X}, (\sigma|_{\overline{X}})^* \mathcal{O}_X(1))$,

(2) $\dim \overline{\mathbf{L}}^* = 3$ *and the projective plane curves* $\overline{D_j}^* := \tau_*(\overline{D_j}) \subset \mathbf{P}^2$ $(0 \leq j \leq 3)$ *are the basis as elements of the associated* $\mathbf{C}$*-vector space, say* $\overline{\mathbf{M}}^*$.

Proof. The proof is obvious. □

Let $F_j(z_0, z_1, z_2)$ be the homogeneous polynomial defining the projective plane curve $\overline{D_j}^* \subset \mathbf{P}^2$ $(0 \leq j \leq 3)$, where $(z_0 : z_1 : z_2)$ is the homogeneous coordinate on $\mathbf{P}^2$. Then we have:

Lemma 4.3. *With the notations as above, the birational map* $\phi : \mathbf{P}^2 \cdots \to X \subset \mathbf{P}^3$ (*cf. Figure* 1) *is defined as follows*:

$$\begin{cases} x_0 = F_0(z_0, z_1, z_2), \\ x_1 = F_1(z_0, z_1, z_2), \\ x_2 = F_2(z_0, z_1, z_2), \\ x_3 = F_3(z_0, z_1, z_2). \end{cases}$$

Proof. The proof is obvious. $\square$

Remark 4. By Lemma 4.3, we have to determine $F_j(z_0, z_1, z_2)$ defining the projective plane curves $\overline{D_j}^* \subset \mathbf{P}^2$ for $0 \leq j \leq 3$ in order to determine the defining polynomial of the hypersurface $X \subset \mathbf{P}^3$, say $F(x_0, x_1, x_2, x_3)$. Since we want to obtain the concrete form of $F(x_0, x_1, x_2, x_3)$ up to the projective transformations of $\mathbf{P}^3$, we may and shall replace, if necessary, the hyperplane H_j by a suitable hyperplane (cf. the arguments in the proofs of Lemmas 4.5 and 4.6). Note that $F(1, x_1, x_2, x_3)$ coincides with the irreducible polynomial $f \in \mathbf{C}[x_1, x_2, x_3]$ defining the hypersurface $\mathbf{A}^2 \cong S \subset \mathbf{A}^3 = \operatorname{Spec}(\mathbf{C}[x_1, x_2, x_3])$.

In the subsequent, we shall determine the detailed forms of $F_j(z_0, z_1, z_2)$ $(0 \leq j \leq 3)$, and consequently the detailed form of the polynomial f of $\mathbf{TYPE}(d, \lambda)$ with $d - 2\lambda \geq 0$ (cf. Lemma 3.3). First of all we have the following:

Lemma 4.4. *With the notations as above, suppose that $d - 2\lambda \geq 0$ (see Lemma 3.3 for the notations). Then we have:*

(1) $\overline{D_0} = \overline{C_\lambda} + (d - \lambda)\overline{L}$ and $\overline{D_1} = \overline{C_\lambda} + (d - \lambda - 1)\overline{L} + \overline{L'}$, where $\overline{L'}$ is the fiber distinct from $\overline{L}$ on a Hirzebruch surface $\overline{X}$ (cf. Lemma 3.4),

(2) The degree N of a Hirzebruch surface $\overline{X}$ is $d - 2\lambda$, i.e., $\overline{X} \cong \mathbf{F}_{d-2\lambda}$ and $\overline{C_\lambda}$ is the minimal section on $\overline{X}$,

(3) $\overline{D_j} \sim M_{d-2\lambda} + (d - \lambda) f_{d-2\lambda}$ $(0 \leq j \leq 3)$, where $\overline{C_\lambda} = M_{d-2\lambda}$ (resp. $f_{d-2\lambda}$) is the minimal section (resp. a fiber) on $\overline{X} \cong \mathbf{F}_{d-2\lambda}$.

Proof. (1) Note that $\overline{X} \cdot \overline{H_0} = \overline{L}$ and $\overline{X} \cdot \overline{E} = \overline{C_\lambda} + (d - \lambda - 1)\overline{L}$ by Lemma 3.3. Hence we have $\overline{D_0} = \sigma^*(H_0)|_{\overline{X}} = (\overline{H_0} + \overline{E})|_{\overline{X}} = \overline{C_\lambda} + (d - \lambda)\overline{L}$. Since $H_1 (\neq H_0)$ is the hyperplane passing through the line L, its proper transform $\overline{H_1}$ on V is linear equivalent to $\overline{H_0}$. Hence $\overline{H_0}|_{\overline{X}} = \overline{L}$, which is a fiber on a Hirzebruch surface $\overline{X}$ (cf. Lemma 3.4), is numerically equivalent to $\overline{H_1}|_{\overline{X}}$. Thus $\overline{H_1}|_{\overline{X}}$ is a fiber, say $\overline{L'}$, on $\overline{X}$. Since $\overline{H_0} \cap \overline{H_1} = \emptyset$, $\overline{L}$ is distinct from $\overline{L'}$. Therefore we have $\overline{D_1} = \sigma^*(H_1)|_{\overline{X}} = (\overline{H_1} + \overline{E})|_{\overline{X}} = \overline{C_\lambda} + (d - \lambda - 1)\overline{L} + \overline{L'}$.

(2) By Lemmas 3.4 and 3.5, $\overline{C_\lambda}$ is a cross-section on a Hirzebruch surface $\overline{X}$ with $(\overline{C_\lambda}^2)_{\overline{X}} = -d + 2\lambda$. Since $d - 2\lambda \geq 0$ by the assumption, $\overline{C_\lambda}$ is the minimal section on $\overline{X}$ and the degree of $\overline{X}$ is $d - 2\lambda$, i.e., $\overline{X} \cong \mathbf{F}_{d-2\lambda}$.

(3) Since $\overline{D_j} \sim \overline{D_0} = \overline{C_\lambda} + (d - \lambda)\overline{L}$ and $\overline{C_\lambda}$ (resp. $\overline{L}$) is the minimal section (resp. a fiber) on $\overline{X} \cong \mathbf{F}_{d-2\lambda}$, we have the assertion. $\square$

We next consider the concrete configurations of $\overline{D_2}$ and $\overline{D_3}$.

Lemma 4.5. *With the notations and the assumptions as above, we may write:*

$$\overline{D_2} = \overline{C_\mu} + \sum_{i=1}^{d-\lambda-\mu} \overline{L_i},$$

where $\overline{C_\mu}$ is an irreducible curve such that $\overline{C_\mu} \sim M_{d-2\lambda} + \mu f_{d-2\lambda}$ on $\overline{X} \cong \mathbf{F}_{d-2\lambda}$ with $d - 2\lambda \leq \mu \leq d - \lambda$, and the $\overline{L_i}$ are mutually distinct fibers on

$\overline{X}$. Furthermore, we may assume that $\overline{L_i} \neq \overline{L}, \overline{L'}$ and $\overline{L_i} \cap \overline{C_\mu} \cap \overline{C_\lambda} = \emptyset$ for all i and that $\overline{C_\mu}$ intersects the minimal section $\overline{C_\lambda}$ on $\overline{X}$ at the distinct $n := \mu - (d - 2\lambda)$ points.

Proof. Since $\overline{D_2} = \sigma^*(H_2)|_{\overline{X}} = \overline{H_2}|_{\overline{X}} \sim M_{d-2\lambda} + (d-\lambda)f_{d-2\lambda}$ on $\overline{X} \cong \mathbf{F}_{d-2\lambda}$ (cf. Lemma 4.4), one irreducible component of $\overline{D_2}$ is a cross-section, where $\overline{H_2}$ is the proper transform on V of the hyperplane H_2. Note that $\overline{H_2}$ does not contain the curve $\overline{C_\lambda}$. Hence the cross-section contained in $\overline{D_2}$ is distinct from the minimal section $M_{d-2\lambda} = \overline{C_\lambda}$ and we can write $\overline{D_2} = \overline{C_\mu} + \sum_{i=1}^{d-\lambda-\mu} \overline{L_i}$, where $\overline{C_\mu}$ is a cross-section on $\overline{X}$ such that $\overline{C_\mu} \sim M_{d-2\lambda} + \mu f_{d-2\lambda}$ with $d - 2\lambda \leq \mu \leq d - \lambda$ and the $\overline{L_i}$ are fibers on $\overline{X}$. We put $n := \mu - (d - 2\lambda)$. In the subsequent, we shall prove that we may assume that $\overline{C_\mu}$ and $\overline{C_\lambda}$ meet each other in the distinct n points and that $\overline{L_i} \neq \overline{L_j}$ $(i \neq j)$, $\overline{L_i} \neq \overline{L}, \overline{L'}$ and, furthermore, that $\overline{L_i} \cap \overline{C_\mu} \cap \overline{C_\lambda} = \emptyset$. In order to do this, we consider the linear pencil $\Lambda := \{H(s,t) | (s : t) \in \mathbf{P}^1\}$ on $\mathbf{P}^3$, where $H(s,t) := \{sx_2 + tx_3 = 0\}$ for $(s : t) \in \mathbf{P}^1$. (Note that $H(1,0) = H_2$ and $H(0,1) = H_3$.) The proper transform $\overline{\Lambda}$ by $\sigma : V \to \mathbf{P}^3$ of Λ is composed of the proper transforms $\overline{H(s,t)}$ on V of the hyperplanes $H(s,t)$. The member $\overline{H(s,t)}$ meets the exceptional divisor $\overline{E} \cong \mathbf{P}^1 \times \mathbf{P}^1$ of σ along the curve, say $\overline{l(s,t)}$, which is linear equivalent to f_0 on $\overline{E}$ (cf. Lemma 3.1), and $\overline{l(s,t)} \cap \overline{l(s',t')} = \emptyset$ if $(s,t) \neq (s',t')$. Since $\overline{C_\lambda}$ is linear equivalent to $\lambda M_0 + f_0$ on $\overline{E}$ (cf. Lemma 3.3), $\overline{l(s,t)}$ meets $\overline{C_\lambda}$ in the distinct λ points, each of which is different from the points $\overline{L} \cap \overline{C_\lambda}$ and $\overline{L'} \cap \overline{C_\lambda}$, for a general $(s : t) \in \mathbf{P}^1$. By replacing $\overline{H_2} = \overline{H(1,0)}$ by $\overline{H(s,t)}$ for a general $(s,t) \in \mathbf{P}^1$, if necessary, we may assume that $\overline{l(1,0)} = \overline{H_2}|_{\overline{E}}$ meets $\overline{C_\lambda}$ at the distinct λ points, say $Q_1, \ldots, Q_\lambda$, and that these points are different from the points $\overline{L} \cap \overline{C_\lambda}$ and $\overline{L'} \cap \overline{C_\lambda}$. Since $\overline{D_2} (= \overline{H_2}|_{\overline{X}})$ passes through all the points $Q_1, \ldots, Q_\lambda$, we can deduce that $\overline{C_\mu}$ meets $\overline{C_\lambda}$ in the distinct n points and that $\overline{L_i} \neq \overline{L}, \overline{L'}$ are mutually distinct, moreover, $\overline{L_i} \cap \overline{C_\mu} \cap \overline{C_\lambda} = \emptyset$ $(1 \leq i \leq d - \lambda - \mu)$. This completes the proof. □

Similarly, we have the following:

Lemma 4.6. *With the notations and the assumptions as above, we may write:*

$$\overline{D_3} = \overline{C_\nu} + \sum_{j=1}^{d-\lambda-\nu} \overline{L'_j},$$

where $\overline{C_\nu}$ is an irreducible curve such that $\overline{C_\nu} \sim M_{d-2\lambda} + \nu f_{d-2\lambda}$ on $\overline{X} \cong \mathbf{F}_{d-2\lambda}$ with $d - 2\lambda \leq \nu \leq d - \lambda$, and the $\overline{L'_j}$ are mutually distinct fibers on $\overline{X}$. We may assume that $\overline{L'_j} \neq \overline{L}, \overline{L'}$ and $\overline{L'_j} \cap \overline{C_\nu} \cap \overline{C_\lambda} = \emptyset$ for all j and that $\overline{C_\nu}$ intersects the minimal section $\overline{C_\lambda}$ at the distinct $m := \nu - (d - 2\lambda)$ points. Furthermore, $\overline{D_2} \cap \overline{D_3} \cap \overline{C_\lambda} = \emptyset$.

Proof. By the same argument as in the proof of Lemma 4.5, $\overline{D_3}$ may be written as $\overline{D_3} = \overline{C_\nu} + \sum_{j=1}^{d-\lambda-\nu} \overline{L'_j}$, where $\overline{C_\nu}$ is an irreducible curve such that $\overline{C_\nu} \sim M_{d-2\lambda} + \nu f_{d-2\lambda}$ on $\overline{X} \cong \mathbf{F}_{d-2\lambda}$ for some $d - 2\lambda \leq \mu \leq d - \lambda$ and it intersects the minimal section $\overline{C_\lambda} = M_{d-2\lambda}$ on $\overline{X}$ at the distinct $m := \nu - (d - 2\lambda)$ points, and the $\overline{L'_j}(\neq \overline{L}, \overline{L'})$ are mutually distinct fibers on $\overline{X}$ with $\overline{L'_j} \cap \overline{C_\nu} \cap \overline{C_\lambda} = \emptyset$. Hence the remaining we have to do is to show $\overline{D_2} \cap \overline{D_3} \cap \overline{C_\lambda} = \emptyset$. But this is easy to see because $\overline{H_2}|_{\overline{E}}$ and $\overline{H_3}|_{\overline{E}}$ are mutually disjoint. □

Thus, by Lemmas 4.4, 4.5 and 4.6, the linear system $\overline{\mathbf{L}}$ on $\overline{X} \cong \mathbf{F}_{d-2\lambda}$ is determined. Next, we consider the proper transform $\overline{\mathbf{L}}^*$ of $\overline{\mathbf{L}}$ via $\tau : \overline{X} \cdots \to \mathbf{P}^2$ (cf. Figure 1 and Lemma 4.2).

Lemma 4.7. *With the notations and the assumptions as above, we have the following*:

(1) $\overline{\mathbf{L}}^* \subseteq |\mathcal{O}_{\mathbf{P}^2}(d-\lambda)|$ (*resp.* $\overline{\mathbf{L}}^* \subseteq |\mathcal{O}_{\mathbf{P}^2}(\lambda+1)|$) *if* $d - 2\lambda > 0$ (*resp. if* $d - 2\lambda = 0$),

(2) $\overline{D_0}^* = (d-\lambda)\overline{L}^*$ (*resp.* $\overline{D_0}^* = (\lambda+1)\overline{L}^*$) *if* $d - 2\lambda > 0$ (*resp. if* $d - 2\lambda = 0$),

(3) $\overline{D_1}^* = (d-\lambda-1)\overline{L}^* + \overline{L'}^*$ (*resp.* $\overline{D_1}^* = \lambda\overline{L}^* + \overline{L'}^*$) *if* $d-2\lambda > 0$ (*resp. if* $d - 2\lambda = 0$), *where* $\overline{L'}^*$ *is the proper transform of the fiber* $\overline{L'}$ (*cf. Lemma* 4.4) *via* $\tau : \overline{X} \cdots \to \mathbf{P}^2$, *which is a line on* $\mathbf{P}^2$ *distinct from* $\overline{L}^*$.

Proof. We first consider the case where $d - 2\lambda = 0$. Then $\overline{X} \cong \mathbf{F}_0$ and $\tau : \overline{X} \cdots \to \mathbf{P}^2$ is factored as $\tau = \tau_1 \circ \tau_0$, where $\tau_0 : \overline{X} \cong \mathbf{F}_0 \cdots \to \mathbf{F}_1$ is the composite $\beta_0 \circ \alpha_0^{-1}$ of the blowing-up $\alpha_0 : Y_0 \to \overline{X_0}$ at the point $p_0 := \overline{C_\lambda} \cap \overline{L}$ and the contraction $\beta_0 : Y_0 \to \mathbf{F}_1$ of the proper transform on Y_0 of $\overline{L}$, and $\tau_1 : \mathbf{F}_1 \to \mathbf{P}^1$ is the contraction of the minimal section which is the proper transform on $\mathbf{F}_1$ of $\overline{C_\lambda}$. We denote by $\overline{D_0}^{(1)} := \tau_{0*}(\overline{D_0})$ (resp. $\overline{D_1}^{(1)} := \tau_{0*}(\overline{D_1})$) the proper transform on $\mathbf{F}_1$ of $\overline{D_0}$ (resp. $\overline{D_1}$). Then it is easy to see that $\overline{D_0}^{(1)} = \overline{C_\lambda}^{(1)} + (\lambda+1)\overline{L}^{(1)}$ and $\overline{D_1}^{(1)} = \overline{C_\lambda}^{(1)} + \lambda\overline{L}^{(1)} + \overline{L'}^{(1)}$, where $\overline{C_\lambda}^{(1)}$ (resp. $\overline{L'}^{(1)}$) is the proper transform of $\overline{C_\lambda}$ (resp. $\overline{L'}$) and $\overline{L}^{(1)}$ is the image on $\mathbf{F}_1$ of the exceptional curve of α_0. By contracting the minimal section $\overline{C_\lambda}^{(1)}$ on $\mathbf{F}_1$, we obtain $\overline{D_0}^* = (\lambda+1)\overline{L}^*$ and $\overline{D_1}^* = \lambda\overline{L}^* + \overline{L'}^*$ as desired.

Next we consider the case where $N = d - 2\lambda > 0$. Then the birational map $\tau : \overline{X} \cong \mathbf{F}_N \cdots \to \mathbf{P}^2$ is factored as $\tau = \tau_1 \circ \cdots \circ \tau_N$, where $\tau_j : \mathbf{F}_j \cdots \to \mathbf{F}_{j-1}$ is the elementary transformation performed outside S (cf. Lemma 4.1). More precisely, each τ_j $(2 \leq j \leq N)$ is described as follows. First elementary transformation $\tau_N : \overline{X} \cong \mathbf{F}_N \cdots \to \mathbf{F}_{N-1}$ is the composite $\tau_N = \beta_N \circ \alpha_N^{-1}$, where $\alpha_N : Y_N \to \overline{X} \cong \mathbf{F}_N$ is the blowing-up at the point, say $p_N \in \overline{L} - (\overline{L} \cap \overline{C_\lambda})$, and $\beta_N : Y_N \to \mathbf{F}_{N-1}$ is the contraction of the proper transform on Y_N of $\overline{L}$. Then the proper transform $\overline{D_0}^{(N-1)} := \tau_{N*}(\overline{D_0})$ (resp. $\overline{D_1}^{(N-1)} := \tau_{N*}(\overline{D_1})$) of $\overline{D_0}$ (resp. $\overline{D_1}$) is written as $\overline{D_0}^{(N-1)} = \overline{C_\lambda}^{(N-1)} + (d-\lambda)\overline{L}^{(N-1)}$ (resp. $\overline{D_1}^{(N-1)} = \overline{C_\lambda}^{(N-1)} + (d-\lambda-1)\overline{L}^{(N-1)} + \overline{L'}^{(N-1)}$), where $\overline{C_\lambda}^{(N-1)}$ and $\overline{L'}^{(N-1)}$ are the proper transforms on $\mathbf{F}_{N-1}$ of $\overline{C_\lambda}$ and $\overline{L'}$, respectively, and $\overline{L}^{(N-1)} := \beta_N(\alpha_N^{-1}(p_N))$. We constructed elementary transformation τ_j :

$\mathbf{F}_j \cdots \to \mathbf{F}_{j-1}$, inductively, as the composite $\tau_j = \beta_j \circ \alpha_j^{-1}$, where $\alpha_j : Y_j \to \mathbf{F}_j$ is the blowing-up at the point, say $p_j \in \overline{L}^{(j)} - (\overline{L}^{(j)} \cap \overline{C_\lambda}^{(j)})$, and $\beta_j : Y_j \to \mathbf{F}_{j-1}$ is the contraction of the proper transform on Y_j of $\overline{L}^{(j)}$ and we denote by $\overline{C_\lambda}^{(j-1)}$ the proper transform on $\mathbf{F}_{j-1}$ of $\overline{C_\lambda}^{(j)}$ and we put $\overline{L}^{(j-1)} := \beta_j(\alpha_j^{-1}(p_j))$ for $2 \le j \le N$ (we put here $\overline{C_\lambda}^{(N)} := \overline{C_\lambda}$ and $\overline{L}^{(N)} := \overline{L}$). Then we can easily see that the proper transform $\overline{D_0}^{(j)}$ (resp. $\overline{D_1}^{(j)}$) on $\mathbf{F}_j$ of $\overline{D_0}$ (resp. $\overline{D_1}$) is written as $\overline{D_0}^{(j)} = \overline{C_\lambda}^{(j)} + (d-\lambda)\overline{L}^{(j)}$ (resp. $\overline{D_1}^{(j)} = \overline{C_\lambda}^{(j)} + (d-\lambda-1)\overline{L}^{(j)} + \overline{L'}^{(j)}$), where $\overline{L'}^{(j)}$ is the proper transform on $\mathbf{F}_j$ of $\overline{L'}$ for $1 \le j \le N$, in particular, $\overline{D_0}^{(1)} = \overline{C_\lambda}^{(1)} + (d-\lambda)\overline{L}^{(1)}$ and $\overline{D_1}^{(1)} = \overline{C_\lambda}^{(1)} + (d-\lambda-1)\overline{L}^{(1)} + \overline{L'}^{(1)}$. Finally $\tau_1 : \mathbf{F}_1 \to \mathbf{P}^2$ is the contraction of the minimal section $\overline{C_\lambda}^{(1)}$ on $\mathbf{F}_1$. The images $\overline{D_0}^*$ and $\overline{D_1}^*$ of $\overline{D_0}^{(1)}$ and $\overline{D_1}^{(1)}$ via τ_1 are $\overline{D_0}^* = (d-\lambda)\overline{L}^*$ and $\overline{D_1}^* = (d-\lambda-1)\overline{L}^* + \overline{L'}^*$. This completes the proof. □

Since we know the configurations of $\overline{D_2}$ and $\overline{D_3}$ on $\overline{X} \cong \mathbf{F}_{d-2\lambda}$ (cf. Lemmas 4.5 and 4.6), the configurations of the projective plane curves $\overline{D_2}^*$ and $\overline{D_3}^*$ can be determined. Namely, we have the following:

Lemma 4.8. *With the notations and the assumptions as above, the projective plane curves $\overline{D_2}^*$ and $\overline{D_3}^*$ are written as follows:*

$$\overline{D_2}^* = \overline{C_\mu}^* + \sum_{i=1}^{d-\lambda-\mu} \overline{L_i}^*, \quad \textit{and,}$$

$$\overline{D_3}^* = \overline{C_\nu}^* + \sum_{j=1}^{d-\lambda-\nu} \overline{L_j'}^*,$$

where $\overline{C_\mu}^*$ (resp. $\overline{C_\nu}^*$, $\overline{L_i}^*$ and $\overline{L_j'}^*$) is the proper transform via $\tau : \overline{X} \cong \mathbf{F}_{d-2\lambda} \cdots \to \mathbf{P}^2$ of $\overline{C_\mu}$ (resp. $\overline{C_\nu}$, $\overline{L_i}$ and $\overline{L_j'}$) (cf. Lemmas 4.5 *and* 4.6) *and $\overline{L_i}^*, \overline{L_j'}^* (\neq \overline{L}^*, \overline{L'}^*)$ are mutually distinct lines on $\mathbf{P}^2$ passing through the common point $p := \overline{L}^* \cap \overline{L'}^*$.*

Proof. We may assume that the centers p_j of the blowing-ups $\alpha_j : Y_j \to \mathbf{F}_j$ (see the proof of Lemma 4.7) are different from the intersection points $\overline{C_\mu}^{(j)} \cap \overline{L}^{(j)}$ and $\overline{C_\nu}^{(j)} \cap \overline{L}^{(j)}$, where $\overline{C_\mu}^{(j)}$ and $\overline{C_\nu}^{(j)}$ are the proper transforms on $\mathbf{F}_j$ of $\overline{C_\mu}$ and $\overline{C_\nu}$, respectively. Then the present lemma is easy to prove. □

Now we shall determine the homogeneous polynomials $F_j(z_0, z_1, z_2)$ defining the projective plane curves $\overline{D_j}^* \subset \mathbf{P}^2$ $(0 \le j \le 3)$. We may assume that the lines $\overline{L}^*$ and $\overline{L'}^*$ are defined by $z_0 = 0$ and $z_1 = 0$, respectively. Then we have:

Lemma 4.9. *With the notations and the assumptions as above, the defining polynomials $F_0(z_0, z_1, z_2)$ and $F_1(z_0, z_1, z_2)$ of the projective plane* *curves* $\overline{D_0}^*$ *and* $\overline{D_1}^*$ *are written as* $F_0(z_0, z_1, z_2) = z_0^{d-\lambda}$ (*resp.* $z_0^{\lambda+1}$) *and* $F_1(z_0, z_1, z_2) = z_0^{d-\lambda-1}z_1$ (*resp.* $z_0^{\lambda}z_1$) *if* $d - 2\lambda > 0$ (*resp. if* $d - 2\lambda = 0$).

Proof. The proof is obvious by Lemma 4.7. □

The remainings we have to determine are the defining polynomials $F_2(z_0, z_1, z_2)$ and $F_3(z_0, z_1, z_2)$ of the projective plane curves $\overline{D_2}^*$ and $\overline{D_3}^*$.

Lemma 4.10. *With the notations as above, the defining polynomials* $F_2(z_0, z_1, z_2)$ *and* $F_3(z_0, z_1, z_2)$ *of the plane curves* $\overline{D_2}^*$ *and* $\overline{D_3}^*$ *are written as* $F_2(z_0, z_1, z_2) = G(z_0, z_1, z_2)(\prod_{i=1}^{d-\lambda-\mu}(a_i z_0 + z_1))$ *and* $F_3(z_0, z_1, z_2) = H(z_0, z_1, z_2)(\prod_{j=1}^{d-\lambda-\mu}(b'_j z_0 + z_1))$, *where* $G(z_0, z_1, z_2)$ (*resp.* $H(z_0, z_1, z_2)$) *is the homogeneous polynomial defining the projective plane curve* $\overline{C_\mu}^*$ (*resp.* $\overline{C_\nu}^*$) *and* a_i, $b'_j \in \mathbf{C}^*$ *are mutually distinct non-zero constants.*

Proof. By Lemma 4.8, $\overline{L_i}^*$ and $\overline{L'_j}^*$ are mutually distinct lines passing through the common point $p := \overline{L}^* \cap \overline{L'}^*$, furthermore, they are different from $\overline{L}^*$ and $\overline{L'}^*$. Since the lines $\overline{L}^*$ and $\overline{L'}^*$ are defined by $z_0 = 0$ and $z_1 = 0$, respectively, we may assume that $\overline{L_i}^*$ and $\overline{L'_j}^*$ are defined by $a_i z_0 + z_1 = 0$ and $b'_j z_0 + z_1 = 0$, respectively, where a_i and b'_j are mutually distinct non-zero constants. It is then easy to obtain the assertion. □

We shall determine the defining polynomials $G(z_0, z_1, z_2)$ and $H(z_0, z_1, z_2)$ of the plane curves $\overline{C_\mu}^*$ and $\overline{C_\nu}^*$. We put $g(z_1, z_2) := G(1, z_1, z_2)$ and $h(z_1, z_2) := H(1, z_1, z_2)$. It is clear that the affine plane curve $C_\mu := \overline{C_\mu}^* - (\overline{C_\mu}^* \cap \overline{L}^*)$ (resp. $C_\nu := \overline{C_\nu}^* - (\overline{C_\nu}^* \cap \overline{L}^*)$) on $\mathbf{A}^2 := \mathrm{Spec}(\mathbf{C}[z_1, z_2]) = \mathbf{P}^2 - \overline{L}^*$ is defined by g (resp. h).

We shall recall the definition of the *generically rational polynomials* (cf. Miyanishi-Sugie [Mi-Su80]).

Definition 4.1. Let $p \in \mathbf{C}[z_1, z_2]$ be an irreducible polynomial of degree $d := \deg(p) > 0$ in the polynomial ring in two complex variables z_1 and z_2, and let $\Lambda_0(p) := \{C_\alpha | \alpha \in \mathbf{C}\}$ be the irreducible linear pencil on $\mathbf{A}^2 := \mathrm{Spec}(\mathbf{C}[z_1, z_2])$ defined by p, where $C_\alpha := (p - \alpha = 0) \subset \mathbf{A}^2$ is an affine plane curve defined by $p - \alpha$ for $\alpha \in \mathbf{C}$. We embed the affine plane $\mathbf{A}^2$ into the projective plane $\mathbf{P}^2$ as the complement of the line l_∞. The linear pencil $\Lambda_0(p)$ can be extended to the linear pencil $\Lambda(p) := \{F_\alpha | \alpha \in \mathbf{C}\} \cup \{F_\infty := dl_\infty\}$ on $\mathbf{P}^2$, where F_α is the closure in $\mathbf{P}^2$ of C_α for $\alpha \in \mathbf{C}$. The base points set $\mathrm{Bs}\Lambda(p)$ is located on l_∞. Let $\varphi : W \to \mathbf{P}^2$ be the shortest succession of blowing-ups with centers at the points $\mathrm{Bs}\,\Lambda(p)$ (including infinitely near points) such that the proper transform $\Lambda := \varphi_*^{-1}(\Lambda(p))$ by φ is base point free. We say p to be a *generically rational polynomial* if the general members of Λ are isomorphic to the smooth rational curve $\mathbf{P}^1$. Since the process φ is performed outside $\mathbf{A}^2$, W contains the open subset U which is isomorphic to $\mathbf{A}^2$. Every component in $W - U$ is either contained in some member of Λ or a quasi-section of Λ. We denote by $k \in \mathbf{Z}_{\geq 0}$ a non-negative integer such that the general affine plane curve

C_α has $(k+1)$-places at infinity, namely, $F'_\alpha - C_\alpha$ consists of $(k+1)$-points, where F'_α is the member of Λ corresponding to the member F_α of $\Lambda(p)$. Then the boundary divisor $W - U$ contains at most $(k+1)$ quasi-sections of Λ. The generically rational polynomial p is called *of simple type with* $(k+1)$*-places at infinity* if the boundary divisor $W - U$ contains exactly $(k+1)$ quasi-sections of Λ. It is clear that the generically rational polynomial p is of simple type if and only if all the quasi-sections of Λ contained in $W - U$ are cross-sections of Λ.

Now we prove the following:

Lemma 4.11. *With the notations and the assumptions as above, g (resp. h) is a generically rational polynomial of simple type with $(n+1)$-places (resp. $(m+1)$-places) at infinity, where we put $n := \mu - (d - 2\lambda)$ (resp. $m := \nu - (d - 2\lambda)$); see Definition* 4.1.

Proof. We shall deal with only the polynomial g, because the argument to see that h is a generically rational polynomial of simple type is similar. Let $\Lambda_0(g)$ be the linear pencil on $\mathbf{A}^2 = \operatorname{Spec}(\mathbf{C}[z_1, z_2])$ defined by g and let $\Lambda(g)$ be the extension of $\Lambda_0(g)$ to $\mathbf{P}^2$ (cf. Definition 4.1). We denote by $\overline{\Lambda}$ the proper transform by $\tau : \overline{X} \cdots \to \mathbf{P}^2$ of $\Lambda(g)$. Then we can easily see that $\overline{\Lambda}$ is the linear pencil spanned by $\overline{C_\mu}$ and $\overline{C_\lambda} + \mu\overline{L}$ with the base points set $\operatorname{Bs}\overline{\Lambda} = \{Q_0, Q_1, \ldots, Q_n\}$, where $Q_0 := \overline{C_\mu} \cap \overline{L}$ and $\{Q_1, \ldots, Q_n\} = \overline{C_\mu} \cap \overline{C_\lambda}$. Note that $\overline{C_\mu}$ and $\overline{C_\lambda}$ meet each other at the distinct n points $Q_1, \ldots, Q_n$, and that $i(\overline{C_\mu} \cdot \overline{C_\lambda}; Q_i) = 1$ for $1 \le i \le n$ (cf. Lemma 4.5). Let $\varphi : \widetilde{X} \to \overline{X}$ denote the shortest succession of the blowing-ups with centers at $\operatorname{Bs}\overline{\Lambda}$ (including infinitely near points) such that the proper transform $\widetilde{\Lambda}$ by φ of $\overline{\Lambda}$ is base point free. In fact, the process of φ is described as follows: Let $b_0^{(1)} : X^{(1)} \to X^{(0)} := \overline{X}$ be the blowing-up at the point Q_0 and let $E_1^{(1)}$ be the exceptional curve. We denote by $C_\mu^{(1)}$ (resp. $C_\lambda^{(1)}$ and $L^{(1)}$) the proper transform on $X^{(1)}$ of $\overline{C_\mu}$ (resp. $\overline{C_\lambda}$ and $\overline{L}$). Then the proper transform $\Lambda^{(1)}$ on $X^{(1)}$ of $\overline{\Lambda}$ has $\operatorname{Bs}\Lambda^{(1)} = \{Q_0^{(1)}, Q_1, \ldots, Q_n\}$, where we put $Q_0^{(1)} := C_\mu^{(1)} \cap E_1^{(1)}$. We define $b_0^{(l)} : X^{(l)} \to X^{(l-1)}$ for $1 \le l \le \mu$, inductively, as the blowing-up at the point $Q_0^{(l-1)}$ with the exceptional curve $E_l^{(l)}$. We denote by $C_\mu^{(l)}$ (resp. $C_\lambda^{(l)}$, $L^{(l)}$ and $E_k^{(l)}$ for $1 \le k < l$) the proper transform on $X^{(l)}$ of $C_\mu^{(l-1)}$ (resp. $C_\lambda^{(l-1)}$, $L^{(l-1)}$ and $E_k^{(l-1)}$ for $1 \le k \le l$) and put $Q_0^{(l)} := C_\mu^{(l)} \cap E_l^{(l)}$. We put $b_0 := b_0^{(\mu)} \circ \cdots \circ b_0^{(1)} : X^{(\mu)} \to X^{(0)}$. The proper transform $\Lambda^{(\mu)}$ by b_0 of $\overline{\Lambda}$ has the base points set $\operatorname{Bs}\Lambda^{(\mu)} = \{Q_1, \ldots, Q_n\}$, namely, the base points over Q_0 are eliminated. Let b_j be the blowing-up at the point Q_j for $1 \le j \le n$. Then the composite $b_n \circ \cdots \circ b_1 \circ b_0$ gives rise to the desired elimination φ of $\operatorname{Bs}\overline{\Lambda}$. The proper tranform $\widetilde{\Lambda}$ on $\widetilde{X}$ is the base point free linear pencil spanned by $\widetilde{C_\mu}$ and $\widetilde{F_\infty} := \widetilde{C_\lambda} + \mu\widetilde{L} + \sum_{l=1}^{\mu-1}(\mu - l)\widetilde{E_l}$, where $\widetilde{C_\mu}$ (resp. $\widetilde{C_\lambda}$, $\widetilde{L}$ and $\widetilde{E_l}$ for $1 \le l \le \mu$) is the proper transform on $\widetilde{X}$ of $\overline{C_\mu}$ (resp. $\overline{C_\lambda}$, $\overline{L}$ and $E_l^{(\mu)}$ for $1 \le l \le \mu$). Since $\widetilde{C_\mu}$ is isomorphic to the smooth rational curve $\mathbf{P}^1$, $\widetilde{\Lambda}$ defines the $\mathbf{P}^1$-fibration $\widetilde{\rho} : \widetilde{X} \to \mathbf{P}^1$. Hence, g is a generically rational polynomial (cf. Definition 4.1). Let $\widetilde{H_j}$ be the proper transform on $\widetilde{X}$ of the exceptional curve of b_j for $1 \le j \le n$. Then $\widetilde{H_1}, \ldots, \widetilde{H_n}$ and the last exceptional curve $\widetilde{E_\mu}$ over the point Q_0 exhaust all the quasi-sections of $\widetilde{\Lambda}$ contained in the boundary $\widetilde{X} - \widetilde{U}$, where $\widetilde{U} := \varphi^{-1}(\overline{X} - (\overline{C_\lambda} \cup \overline{L}))$ is the Zariski open open subset of $\widetilde{X}$ which is isomorphic to $\mathbf{A}^2 = \mathrm{Spec}(\mathbf{C}[z_1, z_2])$. Since all $\widetilde{E_\mu}, \widetilde{H_1}, \ldots, \widetilde{H_n}$ are cross-sections of $\widetilde{\Lambda}$, g is of simple type with $(n+1)$-places at infinity (cf. Definition 4.1). $\square$

From now on, for the time being, we devote ourselves to determining the concrete form of the generically rational polynomial $g \in \mathbf{C}[z_1, z_2]$. As in the proof of Lemma 4.11, we define the several notations as follows: Let $\Lambda_0(g)$ be the linear pencil on $\mathbf{A}^2 := \mathrm{Spec}(\mathbf{C}[z_1, z_2])$ defined by g and let $\Lambda(g)$ be the extension of $\Lambda_0(g)$ to $\mathbf{P}^2$ (cf. Definition 4.1). The proper transform $\overline{\Lambda}$ of $\Lambda(g)$ via $\tau : \overline{X} \cdots \to \mathbf{P}^2$ is the linear pencil spanned by $\overline{C_\mu}$ and $\overline{F_\infty} := \overline{C_\lambda} + \mu\overline{L}$ with the base points set $\mathrm{Bs}\,\overline{\Lambda} = \{Q_0, Q_1, \ldots, Q_n\}$, where $Q_0 := \overline{C_\mu} \cap \overline{L}$ and $\{Q_1, \ldots, Q_n\} = \overline{C_\mu} \cap \overline{C_\lambda}$, here we put $n := \mu - (d - 2\lambda)$. Let $\varphi : \widetilde{X} \to \overline{X}$ be the shortest succession of the blowing-ups with the centers at $\mathrm{Bs}\,\overline{\Lambda}$ (including infinitely near points) such that the proper transform $\widetilde{\Lambda}$ by φ of $\overline{\Lambda}$ is base point free. More precisely, the process of φ is factored as $\varphi = b_n \circ \cdots \circ b_1 \circ b_0$, where $b_0 = b_0^{(\mu)} \circ \cdots \circ b_0^{(1)}$ is the composite of the successive μ blowing-ups over Q_0, and b_j is the blowing-up at Q_j for $1 \le j \le n$ (cf. the proof of Lemma 4.11). We denote by $\widetilde{E_l}$ $(1 \le l \le \mu)$, and by $\widetilde{H_j}$ $(1 \le j \le n)$ the proper transforms on $\widetilde{X}$ of the exceptional curve of $b_0^{(l)}$, and the exceptional curve of b_j. The base point free linear pencil $\widetilde{\Lambda}$ on $\widetilde{X}$ is spanned by $\widetilde{C_\mu}$ and $\widetilde{F_\infty} := \widetilde{C_\lambda} + \mu\widetilde{L} + \sum_{l=1}^{\mu-1}(\mu - l)\widetilde{E_l}$ and it defines the $\mathbf{P}^1$-fibration $\widetilde{\rho} : \widetilde{X} \to \mathbf{P}^1$, where $\widetilde{C_\mu}$ (resp. $\widetilde{C_\lambda}$ and $\widetilde{L}$) is the proper transform on $\widetilde{X}$ of $\overline{C_\mu}$ (resp. $\overline{C_\lambda}$ and $\overline{L}$). The open subset $\widetilde{U} := \varphi^{-1}(\overline{X} - (\overline{C_\lambda} \cup \overline{L})) \subset \widetilde{X}$ is isomorphic to the affine plane $\mathbf{A}^2 = \mathrm{Spec}(\mathbf{C}[z_1, z_2])$ and the complement $\widetilde{X} - \widetilde{U}$ consists of $\widetilde{C_\lambda}$, $\widetilde{L}$, $\widetilde{E_l}$ $(1 \le l \le \mu)$ and $\widetilde{H_j}$ $(1 \le j \le n)$. Among these boundary components, $\widetilde{E_\mu}$, $\widetilde{H_1}, \ldots, \widetilde{H_n}$ are cross-sections of $\widetilde{\rho}$ and the remaining boundary components $\widetilde{E_1}, \ldots, \widetilde{E_{\mu-1}}$ are contained in the reducible fiber $\widetilde{F_\infty}$ of $\widetilde{\rho}$.

We shall investigate the polynomial map $g : \mathbf{A}^2 = \mathrm{Spec}(\mathbf{C}[z_1, z_2]) \to \mathbf{A}^1 := \mathrm{Spec}(\mathbf{C}[g])$ associated to the canonical inclusion $\mathbf{C}[g] \hookrightarrow \mathbf{C}[z_1, z_2]$ and the singular fibers of it (if there exist at all). As seen below (cf. Lemma 4.12, (3)), if $n(= \mu - (d - 2\lambda))$ is a positive integer, then this polynomial map g has at least one singular fiber. Note that the restriction of the $\mathbf{P}^1$-fibration $\widetilde{\rho} : \widetilde{X} \to \mathbf{P}^1$ to $\widetilde{U}$ coincides with the polynomial map $g : \mathbf{A}^2 \to \mathbf{A}^1$, namely, $\widetilde{F} \cap \widetilde{U} \cong C$, where we denote by $\widetilde{F}$ the fiber of $\widetilde{\rho}$ corresponding to the member C of $\Lambda_0(g)$. We have the following:

Lemma 4.12. *With the notations and the assumptions as above, we have the following*:

(1) *The general members C of $\Lambda_0(g)$ are isomorphic to $\mathbf{C}^{(*n)}$, where we denote by $\mathbf{C}^{(*n)}$ the affine line with n-point punctured,*

(2) *Let* $\widetilde{F_1}, \dots, \widetilde{F_r}$ *exhaust all the singular fibers of* $\widetilde{\rho} : \widetilde{X} \to \mathbf{P}^1$ *distinct from* $\widetilde{F_\infty}$ (*if there exist at all*) *and let* $\widetilde{F_i} = \sum_{j=1}^{m_i} a_{i,j} \widetilde{F_{i,j}}$ *be the decomposition of* $\widetilde{F_i}$ *into the irreducible components with* $a_{i,j} \in \mathbf{N}$. *Then* $a_{i,j} = 1$ *for all* $1 \le i \le r$ *and* $1 \le j \le m_i$,

(3) $n + r = \sum_{i=1}^{r} m_i$. *In particular, if* $n \ge 1$, *then we have* $r \ge 1$, *namely,* $\widetilde{\rho} : \widetilde{X} \to \mathbf{P}^1$ *has at least one singular fiber distinct from* $\widetilde{F_\infty}$.

Proof. (1) As remarked in the proof of Lemma 4.11, the boundary $\widetilde{X} - \widetilde{U}$ contains exactly $(n+1)$ cross-sections $\widetilde{E_\mu}, \widetilde{H_1}, \dots, \widetilde{H_n}$. Hence we have $C \cong \widetilde{F} \cap \widetilde{U} \cong \mathbf{P}^1 - \{(n+1)\text{-points}\} = \mathbf{C}^{(*n)}$ for a general member $C \in \Lambda_0(g)$, where $\widetilde{F}$ is a fiber of $\widetilde{\rho}$ corresponding to C. Hence we obtain the assertion (1).

(2) Note that every fiber of $\widetilde{\rho}$ distinct from $\widetilde{F_\infty}$ does not contain any component in $\widetilde{X} - \widetilde{U}$. Hence each component $\widetilde{F_{i,j}}$ of the singular fiber $\widetilde{F_i}$ meets $\widetilde{U}$. Since $\widetilde{U}$ is affine, $\widetilde{F_{i,j}}$ is not contained in $\widetilde{U}$, hence it meets at least one of the cross-sections $\widetilde{E_\mu}, \widetilde{H_1}, \dots, \widetilde{H_n}$ of $\widetilde{\rho}$. Therefore the multiplicity $a_{i,j}$ of $\widetilde{F_{i,j}}$ in the fiber $\widetilde{F_i}$ is 1.

(3) By [Mi-Su80, Lemma 1.6], we obtain the desired equation. □

We shall determine the concrete figure of the polynomial $g \in \mathbf{C}[z_1, z_2]$ by observing the singular fibers of $\widetilde{\rho} : \widetilde{X} \to \mathbf{P}^1$ distinct from $\widetilde{F_\infty}$. Namely, we have the following proposition:

Proposition 4.1. *Let* $g \in \mathbf{C}[z_1, z_2]$ *be the generically rational polynomial defining the affine plane curve* $C_\mu \subset \mathbf{A}^2 := \operatorname{Spec}(\mathbf{C}[z_1, z_2])$ (*see the argument before Definition* 4.1). *We put* $n := \mu - (d - 2\lambda) \ge 0$. *Then* g *is written in the following fashion*:

$$g = (z_2 + A(z_1)) \left(\prod_{j=1}^{n} (z_1 + d_j) \right) + \sum_{j=1}^{n-1} c_j \left(\prod_{k=j+1}^{n} (z_1 + d_k) \right) + c_n,$$

where $d_j \in \mathbf{C}^*$ $(1 \le j \le n)$ *are mutually distinct non-zero constants,* $c_j \in \mathbf{C}$ $(1 \le j \le n)$ *and* $c_n \ne 0$, *and* $A(z_1) \in \mathbf{C}[z_1]$ *is of degree* $\deg A(z_1) = d - 2\lambda$.

Remark 5. The mutually distinct non-zero constants d_j $(1 \le j \le n)$ in the statement of Proposition 4.1 are, in fact, defined in the following way: Let $Q_1, \dots, Q_n$ be the intersection points of the curves $\overline{C_\mu}$ and $\overline{C_\lambda}$, and let $\overline{l_j}$ be the fibers on $\overline{X} \cong \mathbf{F}_{d-2\lambda}$ passing through Q_j for $1 \le j \le n$. Then the proper transforms $\overline{l_j}^* \subset \mathbf{P}^2$ of $\overline{l_j}$ via $\tau : \overline{X} \cdots \to \mathbf{P}^2$ are mutually distinct lines passing through the common point $p := \overline{L}^* \cap \overline{L'}^* = (z_0 = 0) \cap (z_1 = 0) = (0 : 0 : 1)$, hence $\overline{l_j}^*$ are defined by $d_j z_0 + z_1 = 0$ for some mutually distinct non-zero constants d_j for $1 \le j \le n$. It is not hard to see that the d_j $(1 \le j \le n)$ defined so appear in the formula of the polynomial g (cf. Proposition 4.1) by observing the argument in Lemmas 4.13 through 4.16 below carefully.

Our proof of Proposition 4.1 consists of several steps. We shall prove Proposition 4.1 by the induction on the non-negative integer $n := \mu-(d-2\lambda) \in \mathbf{Z}_{\geq 0}$. First of all, we consider the case $n = 0$.

Lemma 4.13. *Let the notations be as above. Suppose that $n = 0$. Then the polynomial $g \in \mathbf{C}[z_1, z_2]$ is written as $g = z_2 + A(z_1)$, where $A(z_1) \in \mathbf{C}[z_1]$ is of degree* $\deg A(z_1) = d - 2\lambda$.

Proof. Suppose that $n = 0$, then $C_\mu \cong \mathbf{A}^1$. The curve $\overline{C_\mu}$ does not meet the minimal section $\overline{C_\lambda} = M_{d-2\lambda}$ on $\overline{X} \cong \mathbf{F}_{d-2\lambda}$ and it meets the fiber $\overline{L}$ on $\overline{X}$ at a point Q_0. Since we can easily prove the present lemma for the case where $d - 2\lambda$ is either 0 or 1, we shall consider only the case $d - 2\lambda \geq 2$ in the subsequent. It is then not hard, by observing the process of $\tau : \overline{X} \cdots \to \mathbf{P}^2$, to see that the projective plane curve $\overline{C_\mu}^* \subset \mathbf{P}^2$ meets the line $\overline{L}^*$ at a single point $p := \overline{L}^* \cap \overline{L'}^* = (0 : 0 : 1)$ with order $d-2\lambda$ and that $\mathrm{mult}_{\,p}\overline{C_\mu}^* = d-2\lambda-1$. On the other hand, $\overline{C_\mu}^*$ and the line $\overline{L'}^*$ meet each other at two points, one of which is the point p and we denote by p' the another intersection point. It is not hard to see that $i(\overline{C_\mu}^* \cdot \overline{L'}^*; p') = 1$. Hence the affine plane curves C_μ and $L' := \overline{L'}^* - \{p\}$, both of which are isomorphic to the affine line $\mathbf{A}^1$, meet each other at a single point p' transversally. Hence we have $\mathbf{C}[z_1, g_{1,1}] = \mathbf{C}[z_1, z_2]$ (cf. Miyanishi [Miy78b]), in particular, the Jacobian determinant $\mathbf{J}((z_1, g_{1,1})/(z_1, z_2))$ of $g_{1,1}$ and z_1 is a non-zero constant, so we may assume that $\partial g_{1,1}/\partial z_2 = 1$. Therefore we can write $g_{1,1}$ as $g_{1,1} = z_2 + A(z_1)$, where $A(z_1) \in \mathbf{C}[z_1]$ is of degree $d - 2\lambda$. □

In the subsequent, we assume that $n > 0$. Then by the equality $n + r = \sum_{i=1}^{r} m_i$ (cf. Lemma 4.12) we have $r \geq 1$, i.e., the $\mathbf{P}^1$-fibration $\widetilde{\rho} : \widetilde{X} \to \mathbf{P}^1$ has at least one singular fiber distinct from $\widetilde{F_\infty}$. Let $\widetilde{F_1} = \sum_{j=1}^{m_1} \widetilde{F_{1,j}}$ be the singular fiber of $\widetilde{\rho}$ distinct from $\widetilde{F_\infty}$, where $\widetilde{F_{1,j}}$ is the irreducible component. Note that $\widetilde{F_{1,j}}$ is contained in the fiber $\widetilde{F_1}$ with multiplicity 1 by Lemma 4.12. More precisely, we have the following:

Lemma 4.14. *With the notations and the assumptions as above, we have the following result concerning the configuration of the singular fiber $\widetilde{F_1} = \sum_{j=1}^{m_1} \widetilde{F_{1,j}}$ of $\widetilde{\rho} : \widetilde{X} \to \mathbf{P}^1$:*

(1) $2 \leq m_1 \leq n + 1$,

(2) *By reordering the indices, if necessary, we may assume that the component $\widetilde{F_{1,1}}$ meets the cross-sections $\widetilde{E_\mu}, \widetilde{H_1}, \ldots, \widetilde{H_s}$ with $s := n+1-m_1$, and the remaining fiber component $\widetilde{F_{1,j}}$ meets the cross-section $\widetilde{H_{j+s-1}}$ for $2 \leq j \leq m_1$,*

(3) *The fiber components $\widetilde{F_{1,j}}$ $(2 \leq j \leq m_1)$ are mutually disjoint and each of them meet $\widetilde{F_{1,1}}$ at one point transversally.*

Proof. The assertion (1) is obvious from the equality $n + r = \sum_{i=1}^{r} m_i$ by noting that $m_i \geq 2$ for $1 \leq i \leq r$. Note that no component in the boundary $\widetilde{X} - \widetilde{U}$ is contained in $\widetilde{F_1}$ (see the proof of Lemma 4.11). By reordering the indices, if necessary, we may assume that the fiber component $\widetilde{F_{1,1}}$ meets the cross-section $\widetilde{E_\mu}$. Assume that a fiber component $\widetilde{F_{1,j_0}}$ does not meet $\widetilde{F_{1,1}}$ for some j_0. Let $\widetilde{H_1}, \dots, \widetilde{H_t}$ exhaust all the cross-sections in $\widetilde{X} - \widetilde{U}$ intersecting $\widetilde{F_{1,j_0}}$. Note that since $\widetilde{F_{1,j_0}} \cap \widetilde{U} \neq \emptyset$ and $\widetilde{U}$ is affine, there exists at least one cross-section in $\widetilde{X} - \widetilde{U}$ intersecting $\widetilde{F_{1,j_0}}$. Let $\overline{F_{1,j}} := \varphi(\widetilde{F_{1,j}})$ denote the proper transform on $\overline{X}$ of $\widetilde{F_{1,j}}$. It is easy to see that $\overline{F_{1,1}}$ meets the fiber $\overline{L}$ at a point Q_0 transversally, so $\overline{F_{1,1}}$ is a cross-section on $\overline{X} \cong \mathbf{F}_{d-2\lambda}$. On the other hand, since $\widetilde{F_{1,1}} \cap \widetilde{F_{1,j_0}} = \emptyset$, we have $\overline{F_{1,1}} \cap \overline{F_{1,j_0}} = \emptyset$. Hence $\overline{F_{1,j_0}}$ is a quasi-section on $\overline{X}$. But since $\overline{F_{1,j_0}}$ does not meet the fiber $\overline{L}$, this is a contradiction. Hence the fiber components $\widetilde{F_{1,j}}$ intersect $\widetilde{F_{1,1}}$ for all $2 \leq j \leq m_1$. Note that $\widetilde{F_{1,j}}$ meets $\widetilde{F_{1,1}}$ at only one point transversally by the general theory of the singular fibers of $\mathbf{P}^1$-fibrations (cf. Miyanishi [Miy01]). Suppose that the number, say t, of the cross-sections in $\widetilde{X} - \widetilde{U}$ meeting a fiber component $\widetilde{F_{1,j_0}}$ is equal to or more than 2 for some $2 \leq j_0 \leq m_1$. Then the proper transform $\overline{F_{1,j_0}}$ meets the minimal section $M_{d-2\lambda} = \overline{C_\lambda}$ at the distinct t-points and it does not meet the fiber $\overline{L}$ on $\overline{X} \cong \mathbf{F}_{d-2\lambda}$. This is absurd. Thus every fiber component $\widetilde{F_{1,j}}$ $(2 \leq j \leq m_1)$ meets exactly one cross-section contained in $\widetilde{X} - \widetilde{U}$. Suppose that the fiber components $\widetilde{F_{1,j_1}}$ and $\widetilde{F_{1,j_2}}$ meet each other for some $2 \leq j_1, j_2 \leq m_1$. Then the fiber $\widetilde{F_1}$ contains a loop or $\widetilde{F_{1,1}} \cap \widetilde{F_{1,j_1}} \cap \widetilde{F_{1,j_2}} \neq \emptyset$, which is a contradiction (cf. [Miy01]). This completes the proof of Lemma 4.14. □

We shall keep the above notations. We consider the image $\overline{F_1} = \sum_{j=1}^{m_1} \overline{F_{1,j}}$ via $\varphi : \widetilde{X} \to \overline{X}$ of the singular fiber $\widetilde{F_1} = \sum_{j=1}^{m_1} \widetilde{F_{1,j}}$, where $\overline{F_{1,j}} := \varphi(\widetilde{F_{1,j}})$ is the image on $\overline{X}$ of $\widetilde{F_{1,j}}$. Then we have the following:

Lemma 4.15. *With the notations as above, we have*:

(1) $\overline{F_{1,1}}$ *is a cross-section on* $\overline{X} \cong \mathbf{F}_{d-2\lambda}$ *passing through the point* $Q_0, Q_1, \dots, Q_s$, *here* $s := n + 1 - m_1$. *Furthermore,* $\overline{F_{1,1}}$ *meets* $\overline{C_\lambda}$ *at* Q_j *transversally for* $1 \leq j \leq s$,

(2) $\overline{F_{1,j}}$ *is a fiber on* $\overline{X} \cong \mathbf{F}_{d-2\lambda}$ *passing through the point* Q_{j+s-1} *for* $2 \leq j \leq m_1$.

Proof. (1) By Lemma 4.14, the component $\widetilde{F_{1,1}}$ of the singular fiber $\widetilde{F_1}$ meets the cross-sections $\widetilde{E_\mu}, \widetilde{H_1}, \dots, \widetilde{H_s}$ of $\widetilde{\rho}$. Note that the process $\varphi : \widetilde{X} \to \overline{X}$ is obtained as the successive contractions of $\widetilde{E_\mu}, \widetilde{E_{\mu-1}}, \dots, \widetilde{E_1}$ in this order and the contractions of $\widetilde{H_j}$ for $1 \leq j \leq n$ (see the proof of Lemma 4.11). Hence $\overline{F_{1,1}}$ intersects $\overline{L}$, which is a fiber on $\overline{X} \cong \mathbf{F}_{d-2\lambda}$, at Q_0 transversally. This means that $\overline{F_{1,1}}$ is a cross-section on $\overline{X}$. Furthermore, since $\widetilde{F_{1,1}}$ meets $\widetilde{H_1}, \dots, \widetilde{H_s}$, $\overline{F_{1,1}}$ meets $\overline{C_\lambda}$ at the points $Q_1, \dots, Q_s$ and $i(\overline{F_{1,1}} \cdot \overline{C_\lambda}; Q_j) = 1$ for $1 \leq j \leq s$.

(2) Note that each of the remaining fiber components $\widetilde{F_{1,j}}$ $(2 \leq j \leq m_1)$ meets $\widetilde{F_{1,1}}$ in one point, which is not the point $\widetilde{F_{1,1}} \cap \widetilde{E_\mu}$. Hence $\overline{F_{1,j}}$ does not meet the fiber $\overline{L}$ on $\overline{X} \cong \mathbf{F}_{d-2\lambda}$, so $\overline{F_{1,j}}$ is a fiber on $\overline{X}$ for $2 \leq j \leq m_1$. Furthermore, since $\widetilde{F_{1,j}}$ meets the cross-section $\widetilde{H_{j+s-1}}$, it follows that $\overline{F_{1,j}}$ passes through the point Q_{j+s-1} for $2 \leq j \leq m_1$. □

Next we consider the proper transform $\overline{F_1}^* = \sum_{j=1}^{m_1} \overline{F_{1,j}}^*$ via $\tau : \overline{X} \cong \mathbf{F}_{d-2\lambda} \cdots \to \mathbf{P}^2$ of $\overline{F_1}$, where $\overline{F_{1,j}}^*$ is the proper transform on $\mathbf{P}^2$ of the component $\overline{F_{1,j}}$ for $1 \leq j \leq m_1$. $\overline{F_1}^*$ is a reducible member of $\Lambda(g)$. We put $C_{1,j} := \overline{F_{1,j}}^* - (\overline{F_{1,j}}^* \cap \overline{L}^*) \subset \mathbf{A}^2 = \mathrm{Spec}(\mathbf{C}[z_1, z_2]) = \mathbf{P}^2 - \overline{L}^*$ and denote by $g_{1,j} \in \mathbf{C}[z_1, z_2]$ the defining polynomial of the affine plane curve $C_{1,j}$ for $1 \leq j \leq m_1$. Then we have:

Lemma 4.16. *With the notations as above, we have*:

(1) $g_{1,1}$ *is a generically rational polynomial of simple type with* $(s+1)$*-places at infinity* (*cf. Definition* 4.1),

(2) $g_{1,j} = z_1 + d_{j+s-1}$ *for* $2 \leq j \leq m_1$, *where the* d_{j+s-1} *are mutually distinct non-zero constants defined in Remark* 5.

Proof. (1) Let $\Lambda_0(g_{1,1})$ be the linear pencil on $\mathbf{A}^2 = \mathrm{Spec}(\mathbf{C}[z_1, z_2])$ defined by $g_{1,1}$ and let $\Lambda(g_{1,1})$ be the extension of $\Lambda_0(g_{1,1})$ to $\mathbf{P}^2$. The proper transform $\overline{\Lambda_{1,1}}$ via $\tau : \overline{X} \cong \mathbf{F}_{d-2\lambda} \cdots \to \mathbf{P}^2$ of $\Lambda(g_{1,1})$ is the linear pencil spanned by $\overline{F_{1,1}}$ and $\overline{C_\lambda} + (d+s-2\lambda)\overline{L}$ with the base points set $\mathrm{Bs}\, \overline{\Lambda_{1,1}} = \{Q_0, Q_1, \ldots, Q_s\}$. Let $\psi : X' \to \overline{X}$ be the shortest succession of the blowing-ups at the points in $\mathrm{Bs}\, \overline{\Lambda_{1,1}}$ (including infinitely near points) such that the proper transform $\Lambda'_{1,1}$ by ψ of $\overline{\Lambda_{1,1}}$ is base point free. The Zariski open subset $U' := \psi^{-1}(\overline{X} - (\overline{C_\lambda} \cup \overline{L}))$ of X' is isomorphic to the affine plane $\mathbf{A}^2 = \mathrm{Spec}(\mathbf{C}[z_1, z_2])$. This process ψ to eliminate $\mathrm{Bs}\, \overline{\Lambda_{1,1}}$ is similar to that to eliminate $\mathrm{Bs}\, \overline{\Lambda}$ (cf. the proof of Lemma 4.11), and we can see that $\Lambda'_{1,1}$ defines the $\mathbf{P}^1$-fibration $\rho' : X' \to \mathbf{P}^1$ and that the boundary $X' - U'$ contains the $(s+1)$ cross sections of ρ'. Thus $g_{1,1}$ is a generically rational polynomial of simple type with $(s+1)$-points at infinity.

(2) By Lemma 4.15, the $\overline{F_{1,j}}$ are mutually distinct fibers passing through the points Q_{j+s-1} on $\overline{X} \cong \mathbf{F}_{d-2\lambda}$ for $2 \leq j \leq m_1$. After the process $\tau : \overline{X} \cdots \to \mathbf{P}^2$, these fibers $\overline{F_{1,j}}$ are brought to the mutually distinct lines $\overline{F_{1,j}}^*$ passing through the common point $p = \overline{L}^* \cap \overline{L'}^* = (z_0 = 0) \cap (z_1 = 0) = (0 : 0 : 1)$. Hence we can write $g_{1,j} = z_1 + d_{j+s-1}$ for $2 \leq j \leq m_1$, where the d_{j+s-1} are mutually distinct non-zero constants defined in Remark 5. $\square$

By Lemmas 4.15 and 4.16, the curve $\overline{F_{1,1}}$ is a cross-section on $\overline{X} \cong \mathbf{F}_{d-2\lambda}$ passing through the points $Q_0, Q_1, \ldots, Q_s$, and $g_{1,1}$, which is a defining polynomial of the affine plane curve $C_{1,1}$, is a generically rational polynomial of simple type with $(s+1)$-places at infinity. Note that $0 \leq s \leq n-1$ because $2 \leq m_1 \leq n+1$. Hence we can write $g_{1,1}$ in the following fashion by the inductive hypothesis:

$$g_{1,1} = (z_2 + A(z_1)) \left(\prod_{j=1}^{s} (z_1 + d_j) \right) + \sum_{j=1}^{s-1} c_j \left(\prod_{k=j+1}^{s} (z_1 + d_k) \right) + c_s,$$

where $d_j \in \mathbf{C}^*$ $(1 \leq j \leq s)$ are mutually distinct non-zero constants defined in Remark 5, $c_j \in \mathbf{C}$ $(1 \leq j \leq s)$ and $c_s \neq 0$, and $A(z_1) \in \mathbf{C}[z_1]$ is of degree

$\deg A(z_1) = d - 2\lambda$. Note that the restriction $\widetilde{F_1} \cap \widetilde{U}$ of the fiber $\widetilde{F_1}$ to $\widetilde{U}$ coincides with $\sum_{j=1}^{m_1} C_{1,j}$, which is the member of $\Lambda_0(g)$. Hence the affine plane curve $\sum_{j=1}^{m_1} C_{1,j}$ is defined by $g - c_n$ for some $c_n \in \mathbf{C}^*$, so we have $g - c_n = \prod_{j=1}^{m_1} g_{1,j}$. Hence g is written as follows:

$$g = (z_2 + A(z_1)) \left(\prod_{j=1}^{n} (z_1 + d_j) \right) + \sum_{j=1}^{s} c_j \left(\prod_{k=j+1}^{n} (z_1 + d_k) \right) + c_n.$$

Therefore we complete the proof of Proposition 4.1.

We can determine the concrete form of another generically rational polynomial h (see the argument before Definition 4.1) by the same argument as that to determine the form of g. Namely, we have the following:

Proposition 4.2. *Let $h \in \mathbf{C}[z_1, z_2]$ be the generically rational polynomial defining the affine plane curve $C_\nu \subset \mathbf{A}^2 := \mathrm{Spec}(\mathbf{C}[z_1, z_2])$ (see the argument before Definition* 4.1*). We put $m := \nu - (d - 2\lambda) \geq 0$. Then h is written in the following fashion*:

$$h = (z_2 + B(z_1)) \left(\prod_{j=1}^{m} (z_1 + d'_j) \right) + \sum_{j=1}^{m-1} c'_j \left(\prod_{k=j+1}^{m} (z_1 + d'_k) \right) + c'_m,$$

where $d'_j \in \mathbf{C}^*$ $(1 \leq j \leq m)$ are mutually distinct non-zero constants, $c'_j \in \mathbf{C}$ $(1 \leq j \leq m)$ and $c'_m \neq 0$, and $B(z_1) \in \mathbf{C}[z_1]$ is of degree $\deg B(z_1) = d - 2\lambda$. Furthermore, each of d'_j is different from d_i for all $1 \leq i \leq n$ (see Proposition 4.1 *and Remark* 5*).*

Proof. By the same argument as that to prove Proposition 4.1, the polynomial h can be written as in the statement except for the possibility that some d'_j coincides with d_i for some $1 \leq i \leq n, 1 \leq j \leq m$. But this possibility is excluded. Indeed, the mutually distinct non-zero constants d'_j $(1 \leq j \leq m)$ are determined as follows (cf. Remark 5): Let $Q'_1, \ldots, Q'_m$ be the intersection points of $\overline{C_\nu}$ and $\overline{C_\lambda}$ and let $\overline{l'_j}$ be the fiber on $\overline{X} \cong \mathbf{F}_{d-2\lambda}$ passing through Q'_j for $1 \leq j \leq m$. Then the proper transforms $\overline{l'_j}^*$ of $\overline{l'_j}$ via $\tau : \overline{X} \cdots \to \mathbf{P}^2$ are mutually distinct lines passing through the common point $p = \overline{L}^* \cap \overline{L'}^* = (z_0 = 0) \cap (z_1 = 0) = (0 : 0 : 1)$, hence they are defined by $d'_j z_0 + z_1 = 0$ for some mutually distinct non-zero constants for $1 \leq j \leq m$. Note that each of these intersection point Q'_j is different from the intersection points $Q_1, \ldots, Q_n$ of $\overline{C_\mu}$ and $\overline{C_\lambda}$ because of $\overline{D_2} \cap \overline{D_3} \cap \overline{C_\lambda} = \emptyset$ (cf. Lemma 4.6). Hence d'_j is different from $d_1, \ldots, d_n$. $\square$

We have completed the preparation to determine the detailed form of the irreducible polynomial $f \in \mathbf{C}[x_1, x_2, x_3]$. Note that $\overline{D_2} \cap \overline{D_3} \cap \overline{C_\lambda} = \emptyset$ (cf. Lemma 4.6). We denote by $Q_1, \ldots, Q_\lambda$ (resp. $Q'_1, \ldots, Q'_\lambda$) the intersection points $\overline{D_2} \cap$

$\overline{C_\lambda}$ (resp. $\overline{D_3} \cap \overline{C_\lambda}$) such that $\overline{C_\mu}$ (resp. $\overline{C_\nu}$) passes through $Q_1, \ldots, Q_n$ (resp. $Q'_1, \ldots, Q'_m$) and $\overline{L_i}$ for $1 \le i \le d-\lambda-\mu$ (resp. $\overline{L'_j}$ for $1 \le j \le d-\lambda-\nu$) passes through Q_{i+n} (resp. Q'_{j+m}). Let $\overline{l_i}$ (resp. $\overline{l'_j}$) be the fiber on $\overline{X} \cong \mathbf{F}_{d-2\lambda}$ passing through the point Q_i (resp. Q'_j) and let $\overline{l_i}^*$ (resp. $\overline{l'_j}^*$) be the its proper transform on $\mathbf{P}^2$ via $\tau : \overline{X} \cdots \to \mathbf{P}^2$ for $1 \le i, j \le \lambda$. The $\overline{l_i}^*$ and $\overline{l'_j}^*$ are mutually distinct lines passing through the common point $p = \overline{L}^* \cap \overline{L'}^* = (z_0 = 0) \cap (z_1 = 0) = (0 : 0 : 1)$, and $\overline{l_i}^*, \overline{l'_j}^*$ are defined by $d_i z_0 + z_1 = 0, d'_j z_0 + z_1 = 0$, respectively, for the mutually distinct non-zero constants d_i and d'_j for $1 \le i, j \le \lambda$. Note that $\overline{L_i} = \overline{l_{i+n}}$ for $1 \le i \le d-\lambda-\mu$ and $\overline{L'_j} = \overline{l'_{j+m}}$ for $1 \le j \le d-\lambda-\nu$, hence we have $a_i = d_{i+n}$ and $b'_j = d'_{j+m}$ in the notations of Lemma 4.10. We write $A(z_1) = \alpha_0 z_1^{d-2\lambda} + \alpha_1 z_1^{d-2\lambda-1} + \cdots + \alpha_{d-2\lambda}$ and $B(z_1) = \beta_0 z_1^{d-2\lambda} + \beta_1 z_1^{d-2\lambda-1} + \cdots + \beta_{d-2\lambda}$, where $\alpha_k, \beta_k \in \mathbf{C}$ and $\alpha_0\beta_0 \neq 0$ for $0 \le k \le d-2\lambda$ (cf. Propositions 4.1 and 4.2). Then the birational map $\phi : \mathbf{P}^2 \cdots \to X \subset \mathbf{P}^3$ is defined as follows (cf. Lemmas 4.3, 4.9, 4.10 and Propositions 4.1, 4.2): If $d - 2\lambda > 0$, then

$$\begin{cases} x_0 = z_0^{d-\lambda}, \\ x_1 = z_0^{d-\lambda-1} z_1, \\ x_2 = \left(\displaystyle\prod_{i=1}^{\lambda} (d_i z_0 + z_1)\right) \left(z_0^{d-2\lambda-1} z_2 + \displaystyle\sum_{k=0}^{d-2\lambda} \alpha_k z_0^k z_1^{d-2\lambda-k}\right) \\ \qquad + \displaystyle\sum_{j=1}^{n} c_j \left(\prod_{l=j+1}^{\lambda} (d_l z_0 + z_1)\right) z_0^{j+d-2\lambda}, \\ x_3 = \left(\displaystyle\prod_{i=1}^{\lambda} (d'_i z_0 + z_1)\right) \left(z_0^{d-2\lambda-1} z_2 + \displaystyle\sum_{k=0}^{d-2\lambda} \beta_k z_0^k z_1^{d-2\lambda-k}\right) \\ \qquad + \displaystyle\sum_{j=1}^{m} c'_j \left(\prod_{l=j+1}^{\lambda} (d'_l z_0 + z_1)\right) z_0^{j+d-2\lambda}, \end{cases}$$

on the other hand, if $d - 2\lambda = 0$, then

$$\begin{cases} x_0 = z_0^{\lambda+1}, \\ x_1 = z_0^{\lambda} z_1, \\ x_2 = \left(\displaystyle\prod_{i=1}^{\lambda} (d_i z_0 + z_1)\right) (z_2 + \alpha_0) \\ \qquad + \displaystyle\sum_{j=1}^{n} c_j \left(\prod_{l=j+1}^{\lambda} (d_l z_0 + z_1)\right) z_0^{j+1}, \\ x_3 = \left(\displaystyle\prod_{i=1}^{\lambda} (d'_i z_0 + z_1)\right) (z_2 + \beta_0) \\ \qquad + \displaystyle\sum_{j=1}^{m} c'_j \left(\prod_{l=j+1}^{\lambda} (d'_l z_0 + z_1)\right) z_0^{j+1}. \end{cases}$$

Hence the defining equation $F(x_0, x_1, x_2, x_3)$ of the hypersurface $X \subset \mathbf{P}^3$ is given as follows:

$$\begin{aligned} F(x_0,x_1,x_2,x_3) &= x_0^{d-\lambda-1}\left\{\left(\prod_{i=1}^{\lambda}(d_i'x_0+x_1)\right)x_2-\left(\prod_{i=1}^{\lambda}(d_ix_0+x_1)\right)x_3\right\} \\ &+\left(\prod_{i=1}^{\lambda}(d_ix_0+x_1)\right)\left(\prod_{i=1}^{\lambda}(d_i'x_0+x_1)\right)\left(\sum_{k=0}^{d-2\lambda}\gamma_k x_0^k x_1^{d-2\lambda-k}\right) \\ &-\left(\prod_{i=1}^{\lambda}(d_j'x_0+x_1)\right)\left\{\sum_{j=1}^{n}c_j x_0^{d-2\lambda+j}\left(\prod_{l=j+1}^{\lambda}(d_lx_0+x_1)\right)\right\} \\ &+\left(\prod_{i=1}^{\lambda}(d_jx_0+x_1)\right)\left\{\sum_{j=1}^{m}c_j' x_0^{d-2\lambda+j}\left(\prod_{l=j+1}^{\lambda}(d_l'x_0+x_1)\right)\right\}=0, \end{aligned}$$

where $\gamma_k := \beta_k - \alpha_k$ for $0 \le k \le d-2\lambda$. Since $f = F(1, x_1, x_2, x_3)$, we have the following theorem consequently. Note that $\gamma_0 \neq 0$ because the degree $\deg(f)$ of f is d.

Theorem 4.1. *Let $f \in \mathbf{C}[x_1, x_2, x_3]$ be the irreducible polynomial of degree $d := \deg(f) \ge 2$ in three complex variables x_1, x_2 and x_3. Suppose that the hypersurface $S := (f = 0) \subset \mathbf{A}^3 := \mathrm{Spec}(\mathbf{C}[x_1, x_2, x_3])$ defined by f is isomorphic to the affine plane; $S \cong \mathbf{A}^2$, f satisfies the condition* (†) (*see Section* 1) *and that f is of* $\mathbf{TYPE}(d, \lambda)$ (*cf. Lemma* 3.3) *with $d - 2\lambda \ge 0$. Then f is written in the following fashion*:

$$\begin{aligned} f &= \left(\prod_{i=1}^{\lambda}(x_1+d_i')\right)x_2-\left(\prod_{i=1}^{\lambda}(x_1+d_i)\right)x_3 \\ &+\left(\prod_{i=1}^{\lambda}(x_1+d_i)\right)\left(\prod_{i=1}^{\lambda}(x_1+d_i')\right)\left(\sum_{k=0}^{d-2\lambda}\gamma_k x_1^{d-2\lambda-k}\right) \\ &-\left(\prod_{i=1}^{\lambda}(x_1+d_i')\right)\left\{\sum_{j=1}^{n}c_j\left(\prod_{l=j+1}^{\lambda}(x_1+d_l)\right)\right\} \\ &+\left(\prod_{i=1}^{\lambda}(x_1+d_i)\right)\left\{\sum_{j=1}^{m}c_j'\left(\prod_{l=j+1}^{\lambda}(x_1+d_l')\right)\right\}, \end{aligned}$$

where d_i and d_j' $(1 \le i, j \le \lambda)$ are mutually distinct non-zero constants, $0 \le n, m \le \lambda$, $c_i \in \mathbf{C}$ for $1 \le i \le n$ and $c_j' \in \mathbf{C}$ for $1 \le j \le m$ with $c_n c_m' \neq 0$ and $\gamma_k \in \mathbf{C}$ for $0 \le k \le d - 2\lambda$ with $\gamma_0 \neq 0$.

Remark 6. We can see that the hypersurface $S := (f = 0) \subset \mathbf{A}^3 = \mathrm{Spec}(\mathbf{C}[x_1, x_2, x_3])$ defined by f in Theorem 4.1 is isomorphic to the affine plane; $S \cong \mathbf{A}^2$. First of all we note that S is smooth by Jacobian criterion of smoothness. In order to see that $S \cong \mathbf{A}^2$, we consider the projection $pr : \mathbf{A}^3 \to \mathbf{A}^1 = \mathrm{Spec}(\mathbf{C}[x_1])$ to the x_1-axis and the restriction $p := pr|_S : S \to \mathbf{A}^1$. The fiber $p^*(\alpha)$ over the point $(x_1 = \alpha) \in \mathbf{A}^1$ is isomorphic to the affine plane curve $p^*(\alpha) \cong (f(\alpha, x_2, x_3) = 0) \subset \mathbf{A}^2 = \mathrm{Spec}(\mathbf{C}[x_2, x_3])$. Since d_i and d'_j $(1 \leq i, j \leq \lambda)$ are mutually distinct non-zero constants, $p^*(\alpha)$ is isomorphic to the affine line; $p^*(\alpha) \cong \mathbf{A}^1$ for every $\alpha \in \mathbf{C}$. Hence all the fibers of $p : S \to \mathbf{A}^1$ are isomorphic to the affine line. By [Kam-Miy78] and the fact $\mathrm{Pic}(\mathbf{A}^1) = 0$, it follows that p is a trivial $\mathbf{A}^1$-bundle structure over the base curve $\mathbf{A}^1$, so we have $S \cong \mathbf{A}^1 \times \mathbf{A}^1 \cong \mathbf{A}^2$. It is easy to see that the closure X of S in $\mathbf{P}^3$ satisfies the condition (†), i.e., $L := X \cap H_0$ is a line in $\mathbf{P}^3$ and $\mathrm{mult}_L X = d - 1$. Therefore f is a variable of $\mathbf{C}[x_1, x_2, x_3]$ by Theorem 1.1. In the former time, we do not know how to construct an automorphism of $\mathbf{C}[x_1, x_2, x_3]$ sending f to a standard variable. But Professor Joost Berson told me simple method to construct such an automorphism in the following fashion: By Theorem 4.1, a polynomial f is of the form $f = a(x_1)x_2 + b(x_1)x_3 + p(x_1)$, where $a(x_1)$ and $b(x_1)$ have no common roots. Then $\mathbf{C}[x_1]a(x_1) + \mathbf{C}[x_1]b(x_1) = \mathbf{C}[x_1]$, so we have $c(x_1)a(x_1) + d(x_1)b(x_1) = 1$ for some $c(x_1), d(x_1) \in \mathbf{C}[x_1]$. Then we can easily see that $\mathbf{C}[f, -c(x_1)x_2 + d(x_1)x_3, x_1] = \mathbf{C}[x_1, x_2, x_3]$, hence the automorphsim $x_1 \mapsto x_1,\ x_2 \mapsto -c(x_1)x_2 + d(x_1)x_3,\ x_3 \mapsto f$ is a desired one.

Remark 7. When $d = \deg(f) \leq 4$, we have $d - 2\lambda \geq 0$ for all $0 \leq \lambda \leq d - 2$. Hence, we can determine all the standard forms of the irreducible polynomials $f \in \mathbf{C}[x_1, x_2, x_3]$ defining the hypersurfaces which are isomorphic to the affine plane $\mathbf{A}^2$ and satisfying (†). For instance, if f is of degree $d = 3$, then f is either of **TYPE**(3,0) or of **TYPE**(3,1). If f is of **TYPE**(3,0) (resp. **TYPE**(3,1)), then f coincides with the polynomial in [Oh99, Theorem 1] of type (VII) (resp. (VIII)) up to an affine transformation.

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