Samelson products in the exceptional Lie group of rank 2^*

By

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1. Introduction

Samelson products [7] have been studied extensively for classical Lie groups [2], [11], [12], but few results are known for exceptional Lie groups. The purpose of this note is to study the Samelson products

$$(1.1) \qquad \langle \ , \ \rangle : \pi_n(\mathrm{G}_2) \times \pi_{11}(\mathrm{G}_2) \to \pi_{n+11}(\mathrm{G}_2) \quad (n \in \{3, 11\}),$$

where G_2 is the exceptional Lie group of rank 2. Note that $\pi_m(\mathrm{G}_2)$ is infinite if and only if $m \in \{3, 11\}$, and that $\langle \pi_3(\mathrm{G}_2), \pi_3(\mathrm{G}_2) \rangle = \pi_6(\mathrm{G}_2) = \mathbb{Z}_3$ by [15]. We determine (1.1) when $n = 3$, and the odd component of (1.1) when $n = 11$. We have two applications: (1) we determine the nilpotency class of $\mathcal{H}(\mathrm{G}_2)_{(p)}$, where $\mathcal{H}(\mathrm{G}_2)$ is the group of homotopy classes of self maps of G_2 and $(\ )_{(p)}$ is the localization at a prime number p; (2) we decide odd primes p for which the typical map $\varphi : \mathrm{S}^3 \times \mathrm{S}^{11} \to \mathrm{G}_2$ (see (2.17) below) is a mod p H-map with respect to a product multiplication for $\mathrm{S}^3_{(p)} \times \mathrm{S}^{11}_{(p)}$ and the canonical multiplication for $\mathrm{G}_{2(p)}$.

In Section 2, we list up known results on homotopy groups of spheres and Lie groups we need, and we state our results. In Section 3, we prove that the image of (1.1) contains the odd component. In Section 4, we complete the proof of our main theorem (Theorem 2.1). In Section 5, we determine the nilpotency class of $\mathcal{H}(\mathrm{G}_2)_{(p)}$ for every prime p (Corollary 2.1). In Section 6, we decide whether or not $\varphi_{(p)}$ is an H-map (Corollary 2.2). In Section 7, we give a remark on $\langle \pi_3(G), \pi_{11}(G) \rangle$ for $G = \mathrm{Spin}(7), \mathrm{Spin}(8), \mathrm{Spin}(9), \mathrm{F}_4$.

2. Notations and results

We do not distinguish in notation between a map and its homotopy class. Let ι_n denote the identity map of S^n. We use the following fibrations (cf. [23,

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Appendix A])

$$S^3 \xrightarrow{\ i'_3\ } SU(3) \xrightarrow{\ p'\ } S^5,$$

(2.1) $$SU(3) \xrightarrow{\ j\ } G_2 \xrightarrow{\ p\ } S^6,$$

(2.2) $$G_2 \xrightarrow{\ i\ } Spin(7) \longrightarrow S^7,$$

and the following elements of [22]:

$$\nu_n \in \pi_{n+3}(S^n) \ (n \geq 4), \quad \varepsilon_n \in \pi_{n+8}(S^n) \ (n \geq 3), \quad \overline{\nu}_n \in \pi_{n+8}(S^n) \ (n \geq 6).$$

The following results are contained in [14], [15], [22]:

(2.3) $$\pi_{n+3}(S^n) = \mathbb{Z}_8\{\nu_n\} \oplus \mathbb{Z}_3 \ \text{ for } n \geq 5, \quad \pi_k(S^7) = 0 \ \text{ for } k = 11, 12,$$

(2.4) $$\pi_{11}(S^6) = \mathbb{Z}\{[\iota_6, \iota_6]\}, \ [\iota_6, \iota_6] \text{ is the Whitehead product},$$

(2.5) $$\pi_{14}(S^6) = \mathbb{Z}_8\{\overline{\nu}_6\} \oplus \mathbb{Z}_2\{\varepsilon_6\} \oplus \mathbb{Z}_3, \quad \pi_{14}(S^7) = \mathbb{Z}_{120},$$

(2.6) $$\pi_3(SU(3)) = \mathbb{Z}\{i'_3\}, \quad \pi_{13}(SU(3)) = \mathbb{Z}_6,$$

(2.7) $$\pi_{10}(SU(3)) = \mathbb{Z}_{30}, \quad \pi_{11}(SU(3)) = \mathbb{Z}_4\{[\nu_5^2]\}, \quad p'_*[\nu_5^2] = \nu_5 \circ \nu_8,$$

(2.8) $$\pi_{14}(Spin(7)) = (\mathbb{Z}_8)^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{9\cdot5\cdot7},$$

(2.9) $$\pi_k(G_2) = 0 \ \text{ for } k = 10, 13,$$

(2.10) $$\pi_3(G_2) = \mathbb{Z}\{i_3\}, \ i_3 = j \circ i'_3, \quad \pi_{11}(G_2) = \mathbb{Z}\{\gamma\} \oplus \mathbb{Z}_2\{j_*[\nu_5^2]\},$$

(2.11)

$$\pi_{14}(G_2) = \mathbb{Z}_8\{[\overline{\nu}_6 + \varepsilon_6]\} \oplus \mathbb{Z}_2\{j_*[\nu_5^2] \circ \nu_{11}\} \oplus \mathbb{Z}_{3\cdot7}, \ p_*[\overline{\nu}_6 + \varepsilon_6] = \overline{\nu}_6 + \varepsilon_6,$$

(2.12) $$\pi_{22}(G_2) = \mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{9\cdot7\cdot11},$$

(2.13) $$\pi_{22}(Spin(7)) = (\mathbb{Z}_8)^2 \oplus (\mathbb{Z}_2)^4 \oplus \mathbb{Z}_{27\cdot5\cdot7\cdot11},$$

(2.14) $$\pi_{22}(S^7) = \mathbb{Z}_8 \oplus (\mathbb{Z}_2)^3 \oplus \mathbb{Z}_{3\cdot5}, \quad \pi_{22}(S^{11}) = \mathbb{Z}_8 \oplus \mathbb{Z}_{9\cdot7}.$$

By (2.5), (2.6), (2.9) and (2.11), we have

(2.15) $$\mathrm{Ker}\{p_* : \pi_{14}(G_2) \to \pi_{14}(S^6)\} = \mathbb{Z}_2\{j_*[\nu_5^2] \circ \nu_{11}\} \oplus \mathbb{Z}_{21}.$$

Choose $\gamma \in \pi_{11}(G_2)$ so as to satisfy [18, Lemma 5.8] (cf. [20, Proposition 2.1(4)]). Then γ is unique up to sign. Our main result is

Theorem 2.1.   *The order of* $\langle i_3, \gamma \rangle$ *is* $3 \cdot 7$.

This theorem contains a known result that the order of $\langle i_3, \gamma \rangle$ is a multiple of 7 (see [9, Corollary 2.6]). But our proof is new even for 7-component. Furukawa [5] proved the identity:

(2.16) $$\langle i_3, j_*[\nu_5^2] \rangle = j_*[\nu_5^2] \circ \nu_{11}.$$

By this identity and Theorem 2.1, we know the image of (1.1) for $n = 3$ as follows:

$$\langle \pi_3(G_2), \pi_{11}(G_2) \rangle = \mathbb{Z}_2\{j_*[\nu_5^2] \circ \nu_{11}\} \oplus \mathbb{Z}_{21}.$$

Let $\mathcal{H}(G_2) = [G_2, G_2]$ be the set of all homotopy classes of based self maps of G_2. It inherits a group structure from G_2. Let $\mathrm{nil}\,\Gamma$ denote the nilpotency class of the group Γ. We proved in [20] (and will recall in Section 5) that $\mathrm{nil}\,\mathcal{H}(G_2) = 3$.

Corollary 2.1. $\mathrm{nil}(\mathcal{H}(G_2))_{(p)}$ *is 3, 2 or 1 according as* $p = 2$, $p \in \{3, 7\}$ *or* $p \notin \{2, 3, 7\}$.

Let us define

$$(2.17) \qquad \varphi : S^3 \times S^{11} \to G_2, \quad \varphi(x, y) = i_3(x)\gamma(y).$$

Recall from [10], [21] that, for a prime p, $\varphi_{(p)} : (S^3 \times S^{11})_{(p)} = S^3_{(p)} \times S^{11}_{(p)} \to G_{2(p)}$ is a homotopy equivalence if and only if $p \geq 7$. As is well-known [1], for a prime p, $S^{2n+1}_{(p)}$ ($n \neq 0, 1, 3$) has an H-multiplication if and only if p is odd. For an odd prime p, let μ' and μ'' be H-multiplications for $S^3_{(p)}$ and $S^{11}_{(p)}$, respectively, and let μ be the usual multiplication for G_2. Let $\mu' \times \mu''$ be the product multiplication for $S^3_{(p)} \times S^{11}_{(p)}$, that is, $(\mu' \times \mu'')((x_1, y_1), (x_2, y_2)) = (\mu'(x_1, x_2), \mu''(y_1, y_2))$.

Corollary 2.2. *Let p be an odd prime. Then the following statements are equivalent*:

(1) $\varphi_{(p)} : (S^3_{(p)} \times S^{11}_{(p)}, \mu' \times \mu'') \to (G_{2(p)}, \mu_{(p)})$ *is an H-map for some μ' and μ'';*

(2) $\varphi_{(p)} : (S^3_{(p)} \times S^{11}_{(p)}, \mu''') \to (G_{2(p)}, \mu_{(p)})$ *is an H-map for all H-multiplications μ''' on $S^3_{(p)} \times S^{11}_{(p)}$;*

(3) $p = 5$ *or* $p \geq 13$;

(4) $S^3_{(p)} \times S^{11}_{(p)}$ *has only one H-multiplication;*

(5) $G_{2(p)}$ *has only one H-multiplication.*

Remark. By [13], the above statement (3) is equivalent to

(6) $\mu_{(p)}$ *is homotopy commutative.*

3. Odd components of $\langle i_3, \gamma \rangle$ and $\langle \gamma, \gamma \rangle$

The purpose of this section is to prove

Proposition 3.1. *The orders of $\langle i_3, \gamma \rangle$ and $\langle \gamma, \gamma \rangle$ are multiples of $3 \cdot 7$ and $9 \cdot 7 \cdot 11$, respectively.*

We need

Lemma 3.1. *The homomorphism $i_* : \pi_k(G_2) \to \pi_k(\mathrm{Spin}(7))$ is an isomorphism for $k = 3, 11$ and a monomorphism for $k = 14, 22$.*

Proof. Consider the homotopy exact sequence of (2.2). Then the assertions follow from (2.3), (2.5), (2.8), (2.11)~(2.14). $\square$

Proof of Proposition 3.1.   Let $\pi : \mathrm{Spin}(7) \to \mathrm{SO}(7)$ be the covering map. Let n be 3 or 11. We have the following commutative diagram:

$$
\begin{array}{ccc}
\pi_n(\mathrm{G}_2) \times \pi_{11}(\mathrm{G}_2) & \xrightarrow{\ \langle\,,\,\rangle\ } & \pi_{n+11}(\mathrm{G}_2) \\[4pt]
{\scriptstyle i_* \times i_*}\big\downarrow{\scriptstyle \cong} & & \big\downarrow{\scriptstyle i_*} \\[4pt]
\pi_n(\mathrm{Spin}(7)) \times \pi_{11}(\mathrm{Spin}(7)) & \xrightarrow{\ \langle\,,\,\rangle\ } & \pi_{n+11}(\mathrm{Spin}(7)) \\[4pt]
{\scriptstyle \pi_* \times \pi_*}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \pi_*} \\[4pt]
\pi_n(\mathrm{SO}(7)) \times \pi_{11}(\mathrm{SO}(7)) & \xrightarrow{\ \langle\,,\,\rangle\ } & \pi_{n+11}(\mathrm{SO}(7)).
\end{array}
$$

(3.1)

By [4, Theorem 2.1], for every natural number m, there exists a homotopy equivalence between localized classifying spaces

$$
\Phi : (B\,\mathrm{SO}(2m+1))_{\frac{1}{2}} \to (B\,\mathrm{Sp}(m))_{\frac{1}{2}},
$$

where $(\ )_{\frac{1}{2}}$ denotes localization away from the prime 2. Hence

$$
\Omega\Phi : \Omega(B\,\mathrm{SO}(2m+1))_{\frac{1}{2}} \to \Omega(B\,\mathrm{Sp}(m))_{\frac{1}{2}}
$$

is an H-equivalence (i.e. a homotopy equivalence which is an H-map), where ΩX is the space of loops in the space X. Therefore we have an H-equivalence

$$
\phi : \mathrm{SO}(2m+1)_{\frac{1}{2}} \to \mathrm{Sp}(m)_{\frac{1}{2}},
$$

and so the commutative square

$$
\begin{array}{ccc}
\pi_n(\mathrm{SO}(7)_{\frac{1}{2}}) \times \pi_{11}(\mathrm{SO}(7)_{\frac{1}{2}}) & \xrightarrow{\ \langle\,,\,\rangle\ } & \pi_{n+11}(\mathrm{SO}(7)_{\frac{1}{2}}) \\[4pt]
{\scriptstyle \phi_* \times \phi_*}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \phi_*} \\[4pt]
\pi_n(\mathrm{Sp}(3)_{\frac{1}{2}}) \times \pi_{11}(\mathrm{Sp}(3)_{\frac{1}{2}}) & \xrightarrow{\ \langle\,,\,\rangle\ } & \pi_{n+11}(\mathrm{Sp}(3)_{\frac{1}{2}}).
\end{array}
$$

By combining the last square with (3.1), we have the following commutative square

(3.2)

$$
\begin{array}{ccc}
\pi_n(\mathrm{G}_2)_{\frac{1}{2}} \times \pi_{11}(\mathrm{G}_2)_{\frac{1}{2}} & \xrightarrow{\ \langle\,,\,\rangle\ } & \pi_{n+11}(\mathrm{G}_2)_{\frac{1}{2}} \\[4pt]
{\scriptstyle \cong}\big\downarrow & & \big\downarrow{\scriptstyle \psi} \\[4pt]
\pi_n(\mathrm{Sp}(3))_{\frac{1}{2}} \times \pi_{11}(\mathrm{Sp}(3))_{\frac{1}{2}} & \xrightarrow{\ \langle\,,\,\rangle\ } & \pi_{n+11}(\mathrm{Sp}(3))_{\frac{1}{2}},
\end{array}
$$

where $\psi = \phi_* \circ (\pi_* \circ i_*)_{\frac{1}{2}}$ is a monomorphism by Lemma 3.1. Since the inclusion induces a monomorphism $\pi_{22}(\mathrm{Sp}(3))_{\frac{1}{2}} \to \pi_{22}(\mathrm{Sp}(5))_{\frac{1}{2}}$ (see [16]), it follows from [2, Theorem 2] that the image of

$$
\langle\,,\,\rangle : \pi_{11}(\mathrm{Sp}(3))_{\frac{1}{2}} \times \pi_{11}(\mathrm{Sp}(3))_{\frac{1}{2}} \to \pi_{22}(\mathrm{Sp}(3))_{\frac{1}{2}}
$$

is a subgroup of order $9 \cdot 7 \cdot 11$ so that $\langle \gamma, \gamma \rangle$ is of order $9 \cdot 7 \cdot 11$ in $\pi_{22}(G_2)_{\frac{1}{2}}$ by (3.2). For similar reasons, $\langle i_3, \gamma \rangle$ is of order $3 \cdot 7$ in $\pi_{14}(G_2)_{\frac{1}{2}}$. Hence Proposition 3.1 follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

4.   The 2-component of $\langle i_3, \gamma \rangle$

The purpose of this section is to prove

Proposition 4.1.   *The order of* $\langle i_3, \gamma \rangle$ *is a divisor of* $3 \cdot 7$.

If this is true, then the order of $\langle i_3, \gamma \rangle$ is $3 \cdot 7$ by Proposition 3.1 and so Theorem 2.1 follows.

The relative Samelson products $\langle \ , \ \rangle_r$ associated with (2.1) are pairings

$$\langle \ , \ \rangle_r : \pi_s(\mathrm{SU}(3)) \times \pi_t(\mathrm{S}^6) \to \pi_{s+t}(\mathrm{S}^6),$$
$$\langle \ , \ \rangle_r : \pi_t(\mathrm{S}^6) \times \pi_s(\mathrm{SU}(3)) \to \pi_{s+t}(\mathrm{S}^6)$$

and they satisfy $\langle \alpha, \beta \rangle_r = (-1)^{st-1} \langle \beta, \alpha \rangle_r$ for $\alpha \in \pi_s(\mathrm{SU}(3))$, $\beta \in \pi_t(\mathrm{S}^6)$ (cf. [7]).

Let $\Delta : \pi_{s+1}(\mathrm{S}^6) \to \pi_s(\mathrm{SU}(3))$ be the connecting homomorphism of (2.1). The homotopy exact sequence of (2.1) implies the identity $p_*(\gamma) = \pm 30[\iota_6, \iota_6]$ by (2.4), (2.7), (2.9) and (2.10). On the other hand, we have $p_*\langle i_3, \gamma \rangle = \langle i_3', p_*\gamma \rangle_r$ by [7, (15.14)]. Hence

$$(4.1) \qquad\qquad p_*\langle i_3, \gamma \rangle = \pm 30 \langle i_3', [\iota_6, \iota_6] \rangle_r.$$

Lemma 4.1. $6\langle i_3', [\iota_6, \iota_6] \rangle_r = 0$.

If Lemma 4.1 holds, then $\langle i_3, \gamma \rangle \in \mathbb{Z}_2\{j_*[\nu_5^2] \circ \nu_{11}\} \oplus \mathbb{Z}_{21}$ by (4.1) and (2.15) and hence $21\langle i_3, \gamma \rangle = 0$ by (2.11), since $\langle i_3, \gamma \rangle$ is divisible by 4, as was proved in [20] and will be recalled in the next section, and so Proposition 4.1 follows.

We devote the rest of this section to the proof of Lemma 4.1. We have

$$(4.2) \qquad\qquad\qquad [\iota_6, \iota_6] = \pm \langle \Delta\iota_6, \iota_6 \rangle_r$$

by [7, (16.3)], and so

$$(4.3) \qquad\qquad \langle i_3', [\iota_6, \iota_6] \rangle_r = \pm \langle i_3', \langle \Delta\iota_6, \iota_6 \rangle_r \rangle_r.$$

The following Jacobi identity holds (cf. [7, (15.12)]):

$$\langle \iota_6, \langle i_3', \Delta\iota_6 \rangle \rangle_r - \langle \Delta\iota_6, \langle \iota_6, i_3' \rangle_r \rangle_r + \langle i_3', \langle \Delta\iota_6, \iota_6 \rangle_r \rangle_r = 0.$$

Since $\langle i_3', \Delta\iota_6 \rangle = \langle \Delta\iota_6, i_3' \rangle$ (cf. [7, (15.10)]), we then have

$$(4.4) \qquad \langle i_3', \langle \Delta\iota_6, \iota_6 \rangle_r \rangle_r = \langle \Delta\iota_6, \langle \iota_6, i_3' \rangle_r \rangle_r - \langle \iota_6, \langle \Delta\iota_6, i_3' \rangle \rangle_r.$$

Lemma 4.2.   *There exists an integer* m *such that, in* $\pi_{14}(\mathrm{S}^6)_{(2)}$, *we have*

$$(4.5) \qquad\qquad\qquad \langle \Delta\iota_6, \langle \iota_6, i_3' \rangle_r \rangle_r = 2m\overline{\nu}_6,$$
$$(4.6) \qquad\qquad\qquad \langle \iota_6, \langle \Delta\iota_6, i_3' \rangle \rangle_r = \pm 2m\overline{\nu}_6.$$

If Lemma 4.2 holds, then $\langle i_3', [\iota_6, \iota_6] \rangle_r = 2m(1 \pm 1)\overline{\nu}_6$ in $\pi_{14}(S^6)_{(2)}$ by (4.3) and (4.4), and so $6\langle i_3', [\iota_6, \iota_6] \rangle_r = 0$ by (2.5), and hence Lemma 4.1 follows.

Proof of Lemma 4.2.   In this proof, all groups are localized at 2. By (2.3), we can write

$$(4.7) \qquad\qquad \langle \iota_6, i_3' \rangle_r = k\nu_6, \quad k \in \mathbb{Z}.$$

By [2, Corollary], [7, (15.13)], [14] and [15], we can easily show that k is odd. We omit its proof, since we do not use this fact below. Now

$$(4.8) \qquad\qquad \langle \Delta\iota_6, \langle \iota_6, i_3' \rangle_r \rangle_r = k\langle \Delta\iota_6, \nu_6 \rangle_r.$$

We have

$$
\begin{aligned}
\langle \Delta\iota_6, \nu_6 \rangle_r &= \langle \Delta\iota_6 \circ \iota_5, \iota_6 \circ \nu_6 \rangle_r \\
&= \langle \Delta\iota_6, \iota_6 \rangle_r \circ \nu_{11} \quad \text{(by [7, (15.11)])} \\
&= \pm[\iota_6, \iota_6] \circ \nu_{11} \quad \text{(by (4.2))} \\
&= \pm 2\overline{\nu}_6 \quad \text{(by [22, Lemma 6.2]),}
\end{aligned}
$$

that is, we have

$$(4.9) \qquad\qquad \langle \Delta\iota_6, \nu_6 \rangle_r = \pm 2\overline{\nu}_6.$$

Hence $\langle \Delta\iota_6, \langle \iota_6, i_3' \rangle_r \rangle_r = \pm 2k\overline{\nu}_6$ by (4.8). Therefore, by taking $m = \pm k$, we obtain (4.5).

We have $\langle \Delta\iota_6, i_3' \rangle = \pm\Delta\langle \iota_6, i_3' \rangle_r$ by [7, (15.10), (15.13)], and

$$
\begin{aligned}
\langle \iota_6, \langle \Delta\iota_6, i_3' \rangle \rangle_r &= \pm\langle \iota_6, \Delta\langle \iota_6, i_3' \rangle_r \rangle_r \\
&= \pm k\langle \iota_6, \Delta\nu_6 \rangle_r \quad \text{(by (4.7)).}
\end{aligned}
$$

Since $\langle \iota_6, \Delta\nu_6 \rangle_r = \pm\langle \Delta\iota_6, \nu_6 \rangle_r$ by [7, (16.3)] and [23, (7.5) on p. 474], it follows from (4.9) that we have $\langle \iota_6, \Delta\nu_6 \rangle_r = \pm 2\overline{\nu}_6$ so that $\langle \iota_6, \langle i_3', \Delta\iota_6 \rangle \rangle_r = \pm 2k\overline{\nu}_6$. Therefore we obtain (4.6). This completes the proof of Lemma 4.2. $\qquad\square$

5.   Proof of Corollary 2.1

We use the usual cell structure of G_2 (cf. [18], [20]). Let $G_2^{(11)}$ be the 11-skeleton of G_2. Let $q_{14} : G_2 \to G_2/G_2^{(11)} = S^{14}$ be the quotient map and let $i_{11} : G_2^{(11)} \to G_2$ be the inclusion map.

Theorem 5.1 (Theorem 2.2 of [20]).
(1) *The sequence*

$$0 \longrightarrow \pi_{14}(G_2) \xrightarrow{q_{14}^*} \mathcal{H}(G_2) \xrightarrow{i_{11}^*} [G_2^{(11)}, G_2] \longrightarrow 1$$

is a central extension of groups.
(2) *There exists* $\alpha \in \mathcal{H}(G_2)$ *such that*

(a) $\mathcal{H}(G_2)$ *is generated by* 1, α *and* $\mathrm{Im}(q_{14}^*)$, *where* 1 *denotes the identity map,*

(b) $[1, [1, \alpha]] = q_{14}^*(j_*[\nu_5^2] \circ \nu_{11})$, *where* $[\ ,\ ]$ *denotes the commutator,*

(c) *there exists* $x_0 \in \pi_{14}(G_2)$ *such that* $2[1, \alpha] = 2q_{14}^*(x_0)$ *and* $\langle i_3, \gamma \rangle = \pm 4x_0$.

(3) $\mathrm{nil}[G_2^{(11)}, G_2] = \mathrm{nil}[G_2^{(11)}, G_2]_{(2)} = 2$ *and* $\mathrm{nil}[G_2^{(11)}, G_2]_{(p)} = 1$ *for every odd prime* p.

Hence $\mathrm{nil}\,\mathcal{H}(G_2) = 3 = \mathrm{nil}\,\mathcal{H}(G_2)_{(2)}$ and $\mathrm{nil}\,\mathcal{H}(G_2)_{(p)} \le 2$ for every odd prime p. If $p = 5$ or $p > 7$, then $\mathcal{H}(G_2)_{(p)} \cong [G_2^{(11)}, G_2]_{(p)}$ by the above exact sequence and (2.11), and so $\mathrm{nil}\,\mathcal{H}(G_2)_{(p)} = 1$. It follows from Theorem 2.1 and (2)(c) that the order of $[1, \alpha]$ is a multiple of 21 so that $\mathrm{nil}\,\mathcal{H}(G_2)_{(p)} \ge 2$, and hence $\mathrm{nil}\,\mathcal{H}(G_2)_{(p)} = 2$, for $p = 3, 7$. This completes the proof of Corollary 2.1.

6. Proof of Corollary 2.2

Let $e : Y \to Y_{(p)}$ be the p-localization map for any nilpotent space Y.

"(2) $\Rightarrow$ (1)" is obvious.

To prove "(1) $\Rightarrow$ (3)" by a contradiction, suppose that p is 3, 7 or 11 and that $\varphi_{(p)} : (S_{(p)}^3 \times S_{(p)}^{11}, \mu' \times \mu'') \to (G_{2(p)}, \mu_{(p)})$ is an H-map for some μ' and μ''. Firstly we consider the cases $p = 3, 7$. We denote by $j_3 : S^3 \to S_{(p)}^3 \times S_{(p)}^{11}$ and $j_{11} : S^{11} \to S_{(p)}^3 \times S_{(p)}^{11}$ the compositions of e and the inclusion maps. Then $\langle j_3, j_{11} \rangle = 0$ by the definition of the Samelson product (cf. [19]). Hence we have

$$0 = \varphi_{(p)*}\langle j_3, j_{11} \rangle = e_*\langle i_3, \gamma \rangle \in \pi_{14}(G_2)_{(p)}.$$

This contradicts Proposition 3.1. Secondly we consider the case $p = 11$. As is easily seen, $\gamma_{(11)} : (S_{(11)}^{11}, \mu'') \to (G_{2(11)}, \mu_{(11)})$ is an H-map under the assumption. We have $\langle e \circ \iota_{11}, e \circ \iota_{11} \rangle = 0$, since $\pi_{22}(S^{11})_{(11)} = 0$ by (2.14). Therefore

$$0 = \gamma_{(11)*}\langle e \circ \iota_{11}, e \circ \iota_{11} \rangle = e_*\langle \gamma, \gamma \rangle \in \pi_{22}(G_2)_{(11)}.$$

This contradicts Proposition 3.1.

To prove "(3) $\Rightarrow$ (2)", let $p = 5$ or $p \ge 13$, and let μ''' be any H-multiplication for $S_{(p)}^3 \times S_{(p)}^{11}$. Let

$$D = D(\varphi_{(p)}, \mu''', \mu_{(p)}) \in [X_{(p)}, G_{2(p)}] = [X, G_2]_{(p)}$$

be the H-deviation of $\varphi_{(p)}$ (see [24, 1.4.1]), where $X = (S^3 \times S^{11}) \wedge (S^3 \times S^{11})$. Then $\varphi_{(p)}$ is an H-map with respect to μ''' and $\mu_{(p)}$ if and only if $D = 0$. We prove the assertion by showing $[X, G_2]_{(p)} = 0$. Since X has a cell structure

(6.1) $$X = S^6 \cup e^{14} \cup e^{14} \cup e^{22} \cup e^{28},$$

it suffices to prove that $\pi_m(G_2)_{(p)} = 0$ for $m = 6, 14, 22, 28$. By [15], we have $\pi_m(G_2)_{(p)} = 0$ for $m = 6, 14, 22$. By [22, Chapter XIII] and [6], we have

$$\pi_{28}(G_2)_{(p)} \cong \pi_{28}(\mathrm{Spin}(7))_{(p)} \cong \pi_{28}(\mathrm{Sp}(3))_{(p)} \cong \pi_{28}(\mathrm{Sp}(2))_{(p)} = 0.$$

Hence (2) follows.

Recall from [3, Theorem 5.5A] that if Y is a CW-H-space, then the set of H-multiplications for Y is in $1-1$ correspondence with $[Y \wedge Y, Y]$. We have

$$[(S^3_{(p)} \times S^{11}_{(p)}) \wedge (S^3_{(p)} \times S^{11}_{(p)}), S^3_{(p)} \times S^{11}_{(p)}] = [X, S^3_{(p)}] \times [X, S^{11}_{(p)}].$$

By using (6.1) and [22], we can show that $[X, S^3_{(p)}]$ and $[X, S^{11}_{(p)}]$ are trivial if and only if $p = 5$ or $p \geq 13$. Hence (3) and (4) are equivalent.

In the rest of the proof, we show that (3) and (5) are equivalent. Let $p \geq 7$. Then $S^3_{(p)} \times S^{11}_{(p)} \simeq G_{2(p)}$ as recalled in Section 2, and so $G_{2(p)}$ has only one H-multiplication if and only if $p \geq 13$ by the above discussion. By [17], $G_{2(5)} \simeq B_1(5)_{(5)}$, where $B_1(5)$ is a S^3-bundle over S^{11}. Hence $[G_{2(5)} \wedge G_{2(5)}, G_{2(5)}] \cong [B_1(5) \wedge B_1(5), B_1(5)_{(5)}]$. By using [22, Chapter XIII] and a cell structure $B_1(5) = S^3 \cup e^{11} \cup e^{14}$ (cf. [8]), we can show that $[B_1(5) \wedge B_1(5), B_1(5)_{(5)}]$ is trivial. Hence $G_{2(5)}$ has only one H-multiplication. Let $C : G_2 \wedge G_2 \to G_2$ be the commutator map, that is, $C(x \wedge y) = xyx^{-1}y^{-1}$. Since $C \circ (i_3 \wedge i_3) = \langle i_3, i_3 \rangle$ and $\pi_6(G_2) = \mathbb{Z}_3\{\langle i_3, i_3 \rangle\}$ by [15], it follows that the order of C is a multiple of 3. Hence $[G_{2(3)} \wedge G_{2(3)}, G_{2(3)}]$ is not trivial and so $G_{2(3)}$ has at least three H-multiplications. Therefore (3) and (5) are equivalent. $\square$

7.   A remark

By [15] and (2.16), we can show that inclusions (cf. [23, Appendix A])

(7.1) $G_2 \subset \mathrm{Spin}(7) \subset \mathrm{Spin}(8) \subset \mathrm{Spin}(9) \subset F_4$

induce isomorphisms

(7.2)
$\langle \pi_3(G_2), \pi_{11}(G_2) \rangle \cong \langle \pi_3(\mathrm{Spin}(7)), \pi_{11}(\mathrm{Spin}(7)) \rangle \cong \langle \pi_3(\mathrm{Spin}(8)), \pi_{11}(\mathrm{Spin}(8)) \rangle,$

(7.3) $\langle \pi_3(\mathrm{Spin}(9)), \pi_{11}(\mathrm{Spin}(9)) \rangle \cong \langle \pi_3(F_4), \pi_{11}(F_4) \rangle = \pi_{14}(F_4)$

and an epimorphism

$$\langle \pi_3(\mathrm{Spin}(8)), \pi_{11}(\mathrm{Spin}(8)) \rangle \to \langle \pi_3(\mathrm{Spin}(9)), \pi_{11}(\mathrm{Spin}(9)) \rangle.$$

Groups in (7.2) and (7.3) are, respectively, isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_{21}$ and $\mathbb{Z}_2$. Here, for every G in (7.1), $\mathbb{Z}_2$ is a direct summand of $\pi_{14}(G)$ and generated by the image of $\langle i'_3, [\nu_5^2] \rangle \in \pi_{14}(\mathrm{SU}(3))$ under the inclusion of SU(3) into G.

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