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Regularities of local times of two-parameter martingales

By

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Abstract

We prove that, as Wiener functionals, local times of two-parameter martingales belong to some fractional Sobolev spaces over the Wiener space and, therefore, exist $(2, \infty)$-quasi surely.

1. Introduction

Regularities in various senses of local times of one-parameter Brownian motion or, more generally, semi-martingales, have received much attention (see e.g. [26, 22, 30, 2, 3, 32, 1, 11, 23]). More concretely, let M be a one-parameter Brownian motion and $L(t, a, \omega)$ its local time. Then on the one hand, its regularity with respect to the variable $a \in \mathbb{R}$ has been studied in depth in [2, 3, 23] and this study has been continued and extended to semi-martingales by Zhang in [32] recently. On the other hand, its regularity as a Wiener functional has been obtained for the first time by Nualart and Vives in [22], then improved by Watanabe in [30]. Recently, this latter result of Watanabe has been proved to be true for smooth semi-martingales in [1]. More precisely, it is proved in [1] that $L(t, a) \in D_r^p$ for all $p > 1$ and $r < \frac{1}{2}$, and $L(t, a) \notin D_{\frac{1}{2}}^p$ for any $p > 1$, where D_r^p is the usual Sobolev space over the classic Wiener space in the sense of Malliavin calculus.

Taking this last result into account, it could be surprising at the first glance at the $(2, 1)$-quasi-sure existence result of $L(t, a)$, proved by Shigekawa for Brownian motion in [26], since one must wonder how the differentiability index has been raised. To answer this query for ourselves we were led in [11] to the following comparison result concerning capacities on the Wiener space

$$C_{2,n}^2(A) \leq C_{p,r}(A), \quad \text{if } pr \geq 2n.$$

With this inequality the result of Shigekawa seems to become more natural and has been extended to semi-martingales in [11], thanks to the previous result on the smoothness of L established in [1].

In the present work we shall study the regularity of local times of two-parameter martingales in the sense of Malliavin calculus. Compared with the

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one-parameter case, however, some new phenomena appear in this situation. First, while the quadratic variation process plays a unique role in the one-parameter case, naturally associated with a two-parameter martingale M there are two increasing processes, say $\langle \tilde{M} \rangle$ and $\langle M \rangle$, and it is proved in [21] that none of them alone, but their sum, is the right increasing process to define the local time. Secondly, the powerful Tanaka formula no longer holds for this sum process and, therefore, the arguments in [1] cannot be adopted here.

To overcome these difficulties we shall deal with the local times L_1 with respect to $\langle \tilde{M} \rangle$ and L_2 with respect to $\langle M \rangle$ separately. For L_1, a Tanaka formula holds true and we can therefore use arguments similar to [1]. But L_2 is more troublesome since there is no analogue of Tanaka formula for it and it is quite irregular with respect to the variable (s, t, a) near the origin (see [25]). To deal with it we shall use a truncation technique and start form the occupation density formula. The smoothness of L_1 and L_2 together will give that of L, which in turn will yield the quasi-sure existence of L.

The organization of this paper is as follows. After giving necessary notions and notations in Section 2, we obtain the regularity of L in the sense of Malliavin calculus in Section 3. Finally we prove the quasi-sure existence of L in Section 4.

To simplify notations, we will use two conventions throughout the paper: 1. unimportant constants will be denoted by the same letter C; 2. If f and g are two functions, then "$f \lesssim g$" will mean that "there exists a constant C such that $f \leq Cg$".

2. Notations and Preliminaries

Throughout the paper, the parameter set is $\mathbb{T} = [0,1]^2$ with the usual partial order, that is, for $z_1 = (s_1, t_1)$, $z_2 = (s_2, t_2) \in \mathbb{T}$, $z_1 \leq z_2$ if and only if $s_1 \leq s_2$ and $t_1 \leq t_2$. $(s_1, t_1) < (s_2, t_2)$ means that $s_1 < s_2$ and $t_1 < t_2$. If $z_1 < z_2$, $(z_1, z_2]$ denotes the rectangle $\{z \in \mathbb{T}, z_1 < z \leq z_2\}$ and $\mathbb{R}_z := [0, z]$. When $z = (s, t)$, we also write $\mathbb{R}_{st}$ for $\mathbb{R}_z$. The increment of a function $f : \mathbb{T} \to \mathbb{R}$ on a rectangle $(z_1, z_2]$, $z_1 = (s_1, t_1)$, $z_2 = (s_2, t_2)$, is $f((z_1, z_2]) := f(z_2) - f(s_1, t_2) - f(s_2, t_1) + f(z_1)$.

(X, H, μ) is the two-parameter classical Wiener space (see [20]), namely X is the space of all continuous functions $\omega : \mathbb{T} \to \mathbb{R}$ vanishing on the axes and

$$H = \left\{ h \in X; h \text{ is absolutely continuous}, \right.$$
$$\left. \|h\|_H^2 := \int_{\mathbb{T}} \left(\frac{\partial^2 h(s,t)}{\partial s \partial t} \right)^2 dsdt < \infty \right\}$$

and μ the two parameter Wiener measure.

Denote by D_r^p the (p, r)-Sobolev space over X endowed with the norm

$$\|F\|_{p,2r} := \|(I - L)^r F\|_p,$$

where L is the Ornstein-Uhlenbeck operator. When r is a natural number, this norm is, by Meyer's inequality, equivalent to

$$\|F\|'_{p,2r} := \|D^{2r}F\|_p + \|F\|_p,$$

where D is the gradient operator. Given an open set O of X, its (p, r)-capacity is defined by ([12, 18])

$$C_{p,r}(O) = \inf\left(\|u\|_{p,r}; u \geq 0, u \geq 1 \ \mu\text{-a.e. on } O\right);$$

and for any subset $A \subset X$,

$$C_{p,r}(A) = \inf\left(C_{p,r}(O); O \supset A\right).$$

In terms of complex interpolation theory, for $0 < r < 1$, the fractional Sobolev spaces D^p_r can be regarded as intermediate spaces between L^p and D^p_1.

The other fractional Sobolev spaces E^p_r $(1 < p < \infty, 0 < r < 1)$, introduced by Watanabe in [29, Definition 1.1], are defined by real interpolation method:

$$E^p_r := (L^p, D^p_1)_{r,p},$$

where $(\cdot, \cdot)$ denotes the real interpolation space as in [28]. There are several equivalent norms in E^p_r and the one we shall use is given by Peetre's K-method:

$$\|u\|_{p,r} := \left[\int_0^1 [\epsilon^{-r} K(\epsilon, u)]^p \frac{d\epsilon}{\epsilon}\right]^{1/p},$$

where

$$K(\epsilon, u) := \inf\Big\{\|u_1\|_p + \epsilon\|u_2\|_{p,1}; u_1 + u_2 = u, u_1, u_2 \in L^p\Big\}. \tag{2.1}$$

According to [29, Theorem 1.1] (see also [10, Proposition 10]), for every $r \in \mathbb{R}$, $1 < p < \infty$ and $\epsilon > 0$, we have

$$E^p_{r+\epsilon} \subset D^p_r \subset E^p_{r-\epsilon}. \tag{2.2}$$

Let $w = \{w_z, z \in \mathbb{T}\}$ be the Brownian sheet realized on X as the coordinate process and $(\mathcal{F}_z)_{z\in\mathbb{T}}$ its associated filtration. Let $M = \{M_z, z \in \mathbb{T}\}$ be a two-parameter martingale with respect to $(\mathcal{F}_z)_{z\in\mathbb{T}}$, vanishing on the axes and bounded in L^2. By Wong-Zakai's representation theorem ([4, Theorem 3.1]), M can be written as

$$M_z = \int_{\mathbb{R}_z} \phi(\xi) dw_\xi + \iint_{\mathbb{R}_z\times\mathbb{R}_z} \psi(\xi, \eta) dw_\xi dw_\eta, \tag{2.3}$$

where $\phi = \{\phi(\xi), \xi \in \mathbb{T}\}$ is an $\mathcal{F}_\xi$-adapted process such that $E(\int_{\mathbb{T}} \phi(\xi)^2 d\xi) < \infty$, $\psi = \{\psi(\xi, \eta), (\xi, \eta) \in \mathbb{T} \times \mathbb{T}\}$ is a measurable and $\mathcal{F}_{\xi\vee\eta}$-adapted process, vanishing unless

$$(\xi, \eta) \in D := \{(\xi, \eta) \in \mathbb{T} \times \mathbb{T} : \xi = (u, v), \eta = (u', v'), u \leq u', v \geq v'\},$$

such that $E(\iint_{\mathbb{T}\times\mathbb{T}} \psi(\xi,\eta)^2 d\xi d\eta) < \infty$.

Let M be a two-parameter martingale with the expression (2.3) and write $z = (s,t)$. By standard calculation it follows that

$$\langle M\rangle_{st} = \int_{\mathbb{R}_{st}} g(u,v)dudv$$

with

$$g(u,v) = \phi(u,v)^2 + \int_{\mathbb{R}_{uv}} \psi(x,v;u,y)^2 dxdy; \tag{2.4}$$

$$\langle M_{\cdot t}\rangle_s = \int_0^s f(u,t)du,$$

where

$$f(u,t) = \int_0^t \left(\phi(u,v) + \int_{\mathbb{R}_{ut}} \psi(\xi;u,v)dw_\xi\right)^2 dv;$$

$$\langle M_{s\cdot}\rangle_t = \int_0^t h(s,v)dv,$$

where

$$h(s,v) = \int_0^s \left(\phi(u,v) + \int_{\mathbb{R}_{sv}} \psi(u,v;\eta)dw_\eta\right)^2 du.$$

Set again

$$l_1(z,\eta) = \phi(\eta) + \int_{\mathbb{R}_z} \psi(\xi,\eta)dw_\xi$$

and

$$l_2(z,\xi) = \phi(\xi) + \int_{\mathbb{R}_z} \psi(\xi,\eta)dw_\eta,$$

then M can be rewritten as (see [31, 8, 9] and here we use notations similar to [9] which is a little bit different from those of [31, 8])

$$M_z = \int_{\mathbb{R}_z} l_1(z,\eta)dw_\eta = \int_{\mathbb{R}_z} l_2(z,\xi)dw_\xi.$$

Also write

$$\tilde{M}_z = \iint_{\mathbb{R}_z\times\mathbb{R}_z} l_1(\xi\vee\eta,\eta)l_2(\xi\vee\eta,\xi)dw_\xi dw_\eta,$$

and denote by $\langle\tilde{M}\rangle_z$ the quadratic variation process of $\tilde{M}_z$, i.e.,

$$\langle\tilde{M}\rangle_z = \iint_{\mathbb{R}_z\times\mathbb{R}_z} l_1^2(\xi\vee\eta,\eta)l_2^2(\xi\vee\eta,\xi)d\xi d\eta.$$

The quadratic covariation process of M and $\tilde{M}$ is given by

$$\langle M, \tilde{M}\rangle_z = \int_{\mathbb{R}_z} k(z')dz'$$

with

$$k(z') = \iint_{\mathbb{R}_{z'}\times\mathbb{R}_{z'}} \psi(\xi;\eta) l_1(\xi\vee\eta,\eta) l_2(\xi\vee\eta,\xi) d\xi d\eta. \tag{2.5}$$

We will need the following Stroock type commutation formula between D and stochastic integrals which is remarked by Liang in [15]:

$$\begin{aligned} DM_z = &\int_{\mathbb{R}_z} D\phi(\xi)dw_\xi + \int_{\mathbb{R}_{z\wedge\cdot}} \phi(\xi)d\xi \\ &+ \iint_{\mathbb{R}_z\times\mathbb{R}_z} D\psi(\xi,\eta)dw_\xi dw_\eta + \iint_{\mathbb{R}_z\times\mathbb{R}_{z\wedge\cdot}} \psi(\xi,\eta)dw_\xi d\eta \\ &+ \iint_{\mathbb{R}_{z\wedge\cdot}\times\mathbb{R}_z} \psi(\xi,\eta)d\xi dw_\eta. \end{aligned} \tag{2.6}$$

Now let τ be a finite random measure on the Borel algebra on $\mathbb{T}$. Following [6, 7], a map $L : \mathbb{T}\times\mathbb{R}\times X \to \mathbb{R}$ is called a local time for M at $\omega \in X$ with respect to τ if the following occupation time formula holds:

$$\int_{\mathbb{R}} F(a)L(z,a,\omega)da = \int_{\mathbb{R}_z} F(M_\xi)(\omega)\tau(d\xi,\omega) \tag{2.7}$$

for every positive Borel function F on $\mathbb{R}$ and every $z \in \mathbb{T}$.

If we take $\tau := d\langle\tilde{M}\rangle$ (resp. $\tau := d\langle M\rangle$), we obtain the local time L_1 (resp. L_2) with respect to $\langle\tilde{M}\rangle$ (resp. $\langle M\rangle$). Using a two-parameter Itô formula, Nualart [19] proved the almost sure existence of L_1 and established a Tanaka formula (see also [24]). The existence of L_2 is more delicate and is proved by Sanz in [25].

There are non-zero continuous martingales M with $\langle M\rangle = 0$ and non-zero continuous martingales M with $\langle\tilde{M}\rangle = 0$. But if $\langle M\rangle = \langle\tilde{M}\rangle = 0$, then M will be constantly zero. Thus the natural measure induced by $\langle M\rangle + \langle\tilde{M}\rangle$ is proposed to define the local time of M in [21].

3. Smoothness of the local time

To study the smoothness of L, we treat L_1 and L_2 separately. First we prove the following result concerning L_1.

Theorem 3.1. *Suppose that M satisfies the condition*
(c1) $\phi(\xi), \psi(\xi,\eta) \in \cap_{1<p<\infty} D_1^p$, $\int_{\mathbb{T}} \|\phi(\xi)\|_{p,1}^p d\xi + \iint_{\mathbb{T}\times\mathbb{T}} \|\psi(\xi,\eta)\|_{p,1}^p d\xi d\eta < \infty$ *for all* $p > 1$.
Then for every $(s,t,a) \in \mathbb{T}\times\mathbb{R}$, $L_1 \in D_r^p$ *for all* $p > 1$ *and* $r < 1/2$.

Proof. We may and shall assume $p \geq 2$. Denote $z = (s,t)$, $\xi = (u,v)$. By a Tanaka type formula (see [19, Theorem 3.1] or [24, Theorem 5.2]),

$$\begin{aligned} L_1(s,t,a) = {} & \frac{2}{3}[(-a)^+]^3 - \frac{2}{3}[(M_z - a)^+]^3 + 2\int_{\mathbb{R}_z} [(M_\xi - a)^+]^2 dM_\xi \\ & +4\int_{\mathbb{R}_z} (M_\xi - a)^+ d\tilde{M}_\xi + 2\int_0^t (M_{sv} - a)^+ d\langle M_{s\cdot}\rangle_v \\ & +2\int_0^s (M_{ut} - a)^+ d\langle M_{\cdot t}\rangle_u - 2\int_{\mathbb{R}_z} (M_\xi - a)^+ d\langle M\rangle_\xi \\ & -4\int_{\mathbb{R}_z} 1_{(M_\xi > a)} d\langle M, \tilde{M}\rangle_\xi =: \sum_{i=1}^{8} I_i(s,t,a). \end{aligned} \tag{3.1}$$

By (2.6) and (c1), it is easily known that

$$\sup_{z\in\mathbb{T}} \|M_z\|_{p,1} < \infty \quad \text{and} \quad \sup_{z\in T} \|\langle \tilde{M}\rangle_z\|_{p,1} < \infty. \tag{3.2}$$

Using this, the chain rule of D and (2.6) we trivially have $\sum_{i=1}^7 I_i \in D_1^p$.

It remains to prove $I_8 \in D_r^p$ for $r < 1/2$. By (2.2), it suffices to prove $I_8(s,t,a) \in E_r^p$ for $r < 1/2$. By definition, this will be done if we prove

$$K(\epsilon, I_8) \lesssim \varepsilon^{\frac{1}{2}} \tag{3.3}$$

where the functional K is defined by (2.1).

To this end we set

$$I_8^\epsilon(s,t,a) = \int_{\mathbb{R}_{st}} F_\epsilon(M_\xi) d\langle M, \tilde{M}\rangle_\xi, \quad \epsilon \in (0,1),$$

where

$$F_\epsilon(x) = \begin{cases} 1 & \text{if } x > a+\epsilon, \\ \frac{1}{2\epsilon}(x - a + \epsilon) & \text{if } |x-a| \leq \epsilon, \\ 0 & \text{if } x < a - \epsilon. \end{cases}$$

We then have

$$\begin{aligned} & I_8^\epsilon(s,t,a) - I_8(s,t,a) \\ = & \int_{\mathbb{R}_{st}} [F_\epsilon(M_\xi) - 1_{(a,\infty)}(M_\xi)] d\langle M, \tilde{M}\rangle_\xi \\ = & \frac{1}{2\epsilon}\int_{\mathbb{R}_{st}} [(M_\xi - a + \epsilon)1_{(a-\epsilon,a)}(M_\xi) + (M_\xi - a - \epsilon)1_{(a,a+\epsilon)}(M_\xi)] d\langle M, \tilde{M}\rangle_\xi. \end{aligned}$$

By Kunita-Watanabe inequality (see [5, Proposition 4]),

$$
\begin{aligned}
&|I_8^\epsilon(s,t,a) - I_8(s,t,a)| \\
&\le \frac{1}{2\epsilon}\int_{\mathbb{R}_{st}} |(M_\xi - a + \epsilon)1_{(a-\epsilon,a)}(M_\xi) + (M_z - a - \epsilon)1_{(a,a+\epsilon)}(M_\xi)| d\langle M, \tilde{M}\rangle_\xi \\
&\le \frac{1}{2}\int_{\mathbb{R}_{st}} 1_{(a-\epsilon,a+\epsilon)}(M_\xi) d\langle M, \tilde{M}\rangle_\xi \\
&\le \frac{1}{2}\left(\int_{\mathbb{R}_{st}} 1_{(a-\epsilon,a+\epsilon)}(M_\xi) d\langle \tilde{M}\rangle_\xi\right)^{1/2} \times \langle M\rangle_{st}^{1/2}.
\end{aligned}
$$

By (3.1) it is easily seen that

$$
\sup_{|x|\le K} E|L_1(s,t,x)|^p \le C, \qquad \forall K > 0.
$$

Consequently

$$
\begin{aligned}
&E|I_8^\epsilon(s,t,a) - I_8(s,t,a)|^p \\
&\lesssim \left(E|\int_{\mathbb{R}_{st}} 1_{(a-\epsilon,a+\epsilon)}(M_\xi) d\langle \tilde{M}\rangle_\xi|^p\right)^{1/2} \times (E|\langle M\rangle_{st}|^p)^{1/2} \\
&\lesssim \left(E|\int_{\mathbb{R}_{st}} 1_{(a-\epsilon,a+\epsilon)}(M_\xi) d\langle \tilde{M}\rangle_\xi|^p\right)^{1/2} \\
&= \left(E|\int_{a-\epsilon}^{a+\epsilon} L_1(s,t,x)dx|^p\right)^{1/2} \\
&\lesssim \epsilon^{\frac{p-1}{2}}\left(\int_{a-\epsilon}^{a+\epsilon} E|L_1(s,t,x)|^p dx\right)^{1/2} \lesssim \epsilon^{p/2},
\end{aligned}
$$

where the equality is due to the occupation time formula (2.7). Thus,

$$
\|I_8^\epsilon(s,t,a) - I_8(s,t,a)\|_p \lesssim \epsilon^{1/2}. \tag{3.4}
$$

On the other hand, from the expression (2.5) we know

$$
\int_{\mathbb{T}} \|k(\xi)\|_{p,1} d\xi < \infty, \quad \forall p > 1.
$$

Hence $I_8^\epsilon(s,t,a) \in \cap_p D_1^p$ and we have

$$
DI_8^\epsilon = \int_{\mathbb{R}_{st}} DF_\epsilon(M_\xi)k(\xi)d\xi + \int_{\mathbb{R}_{st}} F_\epsilon(M_\xi)Dk(\xi)d\xi =: J_1 + J_2.
$$

From (c1), we see that $E\|J_2\|_H^p \le C$. For the term J_1, by Kunita-Watanabe inequality and (2.7), we have

$$
\begin{aligned}
&E(\|J_1\|_H^p)\\
&\lesssim E\|\int_{\mathbb{R}_{st}}\frac{1}{2\epsilon}1_{(a-\epsilon,a+\epsilon)}(M_\xi)DM_\xi d\langle M,\tilde{M}\rangle_\xi\|_H^p\\
&\lesssim \epsilon^{-p}E\left(\int_{\mathbb{R}_{st}}1_{(a-\epsilon,a+\epsilon)}(M_\xi)\|DM_\xi\|_H d\langle M,\tilde{M}\rangle_\xi\right)^p\\
&\lesssim \epsilon^{-p}\left(E\left(\int_{\mathbb{R}_{st}}1_{(a-\epsilon,a+\epsilon)}(M_\xi)\|DM_\xi\|_H^2 d\langle\tilde{M}\rangle_\xi\right)^p\right)^{1/2}\times(\langle M\rangle_{st}^p)^{1/2}\\
&\lesssim \epsilon^{-p}(E(\sup_{\xi\in\mathbb{T}}\|DM_\xi\|_H)^{4p})^{1/4}\times\left(E\left(\int_{\mathbb{R}_{st}}1_{(a-\epsilon,a+\epsilon)}(M_\xi)d\langle\tilde{M}\rangle_\xi\right)^{2p}\right)^{1/4}\\
&\lesssim \epsilon^{-p/2}.
\end{aligned}
$$

Hence

$$\|DI_8^\epsilon\|_p\lesssim\epsilon^{-1/2}.$$

Since (3.4) in particular implies $\|I_8^\epsilon\|_p\le C$, we have

$$\|I_8^\epsilon\|_{p,1}\lesssim\varepsilon^{-\frac{1}{2}}. \tag{3.5}$$

(3.4) and (3.5) then yield (3.3) and the proof is completed. □

We will need the following result.

Theorem 3.2. *Suppose for every $p>1$*

$$\int_{\mathbb{T}}\|\phi(\xi)\|_p^p d\xi+\int\int_{\mathbb{T}\times\mathbb{T}}\|\psi(\xi,\eta)\|_p^p d\xi d\eta<\infty.$$

Then for every $N>0$ $p>1$ and $\delta>0$ there exists a constant $C=C(p,N,\delta)$ such that

$$
\begin{aligned}
&\|L_1(s,t,a)-L_1(s',t',a')\|_p\\
&\le C(|s-s'|^{1/2-\delta}+|t-t'|^{1/2-\delta}+|a-a'|^{1/2})
\end{aligned}
\tag{3.6}
$$

for all $(s,t),(s',t')\in\mathbb{T}$ and $0\le|a|,|a'|\le N$.

Proof. By the inequality

$$\|L_1(s,t,a)-L_1(s',t',a')\|_p\lesssim\|L_1(s,t,a)-L_1(s',t',a')\|_q,\quad q>p, \tag{3.7}$$

it suffices to prove

$$
\begin{aligned}
&\|L_1(s,t,a)-L_1(s',t',a')\|_p\\
&\lesssim(|s-s'|^{1/2-1/p}+|t-t'|^{1/2-1/p}+|a-a'|^{1/2})
\end{aligned}
\tag{3.8}
$$

for every $p > 1$ since then we will have

$$\|L_1(s,t,a) - L_1(s',t',a')\|_p \lesssim (|s-s'|^{1/2-1/q} + |t-t'|^{1/2-1/q} + |a-a'|^{1/2}),$$

and this by choosing q sufficiently large will give (3.6).

To prove (3.8) we only need to show that

$$\text{(3.9)}\quad \|I_i(s,t,a) - I_i(s',t',a')\|_p \lesssim (|s-s'|^{1/2-1/p} + |t-t'|^{1/2-1/p} + |a-a'|^{1/2})$$

for $i = 1, \ldots, 8$.

I_1 will pose no problem. For I_2 we notice

$$\text{(3.10)}\quad \begin{aligned} &E|[(M_{st}-a)^+]^3 - [(M_{s't'}-a)^+]^3|^p \\ &\lesssim E[|M_{st}| + |M_{s't'}|]^{2p}[|M_{st} - M_{s't'}|^p] \\ &\lesssim \Big\{E[|M_{st}| + |M_{s't'}|]^{3p}\Big\}^{2/3}\Big\{E[|M_{st} - M_{s't'}|^{3p}]\Big\}^{1/3}. \end{aligned}$$

We only consider the case $(s,t) \le (s',t')$, since the other cases can be treated similarly. In this case,

$$M_{st} - M_{s't'} = \int_{s'}^{s}\int_0^t dM_{uv} - \int_0^{s'}\int_{t'}^{t} dM_{uv}.$$

We have by Burkholder inequality

$$\text{(3.11)}\quad \begin{aligned} &E[|M_{st} - M_{s't'}|^{3p}] \\ &\lesssim \left[E\left(\int_{s'}^{s}\int_0^t g^2(u,v)dudv\right)\right]^{\frac{3p}{2}} + \left[E\left(\int_0^{s'}\int_{t'}^{t} g^2(u,v)dudv\right)\right]^{\frac{3p}{2}} o \\ &\lesssim |s-s'|^{\frac{3p}{2}-1}\int_{s'}^{s}\int_0^t Eg^{3p}(u,v)dudv + |t-t'|^{\frac{3p}{2}-1}\int_0^{s'}\int_{t'}^{t} Eg^{3p}(u,v)dudv \\ &\lesssim |s-s'|^{\frac{3p}{2}-1} + |t-t'|^{\frac{3p}{2}-1}, \end{aligned}$$

where g is defined by (2.4). Moreover it is trivial that

$$\text{(3.12)}\quad E|[(M_{st}-a)^+]^3 - [(M_{st}-a')^+]^3|^p \lesssim |a-a'|^p.$$

Combining (3.10), (3.11), (3.12) gives

$$\text{(3.13)}\quad \|I_2(s,t,a) - I_2(s',t',a')\|_p \lesssim |s-s'|^{\frac{1}{2}-\frac{1}{3p}} + |t-t'|^{\frac{1}{2}-\frac{1}{3p}} + |a-a'|.$$

Next comes I_3. It is easily seen that

$$\text{(3.14)}\quad \|I_3(s,t,a) - I_3(s,t,a')\|_p \lesssim |a-a'|.$$

On the other hand, by Burkholder and Hölder inequalities we have

(3.15)
$$\begin{aligned}
&E\left|\int_0^s\int_0^t[(M_\xi-a)^+]^2dM_\xi-\int_0^{s'}\int_0^{t'}[(M_\xi-a)^+]^2dM_\xi\right|^p\\
\lesssim&E\left(\int_{s'}^s\int_0^t[(M_\xi-a)^+]^4g(\xi)d\xi\right)^{\frac{p}{2}}+E\left(\int_0^{s'}\int_t^{t'}[(M_\xi-a)^+]^4g(\xi)d\xi\right)^{\frac{p}{2}}\\
\lesssim&|s-s'|^{\frac{p}{2}-1}\int_{s'}^s\int_0^t E\big[[(M_\xi-a)^+]^{2p}g^{\frac{p}{2}}(\xi)\big]d\xi\\
&+|t-t'|^{\frac{p}{2}-1}\int_0^{s'}\int_t^{t'}E\big[(M_\xi-a)^+]^{2p}g^{\frac{p}{2}}(\xi)\big]d\xi\\
\lesssim&|s-s'|^{\frac{p}{2}-1}+|t-t'|^{\frac{p}{2}-1}.
\end{aligned}$$

This together with (3.14) gives

$$\|I_3(s,t,a)-I_3(s',t',a')\|_p\lesssim|s-s'|^{\frac12-\frac1p}+|t-t'|^{\frac12-\frac1p}+|a-a'|. \tag{3.16}$$

In a similar way we can show that

$$\begin{aligned}
&\|I_4(s,t,a)-I_4(s',t',a')\|_p+\|I_7(s,t,a)-I_7(s',t',a')\|_p\\
&\lesssim|s-s'|^{\frac12-\frac1p}+|t-t'|^{\frac12-\frac1p}+|a-a'|.
\end{aligned} \tag{3.17}$$

We now turn to I_5. We have

$$\begin{aligned}
&\int_0^t(M_{sv}-a)^+d\langle M_{s\cdot}\rangle_v-\int_0^{t'}(M_{s'v}-a')^+d\langle M_{s'\cdot}\rangle_v\\
&=\int_0^t dv\int_0^s\Psi(s,u,v,a)du-\int_0^{t'}dv\int_0^{s'}\Psi(s',u,v,a)du,
\end{aligned} \tag{3.18}$$

where

$$\Psi(s,u,v,a):=(M_{sv}-a)^+\left(\phi(u,v)+\int_{\mathbb{R}_{sv}}\psi(u,v;\eta)dw_\eta\right)^2.$$

Since

$$\begin{aligned}
&E\left|\int_0^t dv\int_0^s(\Psi(s,u,v,a)-\Psi(s',u,v,a))du\right|^p\\
&\lesssim\int_0^t dv\int_0^s E|\Psi(s,u,v,a)-\Psi(s',u,v,a)|^p du\\
&\lesssim|s-s'|^{\frac{p}{2}-1}
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
&E\left|\int_0^t dv\int_0^s\Psi(s',u,v,a)du-\int_0^{t'}dv\int_0^{s'}\Psi(s',u,v,a)du\right|^p\\
&\lesssim|s-s'|^{p-1}+|t-t'|^{p-1},
\end{aligned} \tag{3.20}$$

we have

$$\|I_5(s,t,a)-I_5(s',t',a)\| \lesssim |s-s'|^{\frac{p}{2}-1}+|s-s'|^{\frac{p}{2}-1}. \tag{3.21}$$

Trivially

$$\|I_5(s,t,a)-I_5(s,t,a')\|_p \lesssim |a-a'|. \tag{3.22}$$

(3.21) and (3.22) imply

$$\|I_5(s,t,a)-I_5(s',t',a')\|_p \lesssim |s-s'|^{\frac{1}{2}-\frac{1}{p}}+|t-t'|^{\frac{1}{2}-\frac{1}{p}}+|a-a'|. \tag{3.23}$$

In the same way

$$\|I_6(s,t,a)-I_6(s',t',a')\|_p \lesssim |s-s'|^{\frac{1}{2}-\frac{1}{p}}+|t-t'|^{\frac{1}{2}-\frac{1}{p}}+|a-a'|. \tag{3.24}$$

Finally we look at I_8. First it is obvious that

$$\|I_8(s,t,a)-I_8(s',t',a)\|_p \lesssim |s-s'|^{\frac{1}{2}}+|t-t'|^{\frac{1}{2}}. \tag{3.25}$$

Assuming $a'\le a$ and using Kunita-Watanabe inequality, Hölder inequality and the fact that $\sup_{s,t\in\mathbb{T},|x|<N}\|L_1(s,t,x)\|_p<\infty$, we have

$$\begin{aligned} &E|I_8(s,t,a)-I_8(s,t,a')|^p \\ &\lesssim E\left|\int_{\mathbb{R}_{st}} 1_{(a>M_\xi>a')}d\langle M,\tilde{M}\rangle_\xi\right|^p \\ &\lesssim (E|\langle M\rangle_{st}|^p)^{1/2}\times\left(E|\int_{\mathbb{R}_{st}} 1_{(a>M_\xi>a')}d\langle\tilde{M}\rangle_\xi|^p\right)^{1/2} \\ &\lesssim \left(E|\int_{a'}^{a} L_1(s,t,x)dx|^p\right)^{1/2} \\ &\lesssim |a-a'|^{p/2}, \end{aligned} \tag{3.26}$$

Adding (3.25) and (3.26) we obtain

$$\|I_8(s,t,a)-I_8(s',t',a')\|_p \lesssim |s-s'|^{\frac{1}{2}-\frac{1}{p}}+|t-t'|^{\frac{1}{2}-\frac{1}{p}}+|a-a'|. \tag{3.27}$$

Finally we combine (3.13), (3.16), (3.17), (3.23), (3.24) and (3.27) to obtain (3.9). □

We now turn to the smoothness of L_2. According to [25], to insure its existence and continuity, we have to impose the following additional conditions.
(c2) $\phi(\xi)$ and $\psi(\xi,\eta)$ are continuous adapted processes.
(c3) For each $v\ge 0$, the processes $g(\cdot,v)$, $f(\cdot,v)$ are strictly positive semi-martingales; If $g(\cdot,v)=m_v(\cdot)+a_v(\cdot)$, $f(\cdot,v)=n_v(\cdot)+b_v(\cdot)$ are their canonical decompositions, then there exist measurable, adapted and jointly continuous

processes ζ, χ, κ such that

$$\langle m_v(\cdot)\rangle_u = \int_0^u \zeta_v(\sigma)d\sigma,$$
$$\langle n_v(\cdot)\rangle_u = \int_0^u \chi_v(\sigma)d\sigma,$$
$$\langle m_v(\cdot), n_v(\cdot)\rangle_u = \int_0^u \kappa_v(\sigma)d\sigma.$$

Moreover, the total variations of a_v and b_v are integrable on $[0,1]$.
(c4) For each $n \geq 1$, $s \geq 1/n$, $t \geq 1/n$ and $p > 4$ we have

$$\int_{\mathbb{R}_{st}^{(n)}} E\left[\left|\frac{\zeta_v(u)}{f(u,v)^2}\right|^p\right] dudv < \infty,$$
$$\int_{\mathbb{R}_{st}^{(n)}} E\left[\left|\frac{\chi_v(u)g(u,v)^2}{f(u,v)^4}\right|^p\right] dudv < \infty,$$

where

$$\mathbb{R}_{st}^{(n)} := \left[\frac{1}{n}, s\right] \times \left[\frac{1}{n}, t\right].$$

Then we have

Lemma 3.1. *If the conditions (c1) through (c4) hold, then there exists an almost surely jointly continuous process* $\{L_2(s,t,a)\}, a \in \mathbb{R} - \{0\}, (s,t) \in \mathbb{T}$ *satisfying*

$$L_2(s,t,a) = \int_{\mathbb{R}_{st}} \frac{g(u,v)}{f(u,v)} L^1(du,v,a)dv = \int_{\mathbb{R}_{st}} \frac{g(u,v)}{h(u,v)} L^2(u,dv,a)du$$

and

$$\int_{\mathbb{R}_{st}} F(M_{uv})d\langle M\rangle_{uv} = \int_{\mathbb{R}} L_2(s,t,a)F(a)da \quad a.s., \tag{3.28}$$

where $L^1(s,t,a)$ *(resp.,* $L^2(s,t,a)$*) is the local time associated to* $M_{\cdot t}$ *(resp.,* $M_{s\cdot}$*).* L_2 *is called the local time of* M *with respect to* $\langle M\rangle$*. Moreover, set*

$$L_{2,n}(s,t,a) := \int_{\mathbb{R}_{st}^{(n)}} \frac{g(u,v)}{f(u,v)} L^{1,(n)}(du,v,a)dv,$$

where

$$L^{1,(n)}(u,v,a) := L^1(u,v,a) - L^1\left(\frac{1}{n}, v, a\right).$$

Then for each $n \geq 1$, $\delta > 0$, $0 < K_1 < K_2 < \infty$, $p > 4$, *there exists a constant* $C = C(n,\delta,K_1,K_2,p)$ *such that*

$$E|L_{2,n}(s,t,a) - L_{2,n}(s',t',a')|^p \leq C(|s-s'|^{p/2-\delta} + |t-t'|^{p/2-\delta} + |a-a'|^{p/2}) \tag{3.29}$$

for $(s,t),(s',t') \in \mathbb{T}^{(n)}$, $K_1 \le |a|, |a'| \le K_2$, *where* $\mathbb{T}^{(n)} = [0,1/n] \times [0,1/n]$. *In particular*

$$\sup_{(s,t)\in\mathbb{T}^{(n)};K_1\le|a|\le K_2} E|L_{2,n}(s,t,a)|^p < \infty.$$

Proof. All the results and proofs except (3.29) are found in [25, Theorem 1 and Theorem 2]. For (3.29) we notice that the estimate

$$E|L_{2,n}(s,t,a) - L_{2,n}(s,t,a')|^p \lesssim |a-a'|^{p/2} \tag{3.30}$$

is implicit in the proof of [25, Theorem 2]. By (see [25, (8)]) we have

$$L_{2,n}(s,t,a) = \int_\varepsilon^t G(s,v)L^{1,n}(s,v,a)dv - \int_{\mathbb{R}_{st}^{(n)}} L^{1,n}(u,v,a)d_1G(u,v)dv,$$

where

$$G(u,v) := \frac{g(u,v)}{f(u,v)}, \quad (u,v) \in \mathbb{T}^{(n)}$$

and

$$\begin{aligned} d_1G(u,v) &= \frac{1}{f(u,v)}d_1g(u,v) - \frac{g(u,v)}{f(u,v)^2}d_1f(u,v) \\ &\quad + \frac{g(u,v)}{f(u,v)^3}\chi_v(u)du - \frac{1}{f(u,v)^2}\kappa_v(u)du. \end{aligned}$$

But by Ito formula we have

$$G(s,v) = G(\varepsilon,v) + \int_\varepsilon^s d_1G(u,v).$$

Thus

$$\begin{aligned} L_{2,n}(s,t,a) &= \int_{\mathbb{R}_{st}^{(n)}} L^{1,n}(s,v,a)d_1G(u,v)dv \\ &\quad + \int_\varepsilon^t G(\varepsilon,v)L^{1,n}(s,v,a)dv - \int_{\mathbb{R}_{st}^{(n)}} L^{1,n}(u,v,a)d_1G(u,v)dv. \end{aligned}$$

By standard stochastic calculus we deduce from this that

$$E|L_{2,n}(s,t,a) - L_{2,n}(s',t',a)|^p \lesssim |s-s'|^{p/2-\delta} + |t-t'|^{p/2-\delta}. \tag{3.31}$$

(3.30) and (3.31)give (3.29) and this ends the proof. □

We can now state

Theorem 3.3. *If* (c1) *through* (c4) *hold, then for every* $n \ge 1$, $L_{2,n} \in D_r^p$ *for all* $p > 1$ *and* $r < 1/3$.

Proof. First notice that by definition it is easily seen that

$$\int_{\mathbb{R}_{st}^{(n)}} \phi(M_\xi) d\langle M\rangle_\xi = \int_{\mathbb{R}} \phi(x) L_{2,n}(s,t,x)dx. \tag{3.32}$$

Set

$$L_{2,n}^\varepsilon(s,t,a) = \int_{\mathbb{R}_{st}^{(n)}} F_\varepsilon(M_\xi) d\langle M\rangle_\xi,$$

where

$$F_\epsilon(x) = \begin{cases} -\varepsilon^{-2}|x-a| + \varepsilon^{-1} & \text{if } x \in (a-\epsilon, a+\epsilon), \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned}
&E|L_{2,n}^\varepsilon(s,t,a) - L_{2,n}(s,t,a)|^p \\
&= E\left|\int_{\mathbb{R}_{st}^{(n)}} F_\varepsilon(M_\xi)d\langle M\rangle_\xi - L_{2,n}(s,t,a)\right|^p \\
&= E\left|\int_{\mathbb{R}} F_\varepsilon(x)(L_{2,n}(s,t,x) - L_{2,n}(s,t,a))dx\right|^p \\
&\lesssim E\left|\int_{\mathbb{R}} \varepsilon^{-1} 1_{(a-\varepsilon,a+\varepsilon)}(x)(L_{2,n}(s,t,x) - L_{2,n}(s,t,a))dx\right|^p \\
&\lesssim \varepsilon^{-p} E\left|\int_{a-\varepsilon}^{a+\varepsilon} (L_{2,n}(s,t,x) - L_{2,n}(s,t,a))dx\right|^p \\
&\lesssim \varepsilon^{-p}\varepsilon^{p-1} \int_{a-\varepsilon}^{a+\varepsilon} E|L_{2,n}(s,t,x) - L_{2,n}(s,t,a)|^p dx \\
&\lesssim \sup_{x\in(a-\varepsilon,a+\varepsilon)} E|L_{2,n}(s,t,x) - L_{2,n}(s,t,a)|^p \\
&\lesssim \sup_{x\in(a-\varepsilon,a+\varepsilon)} |x-a|^{p/2} \lesssim \varepsilon^{p/2}.
\end{aligned}$$

Hence

$$\|L_{2,n}^\epsilon(s,t,a) - L_{2,n}(s,t,a)\|_p \lesssim \varepsilon^{\frac{1}{2}}. \tag{3.33}$$

In particular

$$\sup_\epsilon \|L_{2,n}^\epsilon(s,t,a)\|_p \le C. \tag{3.34}$$

Now we estimate the Sobolev norm of L_2. We have

$$DL_{2,n}^\varepsilon(s,t,a) = \int_{\mathbb{R}_{st}^{(n)}} F'_\varepsilon(M_\xi) DM_\xi d\langle M\rangle_z + \int_{\mathbb{R}_{st}^{(n)}} F_\varepsilon(M_\xi) Dg(\xi)d\xi =: K_1 + K_2.$$

For K_1, we have

$$(3.35)\quad E\|K_1\|_H^p \lesssim E\left|\int_{\mathbb{R}_{st}^{(n)}} |F'_\varepsilon(M_\xi)|\|DM_\xi\|_H d\langle M\rangle_\xi\right|^p$$

$$\lesssim (E[\sup_{\xi\in\mathbb{T}}\|DM_\xi\|_H^{2p}])^{1/2}\left(E\left|\int_{\mathbb{R}_{st}^{(n)}} |F'_\varepsilon(M_\xi)|d\langle M\rangle_\xi\right|^{2p}\right)^{1/2}$$

$$\lesssim \left(E\left|\int_{\mathbb{R}} |F'_\varepsilon(x)|L_{2,n}(s,t,x)dx\right|^{2p}\right)^{1/2} \quad \text{(by (2.7))}$$

$$\lesssim \left(E\left|\int_{a-\varepsilon}^{a+\varepsilon} \varepsilon^{-2}L_{2,n}(s,t,x)dx\right|^{2p}\right)^{1/2}$$

$$\lesssim \varepsilon^{-2p}\varepsilon^{\frac{2p-1}{2}}\left(\int_{a-\varepsilon}^{a+\varepsilon} E|L_{2,n}(s,t,x)|^{2p}dx\right)^{1/2}$$

$$\lesssim \varepsilon^{-p}.$$

For K_2, we have

$$Dg(u,v) = 2\phi(u,v)D\phi(u,v) + \int_{\mathbb{R}_{uv}^{(n)}} 2\psi(x,v,u,y)D\psi(x,v,u,y)dxdy.$$

Consequently, $\|K_2\|_H$ is less than

$$\int_{\mathbb{R}_{st}^{(n)}} 2F_\varepsilon(M_{uv})\left\|\phi(u,v)D\phi(u,v) + \int_{\mathbb{R}_{uv}^{(n)}} \psi(x,v,u,y)D\psi(x,v,u,y)dxdy\right\|_H dudv$$

$$\lesssim \left(\int_{\mathbb{R}_{st}^{(n)}} F_\varepsilon(M_{uv})^2\phi(u,v)^2dudv\right)^{1/2}\left(\int_{\mathbb{R}_{st}} \|D\phi(u,v)\|_H^2dudv\right)^{1/2}$$

$$+\left(\int_{\mathbb{R}_{st}^{(n)}} F_\varepsilon(M_{uv})^2\int_{\mathbb{R}_{uv}^{(n)}} |\psi(x,v,u,y)|^2dxdydudv\right)^{1/2}$$

$$\times\left(\iint_{\mathbb{R}_{st}^{(n)}\times\mathbb{R}_{uv}^{(n)}} \|D\psi(x,v,u,y)\|_H^2dxdydudv\right)^{1/2}.$$

Hence

(3.36)

$$E\|K_2\|_H^p$$

$$\lesssim \left(E\left|\int_{\mathbb{R}_{st}^{(n)}} F_\varepsilon(M_{uv})^2\phi(u,v)^2dudv\right|^p\right)^{1/2}\left(E\left|\int_{\mathbb{R}_{st}^{(n)}} \|D\phi(u,v)\|_H^2dudv\right|^p\right)^{1/2}$$

$$+\left(E\left|\int_{\mathbb{R}_{st}^{(n)}} F_\varepsilon(M_{uv})^2\int_{\mathbb{R}_{uv}^{(n)}} \psi(x,v,u,y)^2dxdydudv\right|^p\right)^{1/2}$$

$$
\begin{aligned}
&\times \left(E \left| \iint_{\mathbb{R}_{st}^{(n)} \times \mathbb{R}_{uv}^{(n)}} \|D\psi(x,v,u,y)\|_H^2 dxdydudv \right|^p \right)^{1/2} \\
&\lesssim \left(E \left| \int_{\mathbb{R}_{st}^{(n)}} F_\varepsilon(M_{uv})^2 g(u,v) dudv \right|^p \right)^{1/2} \\
&\lesssim \left(E \left| \int_{\mathbb{R}} F_\varepsilon(x)^2 L_{2,n}(s,t,x) dx \right|^p \right)^{1/2} \\
&\lesssim \left(E \left| \int_{\mathbb{R}} \varepsilon^{-2} 1_{(a-\varepsilon,a+\varepsilon)} L_{2,n}(s,t,x) dx \right|^p \right)^{1/2} \\
&\lesssim \varepsilon^{-p} \left(E \left| \int_{a-\varepsilon}^{a+\varepsilon} L_{2,n}(s,t,x) dx \right|^p \right)^{1/2} \\
&\lesssim \varepsilon^{-p/2}.
\end{aligned}
$$

A combination of the (3.35) and (3.36) implies

$$E[\|DL_{2,n}^{\epsilon}\|_H^p] \lesssim \epsilon^{-p},$$

which together with (3.34) gives

$$\|L_{2,n}^{\varepsilon}(s,t,x)\|_{p,1} \lesssim \varepsilon^{-1}.$$

Setting $\delta = \varepsilon^{2/3}$ we obtain

$$
\begin{aligned}
K(\varepsilon, L_{2,n}) &\le \|L_{2,n}^{\delta} - L_{2,n}\|_p + \varepsilon \|L_{2,n}^{\delta}\|_{p,1} \\
&\lesssim (\delta^{1/2} + \varepsilon \cdot \delta^{-1}) \\
&\lesssim \varepsilon^{1/3},
\end{aligned}
$$

which implies $L_{2,n} \in D_r^p$ for $r < \frac{1}{3}$. □

4. Quasi sure existence of the local time L with respect to $\langle M\rangle + \langle \tilde{M}\rangle$

Below we shall prove the quasi sure existence of L for martingales satisfying (c1)–(c4). To this end we first notice that the same arguments as in [15, Theorem 3.3], [16, Theorem 4.2] allow to give

Lemma 4.1. *Both the quadratic variation process $\langle M\rangle$ and the product variation process $\langle \tilde{M}\rangle$ of M admit $(\infty,1)$-modifications and we have*

$$\lim_{n\to\infty} \sum_{i,j=0}^{2^n-1} M(\Delta_{ij}^1)^2 M(\Delta_{ij}^2)^2 = \langle \tilde{M}\rangle_{st}$$

and

$$\lim_{n\to\infty} \sum_{i,j=0}^{2^n-1} M(\Delta_{ij})^2 = \langle M\rangle_{st}$$

uniformly in $(s,t)\in\mathbb{T}$, *q.s., where* $\Delta^1_{ij}=(s_i,s_{i+1}]\times(0,t_{j+1}]$. $\Delta^2_{ij}=(0,s_{i+1}]\times(t_j,t_{j+1}]$, $\Delta_{ij}=(s_i,s_{i+1}]\times(t_j,t_{j+1}]$, $s_i=\frac{i}{2^n}\wedge s$ *and* $t_j=\frac{j}{2^n}\wedge t$. *Here "*$(\infty,1)$*" means "*$(p,1)$ *for all* $p>1$*".*

Then by [11, Theorem 2.24] we have

Lemma 4.2. *The above lemma still holds true if "*$(\infty,1)$*" is replaced by "*$(2,\infty)$*", which stands for "*$(2,r)$ *for all* $r>0$*".*

The next result provides necessary estimate to obtain the Hölder exponent of L.

Theorem 4.1. *Assume the martingale* M *satisfies* (c1)–(c4)*. Define*

$$L(s,t,a):=L_1(s,t,a)+L_2(s,t,a),\quad s,t>0.$$
$$L_{(n)}(s,t,a):=L(s,t,a)-L\left(\frac{1}{n},t,a\right)-L\left(s,\frac{1}{n},a\right)+L\left(\frac{1}{n},\frac{1}{n},a\right),s,t\geq\frac{1}{n}.$$

Then, for every $n\geq 1$, *every* $K_2>K_1>0$, *every* $p>1$, *every* $0<r<1/3$ *and* $0<\alpha<1/3-r$ *there exists a constant* $C=C(K_1,K_2,p,r,\alpha)$ *such that*

$$\|L_{(n)}(s,t,a)-L_{(n)}(s',t',a')\|_{p,r}\leq C(|s-s'|^{\frac{3}{2}\alpha}+|t-t'|^{\frac{3}{2}\alpha}+|a-a'|^{\frac{3}{2}\alpha})\tag{4.1}$$

for all $(s,t),(s',t')\in\mathbb{T}^{(n)}$ *and* $K_1\leq|a|,|a'|\leq K_2$.

Proof. In virtue of Theorem 3.1 and Theorem 3.4, we have

$$\sup_{(s,t)\in\mathbb{T}^{(n)},K_1<|a|<K_2}\|L_{(n)}(s,t,a)\|_{p,r}<\infty.\tag{4.2}$$

By [14, Lemma 2.10], for $\theta\in(0,1)$,

$$D^p_\theta=[L^p,D^p_1]_\theta.$$

Hence, by the reiteration theorem (see e.g., [28, Theorem 1.10.2]), we have

$$D^p_r=[L^p,D^p_{r+\alpha}]_{\frac{r}{r+\alpha}}.$$

Consequently, according to [28, Theorem 1.9.3/f] we have by (3.6), (3.29) and (4.2)

$$\begin{aligned}&\|L_{(n)}(s,t,a)-L_{(n)}(s',t',a')\|_{p,r}\\ &\lesssim\|L_{(n)}(s,t,a)-L_{(n)}(s',t',a')\|_p^{1-r/(r+\alpha)}\|L_{(n)}(s,t,a)-L_{(n)}(s',t',a')\|_{p,r+\alpha}^{r/(r+\alpha)}\\ &\lesssim(|s-s'|^{\frac{3}{2}\alpha}+|t-t'|^{\frac{3}{2}\alpha}+|a-a'|^{\frac{3}{2}\alpha})\end{aligned}$$

for all $(s,t),(s',t')\in\mathbb{T}$ and $K_1\leq|a|,|a'|\leq K_2$. □

By Theorem 4.3 and [27, Theorem 3.4], for every $p>1$, $r<1/3$ and $n\geq 1$, $L_{(n)}$ admits a (p,r)-quasi continuous redefinition as a $C(\mathbb{T}^{(n)}\times([-K_2,-K_1]\cup$

$[K_1, K_2]), \mathbb{R})$-valued function. Using the standard diagonal procedure we see that there exists an $\tilde{L}_{(n)}$ such that for every $p > 1$ and $r < 1/3$, $\tilde{L}_{(n)}$ is a (p, r)-redefinitions of $L_{(n)}$ as a $C(\mathbb{T}^{(n)} \times ([-K_2, -K_1] \cup [K_1, K_2]), \mathbb{R})$-valued function. On the other hand, M and $\langle \tilde{M} \rangle$ admit redefinitions which are (p, r)-quasi continuous ones as $C(\mathbb{T}, \mathbb{R})$-valued functions for every $p > 1$ and $r = 1$ respectively.

We still denote these redefinitions by the original symbols themselves for simplifying notations. By the same as that of [11, Theorem 2.24] we see that for every n, there exists a set B_n with $C_{p,r}(B_n) = 0$ for all $(p, r) \in (1, \infty) \times (0, \frac{1}{3})$ such that The equality

$$\int_{\mathbb{R}_{st}^{(n)}} F(M_\xi)(\omega) d(\langle \tilde{M} \rangle_\xi + \langle M \rangle_\xi) = \int_{\mathbb{R}} F(x) L_{(n)}(s, t, x, \omega) dx \tag{4.3}$$

holds for every $\omega \in B_n^c$, every $(s, t) \in \mathbb{T}$ and every positive Borel function F. Now take $B := \cup_n B_n$, then $C_{p,r}(B) = 0$ for $(p, r) \in (1, \infty) \times (0, 1/3)$ and (4.3) holds for all $(s, t, a, \omega) \in (\mathbb{T} \backslash \{(0, 0)\}) \times ([-K_2, -K_1] \cup [K_1, K_2]) \times B^c$. In particular, it follows that $n \mapsto L_{(n)}(s, t, a, \omega)$ is increasing for these (s, t, a, ω) and we can thereby define

$$L(s, t, a, \omega) := \lim_{n \to \infty} L_{(n)}(s, t, a, \omega).$$

If $st = 0$ we define (of course)

$$L(s, t, \cdot, \cdot) \equiv 0.$$

Thus $L(\cdot, \cdot, \cdot, \cdot)$ is well-defined.

Summing up the above observation we obtain

Theorem 4.2. *There is a set B such that*
(1) *For every $p > 1$ and $r < 1/3$, $C_{p,r}(B) = 0$;*
(2) *The equality*

$$\int_{\mathbb{R}_{st}} F(M_\xi)(\omega) d(\langle \tilde{M} \rangle_\xi + \langle M \rangle_\xi) = \int_{\mathbb{R}} F(x) L(s, t, x, \omega) dx \tag{4.4}$$

holds for every $\omega \in B^c$, every $(s, t) \in \mathbb{T}$ and every positive Borel function F.

Proof. Letting $n \to \infty$ in (4.3) gives (4.4). □

As a consequence of Theorem 4.4 and [11, Theorem 2.15], we have the following

Theorem 4.3. *There is a set B such that*
(1) *For every n, $C_{2,n}(B) = 0$;*
(2) *The equality*

$$\int_{\mathbb{R}_{st}} F(M_\xi)(\omega) d(\langle \tilde{M} \rangle_\xi + \langle M \rangle_\xi) = \int_{\mathbb{R}} F(x) L(s, t, x, \omega) dx$$

holds for every $\omega \in B^c$, every $(s, t) \in \mathbb{T}$ and every positive Borel function F.

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