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On a p-adic analogue of Shintani's formula

By

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Abstract

Shintani expressed the first derivative at $s = 0$ of a partial ζ-function of an algebraic number field in terms of the multiple gamma function. Cassou-Noguès constructed a p-adic analogue of the partial ζ-function and calculated the derivative at $s = 0$. In this paper, we will define a p-adic analogue of the multiple gamma function and give a p-adic analogue of Shintani's formula. This formula has a strong resemblance to the original Shintani's formula. Using this formula, we get a partial result toward Gross' conjecture concerning the order at $s = 0$ of the p-adic L-function.

1. Introduction

In this paper, we will give a formula for the first derivative of the p-adic partial ζ-function at $s = 0$ using the p-adic multiple gamma function (Theorem 6.2). For the usual partial ζ-function, Shintani ([18, Proposition 1]) obtained a formula which expresses the first derivative at $s = 0$ in terms of the multiple gamma function (see Section 3). His formula has the following form: Let F be a totally real algebraic number field of degree n, O_F the ring of integers, E^+ the group of all totally positive units, and $\mathfrak{f}$ an integral ideal of F. Then we embed F into $\mathbf{R}^n$ by $x \mapsto (x^{\sigma_1}, \ldots, x^{\sigma_n})$ where $J_F := \{\sigma_1, \ldots, \sigma_n\}$ is the set of all embeddings $F \to \mathbf{R}$. Let $\{a_\mu\}$ be a complete set of representatives for the narrow ideal class group consisting of integral ideals. Let $\infty_1, \ldots, \infty_n$ denote the archimedean primes of F, $C_\mathfrak{f}$ denote the ideal class group modulo $\mathfrak{f}\infty_1 \cdots \infty_n$. For $\mathfrak{c} \in C_\mathfrak{f}$, let $\zeta_\mathfrak{f}(s, \mathfrak{c}) = \sum_{\mathfrak{a} \in \mathfrak{c}} N(\mathfrak{a})^{-s}$ denote the partial ζ-function of the class $\mathfrak{c}$, where $\mathfrak{a}$ extends over all integral ideals in the class $\mathfrak{c}$. For $v_1, \ldots, v_r \in \mathbf{R}^{+n}$, let $C((v_1, \ldots, v_r)) = \{t_1 v_1 + \cdots + t_r v_r \mid (t_1, \ldots, t_r) \in \mathbf{R}^{+r}\}$ be a cone in $\mathbf{R}^{+n}$. We can take a cone decomposition, i.e., there exist a finite set J and $v_j = (v_{j,1}, \ldots, v_{j,r(j)}), j \in J, v_{j,i} \in O_F$ such that $\mathbf{R}^{+n} = \bigsqcup_{j \in J} \bigsqcup_{u \in E^+} uC(v_j)$ where $\bigsqcup$ denotes the disjoint union and $C(v_j) = C((v_{j,1}, \ldots, v_{j,r(j)}))$. Let $R(\mathfrak{c}, C(v_j)) = \{z = \sum_{k=1}^{r(j)} x_k v_{j,k} \in (\mathfrak{a}_\mu \mathfrak{f})^{-1} \cap C(v_j) \mid 0 < x_i \leq 1, (z)\mathfrak{a}_\mu \mathfrak{f} \in \mathfrak{c}\}$ when $\mathfrak{a}_\mu = \mathfrak{c}$ in the narrow ideal class group. Then there exist algebraic num-

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bers a_q, b_q $(q = 1, \ldots, m)$ such that

$$\left[\frac{d}{ds}\zeta_{\mathfrak{f}}(s, \mathfrak{c})\right]_{s=0} = \sum_{k=1}^{n}\sum_{j\in J}\sum_{z\in R(\mathfrak{c}, C(v_j))} \log\left(\frac{\Gamma_{r(j)}(z^{\sigma_k}, v_j^{\sigma_k})}{\rho(v_j^{\sigma_k})}\right) - \log(N\mathfrak{a}_\mu\mathfrak{f})\zeta_{\mathfrak{f}}(0, \mathfrak{c}) + \sum_{q=1}^{m} a_q \log(b_q).$$

We can write explicitly the last term:

$$\sum_{q=1}^{m} a_q \log(b_q) = \sum_{j\in J}\sum_{z\in R(\mathfrak{c}, C(v_j))} \frac{(-1)^{r(j)}}{n} \sum_{l} C_l \left(\begin{pmatrix} v_j^{\sigma_1} \\ \vdots \\ v_j^{\sigma_n} \end{pmatrix}\right) \prod_{i=1}^{r(j)} \frac{B_{l_i}(x_i(z))}{l_i!}.$$

Here Γ_r denotes the r-ple gamma function. For the other notation, see Section 3.

In [6] and [5], Cassou-Noguès constructed a p-adic analogue $\zeta_{p,\mathfrak{f}}(s, \mathfrak{c})$ of the partial ζ-function. Furthermore she expressed the derivative of $\zeta_{p,\mathfrak{f}}(s, \mathfrak{c})$ at $s = 0$ in terms of the p-adic multiple gamma function. But her definition of the p-adic multiple gamma function is rather ad hoc, and may not be regarded as a proper generalization of Shintani's formula. We will define the (natural) p-adic logarithmic multiple gamma function in Section 5. Take $0 < a_1, \ldots, a_r, a \in F$. Let $\mathfrak{P}$ be a prime ideal of F lying over p. Assume that a_i and a are $\mathfrak{P}$-integral. Then we can define the p-adic r-ple ζ-function $\zeta_{p,r}(s, (a_1, \ldots, a_r), a)$ by a p-adic interpolation of the usual r-ple ζ-function (Theorem 5.1 or [4]). Generalizing Barnes' definition [2] to the p-adic case, we define the p-adic logarithmic r-ple gamma function $L\Gamma_{p,r}(a, (a_1, \ldots, a_r))$ as the derivative of $\zeta_{p,r}(s, (a_1, \ldots, a_r), a)$ at $s = 0$. For the p-adic partial ζ-function $\zeta_{p,\mathfrak{f}}(s, \mathfrak{c})$, we obtain a p-adic formula (Theorem 6.2):

$$\left[\frac{d}{ds}\zeta_{p,\mathfrak{f}}(s, \mathfrak{c})\right]_{s=0} = \sum_{k=1}^{n}\sum_{j\in J}\sum_{z\in R(\mathfrak{c}, C(v_j))} L\Gamma_{p,r(j)}(z^{\sigma_k}, v_j^{\sigma_k}) - \log_p(N\mathfrak{a}_\mu\mathfrak{f})\zeta_{p,\mathfrak{f}}(0, \mathfrak{c}) + \sum_{q=1}^{m} a_q \log_p(b_q).$$

Through the analogy between these two formulas, we can show that the derivative at $s = 0$ of the p-adic L-function is equal to 0 in a special case. In fact, we prove a partial result toward Gross' conjecture ([9, Conjecture 2.12]). In Section 7.2, we show the result (Theorem 7.1): Let F be a totally real field, K an abelian extension over F with conductor $\mathfrak{f}_\mathfrak{o}$ and χ an odd character of $\mathrm{Gal}(K/F)$. Put $\mathfrak{f} = \mathfrak{f}_\mathfrak{o} \times \prod_{\mathfrak{p}|(p), \mathfrak{p}\nmid\mathfrak{f}_\mathfrak{o}} \mathfrak{p}$ and assume that the number of prime ideals $\mathfrak{p}$ satisfying $\mathfrak{p}|(p)$, $\chi(\mathfrak{p}) = 1$ is greater than 1. Then the order at $s = 0$ of $L_{p,\mathfrak{f}}(s, \chi_{-1})$ is greater than 1. Here χ_{-1} is a character twisted by the Teichmüller character.

The Chowla-Selberg formula shows that there exists a strong correlation between Shimura's CM-period and the gamma function. In [19], Yoshida formulated a conjecture: For all $\tau \in \mathrm{Gal}(K/F)$,

$$\prod_{\sigma \in J_K} p_K(\sigma, \tau\sigma) \sim \pi^{-[K:\mathbf{Q}]\mu(\tau)/2} \exp\left(\sum_{\chi \in \hat{G}_-} \chi(\tau) \frac{L'(0,\chi)}{L(0,\chi)} \right). \tag{1.1}$$

Here K is a CM-field which is abelian over a totally real algebraic number field F, $\hat{G}_-$ is the set of all odd characters of $\mathrm{Gal}(K/F)$, J_K is the set of all embeddings of K into $\mathbf{C}$. For $\chi \in \hat{G}_-$, $L(s,\omega)$ is the Artin L-function attached to χ. For $\sigma, \tau \in J_K$, $p_K(\sigma,\tau)$ denotes Shimura's CM-period and

$$\mu(\tau) = \begin{cases} 1 & \text{if } \tau = 1, \\ -1 & \text{if } \tau = \text{the complex conjugation}, \\ 0 & \text{otherwise}. \end{cases}$$

We write $a \sim b$ if $b \neq 0$ and $a/b \in \overline{\mathbf{Q}}$. By Shintani's formula, this conjecture can be sharpened so that every Shimura's CM-period can be expressed in terms of the multiple gamma function [19]. We obtained a p-adic analogue of the formula (1.1) in the case of $F = \mathbf{Q}$, $K = \mathbf{Q}(\zeta_m)$, $(p,m) = 1$, where ζ_m is a primitive m-th root of unity. Indeed, we proved:

$$\Omega_p^K(\mathrm{id},\tau) \sim \exp_p\left(\sum_{\chi} \frac{-\chi(\tau)}{[K:\mathbf{Q}]} \frac{L_p'(0,\chi_{-1})}{L(0,\chi)} \right). \tag{1.2}$$

Here χ extends over all odd Dirichlet characters modulo m, χ_{-1} denotes the primitive character associated to the character χ twisted by the Teichmüller character, Ω_p^K denotes the p-adic CM-period symbol, L_p denotes the Kubota-Leopoldt p-adic L-function, σ_p is an element of $\mathrm{Gal}(K/F)$ such that $\zeta_m \mapsto \zeta_m^p$ and $\exp_p$ is the p-adic exponential function. In the succeeding paper, a proof of (1.2) will be given. Now H. Yoshida and I are conducting cooperative research on this theme [12]. We will investigate the generalization of the formula (1.2) and its refinement, that is, a p-adic analogue of Conjecture (1.1).

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2. Notation and Terminology

For an odd prime number p, $\overline{\mathbf{Q}_p}$ denotes an algebraic closure of $\mathbf{Q}_p$, $\mathbf{C}_p$ denotes a completion of $\overline{\mathbf{Q}_p}$, $|\ |_p$ denotes the valuation on $\mathbf{C}_p$ such that $|p|_p = 1/p$. $\mathbf{O}_{\mathbf{C}_p}$(resp. $\mathbf{O}_{\overline{\mathbf{Q}_p}}$) denotes the ring of integers of $\mathbf{C}_p$(resp. $\overline{\mathbf{Q}_p}$) and $\mathbf{M}_p$ denotes the maximal ideal of $\mathbf{O}_{\mathbf{C}_p}$. We fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}_p}$.

For an algebraic number field F of degree n, O_F denotes the ring of integers, J_F denotes the set of all embedding $F \hookrightarrow \overline{\mathbf{Q}}$ and we put $J_F = \{\sigma_i \mid i = 1, 2, \ldots, n\}$. Then $x^{(i)}$ denotes $\sigma_i(x)$ for $x \in F$. Suppose that F is totally real. For an integral ideal $\mathfrak{f}$ of F, let $I_{\mathfrak{f}}$ be the group of all fractional ideals of F which is prime to $\mathfrak{f}$, $F_{\mathfrak{f}+} = \{\alpha \in F \mid \alpha \gg 0,$ (i.e., totally positive), $v_{\mathfrak{p}}(\alpha - 1) \geq v_{\mathfrak{p}}(\mathfrak{f})$ for all prime ideals $\mathfrak{p}$ dividing $\mathfrak{f}\}$ and $E_{\mathfrak{f}}^{+} = O_F^{\times} \cap F_{\mathfrak{f}+}$. Here $v_{\mathfrak{p}}$ is the normalized $\mathfrak{p}$-adic valuation. Let $P_{\mathfrak{f}+} = \{(\alpha) \mid \alpha \in F_{\mathfrak{f}+}\}$ and $C_{\mathfrak{f}} = I_{\mathfrak{f}}/P_{\mathfrak{f}+}$. For $\mathfrak{a} \in I_{\mathfrak{f}}$, $\overline{\mathfrak{a}}$ denotes the class mod$P_{\mathfrak{f}+}$. For $v_i \in \mathbf{R}^n$, we call $C(v_1, v_2, \ldots, v_r) = \{t_1 v_1 + t_2 v_2 + \cdots + t_r v_r \mid t_i \in \mathbf{R}^+\}$ an open simplicial cone. We embed F into $\mathbf{R}^n$ by $x \mapsto (x^{(1)}, x^{(2)}, \ldots, x^{(n)})$.

We define the Teichmüller character $\theta_p : \mathbf{O}_{\mathbf{C}_p} \to \mathbf{C}_p{}^{\times}$ as follows: Let $x \in \mathbf{O}_{\overline{\mathbf{Q}_p}}$. If $|x|_p < 1$ then we put $\theta_p(x) = 0$. If $|x|_p = 1$ then we take a finite extension K over $\mathbf{Q}_p$ such that $K \ni x$. Let f be the degree of the residue field of K over $\mathbf{Z}/p\mathbf{Z}$. Then we put $\theta_p(x) = \lim_{l \to \infty} x^{p^{fl}}$ and extend θ_p continuously. We define: $\theta_p^0(x) = 0$ if $|x|_p < 1$, $= 1$ if $|x|_p = 1$. Note that $\theta_p(x)$ and $\theta_p^0(x)$ are $(p^f - 1)$-th roots of unity, or 0. If $|x|_p = 1$ then $\theta_p(x)^{-1} x \equiv 1 \bmod \mathbf{M}_p$, and we put $\langle x \rangle = \theta_p(x)^{-1} x$.

For a Dirichlet character χ, χ_* denotes the associated primitive character and χ_n denotes $(\chi \theta_p{}^{-n})_*$. For a character χ of C_f, χ_n denotes $\chi(\theta_p \circ N)^{-n}$ where N denotes the norm.

We use the Iwasawa logarithm function: $\log_p : \mathbf{C}_p^{\times} \to \mathbf{C}_p$ defined by the usual power series on $1 + \mathbf{M}_p$ and extended to $\mathbf{C}_p^{\times}$ by requiring $\log_p p = 0$. Γ_p denotes Morita's p-adic Γ-function [13] defined by $\Gamma_p(n) = (-1)^n \prod_{i=1, p \nmid i}^{n-1} i$ on $\mathbf{N}$ and extended to a continuous function on $\mathbf{Z}_p$.

Let $B_m(X)$ denote the m-th Bernoulli polynomial and $B_m = B_m(0)$ denote the m-th Bernoulli number. We define for $a, b \in \mathbf{N} \cup 0$:

$$\left\{ \begin{matrix} a \\ b \end{matrix} \right\} = \begin{cases} (-1)^{b-1} \dbinom{a-1}{b-1} & \text{if } a, b \geq 1, \\ 1 & \text{if } a = 0, \\ 0 & \text{if } a \geq 1, b = 0. \end{cases}$$

As the set, $\bigsqcup$ denotes the disjoint union.

3. Shintani's formula

First we review the complex case [16], [18]. We define the multiple Riemann ζ-function on $\mathbf{C}$ by

$$\zeta_r(s, (a_1, \ldots, a_r), x) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \left\{ x + \sum_{i=1}^{r} a_i m_i \right\}^{-s},$$

where $a_i, x \in \mathbf{R}^+$. This series converges when $\mathrm{Re}(s) > r$ and can be continued meromorphically to the whole s-plane. We define the multiple gamma function:

$$\log\left(\rho_r\left((a_1, \ldots, a_r)\right)\right) = -\lim_{x \to 0} \left\{ \left[\frac{d}{ds} \zeta_r(s, (a_1, \ldots, a_r), x) \right]_{s=0} + \log x \right\},$$

$$\left[\frac{d}{ds} \zeta_r(s, (a_1, \ldots, a_r), x) \right]_{s=0} = \log\left(\frac{\Gamma_r\left(x, (a_1, \ldots, a_r)\right)}{\rho_r\left((a_1, \ldots, a_r)\right)} \right).$$

We also consider a meromorphic function on $\mathbf{C}$, such that for $\mathrm{Re}(s) > r/n$,

$$\zeta_r(s, A, \xi, x) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \prod_{i=1}^{r} \xi_i^{m_i} \prod_{j=1}^{n} \left\{ \sum_{i=1}^{r} a_{j,i}(m_i + x_i) \right\}^{-s},$$

where $A = (a_{j,i})$ is an $n \times r$-matrix, $x = (x_1, \ldots, x_r)$, $\xi = (\xi_1, \ldots, \xi_r)$, $a_{j,i} > 0$, $x_i \geq 0$, $x \neq 0$ and ξ_i are roots of unity. Let $\zeta_r(s, A, x)$ denote $\zeta_r(s, A, (1, \ldots, 1), x)$. According to [18, Proposition 1],
(3.1)

$$\left[\frac{d}{ds}\zeta_r(s, A, x)\right]_{s=0} = \sum_{j=1}^{n} \log\left(\frac{\Gamma_r\left(x^t A_j, A_j\right)}{\rho_r(A_j)}\right) + \frac{(-1)^r}{n} \sum_{l} C_l(A) \prod_{i=1}^{r} \frac{B_{l_i}(x_i)}{l_i!}.$$

Here l extends over all $l = (l_1, \ldots, l_r)$ such that $l_1 + \cdots + l_r = r$, $0 \leq l_i \in \mathbf{Z}$, $A_j = (a_{j,1}, \ldots, a_{j,r})$, $C_l(A) = \sum_{1 \leq j,k \leq n, j \neq k} C_{l,j,k}(A)$ and

$$C_{l,j,k}(A) = \int_0^1 \left\{ \prod_{m=1}^{r} (a_{j,m} + a_{k,m}u)^{l_m - 1} - \prod_{m=1}^{r} a_{j,m}^{l_m - 1} \right\} \frac{du}{u}.$$

Let F be a totally real number field of degree n, $J_F = \{\sigma_1, \ldots, \sigma_n\}$, $\mathfrak{f}$ an integral ideal, $\mathfrak{c} \in C_{\mathfrak{f}}$ and $\{\mathfrak{a}_\mu\}$ a complete set of representatives for $C_{(1)}$. We define the partial ζ-function of the class $\mathfrak{c}$:

$$\zeta_{\mathfrak{f}}(s, \mathfrak{c}) = \sum_{\mathfrak{g}} N\mathfrak{g}^{-s},$$

where $\mathfrak{g}$ extends over all integral ideals in the class of $\mathfrak{c}$. The next lemmas are proved in [16].

Lemma 3.1. *We can take a finite set J and totally positive elements $v_{j,i} \in O_F$ for $j \in J, i = 1, 2, \ldots, r(j)$ so that $v_{j,1}, \ldots, v_{j,r(j)}$ are linearly independent for each j and*

$$\mathbf{R}^{+n} = \bigsqcup_{j \in J} \bigsqcup_{u \in E_{(1)}^+} uC(v_{j,1}, v_{j,2}, \ldots, v_{j,r(j)}).$$

Lemma 3.2. *With the above notation, we define an $n \times r(j)$-matrix $A_j = (v_{j,m}^{(l)})$ $(1 \leq l \leq n, 1 \leq m \leq r(j))$, $z = x(z)^t v_j$, $x(z) = (x_1(z), \ldots, x_{r(j)}(z))$ for $z \in C(v_j)$ and*

$$\begin{aligned} R(\mathfrak{c}, j) &= R(\mathfrak{c}, C(v_j)) \\ &= \{z \in C(v_j) \cap (\mathfrak{a}_\mu \mathfrak{f})^{-1} \mid x_i(z) \in \mathbf{Q}, 0 < x_i(z) \leq 1, (z)\mathfrak{a}_\mu \mathfrak{f} \in \mathfrak{c}\}, \end{aligned}$$

for $\mathfrak{a}_\mu$ satisfying that $\mathfrak{c} = \mathfrak{a}_\mu \mathfrak{f}$ in $C_{(1)}$. Then $R(\mathfrak{c}, j)$ is finite and

$$\zeta_{\mathfrak{f}}(s, \mathfrak{c}) = \sum_{j \in J} \sum_{z \in R(\mathfrak{c}, j)} N(\mathfrak{a}_\mu \mathfrak{f})^{-s} \zeta_{r(j)}(s, A_j, x(z)).$$

By (3.1) and these lemmas, we get Shintani's formula.

Theorem 3.1.

$$\left[\frac{d}{ds}\zeta_{\mathfrak{f}}(s,\mathfrak{c})\right]_{s=0} = \sum_{k=1}^{n}\sum_{j\in J}\sum_{z\in R(\mathfrak{c},C(v_j))} \log\left(\frac{\Gamma_{r(j)}(z^{(k)},v_j^{(k)})}{\rho_{r(j)}(v_j^{(k)})}\right) - \log(N\mathfrak{a}_\mu\mathfrak{f})\zeta_{\mathfrak{f}}(0,\mathfrak{c}) + T_0,$$

where

$$T_0 = \sum_{j\in J}\sum_{z\in R(\mathfrak{c},C(v_j))} \frac{(-1)^{r(j)}}{n}\sum_{l} C_l(A_j)\prod_{i=1}^{r(j)}\frac{B_{l_i}(x_i(z))}{l_i!}.$$

4. p-adic L-functions, the method by Cassou-Noguès

In this section, we will review the p-adic L-function over a totally real field and calculate the derivatives at $s=0$ following the method by Cassou-Noguès [6], [5]. For other method of construction, we refer the reader to [7]. Imai [10] also defined a p-adic log multiple Γ function and formulated a relationship with a special value of a p-adic analogue of the multiple ζ-function, not with the derivatives. But his functions are different from our functions.

First we will define a p-adic analogue of the ζ-function. We prepare several Lemmas. We assume that p is an odd prime.

Lemma 4.1. *Let $f(s) = \sum_{k=0}^{\infty} a_k\binom{s}{k}$, $a_k \in \mathbf{C}_p$.*

1. *Assume that $|a_k|_p \to 0$. Then $f(s)$ is a continuous function on $\mathbf{Z}_p$.*

2. *Assume that $|a_k/k|_p \to 0$. Then $f(s)$ is differentiable at $s=0$ and $df/ds(0) = \sum_{k=1}^{\infty}(-1)^{k-1}a_k/k$.*

3. *Assume that there exist $C>0, |p|_p^{1/(p-1)} > r > 0$ such that $|a_k|_p \le Cr^k$. Then $f(s)$ is analytic at $s=0$.*

Proof. When $|a_k|_p \to 0$, $f(s)$ converges uniformly. Hence the first assertion is clear. Assume that $|a_k/k|_p \to 0$. Then

$$\frac{df}{ds}(0) = \lim_{s\to 0}\frac{f(s)-f(0)}{s} = \lim_{s\to 0}\sum_{k=1}^{\infty} a_k\binom{s}{k}\frac{1}{s} = \lim_{s\to 0}\sum_{k=1}^{\infty}\frac{a_k}{k}\binom{s-1}{k-1}.$$

By the first assertion, $\sum_{k=1}^{\infty}\frac{a_k}{k}\binom{s-1}{k-1}$ is a continuous function, $\frac{df}{ds}(0) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{a_k}{k}$. To prove the third assertion, it suffices to show that there exist $b_n \in F_{\mathfrak{P}}(n=0,1,\dots)$ such that $f(s) = \sum_{n=0}^{\infty} b_n s^n$ for $s\in\mathbf{Z}_p$. Let b_n be the coefficient of s^n in $\sum_{k=0}^{\infty} a_k\binom{s}{k}$, i.e., $b_n = \sum_{k=n}^{\infty} a_k\frac{c_{n,k}}{k!}$ where $c_{n,k}$ is the coefficient of s^n for $s(s-1)\cdots(s-k+1)$. Since $|k!|_p \ge |p|_p^{k/(p-1)}$, the sum converges. Furthermore

$$|b_n|_p \le \sup_{k=n,n+1,\dots}\left|a_k\frac{c_{n,k}}{k!}\right|_p \le \sup_{k=n,n+1,\dots} C\left(r|p|_p^{\frac{-1}{p-1}}\right)^k = C\left(r|p|_p^{\frac{-1}{p-1}}\right)^n.$$

Hence $\sum_{n=0}^{\infty} b_n s^n$ converges. We have, for every $N \geq 1$,

$$\begin{aligned}\left|f(s)-\sum_{n=0}^{\infty} b_n s^n\right|_p &\leq \left|\sum_{k=N+1}^{\infty} a_k \binom{s}{k}\right|_p \\ &\quad + \left|\sum_{n=N+1}^{\infty} b_n s^n\right|_p + \left|\sum_{n=0}^{N}\sum_{k=N+1}^{\infty} a_k \frac{c_{n,k}}{k!} s^n\right|_p \\ &\leq C r^{N+1} + C\left(r|p|_p^{\frac{-1}{p-1}}\right)^{N+1} + C\left(r|p|_p^{\frac{1}{p-1}}\right)^{N+1}.\end{aligned}$$

The right hand side tends to 0 when $N \to \infty$ and the third assertion follows. □

Lemma 4.2. *Let $f_n(s) = \sum_{k=0}^{\infty} a_{n,k}\binom{s}{k}$ with $a_{n,k} \in \mathbf{C}_p$ and $f(s) = \sum_{n=1}^{\infty} f_n(s)$ such that $|a_{n,k}|_p \to 0$ when $k \to \infty$ and $\sup_{s\in\mathbf{Z}_p} |f_n(s)|_p \to 0$ when $n \to \infty$. Then*

1. *$f(s)$ is a continuous function on $\mathbf{Z}_p$.*
2. *Assume that $\sup_{n=1,2,\dots} |a_{n,k}/k|_p \to 0$ when $k \to \infty$. Then $f(s)$ is differentiable at $s = 0$ and $\frac{df}{ds}(0) = \sum_{n=1}^{\infty} \frac{df_n}{ds}(0) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{a_{n,k}}{k}$.*
3. *Assume that there exist $C > 0, |p|_p^{1/(p-1)} > r > 0$ such that $\sup_{n=1,2,\dots} |a_{n,k}|_p \leq Cr^k$. Then $f(s)$ is analytic at $s = 0$.*

Proof. Since $\sup_{k=0,1,\dots} |a_{n,k}|_p = \sup_{s\in\mathbf{Z}_p} |f_n(s)|_p \to 0$ when $n \to \infty$,

$$f(s) = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} a_{n,k}\binom{s}{k} = \sum_{k=0}^{\infty}\left(\sum_{n=1}^{\infty} a_{n,k}\right)\binom{s}{k}.$$

By Lemma 4.1, the assertions are clear. □

Lemma 4.3. *Let $P(X), Q(X) \in \mathbf{O}_{\mathbf{C}_p}[X_1, X_2, \dots, X_r]$, $b \geq 1$, $(b,p) = 1$ and $\xi_i \neq 1$ $(1 \leq i \leq r)$ be b-th roots of unity. Assume that $P(X) = Q(\Pi X)$ with an element $\Pi \in \mathbf{M}_p$ and $P(0) \equiv 1 \bmod \Pi\mathbf{O}_{\mathbf{C}_p}$. We can define continuous functions on $\mathbf{Z}_p$:*

$$\lambda_{k,P}(s) = \lambda_k(P^s) := \sum_{l_1=0}^{k_1}\cdots\sum_{l_r=0}^{k_r} \left\{ {k_1 \atop l_1} \right\} \cdots \left\{ {k_r \atop l_r} \right\} P(-l)^s,$$

$$\zeta_{p,r}(s, P, \xi) := \sum_{k\in\{0,1,\dots\}^r, k\neq 0} \frac{\lambda_{k,P}(-s)}{(1-\xi)^k},$$

and $\lambda_{k,P}(s)$ and $\zeta_{p,r}(s,P,\xi)$ are analytic at $s=0$. Furthermore

$$\frac{d\lambda_{k,P}}{ds}(0)=\sum_{l_1=0}^{k_1}\cdots\sum_{l_r=0}^{k_r}\left\{ {k_1 \atop l_1} \right\}\cdots\left\{ {k_r \atop l_r} \right\}\log_p(P(-l)),$$

$$\sup_{s\in\mathbf{Z}_p}|\lambda_{k,P}(s)|_p\leq|\Pi|_p^{k_1+\cdots+k_r-r_1},$$

$$\left[\frac{d}{ds}\zeta_{p,r}(s,P,\xi)\right]_{s=0}$$
$$=-\sum_{0\neq k\in\{0,1,\dots\}^r}\frac{\sum_{l_1=0}^{k_1}\cdots\sum_{l_r=0}^{k_r}\left\{ {k_1 \atop l_1} \right\}\cdots\left\{ {k_r \atop l_r} \right\}\log_p(P(-l))}{(1-\xi)^k}.$$

Here $r_1=\#\{j\mid k_j\neq 0, 1\leq j\leq r\}$.

Proof. For all $l\in\mathbf{Z}^r$, $P(l)=Q(\Pi l)\equiv Q(0)=P(0)\equiv 1 \bmod \Pi\mathbf{O}_{\mathbf{C}_p}$. Then we can define $P(-l)^s=\sum_{k=0}^{\infty}(P(-l)^{p^L}-1)^k\binom{s'}{k}$, where $s=p^Ls'$. Let $L=0$. Then

$$\left|\frac{(P(-l)-1)^k}{k}\right|_p\leq\left|\frac{\Pi^k}{k}\right|_p\to 0,\quad \text{when } k\to\infty.$$

By Lemma 4.1, $P(-l)^s$ and $\lambda_{k,P}(s)$ are continuous functions on $\mathbf{Z}_p$ and are differentiable at $s=0$. Furthermore

$$\left.\frac{dP(-l)^s}{ds}\right|_{s=0}=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(P(-l)-1)^k}{k}=\log_p(P(-l)),$$

$$\frac{d\lambda_{k,P}}{ds}(0)=\sum_{l_1=0}^{k_1}\cdots\sum_{l_r=0}^{k_r}\left\{ {k_1 \atop l_1} \right\}\cdots\left\{ {k_r \atop l_r} \right\}\log_p(P(-l)).$$

Let $\epsilon=|p|_p^L\sup_{i=1,2,\dots}\{|\Pi^i/i|_p\}$ and L be large enough to satisfy that $\epsilon<|p|_p^{1/(p-1)}$. Then $|(P(-l)^{p^L}-1)^k|_p=\left|p^L\sum_{i=1}^{p^L}\binom{p^L-1}{i-1}\frac{(P(-l)-1)^i}{i}\right|_p^k\leq\epsilon^k$. Hence, by Lemma 4.1, $P(-l)^s$ and $\lambda_{k,P}(s)$ are analytic at $s=0$. By Lemma 4.2, it suffices to show that $\sup_{s\in\mathbf{Z}_p}|\lambda_{k,P}(s)|_p\leq|\Pi|_p^{k_1+\cdots+k_r-r_1}$. Because $\lambda_{k,P}(s)$ is a continuous function and $\{0,1,2,\dots\}$ is dense in $\mathbf{Z}_p$, it suffices to show $\sup_{n=0,1,\dots}|\lambda_{k,P}(n)|_p\leq|\Pi|_p^{k_1+\cdots+k_r-r_1}$. Furthermore, since $P(X)^n$ is a polynomial if $n=0,1,\dots$ and λ_k has linearity, it suffices to show that

$$\text{for all } Q(X)=\prod_{i=1}^{r}X_i^{n_i}(n_i\geq 0),\ |\lambda_k(Q(\Pi X))|_p\leq|\Pi|_p^{k_1+\cdots+k_r-r_1}.$$

If there exists i such that $k_i=0$, $n_i\neq 0$, then $\lambda_k(Q(\Pi X))=0$. Therefore we may assume that if $k_i=0$ then $n_i=0$. Let $\{k_{i_1},\dots,k_{i_{r_1}}\}$ be the set of all

$k_i \neq 0$. Then

$$\lambda_k(Q(\Pi X)) = (-\Pi)^{n_{i_1}+\cdots+n_{i_{r_1}}} \prod_{j=1}^{r_1} \sum_{l_{i_j}=1}^{k_{i_j}} \binom{k_{i_j}-1}{l_{i_j}-1} (-1)^{l_{i_j}-1} l_{i_j}^{n_{i_j}},$$

$$\sum_{l_{i_j}=1}^{k_{i_j}} \binom{k_{i_j}-1}{l_{i_j}-1} (-1)^{l_{i_j}-1} l_{i_j}^{n_{i_j}} = \frac{-1}{k_{i_j}} \sum_{l_{i_j}=1}^{k_{i_j}} \binom{k_{i_j}}{l_{i_j}} (-1)^{l_{i_j}} l_{i_j}^{n_{i_j}+1} = 0$$
$$\text{if } n_{i_j}+1 < k_{i_j}.$$

Hence $\lambda_k(Q(\Pi X)) = 0$ if $\sum_j n_{i_j} < \sum_j k_{i_j} - r_1$. Therefore, if $\sum_i n_i < \sum_i k_i - r_1$ then $|\lambda_k(Q(\Pi X))|_p = 0 \leq |\Pi|_p^{\sum_i k_i - r_1}$ and if $\sum_i n_i \geq \sum_i k_i - r_1$ then $|\lambda_k(Q(\Pi X))|_p \leq |(-\Pi)^{\sum_i n_i}|_p \leq |\Pi|_p^{\sum_i k_i - r_1}$. $\square$

For an $n \times r$-matrix $A = (a_{j,i})$ $(a_{j,i} \in \mathbf{M}_p)$ and an r-row vector $x = (x_1, \ldots, x_r)$ $(x_i \in \mathbf{O}_{\mathbf{C}_p})$ with $\prod_{j=1}^n \sum_{i=1}^r a_{j,i} x_i \equiv 1 \bmod \mathbf{M}_p$, we define a polynomial

$$P_{A,x}(X) = \prod_{j=1}^n \left\{ \sum_{i=1}^r a_{j,i}(X_i + x_i) \right\}.$$

Then $P_{A,x}$ satisfies the conditions for Lemma 4.3. Let b be a natural number prime to p. For b-th roots of unity $\xi_i \neq 1$, we define a p-adic counterpart of $\zeta_r(s, A, \xi, x)$: for $s \in \mathbf{Z}_p$,

$$\zeta_{p,r}(s, A, \xi, x) = \zeta_{p,r}(s, P_{A,x}, \xi). \tag{4.1}$$

Note that $\lambda_k(P) = 0$ if $k_1 + k_2 + \cdots + k_r > \deg P + r$. According to [6, Theorem 22, Lemma 19], we get the next Theorem.

Theorem 4.1. *Let $F \subset \mathbf{R}$ be an algebraic number field and fix an embedding $F \to \mathbf{C}_p$. Let $0 < a_{j,i} \in F$ and $0 \leq x_i \in F$ such that $|a_{j,i}|_p < 1$, $|a_{j,1}x_1 + \cdots + a_{j,r}x_r - 1|_p < 1$. Let b be a natural number prime to p, $\xi_i \neq 1$ b-th roots of unity and $\xi = (\xi_1, \ldots, \xi_r)$. Then for $0 \leq m \in \mathbf{Z}$,*

$$\zeta_{p,r}(-m, A, \xi, x) = \zeta_r(-m, A, \xi, x),$$

Now we will define a p-adic analogue of the partial ζ-function. Let $F, \mathfrak{f}, \mathfrak{c}$ be as in Section 3 and $\mathfrak{a}$ an integral ideal $\in \mathfrak{c}^{-1}$. Assume that every prime ideal $\mathfrak{p}$ above (p) divides $\mathfrak{f}$. With minor variation from Lemma 3.1, we can take a finite set J and totally positive elements $v_{j,i} \in \mathfrak{a}\mathfrak{f}$ for $j \in J, i = 1, 2, \ldots, r(j)$ so that $v_{j,1}, \ldots, v_{j,r(j)}$ are linearly independent for each j and

$$\mathbf{R}^{+n} = \bigsqcup_{j \in J} \bigsqcup_{u \in E_{\mathfrak{f}}^+} uC(v_{j,1}, v_{j,2}, \ldots, v_{j,r(j)}). \tag{4.2}$$

For integral ideals $\mathfrak{a}, \mathfrak{b}$ prime to $\mathfrak{f}$, we define:

$$\zeta_{\mathfrak{f}}(s, \mathfrak{a}^{-1}, \mathfrak{b}) = N\mathfrak{b}^{1-s}\zeta_{\mathfrak{f}}(s, \overline{\mathfrak{a}^{-1}\mathfrak{b}^{-1}}) - \zeta_{\mathfrak{f}}(s, \overline{\mathfrak{a}^{-1}}).$$

Now we assume that $\mathfrak{b}$ satisfies the following conditions:

$$\text{(4.3)} \qquad \begin{array}{rl} \text{(i)} & (\mathfrak{b}, \mathfrak{f}) = 1, (\mathfrak{b}, \mathfrak{D}) = 1, \\ \text{(ii)} & (\mathfrak{b}, (v_{i,j})) = 1 \text{ for all } j \in J, i = 1, 2, \ldots, r(j), \\ \text{(iii)} & O_F/\mathfrak{b} \simeq \mathbf{Z}/b\mathbf{Z}. \end{array}$$

Here $\mathfrak{D}$ denotes the different of F over $\mathbf{Q}$ and b is the positive generator of $\mathfrak{b} \cap \mathbf{Z}$. Let $\mathbf{e}(x) = \exp(2\pi i x)$. According to [6, Lemmas 2a and 2b, Corollaries 1 and 2],

Lemma 4.4. *Take $\nu \in \mathfrak{D}^{-1}\mathfrak{b}^{-1}$ so that $Tr(\nu) = c/b$ with $0 \neq c \in \mathbf{Z}$, $(b, c) = 1$. Then*

$$\mathbf{e}(Tr(v_{i,j}\nu)) \text{ is a primitive } b\text{-th root of unity for every } i, j \text{ and}$$
$$\sum_{\mu=0}^{b-1} \mathbf{e}(Tr(\alpha\mu\nu)) = \begin{cases} 0 & \text{if } \alpha \in O_F, \notin \mathfrak{b}, \\ N\mathfrak{b} = b & \text{if } \alpha \in \mathfrak{b}. \end{cases}$$

Theorem 4.2. *With the above notation, let $R'(\mathfrak{a}^{-1}, j) = \{z \in C(v_j) \cap O_F \mid 0 < x_i(z) \leq 1, z \in \mathfrak{a}, \equiv 1 \bmod \mathfrak{f}\}$, $\xi_j = (\xi_{j,1}, \ldots, \xi_{r(j)})$, $\xi_{j,i} = \mathbf{e}(Tr(v_{j,i}\nu))$ and $\xi_z = \mathbf{e}(Tr(\nu \sum_{i=1}^{r(j)} x_i(z)v_{j,i}))$ for $z \in R'(\mathfrak{a}^{-1}, j)$. Then*

$$\zeta_{\mathfrak{f}}(s, \mathfrak{a}^{-1}, \mathfrak{b}) = N\mathfrak{a}^s \sum_{\mu=1}^{b-1} \sum_{j \in J} \sum_{z \in R'(\mathfrak{a}^{-1}, j)} \xi_z^{\mu} \zeta_{r(j)}(s, A_j, \xi_j^{\mu}, x(z)).$$

Now we define a function on $s \in \mathbf{Z}_p$:

$$\text{(4.4)} \qquad \zeta_{p,\mathfrak{f}}(s, \mathfrak{a}^{-1}, \mathfrak{b}) = \langle N\mathfrak{a}\rangle^s \sum_{\mu=1}^{b-1} \sum_{j \in J} \sum_{z \in R'(\mathfrak{a}^{-1}, j)} \xi_z^{\mu} \zeta_{p,r(j)}(s, A_j, \xi_j^{\mu}, x(z)).$$

By Theorems 4.2 and 4.1, we have

$$\text{(4.5)} \qquad \zeta_{p,\mathfrak{f}}(-m, \mathfrak{a}^{-1}, \mathfrak{b}) = \theta_p(N\mathfrak{a})^m \zeta_{\mathfrak{f}}(-m, \mathfrak{a}^{-1}, \mathfrak{b}) \quad (m \geq 0).$$

We can take $\mathfrak{b}$ in $P_{\mathfrak{f}+}$. Then there exists $\alpha \in F_{\mathfrak{f}+} \cap O_F$ such that $\mathfrak{b} = (\alpha)$. Since $\overline{\mathfrak{a}\mathfrak{b}} = \overline{\mathfrak{a}}$ in $C_{\mathfrak{f}}$, $\zeta_{\mathfrak{f}}(s, \mathfrak{a}^{-1}, \mathfrak{b}) = (b^{1-s} - 1)\zeta_{\mathfrak{f}}(s, \overline{\mathfrak{a}}^{-1})$. We define the p-adic partial ζ-function of the class $\mathfrak{c}$:

$$\text{(4.6)} \qquad \zeta_{p,\mathfrak{f}}(s, \mathfrak{c}) = \frac{\zeta_{p,\mathfrak{f}}(s, \mathfrak{a}^{-1}, \mathfrak{b})}{(b^{1-s} - 1)},$$

for $\mathfrak{a} \in \mathfrak{c}$, $\mathfrak{b} \in P_{\mathfrak{f}+}$. Here b^{1-s} is well-defined since $b = N(\alpha) \equiv 1 \bmod (p)$. By (4.5),

$$\zeta_{p,\mathfrak{f}}(-m, \mathfrak{c}) = \zeta_{\mathfrak{f}}(-m, \mathfrak{c}) \quad \text{for } m \geq 0, m \equiv 0 \bmod (p-1). \tag{4.7}$$

Let χ be a character of $C_{\mathfrak{f}}$. Then we define the L-function:

$$L(s, \chi) = L_{\mathfrak{f}}(s, \chi) = \sum_{\mathfrak{c} \in C_{\mathfrak{f}}} \chi(\mathfrak{c}) \zeta_{\mathfrak{f}}(s, \mathfrak{c}).$$

We define the p-adic L-function:

$$L_{p,\mathfrak{f}}(s, \chi) = \sum_{\mathfrak{d} \in C_{\mathfrak{f}}} \chi_1(\mathfrak{c}) \zeta_{p,\mathfrak{f}}(s, \mathfrak{c}). \tag{4.8}$$

By (4.5), we obtain

$$L_{p,\mathfrak{f}}(1-m, \chi) = L_{\mathfrak{f}}(1-m, \chi_m) \quad (1 \leq m \in \mathbf{Z}). \tag{4.9}$$

Let $\mathfrak{f}_0$ be an integral ideal, $\mathfrak{f} = \mathfrak{f}_0 \times \prod_{\mathfrak{p} | (p), \mathfrak{p} \nmid \mathfrak{f}} \mathfrak{p}$ and χ be a character of $C_{\mathfrak{f}_0}$. Then χ_{-1} is a character of $C_{\mathfrak{f}}$. By (4.8),

$$L_{p,\mathfrak{f}}(s, \chi_{-1}) = \sum_{\mathfrak{c} \in C_{\mathfrak{f}}} \chi(\mathfrak{c}) \zeta_{p,\mathfrak{f}}(s, \mathfrak{c}). \tag{4.10}$$

We get a formula on the derivative of the p-adic partial ζ-function at $s = 0$.

Theorem 4.3. *Let $\mathfrak{f}$ be an integral ideal such that every prime ideal $\mathfrak{p}$ above (p) divides $\mathfrak{f}$, $\mathfrak{a} \in I_{\mathfrak{f}}$ an integral ideal, $\mathfrak{b} \neq (1)$ an integral ideal satisfying* (4.3), *J, $v_{j,i}$ as in* (4.2), *$R'(\mathfrak{a}^{-1}, j)$, ξ_z, ξ_j as in Theorem* 4.2. *Then*

$$\left[\frac{d}{ds} \zeta_{p,\mathfrak{f}}(s, \mathfrak{a}^{-1}, \mathfrak{b})\right]_{s=0} = \log_p(N\mathfrak{a}) \zeta_{p,\mathfrak{f}}(0, \mathfrak{a}^{-1}, \mathfrak{b})$$
$$+ \sum_{\mu=1}^{b-1} \sum_{j \in J} \sum_{z \in R'(\mathfrak{a}^{-1}, j)} \xi_z^{\mu} \sum_{k=1}^{n} \left[\frac{d}{ds} \zeta_{p,r(j)}(s, v_j^{(k)}, \xi_j^{\mu}, x(z))\right]_{s=0}.$$

Proof. By Lemma 4.3,

$$\left[\frac{d}{ds} \zeta_{p,r(j)}(s, A_j, \xi_j^{\mu}, x(z))\right]_{s=0}$$
$$= -\sum_{k=1}^{n} \sum_{0 \neq m \in \{0,1,\dots\}^{r(j)}} \frac{\sum_{l_1=0}^{m_1} \cdots \sum_{l_{r(j)}=0}^{m_{r(j)}} \left\{ {m_1 \atop l_1} \right\} \cdots \left\{ {m_{r(j)} \atop l_{r(j)}} \right\} \log_p(z + v_j^{(k)} {}^t(-l))}{(1 - \xi_j^{\mu})^m}$$
$$= \sum_{k=1}^{n} \left[\frac{d}{ds} \zeta_{p,r(j)}(s, v_j^{(k)}, \xi_j^{\mu}, x(z))\right]_{s=0}.$$

Then the assertion is clear. $\square$

5. p-adic multiple gamma functions

In this section, we will define the p-adic multiple gamma function as the derivative of the p-adic multiple ζ-function [4] at $s = 0$. Let r be an integer and $R[X] = R[X_1, \ldots, X_r]$ denote the polynomial ring over the ring R. For a map $f(X) : \mathbf{Z}_p^r \to \mathbf{C}_p$, we define:

$$J_X(f(X)) = \lim_{l_1\to\infty}\lim_{l_2\to\infty}\cdots\lim_{l_r\to\infty}\frac{1}{p^{l_1+l_2+\cdots+l_r}}\sum_{n_1=0}^{p^{l_1}-1}\sum_{n_2=0}^{p^{l_2}-1}\cdots\sum_{n_r=0}^{p^{l_r}-1} f(n). \tag{5.1}$$

Here $n = (n_1, \ldots, n_r)$. If $f(X) \in \mathbf{C}_p[X]$, the right hand side converges and we get the next Lemma.

Lemma 5.1. *For $x_i \in \mathbf{C}_p$ $(i = 1, 2, \ldots, r)$,*

$$J_X\left(\prod_{i=1}^{r}(X_i + x_i)^{m_i}\right) = \prod_{i=1}^{r} B_{m_i}(x_i) \quad (m_i \geq 0).$$

Proof. It suffices to show the assertion for the case $r = 1$. If $m = 0$, it is obvious. Assume that $m \neq 0$. Then

$$\lim_{l\to\infty}\frac{1}{p^l}\sum_{n=0}^{p^l-1} n^m = \lim_{l\to\infty}\frac{B_{m+1}(p^l) - B_{m+1}(0)}{(m+1)p^l}$$
$$= \frac{1}{m+1}\frac{dB_{m+1}(X)}{dX}\Big|_{X=0} = B_m(0),$$
$$B_m(x) = \sum_{l=0}^{m}\binom{m}{l}B_l x^{m-l} = J_X\left(\sum_{l=0}^{m}\binom{m}{l}X^l x^{m-l}\right) = J_X((X + x)^m).$$

$\square$

Thus we can define a $\mathbf{C}_p$-linear map:

$$J_X : \mathbf{C}_p[X] \to \mathbf{C}_p, \text{ sending } P(X) \in \mathbf{C}_p[X] \text{ to } J_X(P(X)) \in \mathbf{C}_p. \tag{5.2}$$

Suppose that $r = 1$ and let $P(X)$ be a polynomial $\in \mathbf{C}_p[X]$, $\binom{X}{m}$ denote the polynomial $\frac{X(X-1)\cdots(X-m+1)}{m!}$ for $m \geq 1$ and $\binom{X}{0} = 1$. Then $P(X)$ is a linear combination of $\left\{\binom{X}{m} \mid m = 0, 1, \ldots\right\}$, that is, there exist $a_m \in \mathbf{C}_p$ such that $P(X) = \sum_{m=0}^{M} a_m\binom{X}{m}$. Note that $\sup_{s\in\mathbf{Z}_p}|P(s)|_p = \sup_{i=0,1,\ldots}|a_i|_p$.

Lemma 5.2.

$$J_X\left(\binom{X}{m}\right) = \frac{(-1)^m}{m+1}.$$

Proof. If $m = 0$, it is obvious. Assume that $m \geq 1$. Since $\binom{X+1}{m+1} - \binom{X}{m+1} = \binom{X}{m}$,

$$J_X\left(\binom{X}{m}\right) = \lim_{l\to\infty}\frac{1}{p^l}\left(\binom{p^l}{m+1} - \binom{0}{m+1}\right) = \frac{d\binom{X}{m+1}}{dX}\Big|_{X=0} = \frac{(-1)^m}{m+1}.$$

□

By this lemma,

$$\begin{aligned}|J_X(P(X))|_p &\leq \sup_{m=0,1,\ldots,M}\left|a_i\frac{(-1)^m}{m+1}\right|_p \leq \sup_{m=0,1,\ldots,M}|a_i|_p \sup_{m=0,1,\ldots,M}\left|\frac{(-1)^m}{m+1}\right|_p \\ &\leq (M+1)\sup_{s\in\mathbf{Z}_p}|P(s)|_p.\end{aligned}$$

For general r, let $P(X) = \sum_{m_1=0,\ldots,m_r=0}^{M_1,\ldots,M_r} a_{m_1,\ldots,m_r} X_1^{m_1}\cdots X_r^{m_r} \in \mathbf{C}_p[X_1,\ldots, X_r]$. In this case, we can show that

$$|J_X(P(X))|_p \leq (M_1+1)\cdots(M_r+1)\sup_{s\in{\mathbf{Z}_p}^r}|P(s)|_p. \tag{5.3}$$

Now we define $\|P(X)\|_r = (M_1+1)\cdots(M_r+1)\sup_{s\in{\mathbf{Z}_p}^r}|P(s)|_p$ where $M_i = \sup_{a_{m_1,\ldots,m_r}\neq 0}\{m_i\}$. If $P(X) = 0$ let $\|P(X)\|_r = 0$. We define a space of continuous functions on $\mathbf{Z}_p^r$:

$$\mathbf{C}_p[X]' = \left\{P(X) : \mathbf{Z}_p^r \to \mathbf{C}_p \;\middle|\; P(X) = \sum_{n=1}^{\infty} P_n(X), \right.$$
$$\left. P_n(X) \in \mathbf{C}_p[X], \|P_n(X)\|_r \to 0 \text{ when } n\to\infty\right\}.$$

Then we can extend J_X to an $\mathbf{C}_p$-linear map on $\mathbf{C}_p[X]'$ sending $P(X) = \sum_{n=1}^{\infty} P_n(X)$ to $\sum_{n=1}^{\infty} J_X(P_n(X))$ where $P_n(X) \in K[X]$, $\|P_n(X)\|_r \to 0$ when $n\to\infty$. $J_X(f(X))$ converges when $J_X(f(pX+n'))$ converges for all $n' = (n'_1,\ldots,n'_r), n'_i = 0,1,\ldots,p-1$ $(i = 1,2,\ldots,r)$. Therefore we put

$$\mathbf{C}_p[X]'' = \{P(X) : \mathbf{Z}_p^r \to \mathbf{C}_p \mid P(pX+n') \in \mathbf{C}_p[X]'$$
$$\text{for all } n_i = 0,1,\ldots,p-1,\ (i = 1,2,\ldots,r)\},$$

and we can extend J_X:

$$J_X : \mathbf{C}_p[X]'' \to \mathbf{C}_p, \text{ sending } P(X) \in \mathbf{C}_p[X]''$$
$$\text{to } \frac{1}{p^r}\sum_{n'_1=0}^{p-1}\cdots\sum_{n'_r=0}^{p-1} J_X(P(pX+n')).$$

Let $a_i, a \in \mathbf{O}_{\mathbf{C}_p}, a_i \neq 0$ $(i = 1,2,\ldots,r)$ and $f(X) = \sum_{i=1}^r a_iX_i + a$. For $s \in \mathbf{Z}_p$, we define $P_s(X) = \theta_p^0(f(X))f(X)^r\left(\theta_p(f(X))^{-1}f(X)\right)^{-s}$, that is, for n' satisfying $\theta_p^0(f(n'))=0$ we put $P_s(pX+n')=0$ and for n' satisfying $\theta_p^0(f(n'))\neq 0$, we put

$$P_s(pX+n')=f(pX+n')^r\sum_{k=0}^{\infty}\binom{-s}{k}(\theta_p(f(n'))^{-1}f(pX+n')-1)^k.$$

Then

$$\begin{aligned}&\left\|f(pX+n')^r\binom{-s}{k}(\theta_p(f(n'))^{-1}f(pX+n')-1)^k\right\|_r\\&\qquad=(r+k+1)^r\sup_{x\in\mathbf{Z}_p^r}|f(px+n')^r\binom{-s}{k}(\theta_p(f(n'))^{-1}f(px+n')-1)^k|_p\\&\qquad\leq(r+k+1)^r|\langle f(n')\rangle-1|_p^k\to 0,\ \text{when } k\to\infty.\end{aligned}$$

Therefore $P_s(X)\in\mathbf{C}_p[X]''$. We define the p-adic multiple ζ-function:

$$\zeta_{p,r}(s,(a_1,a_2,\dots,a_r),a)=\frac{(-1)^rJ_X\left(P_s(X)\right)}{(r-s)(r-1-s)\cdots(1-s)a_1a_2\cdots a_r}. \tag{5.4}$$

In particular, for $0\neq a_i\in\mathbf{M}_p, a\notin\mathbf{M}_p$,

$$\zeta_{p,r}(s,(a_1,a_2,\dots,a_r),a)=\frac{(-1)^rJ_X\left(f(X)^r\left(\theta_p(a)^{-1}f(X)\right)^{-s}\right)}{(r-s)(r-1-s)\cdots(1-s)a_1a_2\cdots a_r}, \tag{5.5}$$

and for $0\neq a_i\in\mathbf{M}_p, a\in\mathbf{M}_p$,

$$\zeta_{p,r}(s,(a_1,a_2,\dots,a_r),a)=0. \tag{5.6}$$

We extend $\zeta_{p,r}(s,(a_1,a_2,\dots,a_r),a)$ on $0\neq a_i\in\mathbf{C}_p$, $a\in\mathbf{C}_p$ as follows. When $|a|_p>1$ or $|a_i|_p>1$, there exists the unique positive rational number α such that $\sup\{|a_1|_p,\dots,|a_r|_p,|a|_p\}=|p^{-\alpha}|_p$. Then we define:

$$\zeta_{p,r}(s,(a_1,a_2,\dots,a_r),a)=\zeta_{p,r}(s,(p^\alpha a_1,p^\alpha a_2,\dots,p^\alpha a_r),p^\alpha a). \tag{5.7}$$

This definition does not depend on the choice of p^α. We will check several properties of the p-adic multiple ζ-function.

Lemma 5.3. *Let $0\neq a_i\in\mathbf{C}_p$, $a\in\mathbf{C}_p$.*

1. *$\zeta_{p,r}(s,(a_1,\dots,a_r),a)$ is analytic at $s=0$, and continuous on $s\in\mathbf{Z}_p$.*
2. *There exist $0\neq a_i^{(k)}\in\mathbf{M}_p$, $a^{(k)}\in{\mathbf{O}_{\mathbf{C}_p}}^\times$ such that*

$$\zeta_{p,r}(s,(a_1,\dots,a_r),a)=\sum_{k=1}^{m}\zeta_{p,r}(s,(a_1^{(k)},\dots,a_r^{(k)}),a^{(k)}).$$

3. *For $c\in{\mathbf{O}_{\mathbf{C}_p}}^\times$*

$$\zeta_{p,r}(s,(ca_1,\dots,ca_r),ca)=\langle c\rangle^{-rs}\zeta_{p,r}(s,(a_1,\dots,a_r),a).$$

Proof. By definition of $\zeta_{p,r}$ and (5.3), the last assertion is clear. By (5.5), (5.6), (5.7) and

$$\zeta_{p,r}(s,(a_1,\ldots,a_r),a)=\sum_{n_1=0}^{p-1}\cdots\sum_{n_r=0}^{p-1}\zeta_{p,r}(s,(pa_1,\ldots,pa_r),a+n_1a_1+\cdots+n_ra_r),$$

the second assetion is also clear. Therefore to prove the first assertion, we can assume that $a_i\in\mathbf{M}_p, a\notin\mathbf{M}_p$. Take a non-negative integer L and put $s=p^Ls',\ s'\in\mathbf{Z}_p$. By definition,

$$\begin{aligned}&\zeta_{p,r}(s,(a_1,a_2,\ldots,a_r),a)(s-r)\cdots(s-1)a_1\cdots a_r\\ &\qquad=\sum_{k=0}^{\infty}J_X\left(f(X)^r((\theta_p(a)^{-1}f(X))^{p^L}-1)^k\right)\binom{-s'}{k}.\end{aligned}\tag{5.8}$$

By (5.3), we get

$$\begin{aligned}&\left|J_X\left(f(X)^r\left(\left(\theta_p(a)^{-1}f(X)\right)^{p^L}-1\right)^k\right)\right|_p\\ &\quad\le(r+p^Lk+1)^r\sup_{x\in\mathbf{Z}_p^r}\left\{\left|\left(p^L\sum_{i=1}^{p^L}\binom{p^L-1}{i-1}\frac{\left(\theta_p(a)^{-1}f(x)-1\right)^i}{i}\right)^k\right|_p\right\}.\end{aligned}\tag{5.9}$$

Now let L be large enough to satisfy $|p|_p^L\sup_{i=1,2,\ldots}\{|\Pi^i/i|_p\}<|p|_p^{1/(p-1)}$. By Lemma 4.1, $J_X\left(f(X)^r\left(\theta_p(a)^{-1}f(X)\right)^{-p^Ls'}\right)$ is analytic at $s'=0$. □

Let F be a totally real algebraic number field and fix an embedding $F\to\mathbf{C}_p$. Let $\mathfrak{P}$ be a prime ideal determined by this embedding and $f_{\mathfrak{P}}=[O_F/\mathfrak{P}:\mathbf{Z}/p\mathbf{Z}]$.

Theorem 5.1. *Let $a_i,a\in F$ be elements such that $a_i,a>0$, $|a_i|_p<1,|a|_p\le 1$ $(i=1,2,\ldots,r)$. Then for $0\le k\in\mathbf{Z}$, $k\equiv 0 \bmod p^{f_{\mathfrak{P}}}-1$,*

$$\zeta_{p,r}(-k,(a_1,a_2,\ldots,a_r),a)=\zeta_r(-k,(a_1,a_2,\ldots,a_r),a).$$

Moreover, if $|a-1|_p<1$, this equation holds for any $0\le k\in\mathbf{Z}$.

Proof. Take $x_i\in F$, $x_i>0$ $(i=1,\ldots,r)$ so that $a_1x_1+\cdots+a_rx_r=a$. (For example, $x_i=a/ra_i$.) According to [16, Corollary to Proposition 1] (or [2, §30]), for $m=1,2,\ldots,$

$$\begin{aligned}&\zeta_r(1-m,(a_1,a_2,\ldots,a_r),a)\\ &\qquad=(-1)^r(m-1)!\sum_{\substack{p_1+\cdots+p_r=m+r-1\\ p_i\ge 0}}\prod_{i=1,\ldots,r}\frac{B_{p_i}(x_i)a_i^{p_i-1}}{p_i!}.\end{aligned}$$

By Lemma 5.1,

$$\zeta_r(1-m,(a_1,a_2,\ldots,a_r),a)=\frac{(-1)^r J_X\left(f(X)^{m+r-1}\right)}{m(m+1)\cdots(m+r-1)a_1\cdots a_r}.$$

Hence the assertion is clear. □

Now we define the p-adic logarithmic r-ple Γ-function for $0\neq a_i\in\mathbf{C}_p$, $a\in\mathbf{C}_p$:

$$L\Gamma_{p,r}(a,(a_1,a_2,\ldots,a_r))=\left.\frac{d\zeta_{p,r}(s,(a_1,a_2,\ldots,a_r),a)}{ds}\right|_{s=0}. \tag{5.10}$$

By Lemma 3, (5.8) and Lemma 4.1 (b), for $0\neq a_i\in\mathbf{O}_{\mathbf{C}_p}, a\in\mathbf{O}_{\mathbf{C}_p}$,

(5.11)

$$\begin{aligned}
&L\Gamma_{p,r}(a,(a_1,a_2,\ldots,a_r))\\
&=\left(1+\frac{1}{2}+\cdots+\frac{1}{r}\right)\zeta_{p,r}(0,(a_1,a_2,\ldots,a_r),a)+\frac{(-1)^r}{p^r r!a_1a_2\cdots a_r}\\
&\times\sum_{\substack{0\leq n_i\leq p-1(i=0,\ldots,r)\\ \theta_p^0(f(n))\neq 0}}\sum_{k=1}^{\infty}\frac{(-1)^k J_X\left(f(pX+n)^r\left(\theta_p\left(f(n)\right)^{-1}f(pX+n)-1\right)^k\right)}{k}.
\end{aligned}$$

In particular, for $0\neq a_i\in\mathbf{M}_p$, $a\equiv 1 \bmod \mathbf{M}_p$,

(5.12)

$$\begin{aligned}
&L\Gamma_{p,r}(a,(a_1,a_2,\ldots,a_r))\\
&=\frac{(-1)^r}{r!a_1a_2\cdots a_r}J_X\left(f(X)^r\left(1+\frac{1}{2}+\cdots+\frac{1}{r}+\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left(f(X)-1\right)^k\right)\right).
\end{aligned}$$

We will check several properties of the p-adic logarithmic multiple Γ-function.

Lemma 5.4.

1. $L\Gamma_{p,r}(a,(a_1,\ldots,a_r))$ *is a continuous function on* $0\neq a_i\in\mathbf{C}_p$, $a\in\mathbf{C}_p$.

2. *There exist* $0\neq a_i^{(k)}\in\mathbf{M}_p$, $a^{(k)}\in 1+\mathbf{M}_p$ $(i=1,\ldots,r,\ k=1,\ldots,m)$ *such that*

$$L\Gamma_{p,r}(a,(a_1,\ldots,a_r))=\sum_{k=1}^{m}L\Gamma_{p,r}(a^{(k)},(a_1^{(k)},\ldots,a_r^{(k)})).$$

3. *For* $c\in{\mathbf{O}_{\mathbf{C}_p}}^{\times}$,

$$L\Gamma_{p,r}(ca,(ca_1,\ldots,ca_r))=L\Gamma_{p,r}(a,(a_1,\ldots,a_r))-\log_p(c)\zeta_{p,r}(0,(a_1,\ldots,a_r),a).$$

Proof. By definition, the last assertion is clear. By Lemma 3, there exist $0 \neq a_i^{(k)} \in \mathbf{M}_p, a^{(k)} \in \mathbf{O}_{\mathbf{C}_p}{}^{\times}$ such that

$$L\Gamma_{p,r}(a,(a_1,\dots,a_r)) = \sum_{k=1}^{m} L\Gamma_{p,r}(a^{(k)},(a_1^{(k)},\dots,a_r^{(k)})).$$

By the last assertion,

$$L\Gamma_{p,r}(a^{(k)},(a_1^{(k)},\dots,a_r^{(k)})) = L\Gamma_{p,r}(\langle a^{(k)}\rangle,(\theta_p(a^k)^{-1}a_1^{(k)},\dots,\theta_p(a^k)^{-1}a_r^{(k)})),$$

therefore we get the second assertion. Since $\log_p$ and $\zeta_{p,r}(0,(a_1,\dots,a_r),a)$ are continuous functions, we may assume that $a_i \in \mathbf{M}_p, a \equiv 1 \bmod \mathbf{M}_p$ to prove the first assertion. In this case, by (5.12), (5.3), we get this assertion. □

Lemma 5.5. *If $r \geq 2$, then*

$$\begin{aligned}\zeta_{p,r}(s,(a_1,a_2,\dots,a_r),a) &- \zeta_{p,r}(s,(a_1,a_2,\dots,a_r),a+a_i)\\ &= \zeta_{p,r-1}(s,(a_1,\dots,a_{i-1},a_{i+1},\dots,a_r),a),\end{aligned}$$

$$\begin{aligned}L\Gamma_{p,r}(a,(a_1,a_2,\dots,a_r)) &- L\Gamma_{p,r}(a+a_i,(a_1,a_2,\dots,a_r))\\ &= L\Gamma_{p,r-1}(a,(a_1,\dots,a_{i-1},a_{i+1},\dots,a_r)).\end{aligned}$$

If $r = 1$, then

$$\zeta_{p,1}(s,(a_1),a) - \zeta_{p,1}(s,(a_1),a+a_1) = \begin{cases}\langle p^{\alpha}a\rangle^{-s} & \text{if } |a|_p > |a_1|_p,\ |a|_p > 1,\\ 0 & \text{if } |a|_p < |a_1|_p \text{ or } |a|_p < 1,\end{cases}$$

$$L\Gamma_{p,1}(a,(a_1)) - L\Gamma_{p,1}(a+a_1,(a_1)) = -\log_p(a),$$

where α is a rational number such that $|a|_p = |p^{-\alpha}|_p$. Furthermore, if $a \in \mathbf{Z}_p$, then

$$L\Gamma_{p,1}(a,(1)) = \log_p(\Gamma_p(a)).$$

Proof. Because of the definition (5.7), it suffices to show the assertions when $a_i, a \in \mathbf{O}_{\mathbf{C}_p}$. We assume that $a_i, a \in \mathbf{O}_{\mathbf{C}_p}$. Let $r \geq 2$, $k \geq 0$ and $f(X) = a_1X_1 + \cdots + a_rX_r + a$. We put $P_k(X) = \theta_p^0(f(X))\, f(X)^r \left(\theta_p(f(X))^{-1} f(X)\right)^{-k}$. Then

$$\begin{aligned}&(\zeta_{p,r}(-k,(a_1,a_2,\dots,a_r),a) - \zeta_{p,r}(-k,(a_1,a_2,\dots,a_r),a+a_i))\\ &\quad\times (r+k)(r+k-1)\cdots(k+1)a_1a_2\cdots a_r\\ &\quad= (-1)^r \lim_{\substack{l_j\to\infty\\ j\neq i}} \frac{1}{p^{\sum_{j\neq i} l_j}}\\ &\qquad\times \sum_{\substack{0\leq x_j\leq p^{l_j}-1\\ j\neq i}} \left(\lim_{l_i\to\infty} \frac{1}{p^{l_i}} \left(P_k(X)\mid_{X_i=0} - P_k(X)\mid_{X_i=p^{l_i}}\right)\right)\Big|_{\substack{X_j=x_j\\ j\neq i}}.\end{aligned}$$

Since

$$\begin{aligned}\lim_{l_i\to\infty}\frac{1}{p^{l_i}}&\left(P_k(X)\mid_{X_i=0}-P_k(X)\mid_{X_i=p^{l_i}}\right)\\&=-\theta_p\left(f(X)\mid_{X_i=0}\right)^{-k}\frac{\partial f(X)^{r+k}}{\partial X_i}\Big|_{X_i=0}\\&=-\theta_p\left(f(X)\mid_{X_i=0}\right)^{-k}(r+k)a_if(X)^{r+k-1}\mid_{X_i=0},\end{aligned}$$

$$\begin{aligned}&\zeta_{p,r}(-k,(a_1,a_2,\dots,a_r),a)-\zeta_{p,r}(-k,(a_1,a_2,\dots,a_r),a+a_{i_0})\\&=\frac{(-1)^{r-1}J_{X_1,\dots,X_{i-1},X_{i+1},\dots,X_r}\left(\theta_p\left(f(X)\mid_{X_i=0}\right)^{-k}f(X)^{r+k-1}\mid_{X_i=0}\right)}{(r+k-1)\cdots(k+1)a_1\cdots a_{i-1}a_{i+1}\cdots a_r}\\&=\zeta_{p,r-1}(-k,(a_1,\dots,a_{i-1},a_{i+1},\dots,a_r),a).\end{aligned}$$

Because $\{n\in\mathbf{Z}\mid n\leq 0\}$ is dense in $\mathbf{Z}_p$, the case of $r\leq 2$ is clear. If $r=1$, similarly we obtain the assertion

$$\zeta_{p,1}(s,(a_1),a)-\zeta_{p,1}(s,(a_1),a+a_1)=\theta_p^0(a)\left(\theta_p(a)^{-1}a\right)^{-s}.$$

By Lemma 4.1,

$$\begin{aligned}L\Gamma_{p,1}(a,(a_1))-L\Gamma_{p,1}(a+a_1,(a_1))&=-\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\left(\theta_p(a)^{-1}a-1\right)^k}{k}\\&=-\log_p(a).\end{aligned}$$

Since $\zeta_{p,1}(0,(1),0)=\frac{-1}{p}\sum_{n=1}^{p-1}J_X(pX+n)=0$, by (5.11),

$$L\Gamma_{p,1}(0,(1))=\frac{-1}{p}\sum_{n=1}^{p-1}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\lim_{l\to\infty}\frac{1}{p^l}\sum_{x=0}^{p^l-1}(px+n)\left(\theta_p(n)^{-1}(px+n)-1\right)^k.$$

Here

$$\begin{aligned}&\sum_{n=1}^{p-1}(px+n)(\theta_p(n)^{-1}(px+n)-1)^k\\&\quad=\sum_{n=1}^{\frac{p-1}{2}}((px+n)(\theta_p(n)^{-1}(px+n)-1)^k\\&\qquad+(px+p-n)(-\theta_p(n)^{-1}(px+p-n)-1)^k),\end{aligned}$$

$$\begin{aligned}&\sum_{x=0}^{p^l-1}(px+p-n)\left(-\theta_p(n)^{-1}(px+p-n)-1\right)^k\\&\quad=\sum_{x=0}^{p^l-1}(px-n)\left(-\theta_p(n)^{-1}(px-n)-1\right)^k+n\left(-\theta_p(n)^{-1}(-n)-1\right)^k\\&\qquad+(pp^l-n)\left(-\theta_p(n)^{-1}(pp^l-n)-1\right)^k,\end{aligned}$$

$$
\begin{aligned}
&(px+n)\left(\theta_p(n)^{-1}(px+n)-1\right)^k+(px-n)\left(-\theta_p(n)^{-1}(px-n)-1\right)^k \\
&\quad=\sum_{0\le i\le k, i:\text{even}} 2\binom{k}{i}\theta_p(n)^{-i}(px)^{i+1}\left(\theta_p(n)^{-1}n-1\right)^{k-i} \\
&\qquad+\sum_{0\le i\le k, i:\text{odd}} 2\binom{k}{i}\theta_p(n)^{-i}(px)^{i}n\left(\theta_p(n)^{-1}n-1\right)^{k-i}, \\
&n\left(-\theta_p(n)^{-1}(-n)-1\right)^k+(pp^l-n)\left(-\theta_p(n)^{-1}(pp^l-n)-1\right)^k \\
&\quad=\sum_{i=0}^{k}\binom{k}{i}\left(-\theta_p(n)^{-1}\right)^i(p^{l+1})^{i+1}\left(\theta_p(n)^{-1}n-1\right)^{k-i} \\
&\qquad-\sum_{i=1}^{k}\binom{k}{i}\left(-\theta_p(n)^{-1}\right)^i(p^{l+1})^{i}n\left(\theta_p(n)^{-1}n-1\right)^{k-i}.
\end{aligned}
$$

Since $B_{2m+1}=0$ for $m\ge 1$ and $\lim_{l\to\infty}(1/p^l)(p^{l+1})^i=0$ for $i\ge 2$,

$$
\begin{aligned}
&L\Gamma_{p,1}(0,(1)) \\
&=\sum_{n=1}^{\frac{p-1}{2}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{pk}\{2pB_1(\theta_p(n)^{-1}n-1)^k+2k\theta_p(n)^{-1}pB_1n(\theta_p(n)^{-1}n-1)^{k-1} \\
&\qquad\qquad+p(\theta_p(n)^{-1}n-1)^k+k\theta_p(n)^{-1}pn(\theta_p(n)^{-1}n-1)^{k-1}\} \\
&=0.
\end{aligned}
$$

Then for $n=0,1,\ldots$, $L\Gamma_{p,1}(n,(1))=\sum_{k=0}^{n-1}\log_p(k)=\log_p(\Gamma_p(n))$. Hence for all $a\in\mathbf{Z}_p$, $L\Gamma_{p,1}(a,(1))=\log_p(\Gamma_p(a))$. $\square$

6. Main Theorem

In this section, we will formulate a p-adic analogue of Shintani's formula, and show that there exists the strong analogy. Let the notations be as in Section 4. We transform Theorem 4.3 to the style of Shintani's formula.

Lemma 6.1. *Let the notations be as in Theorem* 4.3. *Then*

$$
\begin{aligned}
&\left[\frac{d}{ds}\zeta_{p,\mathfrak{f}}(s,\mathfrak{a}^{-1},\mathfrak{b})\right]_{s=0}=\log_p(N\mathfrak{a})\zeta_{p,\mathfrak{f}}(0,\mathfrak{a}^{-1},\mathfrak{b}) \\
&+\sum_{k=1}^{n}\sum_{j\in J}\sum_{z\in R'(\mathfrak{a}^{-1},j)}\left(b\sum_{n'}L\Gamma_{p,r(j)}((z+n'{}^tv_j)^{(k)},bv_j^{(k)})-L\Gamma_{p,r(j)}(z^{(k)},v_j^{(k)})\right),
\end{aligned}
$$

where n' extends over all $n'=(n'_1,\ldots,n'_{r(j)})$ such that $0\le n'_i\le b-1$ and $z+n'{}^tv_j\in\mathfrak{b}$.

Proof. By Theorem 4.3, it suffices to show that for each k, z,

$$\begin{aligned}(6.1)\quad &\sum_{\mu=1}^{b-1}\xi_z^{\mu}\zeta_{p,r(j)}(s,v_j^{(k)},\xi_j^{\mu},x(z))\\ &\qquad = b\sum_{n'}\zeta_{p,r(j)}(s,bv_j^{(k)},(z+n'^{t}v_j)^{(k)})-\zeta_{p,r(j)}(s,v_j^{(k)},z^{(k)}).\end{aligned}$$

By Lemma 4.4,

$$\sum_{\mu=1}^{b-1}\xi_z^{\mu}\xi_j^{\mu n}=\sum_{\mu=1}^{b-1}\mathbf{e}(Tr((x+n)v_j\times\mu\nu))=\begin{cases}-1 & \text{if } (x+n)v_j\notin\mathfrak{b},\\ b-1 & \text{if } (x+n)v_j\in\mathfrak{b}.\end{cases}$$

Therefore, by Theorems 5.1 and 4.1, the assertion is clear. □

Let $\mathfrak{a}, v_{j,i}, R'(\mathfrak{a}^{-1},C(v_j)), \mathfrak{b}=(\alpha), b$ be as for (4.2), (4.6) and Theorem 4.2. Now we consider two cone decompositions:

$$\mathbf{R}^{+n}=\bigsqcup_{j\in J}\bigsqcup_{u\in E_{\mathfrak{f}}^{+}}uC(bv_{j,1},bv_{j,2},\dots,bv_{j,r(j)}),$$

$$\mathbf{R}^{+n}=\bigsqcup_{j\in J}\bigsqcup_{u\in E_{\mathfrak{f}}^{+}}uC(\alpha v_{j,1},\alpha v_{j,2},\dots,\alpha v_{j,r(j)}),$$

and define $R'((\mathfrak{ab})^{-1},C(bv_j))$, $R'((\mathfrak{ab})^{-1},C(\alpha v_j))$ respectively as in Theorem 4.2. According to [19, Lemma 2], we can take a cone decomposition $\{C(v_{j'}) \mid C(v_{j'})=C(v_{j',1},\dots,v_{j',r(j')}), j'\in J'\}$ such that

1. $v_{j',i}\in\mathfrak{abf}$, totally positive.
2. $\{C(v_{j'})\mid j'\in J'\}$ is a refinement of $\{C(\alpha v_j)\mid j\in J\}$.
3. There exist $u_{j'}\in E_{\mathfrak{f}}^{+}$ such that $\{C(u_{j'}v_{j'})\mid j'\in J'\}$ is a refinement of $\{C(bv_j)\mid j\in J\}$.

Now $x^{t}(\alpha v_j)\in\mathfrak{ab}(=\alpha\mathfrak{a})$ is equivalent to $x^{t}v_j\in\mathfrak{a}$. If $x^{t}v_j\in O_F$, then $x^{t}(\alpha v_j)\equiv 1 \bmod \mathfrak{f}$ is equivalent to $x^{t}v_j\equiv 1 \bmod \mathfrak{f}$. Hence we get

$$R'((\mathfrak{ab})^{-1},C(\alpha v_j))=\{\alpha z\mid z\in R'(\mathfrak{a}^{-1},C(v_j))\}.$$

For $\sigma\in J_F$,

$$\begin{aligned}&\sum_{j\in J}\sum_{z\in R'(\mathfrak{a}^{-1},C(v_j))}\zeta_{p,r(j)}(s,v_j^{\sigma},z^{\sigma})\\ (6.2)\quad &\quad=\sum_{j\in J}\sum_{z\in R'((\mathfrak{ab})^{-1},C(\alpha v_j))}(\alpha^{\sigma})^{s}\zeta_{p,r(j)}(s,(\alpha v_j)^{\sigma},z^{\sigma})\\ &\quad=\sum_{j'\in J'}\sum_{z\in R'((\mathfrak{ab})^{-1},C(v_{j'}))}(\alpha^{\sigma})^{s}\zeta_{p,r(j')}(s,v_{j'}^{\sigma},z^{\sigma}).\end{aligned}$$

Let $z\in R'((\mathfrak{ab})^{-1},C(bv_j))$. Then $x(z)^{t}(bv_j)=(bx(z))^{t}v_j\in\mathfrak{a},\equiv 1 \bmod \mathfrak{f}$. Hence there exist $n\in\{0,1,\dots\}^{r(j)}$ and $z'\in R'(\mathfrak{a}^{-1},C(v_j))$ such that $bx(z)=x(z')+$

n. Here $0 < x_i(z) \le 1$ and $0 < x_i(z') \le 1$, then $0 \le n_i \le b-1$. Furthermore $x(z)^t(bv_j) = x(z')^t v_j + n^t v_j \in \mathfrak{b}$. Conversely, let $z' \in R'(\mathfrak{a}^{-1}, C(v_j))$ and $n \in \{0,1,\dots\}^{r(j)}$ satisfying that $0 \le n_i \le b-1$ and $x(z')^t v_j + n^t v_j \in \mathfrak{b}$. Let $x = (x(z')+n)/b$. Then $0 < x_i \le 1$ and $x^t(bv_j) = x(z')^t v_j + n^t v_j \equiv 1 \bmod \mathfrak{f}$. Since $\mathfrak{b}$ is prime to $(v_{j,i})$, $\mathfrak{b}$ is prime to $\mathfrak{a}$. Then $x^t(bv_j) = x(z')^t v_j + n^t v_j \in \mathfrak{a} \cap \mathfrak{b} = \mathfrak{ab}$. Therefore

$$\begin{aligned} &R'((\mathfrak{ab})^{-1}, C(bv_j)) \\ &\qquad = \{z + n^t v_j \mid z \in R'(\mathfrak{a}^{-1}, C(v_j)), 0 \le n_i \le b-1, z + n^t v_j \in \mathfrak{b}\}. \end{aligned}$$

For $\sigma \in J_F$,

(6.3)
$$\begin{aligned} &\sum_{j \in J} \sum_{z \in R'(\mathfrak{a}^{-1}, C(v_j))} \sum_{n'} \zeta_{p,r(j)}(s, bv_j^\sigma, (z + n'^t v_j)^\sigma) \\ &\qquad = \sum_{j \in J} \sum_{z \in R'((\mathfrak{ab})^{-1}, C(bv_j))} \zeta_{p,r(j)}(s, bv_j^\sigma, z^\sigma) \\ &\qquad = \sum_{j' \in J'} \sum_{z \in R'((\mathfrak{ab})^{-1}, C(v_{j'}))} (u_{j'}^\sigma)^{-s} \zeta_{p,r(j')}(s, v_{j'}^\sigma, z^\sigma), \end{aligned}$$

where n' extends over all $n' = (n'_1, \dots, n'_{r(j)})$ such that $0 \le n'_i \le b-1$ and $z + n'^t v_j \in \mathfrak{b}$. The following Lemma is a p-adic analogue of [20, Chapter IV, Theorem 6.2].

Lemma 6.2. *Let the notations be as above. Then for any $\sigma \in J_F$,*

$$\zeta_{p,\mathfrak{f}}(0, \mathfrak{c}) = \sum_{j \in J} \sum_{z \in R'(\mathfrak{a}^{-1}, j)} \zeta_{p,r(j)}(0, v_j^\sigma, z^\sigma).$$

Proof. By definition, for any j, σ, z,

$$\begin{aligned} \zeta_{p,r(j)}(0, A_j, \xi_j^\mu, x(z)) &= \sum_{k \in \{0,1,\dots\}^{r(j)}, k \neq 0} \frac{\sum_{l_1=0}^{k_1} \cdots \sum_{l_{r(j)}=0}^{k_{r(j)}} \left\{ {k_1 \atop l_1} \right\} \cdots \left\{ {k_{r(j)} \atop l_{r(j)}} \right\}}{(1-\xi)^k} \\ &= \zeta_{p,r(j)}(0, v_j^\sigma, \xi_j^\mu, x(z)). \end{aligned}$$

By (6.1), (6.2), (6.3),

$$\begin{aligned} &\sum_{\mu=1}^{b-1} \sum_{j \in J} \sum_{z \in R'(\mathfrak{a}^{-1}, j)} \xi_z^\mu \zeta_{p,r(j)}(0, v_j^\sigma, \xi_j^\mu, x(z)) \\ &\qquad = (b-1) \sum_{j \in J} \sum_{z \in R'(\mathfrak{a}^{-1}, j)} \zeta_{p,r(j)}(0, v_j^\sigma, z^\sigma). \end{aligned}$$

Then by (4.4), (4.6),

$$\begin{aligned}\zeta_{p,\mathfrak{f}}(0,\mathfrak{c}) &= \frac{1}{b-1}\sum_{\mu=1}^{b-1}\sum_{j\in J}\sum_{z\in R'(\mathfrak{a}^{-1},j)} \xi_z^{\mu}\zeta_{p,r(j)}(0,A_j,\xi_j^{\mu},x(z)) \\ &= \frac{1}{b-1}\sum_{\mu=1}^{b-1}\sum_{j\in J}\sum_{z\in R'(\mathfrak{a}^{-1},j)} \xi_z^{\mu}\zeta_{p,r(j)}(0,v_j^{\sigma},z^{\sigma}) \\ &= \sum_{j\in J}\sum_{z\in R'(\mathfrak{a}^{-1},j)} \zeta_{p,r(j)}(0,v_j^{\sigma},z^{\sigma}).\end{aligned}$$

□

Now we get a p-adic analogue of Shintani's formurla which expresses the derivative at $s=0$ of the p-adic partial ζ-function in terms of the p-adic multiple Γ-function and some correction terms.

Theorem 6.1. *Let $\mathfrak{f},\mathfrak{c},\mathfrak{a},v_{j,i},R'(\mathfrak{a}^{-1},C(v_j)),\mathfrak{b}=(\alpha),b$ be as for (4.2), (4.6) and Theorem 4.2. Let $\{C(v_{j'})|j'\in J'\}$ be a cone decomposition such that*

1. *$v_{j',i}\in\mathfrak{af}$, totally positive.*
2. *$\{C(v_{j'})\mid j'\in J'\}$ is a refinement of $\{C(v_j)\mid j\in J\}$.*
3. *There exist $u_{j'}\in E_{\mathfrak{f}}^{+}$ such that $\{C(u_{j'}v_{j'})\mid j'\in J'\}$ is a refinement of $\{C((b/\alpha)v_j)\mid j\in J\}$.*

Then

$$\left[\frac{d}{ds}\zeta_{p,\mathfrak{f}}(s,\mathfrak{c})\right]_{s=0} = \log_p(N\mathfrak{a})\zeta_{p,\mathfrak{f}}(0,\mathfrak{c}) + \sum_{k=1}^{n}\sum_{j\in J}\sum_{z\in R'(\mathfrak{a}^{-1},C_j)} L\Gamma_{p,r(j)}(z^{(k)},v_j^{(k)})$$
$$+\frac{-b}{b-1}\sum_{k=1}^{n}\sum_{j'\in J',u_{j'}\neq 1}\sum_{z'\in R'(\mathfrak{a}^{-1},C_{j'})} \log_p(u_{j'}^{(k)})\zeta_{p,r(j')}(0,v_{j'}^{(k)},z'^{(k)}).$$

Proof. By Lemma 6.1,

$$\left[\frac{d}{ds}\zeta_{p,\mathfrak{f}}(s,\mathfrak{c})\right]_{s=0} = \left(\log_p(N\mathfrak{a})+\frac{b\log_p(b)}{b-1}\right)\zeta_{p,\mathfrak{f}}(0,\mathfrak{c}) + \sum_{k=1}^{n}\sum_{j\in J}\sum_{z\in R'(\mathfrak{a}^{-1},C(v_j))}$$
$$\frac{1}{b-1}\left(b\sum_{n'} L\Gamma_{p,r(j)}((z+n'^{t}v_j)^{(k)},bv_j^{(k)}) - L\Gamma_{p,r(j)}(z^{(k)},v_j^{(k)})\right),$$

where n' extends over all $n'=(n'_1,\dots,n'_{r(j)})$ such that $0\le n'_i\le b-1$ and $z+n'v_j\in\mathfrak{b}$. Let $\{C(v'_{j'})\mid j'\in J'\}$ be as above. Then $\{C(\alpha v_{j'})\mid j'\in J'\}$ satisfies that

1. $\alpha v_{j',i}\in\mathfrak{abf}$, totally positive.
2. $\{C(\alpha v_{j'})\mid j'\in J'\}$ is a refinement of $\{C(\alpha v_j)\mid j\in J\}$.
3. There exist $u_{j'}\in E_{\mathfrak{f}}^{+}$ such that $\{C(u_{j'}\alpha v_{j'})\mid j'\in J'\}$ is a refinement of $\{C(bv_j)\mid j\in J\}$.

Therefore, by (6.2) and (6.3),

$$\begin{aligned}
&\sum_{j\in J}\sum_{z\in R'(\mathfrak{a}^{-1},C(v_j))}\sum_{n'} L\Gamma_{p,r(j)}((z+n'^t v_j)^{(k)}, bv_j^{(k)}) \\
&\quad=\sum_{j\in J}\sum_{z\in R'(\mathfrak{a}^{-1},C(v_j))} L\Gamma_{p,r(j)}(z^{(k)}, v_j^{(k)}) \\
&\quad\quad-\log_p(\alpha^{(k)})\sum_{j\in J}\sum_{z\in R'(\mathfrak{a}^{-1},C(v_j))} \zeta_{p,r(j)}(0, v_j^{(k)}, z^{(k)}) \\
&\quad\quad+\sum_{j'\in J'}\sum_{z\in R'(\mathfrak{a}^{-1},C(v_{j'}))} (-\log_p(u_{j'}^{(k)}))\zeta_{p,r(j')}(0, v_{j'}^{(k)}, z^{(k)}).
\end{aligned}$$

Hence, by Lemma 6.2 and $b=\prod_{k=1}^n \alpha^{(k)}$, the assertion is clear. □

Both of Theorems 3.1 and 6.1 express the derivative of the (p-adic) partial ζ-function using the (p-adic) multiple Γ-function, but the correction terms seem slightly different. However, the next theorem shows the storong analogy between two formulas.

Theorem 6.2. *Let the notations be as in Theorem* 3.1*, and assume that every prime ideal* $\mathfrak{p}$ *above* (p) *divides* $\mathfrak{f}$*. We can take* $a_q, b_q \in \overline{\mathbf{Q}}^\times$ $(q=1,\ldots,m)$ *so that* $T_0=\sum_{q=1}^m a_q\log(b_q)$*. So we get*

$$\begin{aligned}
\left[\frac{d}{ds}\zeta_{\mathfrak{f}}(s,\mathfrak{c})\right]_{s=0} &=\sum_{k=1}^n\sum_{j\in J}\sum_{z\in R(\mathfrak{c},C(v_j))} \log\left(\frac{\Gamma_{r(j)}(z^{(k)}, v_j^{(k)})}{\rho_{r(j)}(v_j^{(k)})}\right) \\
&\quad-\log(N\mathfrak{a}_\mu\mathfrak{f})\zeta_{\mathfrak{f}}(0,\mathfrak{c})+\sum_{q=1}^m a_q\log(b_q).
\end{aligned}$$

Using the same a_q, b_q*, we can transcribe the correction terms of the* p*-adic analogue of Shintani's formula in the same manner as in the original Shintani's formula, that is,*

$$\begin{aligned}
\left[\frac{d}{ds}\zeta_{p,\mathfrak{f}}(s,\mathfrak{c})\right]_{s=0} &=\sum_{k=1}^n\sum_{j\in J}\sum_{z\in R(\mathfrak{c},C(v_j))} L\Gamma_{p,r(j)}(z^{(k)}, v_j^{(k)}) \\
&\quad-\log_p(N\mathfrak{a}_\mu\mathfrak{f})\zeta_{p,\mathfrak{f}}(0,\mathfrak{c})+\sum_{q=1}^m a_q\log_p(b_q).
\end{aligned}$$

Proof. Let $\mathfrak{a}_\mu, v_{j,i}, R(\mathfrak{c}, C(v_j))$ be as for Lemmas 3.1 and 3.2. Let $\{\epsilon\}$ denote a set of units which represent $E_{(1)}^+/E_{\mathfrak{f}}^+$. For each $\mathfrak{c}$ and the corresponding $\mathfrak{a}_\mu$, fix an integral ideal $\mathfrak{a}\in\mathfrak{c}^{-1}$ and an elemnt $\alpha_{\mathfrak{a}}\in\mathfrak{a}_\mu\mathfrak{f}\mathfrak{a}$ such that $(\alpha_{\mathfrak{a}})=\mathfrak{a}_\mu\mathfrak{f}\mathfrak{a}$. For each $z\in R(\mathfrak{c}, C(v_j))$, let ϵ_z denote the unique unit in $\{\epsilon\}$ such that $z\alpha_{\mathfrak{a}}\epsilon_z\equiv 1 \bmod \mathfrak{f}$. Then a cone decomposition:

$$\mathbf{R}^{+n}=\bigsqcup_{(j,\epsilon)\in J\times\{\epsilon\}}\bigsqcup_{u\in E_{\mathfrak{f}}^+} uC(\epsilon\alpha_{\mathfrak{a}}(v_{j,1}, v_{j,2},\ldots,v_{j,r(j)}))$$

satisfies the condition for (4.2) and there exists a bijective map:

$$R(\mathfrak{c}, C(v_j)) \to \bigsqcup_{\epsilon \in \{\epsilon\}} R'(\mathfrak{a}^{-1}, C(\epsilon\alpha_{\mathfrak{a}}v_j)), \tag{6.4}$$
$$z \mapsto \epsilon_z \alpha_{\mathfrak{a}} z.$$

We can take an integral ideal $\mathfrak{b} = (\alpha)$ which satisfies the condition of Theorem 6.1 for all $\mathfrak{c}$ and the corresponding cone decomposition. Now put $b = N\mathfrak{b}$ and take a cone decomposition $\{C(v_{j'})|j' \in J'\}$ such that

1. $v_{j',i} \in O_F$, totally positive.
2. $\{C(v_{j'}) \mid j' \in J'\}$ is a refinement of $\{C(v_j) \mid j \in J\}$.
3. There exist $u_{j'} \in E_{\mathfrak{f}}^{+}$ such that $\{C(u_{j'}v_{j'}) \mid j' \in J'\}$ is a refinement of $\{C((b/\alpha)v_j) \mid j \in J\}$.

Then we can apply $\{C(\epsilon\alpha_{\mathfrak{a}}v_{j'})|(j', \epsilon) \in J' \times \{\epsilon\}\}$ to Theorem 6.1 and get

$$\begin{aligned}
&\left[\frac{d}{ds}\zeta_{p,\mathfrak{f}}(s, \mathfrak{c})\right]_{s=0} \\
&= \log_p(N\mathfrak{a})\zeta_{p,\mathfrak{f}}(0, \mathfrak{c}) + \sum_{k=1}^{n} \sum_{j \in J, \epsilon \in \{\epsilon\}} \sum_{z \in R'(\mathfrak{a}^{-1}, C(\epsilon\alpha_{\mathfrak{a}}v_j))} L\Gamma_{p,r(j)}(z^{(k)}, (\epsilon\alpha_{\mathfrak{a}}v_j)^{(k)}) \\
&\quad + \frac{-b}{b-1} \sum_{k=1}^{n} \sum_{j' \in J', \epsilon \in \{\epsilon\}, u_{j'} \neq 1} \sum_{z' \in R'(\mathfrak{a}^{-1}, C(\epsilon\alpha_{\mathfrak{a}}v_{j'}))} \\
&\qquad \log_p(u_{j'}^{(k)})\zeta_{p,r(j')}(0, (\epsilon\alpha_{\mathfrak{a}}v_{j'})^{(k)}, z'^{(k)}).
\end{aligned}$$

Therefore by (6.4), Lemmas 5.4 and 6.2,

$$\begin{aligned}
\left[\frac{d}{ds}\zeta_{p,\mathfrak{f}}(s, \mathfrak{c})\right]_{s=0} &= \sum_{k=1}^{n} \sum_{j \in J} \sum_{z \in R(\mathfrak{c}, C(v_j))} L\Gamma_{p,r(j)}(z^{(k)}, v_j^{(k)}) - \log_p(N\mathfrak{a}_\mu\mathfrak{f})\zeta_{p,\mathfrak{f}}(0, \mathfrak{c}) \\
&\quad + \frac{-b}{b-1} \sum_{k=1}^{n} \sum_{j' \in J', u_{j'} \neq 1} \sum_{z' \in R(\mathfrak{a}^{-1}, C(v_{j'}))} \log_p(u_{j'}^{(k)})\zeta_{p,r(j')}(0, v_{j'}^{(k)}, z'^{(k)}).
\end{aligned}$$

Put

$$\begin{aligned}
T_0(\mathfrak{c}, \{C(v_j)|j \in J\}) = &\left[\frac{d}{ds}\zeta_{\mathfrak{f}}(s, \mathfrak{c})\right]_{s=0} + \log(N\mathfrak{a}_\mu\mathfrak{f})\zeta_{\mathfrak{f}}(0, \mathfrak{c}) \\
&- \sum_{k=1}^{n} \sum_{j \in J} \sum_{z \in R(\mathfrak{c}, C(v_j))} \log\left(\frac{\Gamma_{r(j)}(z^{(k)}, v_j^{(k)})}{\rho_{r(j)}(v_j^{(k)})}\right).
\end{aligned}$$

Since $\zeta_{p,r(j')}(0, v_{j'}^{(k)}, x^t v_{j'}^{(k)}) = \zeta_{r(j')}(0, v_{j'}^{(k)}, x^t v_{j'}^{(k)})$, to prove the assertion, it suffices to show that

$$\begin{aligned}
&T_0(\mathfrak{c}, \{C(v_j)|j \in J\}) \\
&\quad = \frac{-b}{b-1} \sum_{k=1}^{n} \sum_{j' \in J', u_{j'} \neq 1} \sum_{z' \in R(\mathfrak{a}^{-1}, C(v_{j'}))} \log(u_{j'}^{(k)})\zeta_{r(j')}(0, v_{j'}^{(k)}, z'^{(k)}).
\end{aligned}$$

By (3.1) and $C_l\left(\left(\begin{matrix} v_j^{\sigma_1} \\ \vdots \\ v_j^{\sigma_n}\end{matrix}\right)\right) = C_l\left(\left(\begin{matrix} (\frac{b}{\alpha}v_j)^{\sigma_1} \\ \vdots \\ (\frac{b}{\alpha}v_j)^{\sigma_n}\end{matrix}\right)\right)$,

$$T_0(\mathfrak{c}, \{C(v_j)|j \in J\}) = \sum_{j\in J}\sum_{z\in R(\mathfrak{c},C(v_j))} \frac{(-1)^{r(j)}}{n}\sum_l C_l(A_j)\prod_{i=1}^{r(j)}\frac{B_{l_i}(x_i(z))}{l_i!},$$

$$T_0(\mathfrak{c}, \{C(\tfrac{b}{\alpha}v_j)|j \in J\}) = \sum_{j\in J}\sum_{z\in R(\mathfrak{c},C(\frac{b}{\alpha}v_j))} \frac{(-1)^{r(j)}}{n}\sum_l C_l(A_j)\prod_{i=1}^{r(j)}\frac{B_{l_i}(x_i(z))}{l_i!}.$$

Here

$$\begin{aligned} R(\mathfrak{c}, C(\tfrac{b}{\alpha}v_j)) &= \{z \in (\mathfrak{a}_\mu\mathfrak{f})^{-1}\cap C(\tfrac{b}{\alpha}v_j)|(z)\mathfrak{a}_\mu\mathfrak{f}\in\mathfrak{c}\} \\ &= \alpha^{-1}\{z+n^t v_j | z\in R(\mathfrak{c}, C(v_j)), n\in Q_z\}. \end{aligned}$$

Here we put $Q_z = \{n \mid n_i = 0, 1, \ldots, b-1, (z+n^t v_j)\mathfrak{a}_\mu\mathfrak{f} \subset \mathfrak{b}\}$ for $z \in R(\mathfrak{c}, C(v_j))$. Then we get

$$\sum_{z\in R(\mathfrak{c},C(\frac{b}{\alpha}v_j))}\prod_{i=1}^{r(j)}\frac{B_{l_i}(x_i(z))}{l_i!} = \sum_{z\in R(\mathfrak{c},C(v_j))}\sum_{n\in Q_z}\prod_{i=1}^{r(j)}\frac{B_{l_i}\left(\frac{x_i(z)+n_i}{b}\right)}{l_i!}.$$

Since $\mathfrak{b}$ is prime to $(\alpha_{\mathfrak{a}}v_{i,j})$ and $(\alpha_{\mathfrak{a}}) = \mathfrak{a}_\mu\mathfrak{f}\mathfrak{a}$, $\mathfrak{b}$ is prime to $\mathfrak{a}_\mu\mathfrak{f}$, $O_F/\mathfrak{b} = \mathbf{Z}/b\mathbf{Z}$. Therefore for any i, for each $n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_{r(j)}$, there exists unique n_i such that $n \in Q_z$, $0 \le n_i \le b-1$. Here l extends over all $l = (l_1, \ldots, l_{r(j)})$ such that $0 \le l_i, l_1 + \cdots + l_{r(j)} = r(j)$. If $l = (1, 1, \ldots, 1)$, $C_l(A_j) = 0$. We assume that $l \neq (1, 1, \ldots, 1)$, that is, there exists i_0 such that $l_{i_0} = 0$. In this case

$$\begin{aligned} \sum_{n\in Q_z}\prod_{i=1}^{r}\frac{B_{l_i}\left(\frac{x_i(z)+n_i}{b}\right)}{l_i!} &= \prod_{i\neq i_0}\sum_{n_i=0}^{b-1}\frac{B_{l_i}\left(\frac{x_i(z)+ni}{b}\right)}{l_i!} = \prod_{i\neq i_0}\frac{1}{b^{l_i-1}}\frac{B_{l_i}(x_i(z))}{l_i!} \\ &= \frac{1}{b}\prod_{i\neq i_0}\frac{B_{l_i}(x_i(z))}{l_i!}. \end{aligned}$$

Therefore

$$T_0(\mathfrak{c}, \{C(\tfrac{b}{\alpha}v_j)|j\in J\}) = \frac{1}{b}T_0(\mathfrak{c}, \{C(v_j)|j\in J\}),$$

$$\sum_{k=1}^{n} \sum_{j' \in J', u_{j'} \neq 1} \sum_{z' \in R(\mathfrak{c}, C(v_{j'}))} \log(u_{j'}^{(k)}) \zeta_{r(j')}(0, v_{j'}^{(k)}, z'^{(k)})$$
$$= \sum_{k=1}^{n} \sum_{j \in J} \left(\sum_{z \in R(\mathfrak{c}, C(v_j))} \log\left(\frac{\Gamma_{r(j)}(z^{(k)}, v_j^{(k)})}{\rho_{r(j)}(v_j^{(k)})} \right) \right.$$
$$\left. - \sum_{z \in R(\mathfrak{c}, C(\frac{b}{\alpha} v_j))} \log\left(\frac{\Gamma_{r(j)}(z^{(k)}, v_j^{(k)})}{\rho_{r(j)}(v_j^{(k)})} \right) \right)$$
$$= T_0(\mathfrak{c}, \{C(\tfrac{b}{\alpha} v_j) | j \in J\}) - T_0(\mathfrak{c}, \{C(v_j) | j \in J\})$$
$$= \frac{b-1}{-b} T_0(\mathfrak{c}, \{C(v_j) | j \in J\}).$$

We get

$$T_0(\mathfrak{c}, \{C(v_j) | j \in J\}) = \frac{-b}{b-1} \sum_{k=1}^{n} \sum_{j' \in J', u_{j'} \neq 1} \sum_{z' \in R(\mathfrak{c}, C(v_{j'}))} \log(u_{j'}^{(k)}) \zeta_{r(j')}(0, v_{j'}^{(k)}, z'^{(k)}). \tag{6.5}$$

□

7. Applications

7.1. The case of $F = \mathbf{Q}$

Let $F = \mathbf{Q}$, $d \geq 1$. Then $I_{(d)} = \{(a/b) \mid a, b = 1, 2, \ldots, (a, d) = (b, d) = 1\}$ and $P_{(d)+} = \{(a/b) \mid a, b = 1, 2, \ldots, (a, d) = (b, d) = 1, a - b \in (d)\}$. We can identify

$$C_{(d)} = (\mathbf{Z}/d\mathbf{Z})^\times \text{ by } \overline{(a/b)} \mapsto ab' \bmod d\mathbf{Z}, \overline{(a)} \leftarrow a \bmod d\mathbf{Z},$$

where $a, b \in \mathbf{Z}$ such that $(a, d) = (b, d) = 1$ and $b' \in \mathbf{Z}$ such that $bb' \equiv 1 \bmod d$. Hence we can regard a primitive Dirichlet character χ of conductor d as a character of $C_{(d)}$. Assume that d is prime to p. Let $\mathfrak{f} = (pd)$. Then $\{(a) \mid 1 \leq a \leq pd - 1, (a, pd) = 1\}$ is a complete set of representatives of $C_{(pd)}$. Now $C((1))$ is a cone decomposition. By Lemmas 5.4, 6.2 and Theorem 6.2,

$$\left[\frac{d}{ds} \zeta_{p,(pd)}(s, \overline{(a)})\right]_{s=0} = L\Gamma_{p,1}\left(\frac{a}{ad}, (1)\right) - \log_p(pd) \zeta_{p,1}\left(0, (1), \frac{a}{pd}\right) \tag{7.1}$$
$$= L\Gamma_{p,1}(a, (pd)),$$

Let χ be a primitive Dirichlet character of conductor d. By (4.10),

$$\left[\frac{d}{ds} L_{p,(pd)}(s, \chi_{-1})\right]_{s=0} = \sum_{1 \leq a \leq pd-1, (a,pd)=1} \chi((a)) \, L\Gamma_{p,1}(a, (pd))$$
$$= \sum_{1 \leq a \leq d-1, (a,d)=1} \chi((a)) \, L\Gamma_{p,1}(a, (d)).$$

By Lemmas 5.4 and 5.5,

$$\left[\frac{d}{ds}L_{p,(pd)}(s,\chi_{-1})\right]_{s=0} = \sum_{1\le a\le d-1,(a,d)=1} \chi(a)\log_p(\Gamma_p(a/d)) - \log_p(d)L_{p,(pd)}(0,\chi_{-1}). \tag{7.2}$$

We see that this result agrees with Ferrero-Greenberg [8, Proposition 1].

7.2. The order at $s=0$ of p-adic L-functions, Gross' conjecture

In a special case, through the analogy between Shintani's formula and p-adic Shintani's formula, we get the order at $s=0$ of the p-adic L-function. Gross conjectured formulas for the leading term in their Taylor expansion at $s=0$ ([9, Conjecture 2.12]). Let F be a totally real field, K an abelian extension over F with conductor $\mathfrak{f}_\mathfrak{o}$ and χ a primitive character of $C_{\mathfrak{f}_\mathfrak{o}}$. Put $\mathfrak{f}=\mathfrak{f}_\mathfrak{o}\times\prod_{\mathfrak{p}|(p),\mathfrak{p}\nmid\mathfrak{f}_\mathfrak{o}}\mathfrak{p}$. We assume that χ is odd. In this situation, his first conjecture states:

$$\text{the order at } s=0 \text{ of } L_\mathfrak{f}(s,\chi) \quad = \quad \text{the order at } s=0 \text{ of } L_{p,\mathfrak{f}}(s,\chi_{-1}). \tag{7.3}$$

By [9, (2.8), (3.1)], the statement becomes:

$$\text{the order at } s=0 \text{ of } L_{p,\mathfrak{f}}(s,\chi_{-1}) = \#\{\mathfrak{p}|(p)\mid \chi(\mathfrak{p})=1\}. \tag{7.4}$$

Here # means the number of elements in the set. Obviously,

$$\text{the order at } s=0 \text{ of } L_{p,\mathfrak{f}}(s,\chi_{-1})=0 \quad\Leftrightarrow\quad \#\{\mathfrak{p}|(p)\mid \chi(\mathfrak{p})=1\}=0. \tag{7.5}$$

By Theorem 6.2, we can get a partial result toward (7.3).

Theorem 7.1. *If* $\#\{\mathfrak{p}|(p)\mid\chi(\mathfrak{p})=1\}\geq 2$, *the order at* $s=0$ *of* $L_{p,\mathfrak{f}}(s,\chi_{-1})\geq 2$.

Proof. First we show that $\frac{d}{ds}L_{p,\mathfrak{f}}(s,\chi\theta_p)\mid_{s=0}$ is written as a finite sum of terms which take the form of $a\log_p b$, $a,b\in\overline{\mathbf{Q}}$ when $\#\{\mathfrak{p}|(p)\mid\chi(\mathfrak{p})=1\}\geq 2$. By (4.10) and Theorem 6.2, we get

$$\left[\frac{d}{ds}\text{Ł}_{p,\mathfrak{f}}(s,\chi_{-1})\right]_{s=0} = \sum_{\mathfrak{c}\in C_\mathfrak{f}}\sum_{k=1}^{n}\sum_{j\in J}\sum_{z\in R(\mathfrak{c},C_j)}\chi(\mathfrak{c})L\Gamma_{p,r(j)}(z^{(k)},v_j^{(k)}) + \text{finite sum of terms } a\log_p b,\ a,b\in\overline{\mathbf{Q}}. \tag{7.6}$$

Because $\#\{\mathfrak{p}|(p)\mid\chi(\mathfrak{p})=1\}\geq 2$, for each $\sigma\in J_F$, we can take a prime ideal $\mathfrak{p}_\sigma$ satisfying that $\mathfrak{p}_\sigma|(p)$, $(\mathfrak{p}_\sigma)^\sigma\neq\mathbf{M}_p\cap O_F$, $\chi(\mathfrak{p}_\sigma)=1$. Let $\mathfrak{f}_\sigma=\mathfrak{f}/\mathfrak{p}_\sigma$. For each $\mathfrak{a}_\mu$, there exist $\mathfrak{a}_{\mu'}$ and $\pi_{\mu,\sigma}\in F$ which satisfy $\mathfrak{p}_\sigma\mathfrak{a}_\mu=(\pi_{\mu,\sigma})\mathfrak{a}_{\mu'}$ as ideals. We replace $v_{j,i}$ and we may assume $\pi_{\mu,\sigma}v_{j,i}\in O_F$ for all j,i. Put

$$R(\mathfrak{f},\mathfrak{a}_\mu,C(v_j))=\{z\in(\mathfrak{a}_\mu\mathfrak{f})^{-1}\cap C(v_j)\mid (z\mathfrak{a}_\mu\mathfrak{f})\text{ is prime to }\mathfrak{f}\},$$
$$R(\mathfrak{f}_\sigma,\mathfrak{a}_\mu,C(v_j))=\{z\in(\mathfrak{a}_\mu\mathfrak{f}_\sigma)^{-1}\cap C(v_j)\mid (z\mathfrak{a}_\mu\mathfrak{f}_\sigma)\text{ is prime to }\mathfrak{f}_\sigma\},$$
$$R(\mathfrak{f}_\sigma,\mathfrak{a}_{\mu'},C(\pi_{\mu,\sigma}v_j))=\{z\in(\mathfrak{a}_{\mu'}\mathfrak{f}_\sigma)^{-1}\cap C(\pi_{\mu,\sigma}v_j)\mid (z\mathfrak{a}_{\mu'}\mathfrak{f}_\sigma)\text{ is prime to }\mathfrak{f}_\sigma\}.$$

Then

$$\pi_{\mu,\sigma}R(\mathfrak{f},\mathfrak{a}_\mu,C(v_j))=\{z\in R(\mathfrak{f}_\sigma,\mathfrak{a}_{\mu'},C(\pi_{\mu,\sigma}v_j)\mid (z\mathfrak{a}_{\mu'}\mathfrak{f}_\sigma)\text{ is prime to }\mathfrak{p}_\sigma\},$$
$$\pi_{\mu,\sigma}R(\mathfrak{f}_\sigma,\mathfrak{a}_\mu,C(v_j))=R(\mathfrak{f}_\sigma,\mathfrak{a}_{\mu'},C(\pi_{\mu,\sigma}v_j))\cap\mathfrak{p}_\sigma,$$

where $\pi_{\mu,\sigma}R(\mathfrak{f},\mathfrak{a}_\mu,C(v_j))$ denotes the set $\{\pi_{\mu,\sigma}z\mid z\in R(\mathfrak{f},\mathfrak{a}_\mu,C(v_j))\}$. We get

$$R(\mathfrak{f}_\sigma,\mathfrak{a}_{\mu'},C(\pi_{\mu,\sigma}v_j))=\pi_{\mu,\sigma}R(\mathfrak{f},\mathfrak{a}_\mu,C(v_j))\bigsqcup\pi_{\mu,\sigma}R(\mathfrak{f}_\sigma,\mathfrak{a}_\mu,C(v_j)).$$

Hence the first term of the right hand side of (7.6) is equal to

$$\begin{aligned}&\sum_{\mathfrak{a}_\mu}\sum_{\sigma\in J_F}\sum_{j\in J}\sum_{z\in R(\mathfrak{f},\mathfrak{a}_\mu,C(v_j))}\chi((z)\mathfrak{f}\mathfrak{a}_\mu)L\Gamma_{p,r(j)}(z^\sigma,v_j^\sigma)\\
&=\sum_{\mathfrak{a}_\mu}\sum_{\sigma\in J_F}\sum_{j\in J}\sum_{z\in\pi_{\mu,\sigma}^{-1}R(\mathfrak{f}_\sigma,\mathfrak{a}_{\mu'},C(\pi_{\mu,\sigma}v_j))}\chi((z)\mathfrak{f}\mathfrak{a}_\mu)L\Gamma_{p,r(j)}(z^\sigma,v_j^\sigma)\\
&\quad-\sum_{\mathfrak{a}_\mu}\sum_{\sigma\in J_F}\sum_{j\in J}\sum_{z\in R(\mathfrak{f}_\sigma,\mathfrak{a}_\mu,C(v_j))}\chi((z)\mathfrak{f}\mathfrak{a}_\mu)L\Gamma_{p,r(j)}(z^\sigma,v_j^\sigma).\end{aligned}\tag{7.7}$$

By Lemma 5.4, the first term of the right hand side of (7.7) is equal to

$$\sum_{\mathfrak{a}_\mu}\sum_{\sigma\in J_F}\sum_{j\in J}\sum_{z\in R(\mathfrak{f}_\sigma,\mathfrak{a}_\mu,C(\pi_{\mu,\sigma}v_j))}\chi((z)\mathfrak{f}_\sigma\mathfrak{a}_\mu)L\Gamma_{p,r(j)}(z^\sigma,v_j^\sigma)$$
$$+\text{ finite sum of terms }a\log_p b,\ a,b\in\overline{\mathbf{Q}}.$$

Since $\chi(\mathfrak{p}_\sigma)=1$, the second term of the right hand side of (7.7) is equal to

$$-\sum_{\mathfrak{a}_\mu}\sum_{\sigma\in J_F}\sum_{j\in J}\sum_{z\in R(\mathfrak{f}_\sigma,\mathfrak{a}_\mu,C(v_j))}\chi((z)\mathfrak{f}_\sigma\mathfrak{a}_\mu)L\Gamma_{p,r(j)}(z^\sigma,v_j^\sigma).$$

According to [19, Lemma 2], we can take a cone decomposition:
$\mathbf{R}^{+n}=\bigsqcup_{j'\in J'}\bigsqcup_{u\in E_F^+}uC(v_{j'})$ with $v_{j'}=(v_{j',1},\dots,v_{j',r(j')}),v_{j',i}\in O_F$, such that

1. $v_{j',i}$ are totally positive, $\{v_{j',1},\dots,v_{j',r(j')}\}$ are linearly independent.
2. $\{C(v_{j'})\mid j'\in J'\}$ is a refinement of $\{C(v_j)\mid j\in J\}$.
3. There exist $u_{j'}\in E_F^+$ such that $\{C(u_{j'}v_{j'})\mid j'\in J'\}$ is a refinement of $\{C(\pi_{\mu,\sigma}v_j)\mid j\in J\}$.

Then

$$\begin{aligned}&\sum_{\mathfrak{a}_\mu}\sum_{\sigma\in J_F}\sum_{j\in J}\sum_{z\in R(\mathfrak{f}_\sigma,\mathfrak{a}_\mu,C(\pi_{\mu,\sigma}v_j))}\chi((z)\mathfrak{f}_\sigma\mathfrak{a}_\mu)L\Gamma_{p,r(j)}(z^\sigma,v_j^\sigma)\\
&\qquad-\sum_{\mathfrak{a}_\mu}\sum_{\sigma\in J_F}\sum_{j\in J}\sum_{z\in R(\mathfrak{f}_\sigma,\mathfrak{a}_\mu,C(v_j))}\chi((z)\mathfrak{f}_\sigma\mathfrak{a}_\mu)L\Gamma_{p,r(j)}(z^\sigma,v_j^\sigma)\\
&=\sum_{\mathfrak{a}_\mu}\sum_{\sigma\in J_F}\sum_{j'\in J'}\sum_{z\in R(\mathfrak{f}_\sigma,\mathfrak{a}_\mu,C(v_{j'}))}\chi((z)\mathfrak{f}_\sigma\mathfrak{a}_\mu)\\
&\qquad\times\left\{L\Gamma_{p,r(j)}(z^\sigma,v_j^\sigma)-L\Gamma_{p,r(j)}((u_{j'}z)^\sigma,(u_{j'}v_j)^\sigma)\right\}\\
&=\sum_{\mathfrak{a}_\mu}\sum_{\sigma\in J_F}\sum_{j'\in J'}\sum_{z\in R(\mathfrak{f}_\sigma,\mathfrak{a}_\mu,C(v_{j'}))}\chi((z)\mathfrak{f}_\sigma\mathfrak{a}_\mu)\log_p(u_{j'}^\sigma)\zeta_{p,r(j)}(0,v_j^\sigma,z^\sigma),\end{aligned}$$

where $R(\mathfrak{f}_\sigma, \mathfrak{a}_\mu, C(v_{j'})) = \{z \in (\mathfrak{a}_\mu \mathfrak{f}_\sigma)^{-1} \cap C(v_{j'}) \mid (z\mathfrak{a}_\mu \mathfrak{f}_\sigma)$ is prime to $\mathfrak{f}_\sigma\}$. Summing up the calculations above, we conclude that there exist a finite set I and $a_i, b_i \in \overline{\mathbf{Q}}$ for $i \in I$ such that $\frac{d}{ds}L_{p,\mathfrak{f}}(s, \chi\theta_p)\mid_{s=0} = \sum_{i\in I} a_i \log_p(b_i)$. Furthermore we can transform the formula of $\frac{d}{ds}L_{\mathfrak{f}}(s,\chi)\mid_{s=0}$ in the same manner, and using the same I, a_i, b_i, we get $\frac{d}{ds}L_{\mathfrak{f}}(s,\chi)\mid_{s=0} = \sum_{i\in I} a_i \log(b_i)$ because of Theorem 6.2. Note that in this situation, $\zeta_{p,r(j)}(0, v_j^\sigma, z^\sigma) = \zeta_{r(j)}(0, v_j^\sigma, z^\sigma)$. By [1, Corollary 1.1], for $\alpha_1, \ldots, \alpha_n \in \overline{\mathbf{Q}}$,

$$\begin{aligned} &\text{if } \log\alpha_1, \ldots, \log\alpha_n, 2\pi i \text{ are linearly independent over } \mathbf{Q}, \\ &\quad \text{then } \log\alpha_1, \ldots, \log\alpha_n \text{ are linearly independent over } \overline{\mathbf{Q}}. \end{aligned} \tag{7.8}$$

Let I' be a maximal subset of I such that $\{\log(b_{i'}) \mid i' \in I'\}$ are linearly independent over $\mathbf{Q}$. Then there exisit $m, n_{i,i'} \in \mathbf{Z}$ satisfying $\log(b_i) = \sum_{i'\in I'}(n_{i,i'}/m)\log(b_{i'})$ for all $i \in I, i' \in I'$. We may assume that $m, n_{i,i'}$ are even. Hence $b_i^m = \prod_{i'\in I'} {b_{i'}}^{n_{i,i'}}$ and we get $\log_p(b_i) = \sum_{i'\in I'}(n_{i,i'}/m)\log_p(b_{i'})$. By (7.8), $\sum_{i\in I} a_i n_{i,i'} = 0$. Consequently $\frac{d}{ds}L_{p,\mathfrak{f}}(s,\chi\theta_p)\mid_{s=0} = 0$ and we complete the proof. □

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