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On dense orbits in the boundary of a Coxeter system*

By

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Abstract

In this paper, we study the minimality of the boundary of a Coxeter system. We show that for a Coxeter system (W,S) if there exist a maximal spherical subset T of S and an element $s_0 \in S$ such that $m(s_0,t) \geq 3$ for each $t \in T$ and $m(s_0,t_0)=\infty$ for some $t_0 \in T$, then every orbit $W\alpha$ is dense in the boundary $\partial\Sigma(W,S)$ of the Coxeter system (W,S), hence $\partial\Sigma(W,S)$ is minimal, where $m(s_0,t)$ is the order of s_0t in W.

1. Introduction and preliminaries

The purpose of this paper is to study the minimality of the boundary of a Coxeter system. A *Coxeter group* is a group W having a presentation

$$\langle\, S \mid (st)^{m(s,t)} = 1 \text{ for } s,t \in S \,\rangle,$$

where S is a finite set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

(1) $m(s,t) = m(t,s)$ for each $s,t \in S$,
(2) $m(s,s) = 1$ for each $s \in S$, and
(3) $m(s,t) \geq 2$ for each $s,t \in S$ such that $s \neq t$.

The pair (W,S) is called a *Coxeter system*. Let (W,S) be a Coxeter system. For a subset $T \subset S$, W_T is defined as the subgroup of W generated by T, and called a *parabolic subgroup*. If T is the empty set, then W_T is the trivial group. A subset $T \subset S$ is called a *spherical subset of* S, if the parabolic subgroup W_T is finite.

Every Coxeter system (W,S) determines a *Davis-Moussong complex* $\Sigma(W,S)$ which is a CAT(0) geodesic space ([4], [5], [6], [10]). Here the 1-skeleton of $\Sigma(W,S)$ is the Cayley graph of W with respect to S. The natural action of W on $\Sigma(W,S)$ is proper, cocompact and by isometry. We can consider a certain

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fundamental domain $K(W,S)$ which is called a *chamber* of $\Sigma(W,S)$ such that $WK(W,S) = \Sigma(W,S)$ ([5], [6]). If W is infinite, then $\Sigma(W,S)$ is noncompact and $\Sigma(W,S)$ can be compactified by adding its ideal boundary $\partial\Sigma(W,S)$ ([2], [5, §4]). This boundary $\partial\Sigma(W,S)$ is called the *boundary of* (W,S). We note that the natural action of W on $\Sigma(W,S)$ induces an action of W on $\partial\Sigma(W,S)$.

The following theorem was proved in [8].

Theorem 1.1. *Let (W,S) be a Coxeter system. Suppose that there exist a maximal spherical subset T of S and an element $s_0 \in S$ such that $m(s_0,t) \geq 3$ for each $t \in T$ and $m(s_0,t_0) = \infty$ for some $t_0 \in T$. Then $W\alpha$ is dense in $\partial\Sigma(W,S)$ for some $\alpha \in \partial\Sigma(W,S)$.*

Suppose that a group G acts on a compact metric space X by homeomorphisms. Then X is said to be *minimal*, if every orbit Gx is dense in X.

For a negatively curved group Γ and the boundary $\partial\Gamma$ of Γ, by an easy argument, we can show that $\Gamma\alpha$ is dense in $\partial\Gamma$ for each $\alpha \in \partial\Gamma$, that is, $\partial\Gamma$ is minimal.

We note that Coxeter groups are non-positive curved groups and not negatively curved groups in general. Indeed, there exist examples of Coxeter systems whose boundaris are not minimal as follows.

Example 1.1. Let $S = \{s,t,u\}$ and let

$$W = \langle\, S \mid s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (us)^3 = 1 \,\rangle.$$

Then (W,S) is a Coxeter system and $\Sigma(W,S)$ is the flat Euclidean plane. For any $\alpha \in \partial\Sigma(W,S)$, $W\alpha$ is a finite-points set and not dense in $\partial\Sigma(W,S)$ which is a circle. This example implies that we can not omit the assumption "$m(s_0,t_0) = \infty$" in Theorem 1.1.

Example 1.2. Let $S = \{s_1,s_2,s_3,s_4\}$ and let

$$W = \langle\, S \mid s_1^2 = s_2^2 = s_3^2 = s_4^2 = (s_1s_2)^2 = (s_2s_3)^2 = (s_3s_4)^2 = (s_4s_1)^2 = 1 \,\rangle.$$

Then (W,S) is a Coxeter system and $\Sigma(W,S)$ is the Euclidean plane. For any $\alpha \in \partial\Sigma(W,S)$, $W\alpha$ is a finite-points set and not dense in $\partial\Sigma(W,S)$ which is a circle. Here we note that $\{s_1,s_2\}$ is a maximal spherical subset of S, $m(s_1,s_3) = \infty$ and $m(s_2,s_3) = 2$. This example implies that we can not omit the assumption "$m(s_0,t) \geq 3$" in Theorem 1.1.

The purpose of this paper is to prove the following theorem as an extension of Theorem 1.1.

Theorem 1.2. *Let (W,S) be a Coxeter system which satisfies the condition in Theorem* 1.1. *Then every orbit $W\alpha$ is dense in $\partial\Sigma(W,S)$, that is, $\partial\Sigma(W,S)$ is minimal.*

2. Lemmas on Coxeter groups and the Davis-Moussong complexes

In this section, we recall and prove some lemmas for Coxeter groups and the Davis-Moussong complexes which are used later.

Definition 2.1. Let (W, S) be a Coxeter system and $w \in W$. A representation $w = s_1 \cdots s_l$ $(s_i \in S)$ is said to be *reduced*, if $\ell(w) = l$, where $\ell(w)$ is the minimum length of word in S which represents w.

Definition 2.2. Let (W, S) be a Coxeter system. For each $w \in W$, we define $S(w) = \{s \in S \mid \ell(ws) < \ell(w)\}$. For a subset $T \subset S$, we also define $W^T = \{w \in W \mid S(w) = T\}$.

The following lemma is known.

Lemma 2.1 ([1], [3], [4], [6], [9])**.** *Let (W, S) be a Coxeter system.*
(1) *Let $w \in W$ and let $w = s_1 \cdots s_l$ be a representation. If $\ell(w) < l$, then $w = s_1 \cdots \hat{s_i} \cdots \hat{s_j} \cdots s_l$ for some $1 \leq i < j \leq l$.*
(2) *For each $w \in W$ and $s \in S$, $\ell(ws)$ equals either $\ell(w) + 1$ or $\ell(w) - 1$, and $\ell(sw)$ also equals either $\ell(w) + 1$ or $\ell(w) - 1$.*
(3) *For each $w \in W$, $S(w)$ is a spherical subset of S, i.e., $W_{S(w)}$ is finite.*
(4) *For $w \in W$ and a spherical subset T of S, there exists a unique element of longest length in $W_T w$.*
(5) *For $w \in W$ and a spherical subset T of S, $v \in W_T w$ is the element of longest length in $W_T w$ if and only if $\ell(tv) < \ell(v)$ for any $t \in T$. Moreover, then $\ell(v) = \ell(vw^{-1}) + \ell(w)$.*

We prove the following technical lemma.

Lemma 2.2. *Let (W, S) be a Coxeter system, T be a spherical subset of S, and let $w \in W$ and $s \in S$ satisfy $\ell(ws) = \ell(w) + 1$. Moreover, let $x, x' \in W_T$ be the unique elements such that xw and $x'ws$ are the elements of longest length in $W_T w$ and $W_T ws$ respectively, given by Lemma 2.1 (4). If $x = t_1 \ldots t_m$ is a reduced representation, then either $x' = x$ or $x' = t_1 \cdots \hat{t_i} \cdots t_m$ for some $i \in \{1, \ldots, m\}$. In particular, we have $\ell(x') \leq \ell(x)$.*

Proof. By Lemma 2.1 (2), either $\ell(xws) = \ell(xw) + 1$ or $\ell(xws) = \ell(xw) - 1$.

We first suppose that $\ell(xws) = \ell(xw) + 1$. Since xw is the element of longest length in $W_T w$, $\ell(txw) < \ell(xw)$ for any $t \in T$ by Lemma 2.1 (5). Then for each $t \in T$,

$$\ell(txws) \leq \ell(txw) + 1 < \ell(xw) + 1 = \ell(xws).$$

Hence $\ell(txws) < \ell(xws)$ for any $t \in T$. Thus xws is the element of longest length in $W_T ws$, i.e., $x' = x$.

Next we suppose that $\ell(xws) = \ell(xw) - 1$. Since $\ell(ws) = \ell(w) + 1$ and $\ell(xw) = \ell(x) + \ell(w)$,

$$xws = (t_1 \cdots t_m)ws = (t_1 \cdots \hat{t_i} \cdots t_m)w$$

for some $i \in \{1, \ldots, m\}$ by Lemma 2.1 (1). Now xw is the element of longest length in $W_T w$. Here

$$W_T w = W_T(t_1 \cdots \hat{t_i} \cdots t_m)w = W_T xws = W_T ws.$$

Hence xw is the element of longest length in $W_T ws$. Since $xw = (t_1 \cdots \hat{t_i} \cdots t_m)ws$, we obtain $x' = t_1 \cdots \hat{t_i} \cdots t_m$. □

The following lemma was proved in [8].

Lemma 2.3 ([8, Lemma 2.4]). *Let (W,S) be a Coxeter system, $w \in W$ and $s_0 \in S$. Suppose that $m(s_0,t) \geq 3$ for each $t \in S(w)$ and that $m(s_0,t_0) = \infty$ for some $t_0 \in S(w)$. Then $ws_0 \in W^{\{s_0\}}$.*

We can obtain the following lemma by the same argument as the proof of [7, Lemma 4.2].

Lemma 2.4. *Let (W,S) be a Coxeter system and let $\alpha \in \partial\Sigma(W,S)$. Then there exists a sequense $\{s_i\} \subset S$ such that $s_1 \cdots s_i$ is reduced and*

$$d(s_1 \cdots s_i, \operatorname{Im}\xi_\alpha) \leq N$$

for each $i \in \mathbb{N}$, where N is the diameter of $K(W,S)$ in $\Sigma(W,S)$ and ξ_α is the geodesic ray in $\Sigma(W,S)$ such that $\xi_\alpha(0) = 1$ and $\xi_\alpha(\infty) = \alpha$.

3. Proof of the main theorem

Using the lemmas above, we prove Theorem 1.2.

Proof of Theorem 1.2. Let (W,S) be a Coxeter system. Suppose that there exist a maximal spherical subset T of S and an element $s_0 \in S$ such that $m(s_0,t) \geq 3$ for each $t \in T$ and $m(s_0,t_0) = \infty$ for some $t_0 \in T$. Let $\alpha \in \partial\Sigma(W,S)$. By Lemma 2.4, there exists a sequense $\{s_i\} \subset S$ such that $s_1 \cdots s_i$ is reduced and $d(s_1 \cdots s_i, \operatorname{Im}\xi_\alpha) \leq N$ for each $i \in \mathbb{N}$, where N is the diameter of $K(W,S)$ in $\Sigma(W,S)$ and ξ_α is the geodesic ray such that $\xi_\alpha(0) = 1$ and $\xi_\alpha(\infty) = \alpha$. Let $w_i = s_1 \cdots s_i$. For each i, there exists a unique element $x_i \in W_T$ such that $x_i w_i$ is the element of longest length in $W_T w_i$ by Lemma 2.1 (4). Now $w_{i+1} = w_i s_{i+1}$ and $\ell(w_{i+1}) = \ell(w_i) + 1$. By Lemma 2.2, $\ell(x_{i+1}) \leq \ell(x_i)$ for every i. Hence there exists a number n such that $\ell(x_i) = \ell(x_{i+1})$ for each $i \geq n$. Then $x_i = x_{i+1}$ for every $i \geq n$ by Lemma 2.2. Let $x = x_n$. Then xw_i is the element of longest length in $W_T w_i$ for each $i \geq n$. Since $\ell(txw_i) < \ell(xw_i)$ for any $t \in T$, $T \subset S((xw_i)^{-1})$. Here $S((xw_i)^{-1})$ is a spherical subset of S by Lemma 2.1 (3) and T is a maximal spherical subset of S. Hence $S((xw_i)^{-1}) = T$ for each $i \geq n$. By Lemma 2.3, $(s_0 xw_i)^{-1} \in W^{\{s_0\}}$ and $(t_0 s_0 xw_i)^{-1} \in W^{\{t_0\}}$ for each $i \geq n$. Hence $(W^{\{t_0\}})^{-1}$ contains the sequence $\{t_0 s_0 xw_i\}_{i \geq n}$ which converges to $t_0 s_0 x\alpha$. By the proof of [8, Theorem 4.1], $Wt_0 s_0 x\alpha$ is dense in $\partial\Sigma(W,S)$. Here $Wt_0 s_0 x\alpha = W\alpha$. Hence $W\alpha$ is a dense subset of $\partial\Sigma(W,S)$. Thus every orbit $W\alpha$ is dense in $\partial\Sigma(W,S)$, that is, $\partial\Sigma(W,S)$ is minimal. □

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