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The topology of spaces of maps between real projective spaces

By

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1. Introduction

Let $O(k)$ be the group of orthogonal $(k \times k)$ matrices. For connected spaces X and Y, let $\mathrm{Map}(X, Y)$ denote the space consisting of all continuous maps $f : X \to Y$ with compact-open topology. For $m \leq n$, we define the inclusion map $i_{m,n} : \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$ by

$$i_{m,n}([z_0 : z_1 : \cdots : z_m]) = [z_0 : z_1 : \cdots : z_m : 0 : 0 : \cdots : 0].$$

We denote by $\mathrm{Map}_1(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ the path component of $\mathrm{Map}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ containing $i_{m,n}$. Define the map $\tilde{s}_{m,n} : O(n+1) \to \mathrm{Map}_1(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ by the matrix multiplication

$$\tilde{s}_{m,n}(A)([z_0 : \cdots : z_m]) = [z_0 : \cdots : z_m : 0 : 0 : \cdots : 0] \cdot A.$$

Let Δ_k denote the center of $O(k)$ given by $\Delta_k = \{\epsilon E_k : \epsilon = \pm 1\} \cong \mathbb{Z}/2$, where E_k denotes the $(k \times k)$ identity matrix. Since $\tilde{s}_{m,n}$ is constant on the subgroup $\Delta_{m+1} \times O(n-m) \subset O(n+1)$, it induces the map

$$s_{m,n} : \mathrm{PV}_{n+1,m+1} \to \mathrm{Map}_1(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n),$$

where let $\mathrm{PV}_{n+1,m}$ denote the real projective Stiefel manifold of orthogonal $(m+1)$-frames in $\mathbb{R}^{n+1}$ given by $\mathrm{PV}_{n+1,m+1} = (\Delta_{m+1} \times O(n-m)) \backslash O(n+1)$.

The main purpose of this paper is to prove the following result.

Theorem 1.1. *If $m \leq n$, $s_{m,n} : \mathrm{PV}_{m+1,m+1} \to \mathrm{Map}_1(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ is a homotopy equivalence up to dimension $D(m, n) = 2(n-m) - 1$.*

Remark 1. A map $f : X \to Y$ is called *a homotopy equivalence up to dimension N* if the induced homomorphism $f_* : \pi_i(X) \to \pi_i(Y)$ is bijective when $i < N$ and surjective when $i = N$.

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A similar result for $\mathbb{K}\mathrm{P}^n$ was first obtained in [2] (cf. [1]) for the case $\mathbb{K} = \mathbb{C}$ and in this paper we shall treat the case $\mathbb{K} = \mathbb{R}$. However, a partial result only holds for the case $\mathbb{K} = \mathbb{H}$ (cf. [3]) because the quaternion field $\mathbb{H}$ is not commutative.

2. Reduction of the proof

The inclusion map $i = i_{m-1,m} : \mathbb{R}\mathrm{P}^{m-1} \to \mathbb{R}\mathrm{P}^m$ induces the fibration

$$i^* : \mathrm{Map}_1(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \to \mathrm{Map}_1(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{R}\mathrm{P}^n)$$

given by $i^*(f) = f \circ i$ with fiber F_m, where the space F_m is defined by

$$F_m = \{f \in \mathrm{Map}_1(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) : f \circ i = i_{m-1,n}\}.$$

We remark that the homeomorphism

$$\phi_{m,n} : \frac{\Delta_m \times O(n-m+1)}{\Delta_{m+1} \times O(n-m)} \cong \frac{O(n-m+1)}{O(n-m)} \xrightarrow{\cong} S^{n-m}$$

is induced from the map $O(n-m+1) \to S^{n-m}$ given by

$$A = \begin{pmatrix} z_0 & z_1 & \cdots & z_{n-m} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,n-m} \\ \vdots & \ddots & \ddots & \vdots \\ a_{n-m,0} & a_{n-m,1} & \cdots & a_{n-m,n-m} \end{pmatrix} \mapsto (z_0, \dots, z_{n-m}). \tag{2.0}$$

Then we have the commutative diagram

$$\begin{array}{ccccc} S^{n-m} & \longrightarrow & \mathrm{PV}_{n+1,m+1} & \longrightarrow & \mathrm{PV}_{n+1,m} \\ {\scriptstyle s_m}\downarrow & & {\scriptstyle s_{m,n}}\downarrow & & {\scriptstyle s_{m-1,n}}\downarrow \\ F_m & \longrightarrow & \mathrm{Map}_1(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) & \xrightarrow{i^*} & \mathrm{Map}_1(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{R}\mathrm{P}^n) \end{array} \tag{†}$$

where two horizontal sequences are fibration sequences.

We identify $\mathbb{R}\mathrm{P}^k = \mathbb{R}\mathrm{P}^{k-1} \cup_{\gamma_{k-1}} e^k$, where $\gamma_k : S^k \to \mathbb{R}\mathrm{P}^k$ denotes the usual Hopf fibering. Let $\mu : \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^m \vee S^m$ denote the co-action map given by pinching the hemisphere of the top cell e^m. Then consider a pairing $P : F_m \times \Omega^m \mathbb{R}\mathrm{P}^n \to F_m$ defined by $P(f, \omega) = \nabla \circ (f \vee \omega) \circ \mu$ for $(f, \omega) \in F_m \times \Omega^m \mathbb{R}\mathrm{P}^m$, where $\nabla : \mathbb{R}\mathrm{P}^m \vee \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^m$ denotes a folding map. It also induces a homotopy equivalence $F_m \simeq \Omega^m \mathbb{R}\mathrm{P}^n$ by multiplying by the element $i_{m,n}$. Because there is a homotopy equivalence $\Omega^m \mathbb{R}\mathrm{P}^n \simeq \Omega^m S^n$, we obtain the homotopy commutative diagram

$$\begin{array}{ccccc} S^{n-m} & \longrightarrow & \mathrm{PV}_{n+1,m+1} & \longrightarrow & \mathrm{PV}_{n+1,m} \\ {\scriptstyle s'_m}\downarrow & & {\scriptstyle s_{m,n}}\downarrow & & {\scriptstyle s_{m-1,n}}\downarrow \\ \Omega^m S^n & \longrightarrow & \mathrm{Map}_1(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) & \xrightarrow{i^*} & \mathrm{Map}_1(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{R}\mathrm{P}^n) \end{array} \tag{‡}$$

where two horizontal sequences are fibration sequences.

Now we introduce the following result.

Lemma 2.1. $(s'_m)_* : \pi_{n-m}(S^{n-m}) \xrightarrow{\cong} \pi_{n-m}(\Omega^m S^n) \cong \mathbb{Z}$ *is an isomorphism.*

We postpone the proof of Lemma 2.1 to the next section and complete the proof of Theorem 1.1.

Proof of Theorem 1.1. The proof of Theorem 1.1 is by induction on m keeping n fixed. If $m = 0$, the map $s_{m,n}$ is a homeomorphism and the assertion clearly holds. Suppose $n \geq m \geq 1$ and that the assertion is true for $m-1$. Consider the homotopy exact sequences induced from (‡):

$$\begin{array}{ccccccc}
\to \pi_k(S^{n-m}) & \longrightarrow & \pi_k(\mathrm{PV}_{n+1,m+1}) & \longrightarrow & \pi_k(\mathrm{PV}_{n+1,m}) & \xrightarrow{\partial'} \\
{\scriptstyle (s'_m)_*}\big\downarrow & & {\scriptstyle (s_{m,n})_*}\big\downarrow & & {\scriptstyle (s_{m-1,n})_*}\big\downarrow & \\
\to \pi_k(\Omega^m S^n) & \longrightarrow & \pi_k(\mathrm{Map}_1(\mathbb{RP}^m, \mathbb{RP}^n)) & \xrightarrow{i^*} & \pi_k(\mathrm{Map}_1(\mathbb{RP}^{m-1}, \mathbb{RP}^n)) & \xrightarrow{\partial}
\end{array}$$

It follows from Lemma 2.1 that s'_m is identified with the m-fold suspension $E^m : S^{n-m} \to \Omega^m S^n$ (up to homotopy equivalence). Hence, s'_m is a homotopy equivalence up to dimension $D(m,n) = 2(n-m)-1$. Then the assertion easily follows from the Five Lemma. This completes the proof of Theorem 1.1. □

3. Proof of Lemma 2.1

In this section we prove Lemma 2.1. Since we can identify s_m with s'_m up to homotopy equivalence, it is sufficient to show the following result.

Lemma 3.1. $(s_m)_* : \pi_{n-m}(S^{n-m}) \xrightarrow{\cong} \pi_{n-m}(F_m) \cong \mathbb{Z}$ *is an isomorphism.*

Proof. We note that the map $s_m : S^{n-m} \to F_m$ is the composite of maps

$$S^{n-m} \underset{\cong}{\xleftarrow{\phi_{m,n}}} \frac{\Delta_m \times O(n-m+1)}{\Delta_{m+1} \times O(n-m)} \xrightarrow{j} \frac{O(n+1)}{\Delta_{m+1} \times O(n-m)} \xrightarrow{s_{m,n}} \mathrm{Map}_1(\mathbb{RP}^m, \mathbb{RP}^n),$$

where j denotes the natural inclusion map induced from the inclusion map $\Delta_m \times O(n-m+1) \subset O(n+1)$. Since

$$[w_0 : \cdots : w_m : 0 : \cdots : 0] \cdot \begin{pmatrix} E_m & O \\ O & A \end{pmatrix} = [w_0 : \cdots : w_{m-1} : w_m z_0 : w_m z_1 : \cdots w_m z_{n-m}]$$

for $(z_0, z_1, \ldots, z_{n-m}) \in S^{n-m}$ and

$$A = \begin{pmatrix} z_0 & z_1 & \cdots & z_{n-m} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,n-m} \\ \vdots & \ddots & \ddots & \vdots \\ a_{n-m,0} & a_{n-m,1} & \cdots & a_{n-m,n-m} \end{pmatrix},$$

it follows from (2.0) that the map $S^{n-m} \stackrel{s_m}{\to} F_m \subset \mathrm{Map}_1(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ corresponds to a map $\phi : \mathbb{R}\mathrm{P}^m \times S^{n-m} \to \mathbb{R}\mathrm{P}^n$ given by

$$\phi(\mathbf{w}, \mathbf{z}) = [w_0 : \cdots : w_{m-1} : w_m z_0 : w_m z_1 : \cdots : w_m z_{n-m}]$$

for $(\mathbf{w}, \mathbf{z}) = ([w_0 : \cdots : w_m], (z_0, \ldots, z_{n-m})) \in \mathbb{R}\mathrm{P}^m \times S^{n-m}$.

Let $\epsilon : S^{n-m} \to \mathrm{Map}_1(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ be the constant map at $i_{m,n}$, which is defined by $\epsilon(\mathbf{w}) = i_{m,n}$ for any $\mathbf{w} \in \mathbb{R}\mathrm{P}^m$. Then the map ϵ corresponds to a map $\psi : \mathbb{R}\mathrm{P}^m \times S^{n-m} \to \mathbb{R}\mathrm{P}^n$ given by

$$\psi(\mathbf{w}, \mathbf{z}) = [w_0 : \cdots : w_{m-1} : w_m : 0 : \cdots : 0]$$

for $(\mathbf{w}, \mathbf{z}) = ([w_0 : \cdots : w_m], (z_0, \ldots, z_{n-m})) \in \mathbb{R}\mathrm{P}^m \times S^{n-m}$.

Then two maps ϕ and ψ agree on the subspace

$$(\mathbb{R}\mathrm{P}^{m-1} \times S^{n-m}) \cup (\mathbb{R}\mathrm{P}^m \times (1, 0, \ldots, 0))$$

and we would like to study the difference element between them. For this purpose, it is sufficient to replace the pair $(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^{m-1})$ by the pair (D^m, S^{m-1}) (using a characteristic map of the top cell in $\mathbb{R}\mathrm{P}^m$) and to find the difference element between the two resulting maps on $D^m \times S^{n-m}$.

From now on, we embed $D^m \times S^{n-m}$ in $S^n = \partial(D^m \times D^{n-m+1})$ in the usual way. Let $\mathbf{w} = (w_0, \ldots, w_{m-1})$ and $\mathbf{z} = (z_0, \ldots, z_{n-m})$ run over S^m and S^{n-m}, respectively. Then the points $((\sin\theta)\mathbf{w}, (\cos\theta)\mathbf{z})$ runs over S^n $(0 \le \theta \le \pi/2)$, and the space $D^m \times S^{n-m}$ may be regarded as the subset

$$\{((\sin\theta)\mathbf{w}, (\cos\theta)\mathbf{z}) : \mathbf{w} \in S^m, \mathbf{z} \in S^{n-m}, 0 \le \theta \le 4/\pi\}.$$

We define the map $\overline{\gamma}_m : D^m \to \mathbb{R}\mathrm{P}^m$ by

$$\overline{\gamma}_m((\sin\theta)\mathbf{w}) = ((\sin 2\theta)\mathbf{w}, \cos\theta) \qquad (0 \le \theta \le 4/\pi).$$

Then $\overline{\gamma}_m$ represents the characteristic map of the top cell e^m in $\mathbb{R}\mathrm{P}^m$. In this case, the corresponding two maps $\phi', \psi' : D^m \times S^{n-m} \to \mathbb{R}\mathrm{P}^m$ are given by

$$\begin{cases} \phi'((\sin\theta)\mathbf{w}, (\cos\theta)\mathbf{z}) = [(\sin 2\theta)\mathbf{w} : (\cos 2\theta)\mathbf{z}] \\ \psi'((\sin\theta)\mathbf{w}, (\cos\theta)\mathbf{z}) = [(\sin 2\theta)\mathbf{w} : (\cos 2\theta)(1, 0, \ldots, 0)] \end{cases}$$

for $0 \le \theta \le \pi/4$, $(\mathbf{w}, \mathbf{z}) \in S^m \times S^{n-m}$.

Two maps ϕ' and ψ' agree on $(S^{m-1} \times S^{n-m+1}) \cup (D^m \times (1, 0, \ldots, 0))$ and we wish to know the difference element between them. For this purpose, we extend ϕ' and ψ' over $S^{m-1} \times D^{n-m+1}$ by

$$\phi'((\sin\theta)\mathbf{w}, (\cos\theta)\mathbf{z}) = \psi'((\sin\theta)\mathbf{w}, (\cos\theta)\mathbf{z}) = [\mathbf{w} : \mathbf{0}] \qquad (\pi/4 \le \theta \le \pi/2).$$

Now two maps ϕ' and ψ' agree on $(S^{m-1} \times D^{n-m+1}) \cup (D^m \times (1, 0, \ldots, 0))$, which is a contractible space. Hence their difference element is

$$[\phi'] - [\psi'] \in \pi_n(\mathbb{R}\mathrm{P}^n).$$

However, since ψ' factors through D^n, $[\psi'] = 0$. Thus, the required difference element is $[\phi'] \in \pi_n(\mathbb{R}\mathrm{P}^n)$. Now define the map $\phi'' : S^n \to S^n$ by

$$\phi''((\sin\theta)\mathbf{w}, (\cos\theta)\mathbf{z}) = ((\sin\theta)\mathbf{w}, (\cos\omega(\theta))\mathbf{z}), \quad \text{where}$$

$$\omega(\theta) = \begin{cases} 2\theta & \text{if} \quad 0 \le \theta \le \pi/4 \\ \pi/2 & \text{if} \quad \pi/4 \le \theta \le \pi/2. \end{cases}$$

Then ϕ'' is a lifting of ϕ' to S^n such that $\gamma_n \circ \phi'' = \phi'$. Because the function $\omega(\theta)$ is homotopic to the identity keeping 0 and $\pi/2$ fixed. Hence $\phi'' \simeq \mathrm{id} : S^n \to S^n$. So the required difference element is $[\phi'] = [\gamma_n]$, and it is a generator of $\pi_n(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}$. Because ϕ' and ψ' correspond to s_m and ϵ, respectively, $[s_m] \in \pi_{n-m}(F_m) \cong \mathbb{Z}$ is a generator. □

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