J. Math. Kyoto Univ. (JMKYAZ)
45-2 (2005), 421–426

Boundary identity principle for pseudo-holomorphic curves

By

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Abstract

We prove a boundary version of the Unique Continuation Principle for pseudo-holomorphic curves. It is a consequence of the boundary regularity of pseudo-holomorphic curves. which can be achieved by a bootstrap method.

1. Introduction

In recent years, much interests are focused upon the study of almost complex structures and the properties of pseudo-holomorphic mappings between almost complex manifolds. The goal of this paper is to prove a boundary version of the Unique Continuation Principle for pseudo-holomorphic mappings, which is stated as follows:

Theorem 1.1. *Let S be a connected Riemann surface with smooth boundary ∂S and let M be a smooth manifold with a smooth almost complex structure J. Suppose that a pseudo-holomorphic map $f : S \to M$ is continuous up to the boundary and that f is constant on an open arc γ of ∂S. Then f is constant on S.*

It is known that the interior Unique Continuation Principle is still valid for pseudo-holomorphic mappings, that is, if $f : S \to M$ is constant on an open subset of S, then f is constant on entire S. This is a consequence of the vanishing theorem of a smooth mapping satisfying a partial differential inequality, which is proved by N. Aronszajn ([2]) and Hartman-Wintner ([5]):

Lemma 1.1. *Let Ω be a connected domain in $\mathbf{R}^2$ containing 0. Suppose that a smooth map $f : \Omega \to \mathbf{R}^N$ satisfies that*

$$|\Delta f| \leq C|Df|$$

for a positive constant C and that $f(0) = 0$. Then $f \equiv 0$ on Ω if f vanishes to infinite order at 0.

Received April 18, 2005

Here, Δf and Df represent the Laplacian and the gradient of f. To prove Theorem 1.1., we first show that f is in fact smooth up to γ by a standard bootstrap. This yields a smooth reflection of f, which makes the problem an interior one. Applying Lemma 1.1. to a smooth reflection of f, we can achieve Theorem 1.1.

2. Preliminaries

Throughout this paper, an almost complex manifold means a C^∞ smooth manifold with a C^∞ smooth almost complex structure J.

Let M and M' be almost complex manifolds with almost complex structures J and J', respectively. A smooth map $f : M \to M'$ is called a *pseudo-holomorphic map* if its differential commutes with J and J', that is

$$df \circ J = J' \circ df.$$

Let Ω be a domain in $\mathbf{R}^n$ and let $f : \Omega \to \mathbf{R}^N$ be differentiable to k-th order on Ω for a nonnegative integer k. For a real number $\alpha \in (0,1)$, we define the (k,α)-Hölder norm $\|f\|_{k,\alpha}$ of f by

$$\|f\|_{k,\alpha} = \sum_{|I|\le k} \sup_{\Omega} |D^I f(x)| + \sum_{|I|=k} \sup_{x\ne y} \frac{|D^I f(x) - D^I f(y)|}{|x-y|^\alpha}$$

where $I = (i_1, \ldots, i_n)$ is a multi-index and $D^I = (\partial/\partial x_1)^{i_1} \cdots (\partial/\partial x_n)^{i_n}$. Define the (k,α)-Hölder space $C^{k,\alpha}(\Omega)$ on Ω by

$$C^{k,\alpha}(\Omega) = \{f : \|f\|_{k,\alpha} < \infty\}.$$

We denote by $\mathbf{D}$ and $\mathbf{D}^+$ the unit disc and the upper half disc in $\mathbf{C}$, respectively, that is,

$$\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$$

and

$$\mathbf{D}^+ = \{z \in \mathbf{D} : \operatorname{Im} z > 0\}.$$

3. Proof of Theorem 1.1.

Fix $z_0 \in \gamma$. Let $p = f(z_0)$. Choosing local coordinates, we may assume that f maps $\mathbf{D}^+ \cup \gamma$ into a neighborhood U of 0 in $\mathbf{R}^{2n}$ and that $f \equiv 0$ on γ where $\gamma = \{z \in \mathbf{D} : \operatorname{Im} z = 0\}$. A C^2 function u on U is said to be *strictly J-plurisubharmonic* if its Levi form $L(X) := -d(J^*du)(X, JX)$ is positive definite, where J^* represents the dual operator of J . We can also assume that the complex structure J coincides with the standard complex structure J_{st} at 0 and J is sufficiently close to J_{st} in C^2 sense on U so that the function $u_0 = \sum |w^j|^2$ is strictly J-plurisubharmonic on U, where $(w^1, \ldots, w^n)$ is the standard coordinates of $\mathbf{C}^n = \mathbf{R}^{2n}$. In this situation, the following lemma holds.

Lemma 3.1. *There exists a constant C such that $|Df(z)| \leq C|\mathrm{Im}\, z|^{-1/2}$ for every $z \in \mathbf{D}^+$, that is, $f \in C^{0,1/2}(\mathbf{D}^+ \cup \gamma)$.*

Lemma 3.1. is a consequence of Theorem 1.1. in [3]. In fact, the authors of [3] have proved the theorem in case when the target manifolds have integrable structures. A crucial part of the proof is an estimation of the Kobayashi metric of target manifold, which is also available for every almost complex manifold. (See [4].) This implies that Lemma 3.1. holds by the assumption that u_0 is strictly J-plurisubharmonic.

To prove the boundary regularity of f, we need some basic properties of one variable $\overline{\partial}$-equations. Let Ω be a domain in the complex plane $\mathbf{C}$, and take g and h in L^1_{loc}, the space of locally integrable functions. We say that $\partial g/\partial \bar{z} = h$ *in the weak sense* in Ω if for every smooth function ϕ with compact support in Ω, we have

$$\int_\Omega g(z) \frac{\partial \phi}{\partial \bar{z}}(z) = -\int_\Omega h(z)\phi(z).$$

Lemma 3.2. *A L^1_{loc} function g on a domain Ω is holomorphic if and only if $\partial g/\partial \bar{z} = 0$ in the weak sense.*

Lemma 3.3. *Take $h \in L^\infty(\mathbf{C})$ with compact support. Define a function g by*

$$g(z) = \frac{1}{2\pi i}\int_{\mathbf{C}} \frac{h(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

for every $z \in \mathbf{C}$. Then the followings hold:

(a) *$\partial g/\partial \bar{z} = h$ in the weak sense.*

(b) *$g \in C^{0,\alpha}(\Omega)$ for every $0 < \alpha < 1$ and every bounded domain Ω in $\mathbf{C}$.*

(c) *For every non-negative integer k and every $0 < \alpha < 1$, $g \in C^{k+1,\alpha}(\mathbf{C})$ whenever $h \in C^{k,\alpha}(\mathbf{C})$.*

The proofs of Lemma 3.2. and Lemma 3.3. may be found in [1], for instance.

Decompose the complexified tangent bundle $TU \otimes \mathbf{C}$ into the direct sum of eigen-subspaces of J_{st}, i.e.

$$TU \otimes \mathbf{C} = T^{1,0} \oplus T^{0,1}$$

where $T^{1,0}$ and $T^{0,1}$ are the bundles of subspaces corresponding to the eigenvalues i and $-i$ of J_{st}, respectively. Similarly,

$$TU \otimes \mathbf{C} = T_J^{1,0} \oplus T_J^{0,1}$$

where $T_J^{1,0}$ and $T_J^{0,1}$ are the bundles of eigen-subspaces corresponding to the eigenvalues i and $-i$ of J. Since J is sufficiently close to J_{st} on U, there exists a $\mathbf{R}$-linear bundle map $\mu : T^{1,0} \to T^{0,1}$ such that

$$T_J^{1,0} = \{X + \mu(X) : X \in T^{1,0}\}.$$

Taking conjugates, the bundle $T_J^{0,1}$ is the graph of $\bar{\mu} : T^{0,1} \to T^{1,0}$, that is,

$$T_J^{0,1} = \{Y + \bar{\mu}(Y) : Y \in T^{0,1}\}.$$

Decompose a vector $X \in TU \otimes \mathbf{C}$ by $X = X^{1,0} + X^{0,1}$ with respect to the decomposition $TU \otimes \mathbf{C} = T^{1,0} \oplus T^{0,1}$. Then the $T_J^{0,1}$-component of X is $Y + \bar{\mu}(Y)$ where

$$Y = (I - \mu\bar{\mu})^{-1}(X^{0,1} - \mu(X^{1,0})). \tag{3.1}$$

Note that $I - \mu\bar{\mu}$ is invertible since μ is sufficiently small on U, where I represents the identity operator. Since f is pseudo-holomorphic on $\mathbf{D}^+$, it follows that f satisfies the equation

$$\frac{\partial f}{\partial \bar{z}} - \bar{\mu}\left(\overline{\frac{\partial f}{\partial z}}\right) = 0 \tag{3.2}$$

by (3.1). Define a map $\tilde{f} \in C^{0,1/2}(\mathbf{D})$ by

$$\tilde{f}(z) = \begin{cases} f(z) & \text{if } \mathrm{Im} z \geq 0 \\ \overline{f(\bar{z})} & \text{if } \mathrm{Im} z < 0. \end{cases}$$

Let ϕ be the map defined by

$$\phi(z) = \begin{cases} \frac{\partial \tilde{f}}{\partial \bar{z}}(z) & \text{if } z \in \mathbf{D} \setminus \gamma \\ 0 & \text{if } z \in \gamma. \end{cases}$$

Let μ_w be the restriction of μ on the space $T_w^{1,0}$ for every $w \in U$. Then μ_w is smooth in w and $\mu_0 = 0$. Therefore, we have that

$$|\mu_{f(z)}| = O(|f(z)|) = O(|f(z) - f(\mathrm{Re}\, z)|) = O(|\mathrm{Im}\, z|^{1/2}) \tag{3.3}$$

as $z \to \gamma$, since $f \in C^{0,1/2}(\mathbf{D}^+ \cup \gamma)$. It follows that $\phi \in L^\infty(\mathbf{D})$ by (3.2), (3.3) and Lemma 3.1. For $0 < r < 1$, we denote by $\mathbf{D}_r$ the radius r disc in $\mathbf{C}$. Choose a smooth function χ with compact support in $\mathbf{D}$ such that $\chi \equiv 1$ on $\mathbf{D}_r$. Define a map ψ by

$$\psi(z) = \frac{1}{2\pi i} \int_{\mathbf{C}} \frac{\chi(\zeta)\phi(\zeta)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}.$$

Then $\psi \in C^{0,\alpha}(\mathbf{D})$ for every $0 < \alpha < 1$ and $\tilde{f} - \psi$ is holomorphic on $\mathbf{D}_r \setminus \gamma$ by Lemma 3.2. and Lemma 3.3. Since $\tilde{f} - \psi$ is continuous on $\mathbf{D}$, it is holomorphic on $\mathbf{D}_r$ and hence $\tilde{f} \in C^{0,\alpha}(\mathbf{D}_r)$ for every $0 < \alpha < 1$. Take $\alpha > 1/2$ and let $\beta = \alpha - 1/2 > 0$. Since

$$|\mu_{f(z)}| = O(|f(z) - f(\mathrm{Re}\, z)|) = O(|\mathrm{Im}\, z|^{\alpha})$$

as $z \to \gamma$, it follows that $\phi \in C^{0,\beta}(\mathbf{D}_r)$. Then $\tilde{f}$ and ψ are in $C^{1,\beta}(\mathbf{D}_r)$ by Lemma 3.3.

Let q be a $C^{k,\alpha}$ map on $\mathbf{D}$ for $k \geq 1$ and $0 < \alpha < 1$. Let $\rho < 1$ be a positive real number. If we write $q_\rho(z) = q(\rho z)$ for $|z| < \rho^{-1}$, then

$$\|q_\rho\|_{k,\alpha} \leq \|q\|_{L^\infty} + \rho\|q\|_{k,\alpha}.$$

Now, let $q(z) = \mu_{\tilde{f}(z)}$. Then $q \in C^{1,\beta}(\mathbf{D}_r)$. We have already assume that J is so close to J_{st} that $\|q\|_{L^\infty}$ is small enough. Therefore, taking dilation by a small constant ρ if necessary, we may assume that $\|q\|_{1,\beta}$ is sufficiently small. Then $\tilde{f} \in C^{2,\beta}$ by [6, Proposition 2.3.6]. Therefore, $q \in C^{2,\beta}$, $\tilde{f} \in C^{3,\beta}$ and so on. Altogether, we have proved the following proposition.

Proposition 3.1. *Under the assumption for $f : S \to M$ imposed in Theorem 1, f is smooth up to γ.*

Again, we assume that f maps $\mathbf{D}^+$ into U, a neighborhood of 0 in $\mathbf{R}^{2n}$, $\gamma = \{z \in \mathbf{D} : \operatorname{Im} z = 0\}$ and $f \equiv 0$ on γ. Since f is pseudo-holomorphic, it satisfies that

$$\frac{\partial f}{\partial y} = J(f)\frac{\partial f}{\partial x} \tag{3.4}$$

on $\mathbf{D}^+$, where $z = x + iy$ is the standard coordinate of $\mathbf{C}$. Differentiating (3.4) in y,

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(J(f))\frac{\partial f}{\partial x} + J(f)\frac{\partial^2 f}{\partial x \partial y}. \tag{3.5}$$

In a similar way, we have

$$\frac{\partial^2 f}{\partial x^2} = -\frac{\partial}{\partial x}(J(f))\frac{\partial f}{\partial y} - J(f)\frac{\partial^2 f}{\partial x \partial y}. \tag{3.6}$$

Adding (3.5) to (3.6), it follows that f satisfies the equation

$$\Delta f - \frac{\partial}{\partial y}(J(f))\frac{\partial f}{\partial x} + \frac{\partial}{\partial x}(J(f))\frac{\partial f}{\partial y} = 0 \tag{3.7}$$

on $\mathbf{D}^+$. Since f is smooth up to γ and $f \equiv 0$ on γ, $\partial f/\partial x \equiv 0$ on γ. Therefore, $\partial f/\partial y \equiv 0$ on γ since f is pseudo-holomorphic. The second order derivatives $\partial^2 f/\partial x^2$ and $\partial^2 f/\partial x \partial y$ also vanish on γ. Then $\partial^2 f/\partial y^2$ vanishes on γ by (3.7). Inductively, it follows that all the derivatives of f vanish on γ. Therefore, if we define a map f_1 on $\mathbf{D}$ by

$$f_1(z) = \begin{cases} f(z) & \text{if } \operatorname{Im} z \geq 0 \\ f(\bar{z}) & \text{if } \operatorname{Im} z < 0, \end{cases}$$

then f_1 is smooth on $\mathbf{D}$ and it vanishes to infinite order at 0. Moreover, Taking $C = 2\sup_{\mathbf{D}^+}|D(J(f))|$, f_1 satisfies the differential inequality

$$|\Delta f_1| \leq C|Df_1|.$$

This implies that f_1 vanishes identically on $\mathbf{D}$ by Lemma 1.1.

Now, let f be a pseudo-holomorphic map on a connected Riemann surface S with smooth boundary ∂S. If f is constant on an open arc γ of ∂S, then the previous arguments imply that f should be constant on a neighborhood of a point $z_0 \in \gamma$. Therefore, Theorem 1.1. follows the interior Unique Continuation Principle for pseudo-holomorphic curves.

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