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Comparison theorems for eigenvalues of one-dimensional Schrödinger operators

By

Min-Jei Huang

Abstract

The Schrödinger operator $H = -d^2/dx^2 + V(x)$ on an interval $[0, a]$ with Dirichlet or Neumann boundary conditions has discrete spectrum $E_1[V] < E_2[V] < E_3[V] < \cdots$, for bounded V. In this paper, we apply the perturbation theory of discrete eigenvalues to obtain upper bounds for $\sum_{j=1}^k E_j[V]$, where k is any positive integer. Our results include the following:

(i) $\sum_{j=1}^k E_j[V] \le \sum_{j=1}^k E_j[V_s]$, where $V_s(x) = [V(x) + V(a-x)]/2$, with equality if and only if V is symmetric about $x = a/2$.

(ii) If V is convex, then the Dirichlet eigenvalues satisfy

$$\sum_{j=1}^k E_j[V] \le \sum_{j=1}^k E_j[0] + \frac{k}{a}\int_0^a V(x)dx$$

with equality if and only if V is constant.

(iii) If V is concave, then the Neumann eigenvalues satisfy

$$\sum_{j=1}^k E_j[V] \le \sum_{j=1}^k E_j[0] + \frac{k}{a}\int_0^a V(x)dx$$

with equality if and only if V is constant.

1. The basic theorem

Let Ω be a region in the complex plane, and for each $z \in \Omega$, let $T(z)$ be a closed operator with nonempty resolvent set. $\{T(z)\}$ is called an analytic family of type (A) if the operator domain of $T(z)$ is some set $\mathcal{D}$ independent of z, and for each $\varphi \in \mathcal{D}$, $T(z)\varphi$ is a vector-valued analytic function of z ([1], [3]). Suppose that $\{T(z)\}$ is an analytic family of type (A) in Ω. The Kato-Rellich theorem ([3]) asserts that if $z_0 \in \Omega$ and if $E(z_0)$ is an isolated nondegenerate eigenvalue of $T(z_0)$, then, for z near z_0, there is a unique point $E(z)$ in the

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spectrum of $T(z)$ near $E(z_0)$ which is an isolated nondegenerate eigenvalue. Moreover, $E(z)$ is analytic near $z = z_0$, and there is an analytic eigenvector $u(z)$ near $z = z_0$.

We now consider the eigenvalue problem for one-dimensional Schrödinger operators. Let $V(x)$ be a bounded real-valued function on the interval $[0, a]$, and let H be the selfadjoint operator on $L^2([0, a])$ given by $-d^2/dx^2 + V(x)$ with Dirichlet or Neumann boundary conditions. As we know, H has discrete spectrum

$$E_1 < E_2 < E_3 < \cdots$$

with corresponding normalized eigenfunctions $u_1(x)$, $u_2(x)$, $u_3(x), \dots$. Also, the $u_j(x)$ can be chosen so as to be real-valued and to form a complete orthonormal basis for $L^2([0, a])$.

In this paper, we shall apply the perturbation theory of discrete eigenvalues to obtain upper bounds for $\sum_{j=1}^{k} E_j$, the sum of the k lowest eigenvalues of H, where k is any positive integer. To apply the idea of this theory to eigenvalues, let $V(\cdot, t)$, $t \in \mathbb{R}$, be a one-parameter family of bounded potentials, and consider the selfadjoint operator $H(t) = -d^2/dx^2 + V(x, t)$ on $L^2([0, a])$ with Dirichlet or Neumann boundary conditions. We assume that $H(t)$ has an analytic continuation to a region Ω so that $\{H(z)\}$ is an analytic family of type (A) in Ω. If $E_j(t)$ is the jth eigenvalue of $H(t)$, there is a simple formula for the derivative of $E_j(t)$:

$$\frac{d}{dt} E_j(t) = \int_0^a \frac{\partial V}{\partial t}(x, t) u_j^2(x, t) dx, \tag{1}$$

where $u_j(x, t)$ is the normalized eigenfunction corresponding to the eigenvalue $E_j(t)$. Here we note the following basic formula for the second derivative of $E_j(t)$.

Theorem 1 (the second-order perturbation formula)**.**

$$\frac{d^2}{dt^2} E_j(t) = \int_0^a \frac{\partial^2 V}{\partial t^2}(x, t) u_j^2(x, t) dx + 2 \sum_{n=1,\, n \neq j}^{\infty} \frac{1}{E_j(t) - E_n(t)} \left[\int_0^a \frac{\partial V}{\partial t}(x, t) u_j(x, t) u_n(x, t) dx \right]^2 .$$

Proof. See, for example, [2, Chapter 17] or [3, Chapter XII]. □

The following result, which is an important consequence of Theorem 1, plays a major role in the next section.

Theorem 2. *If* $(\partial^2 V/\partial t^2)(x, t) \leq 0$, *then*

$$\frac{d^2}{dt^2}(E_1 + E_2 + \cdots + E_k)(t) \leq 0 \qquad \textit{for any} \quad k \geq 1.$$

Proof. Since $(\partial^2 V/\partial t^2)(x,t) \leq 0$, we have from Theorem 1 that

$$\frac{d^2}{dt^2}E_j(t) \leq 2\sum_{n\neq j}\frac{1}{E_j(t)-E_n(t)}A_{j,n}^2(t),$$

where $A_{j,n}(t) = \int_0^a (\partial V/\partial t)(x,t)u_j(x,t)u_n(x,t)dx$. It follows that

$$\begin{aligned}\frac{d^2}{dt^2}(E_1+E_2+\cdots+E_k)(t) &\leq 2\sum_{j=1}^{k}\sum_{n\neq j}\frac{1}{E_j(t)-E_n(t)}A_{j,n}^2(t)\\ &= 2\sum_{j=1}^{k}\sum_{n=k+1}^{\infty}\frac{1}{E_j(t)-E_n(t)}A_{j,n}^2(t)\\ &\leq 0,\end{aligned}\tag{2}$$

where we have used the fact that $A_{j,n}(t) = A_{n,j}(t)$ in the second step. □

Theorem 2 indicates that the concavity of $\sum_{j=1}^{k} E_j(t)$ is connected with the concavity of $V(x,t)$ with respect to t. In fact, there is a natural way of approaching this connection based on the min-max principle and basic facts about concave functions. For the linear case $V(x,t) = tV(x)$, it was shown in [4] (pp. 153–154) that $\sum_{j=1}^{k} E_j(t)$ is a concave function of t for any $k \geq 1$. Here we prove a theorem that is a generalization of this result.

Theorem 3. *If $V(x,t)$ is concave with respect to t, then, for any $k \geq 1$, $\sum_{j=1}^{k} E_j(t)$ is a concave function of t.*

Proof. By the min-max principle ([4, p. 152]),

$$\sum_{j=1}^{k} E_j(t) = \inf_{\{\varphi_1,\ldots,\varphi_k\}}\sum_{j=1}^{k}\langle\varphi_j, H(t)\varphi_j\rangle,\tag{3}$$

where the infimum is taken over all orthonormal systems $\{\varphi_1,\ldots,\varphi_k\}$ in $\mathcal{D} \equiv \mathcal{D}(H(t))$, the domain of $H(t)$. For simplicity of notation, write $H(t) = H_0 + V(t)$. Then, by the concavity of $V(t)$, we have

$$V\left(\sum_{i=1}^{n}\alpha_i t_i\right) \geq \sum_{i=1}^{n}\alpha_i V(t_i)$$

for all $\alpha_i \geq 0$ with $\sum_{i=1}^{n}\alpha_i = 1$. So,

$$\left\langle\varphi, H\left(\sum_{i=1}^{n}\alpha_i t_i\right)\varphi\right\rangle \geq \sum_{i=1}^{n}\alpha_i\langle\varphi, H(t_i)\varphi\rangle$$

for all $\varphi \in \mathcal{D}$. This implies that $\sum_{j=1}^{k}\langle\varphi_j, H(t)\varphi_j\rangle$ is a concave function of t. Since the infimum of any collection of concave functions is concave, we conclude that $\sum_{j=1}^{k} E_j(t)$ is concave. □

2. Applications

For a bounded potential V on $[0, a]$, we denote by $E_j[V]$ the jth eigenvalue of the selfadjoint operator $-d^2/dx^2 + V(x)$ on $L^2([0, a])$ with Dirichlet [Neumann] boundary conditions.

We begin with a comparison theorem.

Theorem 4. *If $V(x)$ is a bounded potential on $[0, a]$, then, for any $k \geq 1$,*

$$\sum_{j=1}^{k} E_j[V] \leq \sum_{j=1}^{k} E_j[V_s], \tag{4}$$

where $V_s(x) = [V(x) + V(a-x)]/2$. The equality holds only if $V = V_s$; i.e., only if V is symmetric about $x = a/2$.

Proof. Consider the one-parameter family of potentials: $V(x,t) = tV(x) + (1-t)V_s(x)$. By (1), we have

$$\sum_{j=1}^{k} E_j'(t) = \frac{1}{2} \sum_{j=1}^{k} \int_0^a [V(x) - V(a-x)] u_j^2(x,t) dx.$$

Note that the potential $V_s(x)$ is symmetric about $x = a/2$ with corresponding normalized eigenfunctions $u_j(x,0)$, $j = 1, 2, \ldots$. So,

$$u_{2j-1}(x,0) = u_{2j-1}(a-x,0) \quad \text{and} \quad u_{2j}(x,0) = -u_{2j}(a-x,0).$$

Thus, for each j, $u_j^2(x,0)$ is symmetric about $x = a/2$. On the other hand, the potential $V(x) - V(a-x)$ is antisymmetric about $x = a/2$. It follows that

$$\sum_{j=1}^{k} E_j'(0) = \frac{1}{2} \sum_{j=1}^{k} \int_0^a [V(x) - V(a-x)] u_j^2(x,0) dx = 0.$$

Since $(\partial^2 V/\partial t^2)(x,t) = 0$, we have by Theorem 2 that $\sum_{j=1}^{k} E_j''(t) \leq 0$. Thus, for any $t \geq 0$, we have

$$\sum_{j=1}^{k} E_j'(t) \leq \sum_{j=1}^{k} E_j'(0) = 0.$$

This implies that

$$\sum_{j=1}^{k} E_j[V] = \sum_{j=1}^{k} E_j(1) \leq \sum_{j=1}^{k} E_j(0) = \sum_{j=1}^{k} E_j[V_s].$$

Finally, if the equality holds in (4), then $\sum_{j=1}^{k} E_j(t)$ is constant for $0 \leq t \leq 1$ so that $\sum_{j=1}^{k} E_j''(t) = 0$ for $0 \leq t \leq 1$. Now taking $t = 0$ and using (2), we see that

$$A_{j,n}(0) = \frac{1}{2} \int_0^a [V(x) - V(a-x)] u_j(x,0) u_n(x,0) dx = 0$$

for $1 \le j \le k$ and $n \ge k+1$. Thus, writing $f(x) = [V(x) - V(a-x)]/2$, we have, for $1 \le j \le k$,

$$\begin{aligned} f(x)u_j(x,0) &= \sum_{n=1}^{k} \left[\int_0^a f(x)u_j(x,0)u_n(x,0)dx \right] u_n(x,0) \\ &= \sum_{n=1}^{k} A_{j,n}(0)u_n(x,0). \end{aligned}$$

Since $u_1(x,0)$ has no zeros in the open interval $(0,a)$, we see that $f(x)$ is continuous on $(0,a)$. Moreover, for each $x \in (0,a)$, $f(x)$ is an eigenvalue of the $k \times k$ matrix $[A_{j,n}(0)]$. It follows that $f(x)$ must be constant. Thus, $f(x) = f(a/2) = 0$. This shows that $V = V_s$. $\square$

We remark that there is an alternative proof of the inequality (4) based on the min-max principle rather than the second-order perturbation formula and antisymmetry. In fact, by (3), the sum of the k lowest Dirichlet [Neumann] eigenvalues for any potential V is given by

$$\sum_{j=1}^{k} E_j[V] = \inf_{\{\varphi_1,\dots,\varphi_k\}} \sum_{j=1}^{k} \int_0^a [|\varphi_j'(x)|^2 + V(x)|\varphi_j(x)|^2]dx,$$

where the infimum is taken over all functions $\varphi_1, \dots, \varphi_k \in C^1$ which satisfy $\langle \varphi_i, \varphi_j \rangle = \delta_{i,j}$ and the Dirichlet [Neumann] boundary conditions. Thus,

$$\begin{aligned} \sum_{j=1}^{k} E_j[V_s] &= \inf_{\{\varphi_1,\dots,\varphi_k\}} \sum_{j=1}^{k} \int_0^a \left[|\varphi_j'(x)|^2 + \frac{1}{2}V(x)|\varphi_j(x)|^2 \right. \\ &\qquad \left. + \frac{1}{2}V(a-x)|\varphi_j(x)|^2 \right] dx \\ &\ge \frac{1}{2} \inf_{\{\varphi_1,\dots,\varphi_k\}} \sum_{j=1}^{k} \int_0^a [|\varphi_j'(x)|^2 + V(x)|\varphi_j(x)|^2]dx \\ &\quad + \frac{1}{2} \inf_{\{\varphi_1,\dots,\varphi_k\}} \sum_{j=1}^{k} \int_0^a [|\varphi_j'(x)|^2 + V(a-x)|\varphi_j(x)|^2]dx \\ &= \frac{1}{2}\sum_{j=1}^{k} E_j[V] + \frac{1}{2}\sum_{j=1}^{k} E_j[V] \\ &= \sum_{j=1}^{k} E_j[V] \end{aligned}$$

since the Dirichlet [Neumann] eigenvalues of $-d^2/dx^2 + V(a-x)$ are the same as those of $-d^2/dx^2 + V(x)$.

As an immediate corollary of Theorem 4, we have

Corollary 5. *If $V(x)$ is a concave potential on $[0,a]$, then, for any $k \geq 1$,*

$$\sum_{j=1}^{k} E_j[V] \leq \sum_{j=1}^{k} E_j[0] + kV(a/2) \tag{5}$$

with equality if and only if V is constant.

Proof. If $V(x)$ is concave on $[0,a]$, we have $V_s(x) \leq V(a/2)$ for all $x \in [0,a]$ so that

$$\sum_{j=1}^{k} E_j[V] \leq \sum_{j=1}^{k} E_j[V_s] \leq \sum_{j=1}^{k} E_j[V(a/2)] = \sum_{j=1}^{k} E_j[0] + kV(a/2).$$

From this and Theorem 4, it follows that the equality occurs in (5) if and only if $V(x) = V_s(x) = V(a/2)$. This proves the corollary. □

Remark 1. The eigenvalues $E_j[0]$ for the zero potential are well-known. In the Dirichlet case, $E_j[0] = j^2\pi^2/a^2$. In the Neumann case, $E_j[0] = (j-1)^2\pi^2/a^2$.

Remark 2. An improvement of Corollary 5 in the Neumann case will be given in Theorem 9.

Now, for bounded V, we consider the one-parameter family of potentials: $V(x,t) = tV(x)$. Then, by Theorem 2, we have $\sum_{j=1}^{k} E_j''(t) \leq 0$ so that

$$\sum_{j=1}^{k} E_j(t) \leq \sum_{j=1}^{k} E_j(0) + t\sum_{j=1}^{k} E_j'(0)$$

for all $t \geq 0$. In particular, taking $t = 1$, we get

$$\sum_{j=1}^{k} E_j[V] \leq \sum_{j=1}^{k} E_j[0] + \sum_{j=1}^{k} \int_0^a V(x)u_j^2(x,0)dx. \tag{6}$$

Here the normalized eigenfunctions $u_j(x,0)$ for the zero potential can, for example, be taken as

$$u_j(x,0) = \sqrt{2/a}\sin(j\pi x/a) \tag{7}$$

in the Dirichlet case; and

$$u_j(x,0) = \begin{cases} \sqrt{1/a} & \text{for} \quad j = 1, \\ \sqrt{2/a}\cos[(j-1)\pi x/a] & \text{for} \quad j \geq 2 \end{cases} \tag{8}$$

in the Neumann case.

In the remainder of this section, we shall give two applications of the inequality (6). We first note a useful fact.

Lemma 6. *If $V(x)$ is a convex potential on $[0,a]$, then*

$$2\int_0^a V(x)\sin^2\left(\frac{j\pi x}{a}\right)dx \le \int_0^a V(x)dx \tag{9}$$

for $j = 1, 2, 3, \ldots$.

Proof. We suppose first that V is differentiable. Then an integration by parts gives

$$\begin{aligned}2\int_0^a V(x)\sin^2\left(\frac{j\pi x}{a}\right)dx - \int_0^a V(x)dx &= -\int_0^a V(x)\cos\left(\frac{2j\pi x}{a}\right)dx\\ &= \frac{a}{2j\pi}\int_0^a V'(x)\sin\left(\frac{2j\pi x}{a}\right)dx.\end{aligned}$$

Since V is convex, V' is monotone increasing on $[0,a]$. Thus,

$$\begin{aligned}\int_0^a V'(x)\sin\left(\frac{2j\pi x}{a}\right)dx &= \sum_{n=0}^{j-1}\int_{na/j}^{(n+1)a/j} V'(x)\sin\left(\frac{2j\pi x}{a}\right)dx\\ &\le \sum_{n=0}^{j-1} V'\left(\frac{(2n+1)a}{2j}\right)\int_{na/j}^{(n+1)a/j}\sin\left(\frac{2j\pi x}{a}\right)dx\\ &= \sum_{n=0}^{j-1} V'\left(\frac{(2n+1)a}{2j}\right)\cdot 0\\ &= 0\end{aligned}$$

and (9) follows.

To prove (9) without the assumption that V is differentiable, we introduce the approximate identity $\{\eta_\varepsilon(x)\}$. Let $\eta(x)$ be any positive, infinitely differentiable function with support in $(-1,1)$ so that $\int_{-\infty}^{\infty}\eta(x)dx = 1$. Define $\eta_\varepsilon(x) = \varepsilon^{-1}\eta(x/\varepsilon)$ for $\varepsilon > 0$. Now, let $\tilde V(x)$ be any continuous extension of $V(x)$ to the whole of $(-\infty,\infty)$, and set

$$V_\varepsilon(x) = \int_{-\infty}^{\infty}\eta_\varepsilon(x-t)\tilde V(t)dt.$$

Then

$$\begin{aligned}|V_\varepsilon(x) - V(x)| &\le \int_{-\infty}^{\infty}\eta_\varepsilon(x-t)|\tilde V(t) - \tilde V(x)|dt\\ &\le \left(\sup_{\{t/|x-t|\le\varepsilon\}}|\tilde V(t)-\tilde V(x)|\right)\int_{-\infty}^{\infty}\eta_\varepsilon(x-t)dt\\ &= \sup_{\{t/|x-t|\le\varepsilon\}}|\tilde V(t)-\tilde V(x)|\end{aligned}$$

so $V_\varepsilon \to V$ uniformly on $[0,a]$. Also, if $x, y \in [\delta, a-\delta] \subset (0,a)$ and if $\varepsilon < \delta$, then, by the convexity of V on $[0,a]$, we have

$$\begin{aligned}V_\varepsilon\left(\frac{x+y}{2}\right) &= \int_{-\infty}^{\infty}\tilde{V}\left(\frac{x+y}{2}-t\right)\eta_\varepsilon(t)dt\\ &\le \int_{-\varepsilon}^{\varepsilon}\frac{1}{2}[V(x-t)+V(y-t)]\eta_\varepsilon(t)dt\\ &= \frac{1}{2}[V_\varepsilon(x)+V_\varepsilon(y)],\end{aligned}$$

which implies that V_ε is convex on $[\delta, a-\delta]$ whenever $\varepsilon < \delta$. Since V_ε is differentiable, the first part of the proof gives

$$2\int_\delta^{a-\delta} V_\varepsilon(x)\sin^2\left(\frac{j\pi(x-\delta)}{a-2\delta}\right)dx \le \int_\delta^{a-\delta} V_\varepsilon(x)dx.$$

Taking $\varepsilon \to 0$, we see that

$$2\int_\delta^{a-\delta} V(x)\sin^2\left(\frac{j\pi(x-\delta)}{a-2\delta}\right)dx \le \int_\delta^{a-\delta} V(x)dx.$$

Since this is true for all δ with $0 < \delta < a/2$, (9) is proved in the general case by letting $\delta \to 0$. □

With this lemma, we can now prove the following result for convex potentials.

Theorem 7. *If $V(x)$ is a convex potential on $[0,a]$, then, for any $k \ge 1$, the Dirichlet eigenvalues satisfy*

$$\sum_{j=1}^{k} E_j[V] \le \sum_{j=1}^{k} E_j[0] + \frac{k}{a}\int_0^a V(x)dx. \tag{10}$$

Moreover, the equality holds if and only if V is constant.

Proof. The inequality (10) follows immediately from (6), (7) and Lemma 6. To examine the case of equality, we have from Theorem 4, (6), (7) and Lemma 6 that

$$\begin{aligned}\sum_{j=1}^{k} E_j[V] &\le \sum_{j=1}^{k} E_j[V_s]\\ &\le \sum_{j=1}^{k} E_j[0] + \sum_{j=1}^{k}\frac{2}{a}\int_0^a V_s(x)\sin^2\left(\frac{j\pi x}{a}\right)dx\\ &\le \sum_{j=1}^{k} E_j[0] + \frac{k}{a}\int_0^a V_s(x)dx\\ &= \sum_{j=1}^{k} E_j[0] + \frac{k}{a}\int_0^a V(x)dx,\end{aligned}$$

where $V_s(x) = [V(x) + V(a-x)]/2$ is also convex. Thus, by Theorem 4 and Lemma 6, equality holds in (10) only when $V = V_s$ and $2\int_0^a V_s(x)\sin^2(j\pi x/a)dx = \int_0^a V_s(x)dx$ for all $j = 1, 2, \ldots, k$. To see that these conditions imply that V is constant, we take $j = 1$ and note that V_s is a symmetric single-well potential, i.e., $V_s(x) = V_s(a-x)$ and V_s is monotone decreasing on $[0, a/2]$. Since $\sin^2(\pi x/a)$ is symmetric about $x = a/2$ and monotone increasing on $[0, a/2]$, it follows that

$$2\int_0^a V_s(x)\sin^2\left(\frac{\pi x}{a}\right)dx \leq \frac{2}{a}\int_0^a V_s(x)dx \int_0^a \sin^2\left(\frac{\pi x}{a}\right)dx = \int_0^a V_s(x)dx.$$

Moreover, the equality holds here only when V_s is constant. This together with the condition $V = V_s$ completes the proof of the theorem. □

A fact corresponding to Lemma 6 for concave potentials is given by

Lemma 8. *If $V(x)$ is a concave potential on $[0, a]$, then*

$$2\int_0^a V(x)\cos^2\left(\frac{j\pi x}{a}\right)dx \leq \int_0^a V(x)dx$$

for $j = 1, 2, 3, \ldots$.

Proof. Since V is concave, $-V$ is convex. Hence, by Lemma 6,

$$-2\int_0^a V(x)\sin^2\left(\frac{j\pi x}{a}\right)dx \leq -\int_0^a V(x)dx.$$

So,

$$\begin{aligned} 2\int_0^a V(x)\cos^2\left(\frac{j\pi x}{a}\right)dx &= 2\int_0^a V(x)dx - 2\int_0^a V(x)\sin^2\left(\frac{j\pi x}{a}\right)dx \\ &\leq \int_0^a V(x)dx. \end{aligned}$$

□

As a final application of our comparison techniques, we prove the following result for concave potentials. This improves the result of Corollary 5 in the Neumann case.

Theorem 9. *If $V(x)$ is a concave potential on $[0, a]$, then, for any $k \geq 1$, the Neumann eigenvalues satisfy*

$$\sum_{j=1}^{k} E_j[V] \leq \sum_{j=1}^{k} E_j[0] + \frac{k}{a}\int_0^a V(x)dx. \tag{11}$$

Moreover, the equality holds if and only if V is constant.

Proof. The proof is similar to that of Theorem 7. Since V is concave, so is V_s. Hence, by Theorem 4, (6), (8) and Lemma 8, we have

$$\begin{aligned}\sum_{j=1}^{k} E_j[V] &\le \sum_{j=1}^{k} E_j[V_s] \\ &\le \sum_{j=1}^{k} E_j[0] + \frac{1}{a}\int_0^a V_s(x)dx + \sum_{j=2}^{k} \frac{2}{a}\int_0^a V_s(x)\cos^2\left(\frac{(j-1)\pi x}{a}\right)dx \\ &\le \sum_{j=1}^{k} E_j[0] + \frac{k}{a}\int_0^a V_s(x)dx \\ &= \sum_{j=1}^{k} E_j[0] + \frac{k}{a}\int_0^a V(x)dx\end{aligned}$$

and equality can hold in (11) only when $V = V_s$ and $2\int_0^a V_s(x)\cos^2(j\pi x/a)dx = \int_0^a V_s(x)dx$ for all $j = 1, 2, \ldots, k-1$; i.e., only when $V = V_s$ and $2\int_0^a V_s(x)\times \sin^2(j\pi x/a)dx = \int_0^a V_s(x)dx$ for all $j = 1, 2, \ldots, k-1$. As in the proof of Theorem 7, these conditions imply that V is constant since $-V_s$ is a symmetric single-well potential. $\square$

Department of Mathematics
National Tsing Hua University
Hsinchu 30043, Taiwan
e-mail: mjhuang@math.nthu.edu.tw

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