J. Math. Kyoto Univ. (JMKYAZ)
46-1 (2006), 231–234

Erratum to "On the minimal solution for quasilinear degenerate elliptic equation and its blow-up" (J. Math. Kyoto Univ. Vol. 44 No. 2, 381–439)

By

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Proposition 8.2 in §8, namely the characterization of the behavior of solutions ψ_t as $t \to 0$ for a certain class of quasilinear elliptic equations, needs a correction about the support of their gradients. In the paper we used this property to have the uniform boundedness of $\psi_t, t \in [0,T]$ in th proof of Theorem 8.1, therefore it should be replaced by the next, in which the boundedness is simply given by a method of iteration.

Proposition 8.2. *Let $\varphi \in \tilde{V}_{\lambda,p}(\Omega)$ satisfy $|\nabla\varphi| = 0$ on $F_\varepsilon = \{x \in \Omega : dist(x, F_{\lambda,p}) \le \varepsilon\}$ for some $\varepsilon > 0$. Then there is a unique solution η_t of* (8.14) *for a small $T > 0$ such that $\eta_t = u_\lambda - t\psi_t$ for $\psi_t \in C^0([0,T], V_{\lambda,p}(\Omega))$ and*

$$\sup_{x\in\Omega, t\in[0,T]} |\psi_t| < \infty, \tag{8.1}$$

$$\lim_{t\to 0} ||\psi_t - \varphi||_{V_{\lambda,p}(\Omega\setminus F_\varepsilon)} = 0. \tag{8.2}$$

Proof. Since ∇u_λ does not vanish in $\overline{\Omega \setminus F_\varepsilon}$ and the nonlinearity $f \in C^1([0,\infty))$, first we see $u_\lambda \in C^{2,\sigma}(\overline{\Omega \setminus F_\varepsilon})$ for some $\sigma \in (0,1)$ as a solution to uniformly elliptic equation. By the theory of monotone operator $L_p(\cdot)$, there is a unique solution $\psi_t \in W_0^{1,p}(\Omega)$ for each t and $\nabla\psi_t$ is Hölder continuous function w.r.t. $x \in \Omega$. In §9, it is proved that $\psi_t - \varphi$ satisfies uniformly elliptic equation in $\Omega \setminus F_\varepsilon$ for a sufficiently small t. Hence by the elliptic regularity theory ψ_t can be assumed to be uniformly bounded in $C^2(\overline{\Omega \setminus F_\varepsilon})$ for a fixed small $\varepsilon > 0$. Since $L_p(\cdot)$ is differentiable in $W_0^{1,p}(\Omega)$, we have

$$\frac{L_p(u_\lambda - t\psi_t) - L_p(u_\lambda)}{t} = -\int_0^1 L_p'(u_\lambda - st\psi_t)\psi_t\, ds = -L_p'(u_\lambda)\varphi \in C^2(\overline{\Omega \setminus F_\varepsilon}).$$

1991 *Mathematics Subject Classification*(*s*). 35J65, 35J35
Received June 23, 2005
Revised October 6, 2005

Let us set $w = k\psi_t^{2k-1}$ and $v = \psi_t^k$, $(k = 1, 2, \ldots)$. Assuming that $1 < p \leq 2$, first we shall prove

$$\int_\Omega |\nabla v|^2\, dx \leq C \int_{F_\varepsilon^c} |w|\, dx, \qquad k = 1, 2, \ldots. \tag{8.3}$$

For $k = 1$ the uniform boundedness of ψ_t in $W_0^{1,2}(\Omega)$ w.r.t. t follows from this inequality. By Sobolev type inequality and the definition of v and w we also have

$$\left(\int_\Omega |\psi_t|^{2k}\, dx\right)^{\frac{1}{2k}} \leq (Ck)^{\frac{1}{2k}} \left(\int_{F_\varepsilon^c} |\psi_t|^{2k-1}\, dx\right)^{\frac{1}{2k}} \quad k = 1, 2, \ldots. \tag{8.4}$$

Here C is a positive number independent of each t and k. Letting $k \to \infty$, we immediately have

$$\sup_{x\in\Omega, t\in[0,T]} |\psi_t| \leq \sup_{x\in F_\varepsilon^c, t\in[0,T]} |\psi_t| < +\infty$$

and this proves (8.1) for $1 < p \leq 2$. To establish (8.3) we use w and v as test functions and obtain

$$\begin{aligned}
&\langle L_p'(u_\lambda)\varphi, w\rangle = \frac{1}{t}\langle L_p(u_\lambda) - L_p(\eta_t), w\rangle = \left\langle \int_0^1 L_p'(\eta_t^{(s)})\psi_t\, ds, w \right\rangle \\
&= \int_0^1 ds \int_\Omega |\nabla \eta_t^{(s)}|^{p-2} \left[(\nabla\psi_t, \nabla w) + (p-2)\frac{(\nabla\eta_t^{(s)}, \nabla\psi_t)(\nabla\eta_t^{(s)}, \nabla w)}{|\nabla\eta_t^{(s)}|^2} \right] dx \\
&\geq \frac{(2k-1)(p-1)}{k} \int_0^1 ds \int_\Omega |\nabla\eta_t^{(s)}|^{p-2} |\nabla v|^2\, dx \geq C\frac{(2k-1)(p-1)}{k} \int_\Omega |\nabla v|^2\, dx.
\end{aligned}$$

Here $\eta_t^{(s)} = u_\lambda - st\psi_t$ and C is a positive number independent of each v and w. Since $L_p'(u_\lambda)\varphi$ vanishes on F_ε, we have the inequality (8.3).

Secondly we consider the case $p \geq 2$. Again using v and w we have

$$\begin{aligned}
&||v||^2_{V_{\lambda,p}(\Omega)} = \int_\Omega |\nabla u_\lambda|^{p-2} |\nabla v|^2\, dx \leq C \int_\Omega \big(|\nabla\eta_t| + |\nabla u_\lambda|\big)^{p-2} |\nabla v|^2\, dx \\
&\leq \frac{C}{t}\big|\langle L_p(\eta_t) - L_p(u_\lambda), w\rangle\big| = C\big|\langle L_p'(u_\lambda)\varphi, w\rangle\big| \leq C \int_{F_\varepsilon^c} |w|\, dx \leq C||w||_{V_{\lambda,p}(\Omega)}.
\end{aligned}$$

If we put $k = 1$ in this inequality, we have, for some positive number C independent of each t

$$||\psi_t||_{V_{\lambda,p}(\Omega)} \leq C.$$

Moreover we also have (8.4) by Lemma 4.1 in the original paper. Hence ψ_t is uniformly bounded in $L^\infty \cap V_{\lambda,p}(\Omega)$.

Now we prove the second assertion (8.2) assuming that $1 < p \leq 2$. Noting that ψ_t is uniformly bounded in $W_0^{1,2}(\Omega)$, $L_p'(u_\lambda)$ is elliptic in $\Omega \setminus F_\varepsilon$ and that

$$L_p(u_\lambda - t\psi_t) = L_p(u_\lambda) - tL_p'(u_\lambda)\varphi = L_p(u_\lambda) - tL_p'(u_\lambda)\psi_t + o(t) \quad \text{in } [W_0^{1,p}(\Omega\setminus F_\varepsilon)]',$$

we have

$$L_p'(u_\lambda)(\psi_t - \varphi) = o(1) \quad \text{in } [W_0^{1,p}(\Omega \setminus F_\varepsilon)]' \text{ as } t \to 0.$$

By the compactness we see $\psi_t \to \varphi$ in $L^2(\Omega \setminus F_\varepsilon)$. Then, from Lemma 3.1 and a usual argument using a cut-off function, we have

$$||\psi_t - \varphi||^2_{V_{\lambda,p}(\Omega \setminus F_\varepsilon)} = o(1) \quad \text{as } t \to 0.$$

This proves the assertion provided that $1 < p \leq 2$. When $p > 2$, we use

$$L_p(u_\lambda - t\varphi) = L_p(u_\lambda) - tL_p(u_\lambda)\varphi + o(t), \quad \text{in } [V_{\lambda,p}(\Omega)]' \text{ as } t \to 0.$$

Since $\varphi \in \tilde{V}_{\lambda,p}(\Omega)$, we have

$$\begin{aligned}
||\psi_t - \varphi||^2_{V_{\lambda,p}(\Omega)} &= \int_\Omega |\nabla u_\lambda|^{p-2} |\nabla(\psi_t - \varphi)|^2 \, dx \\
&\leq C \int_\Omega \big(|\nabla \eta_t| + |\nabla(u_\lambda - t\varphi)|\big)^{p-2} |\nabla(\psi_t - \varphi)|^2 \, dx \\
&\leq C \frac{1}{t} \big|\langle L_p(\eta_t) - L_p(u_\lambda - t\varphi), \psi_t - \varphi \rangle\big| \\
&= o(1) ||\varphi||_{V_{\lambda,p}(\Omega)} ||\psi_t - \varphi||_{V_{\lambda,p}(\Omega)}.
\end{aligned}$$

So that we have the desired result. □

For reader's convenience, we shall give a rough sketch of the proof of Theorem 8.1. If we admit Proposition 8.3, then we can prove Theorem 8.1 without changing the argument in the original paper. Therefore it suffices to show Proposition 8.3 using a new Proposition 8.2 in stead of the old one.

By K we denote an arbitrary compact set K contained in $\overline{\Omega} \setminus F_{\lambda.p}$. By $K' \subset K$ we denote another arbitrary compact set satisfying $dist(K', \partial K \cap \Omega) > 0$. Following the argument in the proof of Proposition 8.3, we see that $W_t = \psi_t - \varphi \in L^\infty(K) \cap V_{\lambda,p}(\Omega)$ satisfies uniformly elliptic equation (9.11) in K with a parameter $t \in [0, T]$ for a sufficiently small $T > 0$. Namely,

$$\sum_{j,k} A_{j,k} \partial^2_{j,k} W_t = H(x),$$

where $A_{j,k} \in C^{1,\sigma}(K)$ and $H \in C^{0,\sigma}(K)$ for some $\sigma \in (0,1)$ uniformly in $t \in [0,T]$. Further $H(x)$ can be written in the form

$$H(x) = A_t(x) \cdot \nabla W_t + o(1) B_t(x) \quad \text{as } t \to 0,$$

where $A_t \in [C^{1,\sigma}(K)]^N$ and $B_t \in C^{0,\sigma}(K)$ uniformly in $t \in [0,T]$. Here we used the fact $u_\lambda \in C^{2,\sigma}(K)$, $\eta_t \in C^{2,\sigma}(K)$ and $t\psi_t = u_\lambda - \eta_t \in C^{2,\sigma}(K)$ uniformly in $t \in [0,T]$. In particular, $H(x)$ satisfies the growth condition

$$|H(x)| \leq C(|\nabla W_t| + o(1)) \quad \text{as } t \to 0.$$

Then by the regularity theory of linear elliptic equation, we see that W_t is bounded in $C^{1,\sigma}(K')$ uniformly in $t \in [0,T]$. More precisely we have for some positive number C

$$||W_t||_{C^{1,\sigma}(K')} \leq C||W_t||_{L^\infty(K)} + o(1) \text{ as } t \to 0.$$

Clearly ψ_t is also uniformly bounded in $C^{1,\sigma}(K')$. By Proposition 8.2 we may assume that $\nabla\psi_t - \nabla\varphi$ converges to 0 as $t \to 0$ almost everywhere. Since W_t is uniformly bounded in $C^{1,\sigma}(K')$, for any $q > 0$, $\nabla\psi_t - \nabla\varphi$ converges to 0 in $L^q(K')$. Then it follows from Sobolev imbedding theorem that $\lim_{t\to 0} W_t = 0$ in $L^\infty(K')$ noting that $W_t = 0$ on the boundary $\partial\Omega$. After all, for any compact set $K'' \subset K'$ with $dist(K'', \partial K' \cap \Omega) > 0$ we see

$$||W_t||_{C^{1,\sigma}(K'')} \leq C||W_t||_{L^\infty(K')} + o(1) \to 0 \quad \text{as } t \to 0.$$

This proves Proposition 8.3.

Remark. 1. Iterating this procedure we can show $W_t \in C^{2,\sigma}(K)$ uniformly in $t \in [0,T]$. In particular $\psi_t \in C^{2,\sigma}(\overline{\Omega \setminus F_\varepsilon})$ uniformly in $t \in [0,T]$. Similarly if we assume the nonlinearity $f \in C^\infty$, then $W_t \in C^\infty(K)$ holds.

2. For the sake of simplicity we employed the linearity of H w.r.t. ∇W in the proof of Proposition 8.3. We note that this property is not crucial but the growth condition is sufficient. See the remark just after Theorem 2 in [1] for example.

3. Proposition 8.1 contains a similar mistake. In the statement, "$\varphi_t \in C^0([0,T_0], \tilde{V}_{\lambda,p}(\Omega))$" should be replaced by "$\varphi_t \in C^0([0,T_0], V_{\lambda,p}(\Omega))$". According to this, the description " From the coercivity of $L'_p(u_\lambda)$ we see $\nabla\varphi_t = 0$ in D ." should be removed in the proof.

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References

[1] E. Di Benedetto, $C^{1,\alpha}$ *local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. **7**-8 (1983), 827–850.