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Asymptotics of solutions to the fourth order Schrödinger type equation with a dissipative nonlinearity

By

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Abstract

In this paper, the asymptotic behavior in time of solutions to the one-dimensional fourth order nonlinear Schrödinger type equation with a cubic dissipative nonlinearity $\lambda|u|^2u$, where λ is a complex constant satisfying $\operatorname{Im}\lambda < 0$, is studied. This nonlinearity is a long-range interaction. The local Cauchy problem at infinite initial time (the final value problem) to this equation is solved for a given final state with no size restriction on it. This implies the existence of a unique solution for the equation approaching some modified free dynamics as $t \to +\infty$ in a suitable function space. Our modified free dynamics decays like $(t\log t)^{-1/2}$ as $t \to \infty$.

1. Introduction

We study the asymptotic behavior in time of solutions for the fourth order nonlinear Schrödinger type equation with a cubic dissipative nonlinearity in one space dimension:

$$i\partial_t u - \frac{1}{4}\partial_x^4 u = (\lambda_1 + i\lambda_2)|u|^2 u, \quad t > 0,\ x \in \mathbb{R}, \tag{1.1}$$

where $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, $\lambda_1 \in \mathbb{R}$, $\lambda_2 < 0$, and u is a complex valued unknown function of (t, x). The nonlinearity of the equation (1.1) has a dissipative property, and it is a long-range interaction. In this paper, we solve the local Cauchy problem at infinite initial time (the final value problem) to the equation (1.1) for a given final state u_+ with no size restriction on u_+, which implies the existence of a unique solution u for the equation (1.1) approaching some modified free dynamics u_a as $t \to +\infty$ in a suitable function space. The asymptotics u_a decays like $(t\log t)^{-1/2}$ as $t \to \infty$.

We recall several known results on the asymptotic behavior of solutions to the nonlinear Schrödinger equation

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$$i\partial_t u + \frac{1}{2}\Delta u = \mu |u|^{p-1} u, \quad t > 0,\ x \in \mathbb{R}^n, \tag{1.2}$$

where $p > 1$, $\mu \in \mathbb{C} \setminus \{0\}$ and $\Delta = \sum_{j=1}^{n} \partial^2/\partial x_j^2$ is the Laplace operator with respect to the space variable x. It is well-known that if $p > 1 + 2/n$, then the nonlinearity $\mu|u|^{p-1}u$ is a short-range interaction, that is, contribution of the nonlinearity is negligible for large time. (For results on the short-range scattering for the equation (1.2), see, e.g., Ginibre [4].) On the other hand, if $p \le 1 + 2/n$, then the nonlinearity $\mu|u|^{p-1}u$ is a long-range interaction, that is, contribution of the nonlinear term is not negligible for large time. (More precisely, in Barab [1], it was shown that there does not exist an asymptotically free solution for the equation (1.2) if $1 \le p \le 1+2/n$ and $\mu \in \mathbb{R}\setminus\{0\}$.) Therefore we see that for the equation (1.2), the exponent $p = 1 + 2/n$ is critical between the short range case and the long range one. Recall that the solution to the Cauchy problem of the free Schrödinger equation

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = \phi(x), & x \in \mathbb{R}^n \end{cases}$$

is given by $U(t)\phi$, where $U(t) = e^{it\Delta/2}$, and it decays as $\|U(t)\phi\|_{L^q} \le Ct^{-n(1/2-1/q)}\|\phi\|_{L^{q'}}$, where $q \ge 2$ and $1/q + 1/q' = 1$. We consider the equation (1.2) with the critical exponent $p = 1 + 2/n$ and $\mu \in \mathbb{R} \setminus \{0\}$. In this case, the modified wave operators to the equation (1.2) were constructed by Ozawa [10] for $n = 1$ and Ginibre-Ozawa [5] for $n = 2$ or 3 for small final data u_+ by a suitable phase shift, more precisely, the solution u behaves like the modified free profile $U(t)e^{-iS(t,-i\nabla)}u_+$, where $S(t,x) = \mu|\hat{u}_+(x)|^{2/n}\log t$. Ginibre and Velo [6] proved the existence of modified wave operators to the equation (1.2) in the case $n = 1$ without any size restriction of the final state u_+ and extended the above results. For $p = 1+2/n$ and $n \le 3$, Hayashi and Naumkin [7] showed that the small global solution for the initial value problem of the equation (1.2) with $\mu \in \mathbb{R} \setminus \{0\}$ satisfies the time decay estimate $\|u(t)\|_{L_x^\infty} = O(t^{-n/2})$, and that the solution has a modified free profile with the above phase shift. Furthermore, for $p = 3$ and $n = 1$, Carles [3] proved the existence of the modified scattering operator via the geometric optics. Recently, when $p = 1 + 2/n$ and $n \le 3$, Hayashi and Naumkin [8] constructed the modified scattering operator, and their result is an improvement of [3] in the one-dimensional case.

In the case of $\mu \in \mathbb{C}$ with $\operatorname{Im}\mu < 0$ and the critical exponent $p = 1 + 2/n$, the nonlinearity $\mu|u|^{p-1}u$ of the nonlinear Schrödinger equation (1.2) has a dissipative property. In this case, recently, in [15], a time decay estimate and an asymptotic behavior of the small global solution for the initial value problem of the equation (1.2) were obtained when the space dimension $n = 1, 2$ or 3. According to [15], roughly speaking, there exists a $u_+ \in L_x^2 \cap L_x^\infty$ such that the small solution to the initial value problem to the equation (1.2) with $\mu \in \mathbb{C}$, $\operatorname{Im}\mu < 0$ and $p = 1 + 2/n$ behaves like $U(t)\widetilde{B}(t,-i\nabla)u_+$ in L_x^2 as $t \to +\infty$, where

$$\widetilde{B}(t,x) = \left(1 + \frac{2|\mu_2|}{n}|\hat{u}_+(x)|^{2/n}\log t\right)^{-n/2} \times \exp\left(i\frac{n\mu_1}{2|\mu_2|}\log\left(1 + \frac{2|\mu_2|}{n}|\hat{u}_+(x)|^{2/n}\log t\right)\right),$$

and $\mu = \mu_1 + i\mu_2$ with $\mu_1 \in \mathbb{R}$ and $\mu_2 < 0$. (This means that the solution has a modified free profile not only with a phase shift but also with a correction of amplitude.) Furthermore, in this case, the solution u of the equation (1.2) decays like $\|u(t)\|_{L^\infty_x} = (t\log t)^{-n/2}$ as $t \to +\infty$.

We return to the fourth order nonlinear Schrödinger type equation (1.1). The local well-posedness for the fourth order nonlinear Schrödinger type equation was studied in [11, 12], and by the Strichartz estimate, the global well-posedness in L^2 for the equation (1.1) was proved in the appendix in [13]. We consider the asymptotic behavior in time of solutions to the equation (1.1). The solution to the initial value problem of the free equation

$$\begin{cases} i\partial_t v - \dfrac{1}{4}\partial_x^4 v = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ v(0,x) = v_0(x) \end{cases}$$

is given by $v(t,\cdot) = V(t)v_0$, where $V(t) = e^{-it\partial_x^4/4}$ is the free evolution operator. Ben-Artzi, Koch and Saut [2] showed that the free solution $V(t)v_0$ decays like $t^{-1/4}$ in L^∞_x for $v_0 \in L^1_x$, that is, $\|V(t)v_0\|_{L^\infty_x} \le C\|v_0\|_{L^1_x} t^{-1/4}$. Furthermore, it is well-known that if v_0 satisfies $|\hat{v}_0(\xi)| = O(|\xi|^\alpha)$, as $|\xi| \to 0$, with suitable $\alpha > 0$, then the free solution $V(t)v_0$ behaves as

$$\frac{1}{(3it)^{1/2}}\frac{1}{|\sqrt[3]{x/t}|}\hat{v}_0\left(\sqrt[3]{\frac{x}{t}}\right)\exp\left(\frac{3}{4}ix\sqrt[3]{\frac{x}{t}}\right)$$

in large time, and this function decays like $t^{-1/2}$ in L^∞_x as for the nonlinear Schrödinger equation (see [13]). Here $\hat{v}_0$ denotes the Fourier transform of v_0. In view of this, it is expected that cubic nonlinearities are critical between the short-range and the long-range interaction as in the case of one dimensional the nonlinear Schrödinger equation. In the case of $\lambda_2 = 0$, in [13], the large time behavior of solutions to the equation (1.1) was studied, that is, the existence of the modified wave operator was shown for a given small final state u_+ satisfying $|\hat{u}_+(\xi)| = O(|\xi|^\alpha)$ with some $\alpha > 0$ as $|\xi| \to 0$. The asymptotics at $t = \pm\infty$ of solution in [13] is $\sqrt{2\pi}F(t,x)\hat{u}_+(\sqrt[3]{x/t})e^{-i\widetilde{S}_\pm(t,x/t)}$, where

$$F(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi - it\xi^4/4}\,d\xi$$

is the fundamental solution to the free equation, and

$$\widetilde{S}_\pm(t,x) = \pm\frac{\lambda_1}{3}\frac{|\hat{u}_+(\sqrt[3]{x})|^2}{(\sqrt[3]{x})^2}\log|t|$$

is a phase correction. Furthermore, in [14], the modified wave operator for the fourth order Schrödinger type equation with the non-gauge-invariant nonlinearity $\lambda_0|u|^2u + \lambda_1 u^3 + \lambda_2 u\bar{u}^2 + \lambda_3\bar{u}^3$, where $\lambda_0 \in \mathbb{R}$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$, was constructed.

In this paper, in the case of $\lambda_1 \in \mathbb{R}$ and $\lambda_2 < 0$, we solve the local Cauchy problem at infinite initial time (the final value problem) to the equation (1.1) for a given final state u_+ with no size restriction on u_+, that is, the existence of a unique solution u for the equation (1.1) approaching some modified free dynamics u_a as $t \to +\infty$ in a suitable function space. Our asymptotics u_a of solution u is a modified free profile not only with a phase shift but also with a correction of amplitude (see the definition (1.5)–(1.9) of u_a below). We can solve this problem without assuming smallness condition on the final state u_+ (see (1.5)–(1.9)), since the asymptotics u_a decays like $(t\log t)^{-1/2}$ in L^∞_x, which is faster than the free solution.

Notation. For $\psi \in \mathcal{S}'$, we denote the Fourier transform of ψ by $\hat{\psi}$ or $\mathcal{F}\psi$. For $\psi \in L^1(\mathbb{R}^n)$, $\hat{\psi}$ is represented as

$$\mathcal{F}\psi(\xi) = \hat{\psi}(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\psi(x)e^{-ix\cdot\xi}\,dx.$$

For a space-time variable $(t,x) \in \mathbb{R}^2$, we denote $\partial_0 = \partial_t = \partial/\partial t$ and $\partial = \partial_x = \partial/\partial x$. For $m, s \in \mathbb{R}$, we introduce the weighted Sobolev spaces:

$$H^{m,s} = H^{m,s}(\mathbb{R}) = \{\psi \in \mathcal{S}'; \|\psi\|_{H^{m,s}} = \|(1+|x|^2)^{s/2}(1-\partial_x^2)^{m/2}\psi\|_{L^2} < \infty\}.$$

We denote $H^{m,0}$ by H^m. $\dot{H}^m$ is the homogeneous Sobolev space:

$$\dot{H}^m = \dot{H}^m(\mathbb{R}) = \{\psi \in \mathcal{S}'; \|\psi\|_{\dot{H}^m} = \|(-\partial_x^2)^{m/2}\psi\|_{L^2_x} < \infty\}.$$

We introduce the following operators:

$$V(t) = e^{-it\partial_x^4/4}, \quad \mathcal{L} = i\partial_t - \frac{1}{4}\partial_x^4. \tag{1.3}$$

For $a \in \mathbb{R}$, $\sqrt[3]{a}$ denotes the unique real root of the equation $x^3 = a$. C denotes a constant and so forth. They may differ from line to line, when it does not cause any confusion.

Before stating the main result, we introduce the set of final states and the asymptotic function.

Let

$$\mathcal{D} = \{\psi \in L^2; \|\psi\|_{\mathcal{D}} < \infty\}, \tag{1.4}$$

where

$$\|\psi\|_{\mathcal{D}} = \|\psi\|_{H^{0,4}} + \sum_{k=0}^{4}\|x^k\psi\|_{\dot{H}^{k-12}}.$$

For a final state u_+, we introduce the following asymptotic profile:

$$u_a(t,x) = u_1(t,x)B\left(t, \frac{x}{t}\right), \tag{1.5}$$

where

$$u_1(t,x) = \frac{1}{(3it)^{1/2}} \frac{1}{|\sqrt[3]{x/t}|} \hat{u}_+\left(\sqrt[3]{\frac{x}{t}}\right) \exp\left(\frac{3}{4} ix \sqrt[3]{\frac{x}{t}}\right), \tag{1.6}$$

$$B(t,x) = W(t,x)e^{-iS(t,x)}, \tag{1.7}$$

$$S(t,x) = \frac{\lambda_1}{2|\lambda_2|} \log\left(1 + \frac{2|\lambda_2|}{3} \frac{|\hat{u}_+(\sqrt[3]{x})|^2}{(\sqrt[3]{x})^2} \log t\right), \tag{1.8}$$

$$W(t,x) = \left(1 + \frac{2|\lambda_2|}{3} \frac{|\hat{u}_+(\sqrt[3]{x})|^2}{(\sqrt[3]{x})^2} \log t\right)^{-1/2} \tag{1.9}$$

for $t \geq 1$ and $x \in \mathbb{R}$. u_1 is an approximate solution for the free equation

$$i\partial_t v - \frac{1}{4}\partial_x^4 v = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R} \tag{1.10}$$

with $v(0,x) = u_+(x)$. (See [13].) u_a is a modified free dynamics with the modifier B. We note that S and W are real valued.

The main result in this paper is the following theorem:

Theorem 1.1. *Let $\lambda_1 \in \mathbb{R}$, $\lambda_2 < 0$ and let $u_+ \in \mathcal{D}$ be a final state, where $\mathcal{D}$ is defined by* (1.4). *Then there exist a $T \geq 3$ and a unique solution u for the equation* (1.1) *satisfying*

$$u \in C([T,\infty); L_x^2),$$

$$\sup_{t \geq T} t^d (\|u(t) - u_a(t)\|_{L_x^2} + \|u - u_a\|_{L^8((t,\infty);L_x^\infty)}) < \infty, \tag{1.11}$$

where $3/8 < d < 1$ and the asymptotic profile u_a is defined by (1.5)–(1.9).

Remark 1. In Theorem 1.1, no smallness condition on the final state u_+ is assumed.

Remark 2. We comment on the time decay for the asymptotic profile u_a and the solution u and on the convergence rate of their difference $u - u_a$. Let $\lambda_1 \in \mathbb{R}$, $\lambda_2 < 0$, $u_+ \in \mathcal{D}$ and $\hat{u}_+ \neq 0$. It is easy to see by the definition (1.5)–(1.9) of u_a that

$$\begin{aligned} \|u_a(t)\|_{L_x^2} &= \frac{1}{\sqrt{3t}} \left\| \frac{1}{\sqrt[3]{x/t}} \hat{u}_+\left(\sqrt[3]{\frac{x}{t}}\right) W\left(t, \frac{x}{t}\right) \right\|_{L_x^2} \\ &= \frac{1}{\sqrt{3}} \|(1/\sqrt[3]{x})\hat{u}_+(\sqrt[3]{x})W(t,x)\|_{L_x^2}. \end{aligned} \tag{1.12}$$

From the identity (1.12), the inequality

$$|(1/\sqrt[3]{x})\hat{u}_+(\sqrt[3]{x})W(t,x)| \leq |(1/\sqrt[3]{x})\hat{u}_+(\sqrt[3]{x})|, \tag{1.13}$$

and the equality

$$\|(1/\sqrt[3]{x})\hat{u}_+(\sqrt[3]{x})\|_{L^2_x} = \sqrt{3}\|\hat{u}_+\|_{L^2_x} = \sqrt{3}\|u_+\|_{L^2_x}, \tag{1.14}$$

we have

$$\|u_a(t)\|_{L^2_x} \leq \|u_+\|_{L^2_x}.$$

Furthermore by the identities (1.12) and (1.14), the estimate (1.13) and Lebesgue's dominated convergence theorem, we see that

$$\lim_{t\to+\infty} \|u_a(t)\|_{L^2} = 0. \tag{1.15}$$

On the other hand, there exists a constant $\kappa > 0$ such that

$$\|u_a(t)\|_{L^2_x}(\log t)^{1/2} \geq \kappa \tag{1.16}$$

for sufficiently large $t \geq 3$. The above estimate (1.16) follows from

$$\begin{aligned}
\|u_a(t)\|_{L^2_x} =& \frac{1}{\sqrt{3}}\|(1/\sqrt[3]{x})\hat{u}_+(\sqrt[3]{x})W(t,x)\|_{L^2_x} \\
=& \frac{1}{\sqrt{3}}\left\|\left(1+\frac{2|\lambda_2|}{3}\frac{|\hat{u}_+(\sqrt[3]{x})|^2}{(\sqrt[3]{x})^2}\log t\right)^{-1/2}\frac{\hat{u}_+(\sqrt[3]{x})}{\sqrt[3]{x}}\right\|_{L^2_x} \\
=& \left\|\left(1+\frac{2|\lambda_2|}{3}|x|^{-2}|\hat{u}_+(x)|^2\log t\right)^{-1/2}\hat{u}_+(x)\right\|_{L^2_x} \\
\geq& \left\|\left(1+\frac{2|\lambda_2|}{3}\,\||x|^{-1}\hat{u}_+\|^2_{L^\infty_x}\log t\right)^{-1/2}\hat{u}_+\right\|_{L^2_x} \\
=& \left(1+\frac{2|\lambda_2|}{3}\||x|^{-1}\hat{u}_+\|^2_{L^\infty_x}\log t\right)^{-1/2}\|\hat{u}_+\|_{L^2_x} \\
\geq& \left(1+\frac{2|\lambda_2|}{3}\||x|^{-1}\hat{u}_+\|^2_{L^\infty_x}\right)^{-1/2}\|u_+\|_{L^2_x}(\log t)^{-1/2},
\end{aligned}$$

provided $\log t \geq 1$, $u_+ \in L^2_x$ and $|x|^{-1}\hat{u}_+ \in L^\infty_x$, which follows from $u_+ \in \dot{H}^{-1}\cap\dot{H}^{-2}$, $xu_+ \in \dot{H}^{-1}$ and the embedding $H^1 \hookrightarrow L^\infty_x$. From the asymptotic formula (1.11) and the time decay (1.15) of $\|u_a(t)\|_{L^2_x}$, we see that the solution u for the equation (1.1) obtained in Theorem 1.1 converges to zero in L^2 as $t \to +\infty$:

$$\lim_{t\to+\infty} \|u(t)\|_{L^2} = 0. \tag{1.17}$$

Similarly, by the asymptotic formula (1.11) and the lower boundedness (1.16) of $\|u_a(t)\|_{L^2_x}(\log t)^{1/2}$, we see that

$$\|u(t)\|_{L^2_x}(\log t)^{1/2} \geq \kappa \tag{1.18}$$

for $t \geq 3$. In view of the asymptotic formula (1.11) in Theorem 1.1, the decay (1.17) of $\|u(t)\|_{L^2_x}$ and the lower boundedness (1.18) of $\|u(t)\|_{L^2_x}(\log t)^{1/2}$, $\|u(t) - u_a(t)\|_{L^2_x}$ decays faster than $\|u(t)\|_{L^2_x}$ as $t \to +\infty$. This means that the modified free dynamics u_a approximates the solution u for the equation (1.1) better than "zero" as $t \to +\infty$. Finally we remark the time decay of the asymptotics u_a in L^∞_x:

$$\|u_a(t)\|_{L^\infty_x} \leq \frac{C}{|\lambda_2|^{1/2}(t\log t)^{1/2}}$$

for $t \geq 3$, since $\lambda_2 < 0$. (See Proposition 4.1 below.)

Remark 3. In the case of $\lambda_1 \in \mathbb{R}$ and $\lambda_2 > 0$, a similar result holds for the negative time if we replace the modifiers by

$$\begin{aligned} S(t,x) =& \frac{\lambda_1}{2\lambda_2} \log\left(1 + \frac{2\lambda_2}{3}\frac{|\hat{u}_+(\sqrt[3]{x})|^2}{(\sqrt[3]{x})^2}\log|t|\right), \\ W(t,x) =& \left(1 + \frac{2\lambda_2}{3}\frac{|\hat{u}_+(\sqrt[3]{x})|^2}{(\sqrt[3]{x})^2}\log|t|\right)^{-1/2} \end{aligned}$$

for $t \leq -1$ and $x \in \mathbb{R}$.

We briefly explain the strategy of the proof of Theorem 1.1. Put $f(u) = (\lambda_1 + i\lambda_2)|u|^2u$, where $\lambda_1 \in \mathbb{R}$ and $\lambda_2 < 0$. For $T > 0$ and $\rho > 0$, we introduce the following function spaces

$$\begin{aligned} X_T &= \{w \in C([T,\infty); L^2_x);\ \|w\|_{X_T} < \infty\}, \\ \widetilde{X}_T(\rho) &= \{w \in C([T,\infty); L^2_x);\ \|w\|_{X_T} \leq \rho\}, \end{aligned}$$

where

$$\|w\|_{X_T} = \sup_{t\geq T} t^d(\|w(t)\|_{L^2_x} + \|w\|_{L^8((t,\infty);L^\infty_x)}),$$

where $3/8 < d < 1$. Let A be a function satisfying

$$\|A(t)\|_{L^\infty_x} \leq \eta(t\log t)^{-1/2}, \tag{1.19}$$

$$\|\mathcal{L}A(t) - f(A(t))\|_{L^2_x} \leq \eta t^{-1-d'}, \tag{1.20}$$

where $d' > d$. First, we prove that for any $\eta > 0$, there exist a $T \geq 3$ and a unique solution u to the equation (1.1) satisfying $u - A \in X_T$. The main part of the proof is to show that for any $\rho > 0$ and $\eta > 0$, there exists a sufficiently large $T \geq 3$ such that the equation (1.1) has a unique solution u satisfying $u - A \in \widetilde{X}_T(\rho)$ by using the Strichartz estimate (see Lemma 2.2) and the contraction argument. (See Proposition 3.1.) Next, for a given final state $u_+ \in \mathcal{D}$, we show that our asymptotics u_a defined by (1.5)–(1.9) satisfies the conditions (1.19) and (1.20) with some $d' > d$ (see Proposition 4.1). Note that the asymptotics u_a of solution u is a modified free profile not only with a phase shift but also with a correction of amplitude in order to absorb the long-range effect of $f(u_a)$. (For how to choose an asymptotics u_a, see Remark 5.) These two steps yield Theorem 1.1.

This paper is organized as follows. In Section 2, we derive several lemmas needed for the proof of Theorem 1.1. In Section 3, we solve the abstract final value problem around an asymptotic function which decays like $(t\log t)^{-1/2}$ in L_x^∞ and approximates the equation (1.1) suitably in large time. In Section 4, we show our asymptotics u_a defined by (1.5)–(1.9) satisfies the assumptions of the final value problem in Section 3, and we prove Theorem 1.1.

2. Preliminaries

The following lemma is used in order to solve the Cauchy problem at infinity in Section 3.

Lemma 2.1. *Let $a > 1$ and $b > 0$. Then there exists a constant $C > 0$ such that for $t \geq 2$,*

$$\int_t^\infty s^{-a}(\log s)^{-b}\,ds \leq Ct^{-a+1}(\log t)^{-b}.$$

Proof. By the integration by parts, we see

$$\begin{aligned}
&\int_t^\infty s^{-a}(\log s)^{-b}\,ds\\
&\quad=\left[-\frac{1}{a-1}s^{-a+1}(\log s)^{-b}\right]_t^\infty\\
&\qquad+\frac{1}{a-1}\int_t^\infty s^{-a+1}(-b)(\log s)^{-b-1}s^{-1}\,ds\\
&\quad=\frac{1}{a-1}t^{-a+1}(\log t)^{-b}-\frac{b}{a-1}\int_t^\infty s^{-a}(\log s)^{-b-1}\,ds\\
&\quad\leq\frac{1}{a-1}t^{-a+1}(\log t)^{-b}.
\end{aligned}$$

This proves the lemma. □

We introduce the Strichartz estimate for the free equation obtained by Kenig-Ponce-Vega [9]. We define the linear operator

$$(Gh)(t) = \int_t^\infty V(t-s)h(s)\,ds,$$

where $V(t)$ is the free evolution of the equation operator defined by (1.3), and h is a function of (t, x). The following lemma is needed in order to solve the Cauchy problem at infinity by the contraction argument in the next section.

Lemma 2.2 (Kenig-Ponce-Vega [9])**.** *Let (q, r) and $(\tilde{q}, \tilde{r})$ be pairs of positive numbers satisfying $4/q = 1/2 - 1/r$, $8 \leq q \leq \infty$, $4/\tilde{q} = 1/2 - 1/\tilde{r}$*

and $8 \le \tilde{q} \le \infty$. *Then* G *is a bounded operator from* $L_t^{\tilde{q}'}((T_0,\infty);L_x^{\tilde{r}'}(\mathbb{R}))$ *into* $L_t^q((T_0,\infty);L_x^r(\mathbb{R}))$ *with norm uniformly bounded with respect to* T_0*, where* $(\tilde{q}',\tilde{r}')$ *is a pair of positive numbers satisfying* $1/\tilde{q}+1/\tilde{q}'=1$ *and* $1/\tilde{r}+1/\tilde{r}'=1$. *Furthermore, if* $h \in L_t^{\tilde{q}'}((T_0,\infty);L_x^{\tilde{r}'}(\mathbb{R}))$*, then* $Gh \in C_t([T_0,\infty);L_x^2(\mathbb{R}))$.

To estimate asymptotic functions, the following two lemmas are needed in Section 4.

Lemma 2.3. *Let* $a \ge 1/3$. *Then the identity*

$$\||x|^{-a}\psi(\sqrt[3]{x})\|_{L_x^2} = \sqrt{3}\||x|^{1-3a}\psi\|_{L_x^2}$$

holds if $|x|^{1-3a}\psi \in L_x^2$.

Proof. By the change of variables $y=\sqrt[3]{x}$ in the integral, we have

$$\begin{aligned}
\||x|^{-a}\psi(\sqrt[3]{x})\|_{L_x^2} &= \left(\int_{\mathbb{R}} |x|^{-2a}|\psi(\sqrt[3]{x})|^2\,dx\right)^{1/2} \\
&= \left(\int_{\mathbb{R}} |y|^{-6a}|\psi(y)|^2(3y^2)\,dy\right)^{1/2} \\
&= \sqrt{3}\left(\int_{\mathbb{R}} |y|^{2-6a}|\psi(y)|^2\,dy\right)^{1/2},
\end{aligned}$$

which implies this lemma. □

Lemma 2.4. *Let* g *be a measurable function on* $\mathbb{R}$, *and let* $\tilde{g}(x) = g(\sqrt[3]{x})/|\sqrt[3]{x}|$. *Then the identity*

$$\|\tilde{g}\|_{L_x^2} = \sqrt{3}\|g\|_{L_x^2}$$

holds for $g \in L_x^2$, *and there exists a constant* $C_k > 0$ *such that*

$$\|\partial_x^k \tilde{g}\|_{L_x^2} \le C_k \sum_{j=0}^{k} \||x|^{-3k+j}\partial_x^j g\|_{L_x^2}, \quad k=1,2,3,4$$

if the right hand sides are finite.

Proof. The first identity follows from Lemma 2.3 immediately. Using the Leibniz rule and Lemma 2.3, we can show the second inequality by a simple calculation. □

3. The final value problem

In this section, we solve the abstract Cauchy problem at infinite initial time for the equation (1.1). Let A be a given asymptotic profile of the equation

(1.1), namely an approximate solution for that equation as $t \to +\infty$. We put our nonlinearity by $f(u)$:

$$f(u) = \lambda |u|^2 u, \tag{3.1}$$

where $\lambda = \lambda_1 + i\lambda_2$, $\lambda_1 \in \mathbb{R}$ and $\lambda_2 < 0$. We introduce the following function:

$$R = \mathcal{L}A - f(A), \tag{3.2}$$

where $\mathcal{L}$ is the operator defined by (1.3). The function R is difference between the left hand sides and the right hand ones in the equation (1.1) substituted $u = A$.

For $T > 0$, we introduce the following function space

$$X_T = \{w \in C([T,\infty); L^2_x);\ \|w\|_{X_T} < \infty\},$$

where

$$\|w\|_{X_T} = \sup_{t \geq T} t^d (\|w(t)\|_{L^2_x} + \|w\|_{L^8((t,\infty);L^\infty_x)}),$$

where $3/8 < d < 1$. For $\rho > 0$ and $T > 0$, we define

$$\widetilde{X}_T(\rho) = \{w \in C([T,\infty); L^2_x);\ \|w\|_{X_T} \leq \rho\}.$$

X_T is a Banach space with the norm $\|\cdot\|_{X_T}$ and $\widetilde{X}_T(\rho)$ is a complete metric space with the $\|\cdot\|_{X_T}$-metric.

Proposition 3.1. *Let $\lambda_1 \in \mathbb{R}$ and $\lambda_2 < 0$, and let d be a constant satisfying $3/8 < d < 1$. Assume that there exists a constant $\eta > 0$ such that for $t \geq 3$,*

$$\|A(t)\|_{L^\infty_x} \leq \eta (t \log t)^{-1/2},$$
$$\|R(t)\|_{L^2_x} = \|\mathcal{L}A(t) - f(A(t))\|_{L^2_x} \leq \eta t^{-1-d'},$$

where d' is a constant satisfying $d' > d$. Then there exist a $T \geq 3$ and a unique solution u for the equation (1.1) satisfying

$$u \in C([T,\infty); L^2_x),$$
$$\sup_{t \geq T} t^d (\|u(t) - A(t)\|_{L^2_x} + \|u - A\|_{L^8((t,\infty);L^\infty_x)}) < \infty.$$

Proof. We put $v = u - A$. Then the equation (1.1) is equivalent to

$$\mathcal{L}v = f(v + A) - f(A) - R, \tag{3.3}$$

where R is defined by (3.2). The associate integral equation to the equation (3.3) is

$$v(t) = i \int_t^\infty V(t-s)[\{f(v(s) + A(s)) - f(A(s))\} - R(s)]\, ds. \tag{3.4}$$

It is sufficient to show the existence of a unique solution v to the equation (3.4) in X_T for sufficiently large $T \geq 3$. Our main task is to show that for any $\rho > 0$ and $\eta > 0$, there exists a $T \geq 3$ such that the equation (3.4) has a unique solution in $\widetilde{X}_T(\rho)$ by the contraction argument. Namely we define the nonlinear operator K by

$$(Kv)(t) = i \int_t^\infty V(t-s)[\{f(v(s)+A(s)) - f(A(s))\} - R(s)]\, ds$$

for $v \in \widetilde{X}_T(\rho)$, and we show that for any $\rho > 0$ and $\eta > 0$, K is a contraction map on $\widetilde{X}_T(\rho)$ if $T \geq 3$ is sufficiently large. Let $\rho > 0$ be arbitrary, and $T \geq 3$ which will be determined below. Let $v \in \widetilde{X}_T(\rho)$ and $t \geq T$. By the assumptions, Hölder's inequality, Lemmas 2.1, 2.2 and the fact $3/8 < d < \min\{1, d'\}$, we see

$$\begin{aligned}
&\|(Kv)(t)\|_{L^2_x} + \|Kv\|_{L^8((t,\infty);L^\infty_x)} \\
&\leq C(\||v|^3\|_{L^{8/7}((t,\infty);L^1_x)} + \||A||v|^2\|_{L^1((t,\infty);L^2_x)} \\
&\quad + \||A|^2|v|\|_{L^1((t,\infty);L^2_x)} + \|R\|_{L^1((t,\infty);L^2_x)}) \\
&\leq C(\|v\|_{L^8((t,\infty);L^\infty_x)}\||v|^2\|_{L^{4/3}((t,\infty);L^1_x)} \\
&\quad + \|v\|_{L^8((t,\infty);L^\infty_x)}\|Av\|_{L^{8/7}((t,\infty);L^2_x)} \\
&\quad + \||A|^2 v\|_{L^1((t,\infty);L^2_x)} + \|R\|_{L^1((t,\infty);L^2_x)}) \\
&\leq C\Bigg\{\|v\|_{L^8((t,\infty);L^\infty_x)} \left(\int_t^\infty \|v(s)\|_{L^2_x}^{8/3}\, ds\right)^{3/4} \\
&\quad + \|v\|_{L^8((t,\infty);L^\infty_x)} \left(\int_t^\infty \|A(s)\|_{L^\infty_x}^{8/7}\|v(s)\|_{L^2_x}^{8/7}\, ds\right)^{7/8} \\
&\quad + \int_t^\infty (\|A(s)\|_{L^\infty_x}^2 \|v(s)\|_{L^2_x} + \|R(s)\|_{L^2_x})\, ds\Bigg\} \\
&\leq C\Bigg\{\rho t^{-d}\left(\int_t^\infty \rho^{8/3}s^{-8d/3}\, ds\right)^{3/4} \\
&\quad + \rho t^{-d}\left(\int_t^\infty \eta^{8/7}(s\log s)^{-4/7}\rho^{8/7}s^{-8d/7}\, ds\right)^{7/8} \\
&\quad + \int_t^\infty (\eta^2(s\log s)^{-1}\rho s^{-d} + \eta s^{-1-d'})\, ds\Bigg\} \\
&\leq C t^{-d}(\rho^3 t^{-2d+3/4} + \rho^2\eta t^{-d+3/8}(\log t)^{-1/2} \\
&\quad + \rho\eta^2(\log t)^{-1} + \eta t^{-(d'-d)}).
\end{aligned}$$

Therefore we obtain

$$\|Kv\|_{X_T} \leq C(\rho^3 T^{-2d+3/4} + \rho\eta^2(\log T)^{-1} + \eta T^{-(d'-d)}). \tag{3.5}$$

In the same way as above, for $v_1, v_2 \in \widetilde{X}_T(\rho)$, we can show

$$\begin{aligned}(3.6)\qquad &\|Kv_1 - Kv_2\|_{X_T}\\ \leq& C((\|v_1\|_{X_T}^2 + \|v_2\|_{X_T}^2)T^{-2d+3/4} + \eta^2(\log T)^{-1})\|v_1 - v_2\|_{X_T}\\ \leq& C(\rho^2 T^{-2d+3/4} + \eta^2(\log T)^{-1})\|v_1 - v_2\|_{X_T}.\end{aligned}$$

We note that for $\rho > 0$ and $\eta > 0$, there exists a sufficiently large $T \geq 3$ such that

$$C(\rho^3 T^{-2d+3/4} + \rho\eta^2(\log T)^{-1} + \eta T^{-(d'-d)}) \leq \rho,$$
$$C(\rho^2 T^{-2d+3/4} + \eta^2(\log T)^{-1}) \leq \frac{1}{2},$$

since $3/8 < d < d'$. From this fact, the estimates (3.5) and (3.6), we see that the operator K is a contraction map on $\widetilde{X}_T(\rho)$ for sufficiently large $T \geq 3$. Therefore for any $\rho > 0$ and $\eta > 0$, there exist a $T \geq 3$ and a unique solution to the integral equation (3.4) in $\widetilde{X}_T(\rho)$. The uniqueness of solutions to the equation (3.4) in X_T for sufficiently large $T \geq 3$ (which depends only on $\eta > 0$) follows from the first inequality of the estimate (3.6) for solutions $v_1 \in X_T$ and $v_2 \in X_T$, (i.e., $Kv_j = v_j$, $j = 1, 2$). Hence for any $\eta > 0$, there exists a $T \geq 3$ such that the equation (3.4) has a unique solution in X_T. This completes the proof of this proposition. □

Remark 4. In Proposition 3.1, no size restriction on $\eta > 0$ is assumed.

4. Proof of Theorem 1.1

In this section, we show Theorem 1.1, by proving that the asymptotic profile u_a defined by (1.5)–(1.9) satisfies the assumptions in Proposition 3.1.

Let $u_+ \in \mathcal{D}$ be a final state, where $\mathcal{D}$ is defined by (1.4), and let u_a be the asymptotics defined by (1.5)–(1.9). $f(u)$ is our nonlinearity defined by (3.1).

The main task of this section is to show the following proposition, namely we show that u_a satisfies the assumption of Proposition 3.1. Theorem 1.1 immediately follows from Propositions 3.1 and 4.1.

Proposition 4.1. *Let $\lambda_1 \in \mathbb{R}$, $\lambda_2 < 0$, $u_+ \in \mathcal{D}$ and u_a be the asymptotics defined by* (1.5)–(1.9)*. Then there exist a polynomial $P(\|u_+\|_{\mathcal{D}})$ of $\|u_+\|_{\mathcal{D}}$ and a positive integer k such that for $t \geq 3$,*

$$(4.1)\qquad \|u_a(t)\|_{L_x^\infty} \leq C|\lambda_2|^{-1/2}(t\log t)^{-1/2},$$
$$(4.2)\qquad \|\mathcal{L}u_a(t) - f(u_a(t))\|_{L_x^2} \leq CP(\|u_+\|_{\mathcal{D}})t^{-2}(\log t)^k.$$

The inequality (4.1) follows from the definition of u_a immediately. (See the proof of Proposition 4.1 below.) Our main task is the proof of the estimate (4.2).

Let $t \geq 3$, $x \in \mathbb{R}$, and we denote $(\partial_0 h)(t,x) = (\partial_t h)(t,x)$. By the definition of the asymptotics u_a and the Leibniz rule, we have

$$
\begin{aligned}
\mathcal{L}u_a &= (\mathcal{L}u_1)B\left(t,\frac{x}{t}\right) + u_1 i\partial_t\left(B\left(t,\frac{x}{t}\right)\right)\\
&\quad - (\partial_x^3 u_1)\partial_x\left(B\left(t,\frac{x}{t}\right)\right) - \frac{3}{2}(\partial_x^2 u_1)\partial_x^2\left(B\left(t,\frac{x}{t}\right)\right)\\
&\quad - (\partial_x u_1)\partial_x^3\left(B\left(t,\frac{x}{t}\right)\right) - \frac{1}{4}u_1\partial_x^4\left(B\left(t,\frac{x}{t}\right)\right)\\
&= u_1 i(\partial_0 B)\left(t,\frac{x}{t}\right) + (\mathcal{L}u_1)B\left(t,\frac{x}{t}\right)\\
&\quad - \frac{1}{t}\left(\frac{ix}{t}u_1 + \partial_x^3 u_1\right)(\partial_x B)\left(t,\frac{x}{t}\right)\\
&\quad - \frac{3}{2}\frac{1}{t^2}(\partial_x^2 u_1)(\partial_x^2 B)\left(t,\frac{x}{t}\right) - \frac{1}{t^3}(\partial_x u_1)(\partial_x^3 B)\left(t,\frac{x}{t}\right)\\
&\quad - \frac{1}{4}\frac{1}{t^4}u_1(\partial_x^4 B)\left(t,\frac{x}{t}\right).
\end{aligned}
\tag{4.3}
$$

By the definition (1.7)–(1.9) of the modifier B, we note that B satisfies the following ordinary differential equation:

$$
i\partial_t B(t,x) = (\lambda_1 + i\lambda_2)\frac{1}{3t}\frac{|\hat{u}_+(\sqrt[3]{x})|^2}{(\sqrt[3]{x})^2}|B(t,x)|^2 B(t,x)
\tag{4.4}
$$

for $t > 1$ and $x \in \mathbb{R}$. On the other hand, it follows from the definitions of asymptotics u_a and the nonlinearity $f(u)$ that

$$
\begin{aligned}
f(u_a) =& \frac{\lambda_1 + i\lambda_2}{3t}\frac{1}{(\sqrt[3]{x/t})^2}\left|\hat{u}_+\left(\sqrt[3]{\frac{x}{t}}\right)\right|^2\\
&\times\left|B\left(t,\frac{x}{t}\right)\right|^2 B\left(t,\frac{x}{t}\right)u_1.
\end{aligned}
\tag{4.5}
$$

From the equalities (4.4) and (4.5), we see that

$$
u_1 i(\partial_0 B)\left(t,\frac{x}{t}\right) = f(u_a).
\tag{4.6}
$$

Therefore by the equations (4.3) and (4.6), we obtain

$$
\begin{aligned}
\mathcal{L}u_a - f(u_a) = (\mathcal{L}u_1)B\left(t,\frac{x}{t}\right) &- \frac{1}{t}\left(\frac{ix}{t}u_1 + \partial_x^3 u_1\right)(\partial_x B)\left(t,\frac{x}{t}\right)\\
&- \frac{3}{2}\frac{1}{t^2}(\partial_x^2 u_1)(\partial_x^2 B)\left(t,\frac{x}{t}\right) - \frac{1}{t^3}(\partial_x u_1)(\partial_x^3 B)\left(t,\frac{x}{t}\right)\\
&- \frac{1}{4}\frac{1}{t^4}u_1(\partial_x^4 B)\left(t,\frac{x}{t}\right).
\end{aligned}
\tag{4.7}
$$

Remark 5. Here we remark how to choose a modifier $B = We^{-iS}$. Solving the ordinary differential equation (4.4) with $B(1,x) = 1$, we determine our modifier B so that B satisfies the equality (4.6). By using the equality (4.6), we can cancel contribution of the long-range term $f(u_a)$ in $\mathcal{L}u_a - f(u_a)$.

We estimate the right hand side of the equality (4.7).

Lemma 4.1. *Let $\lambda_1 \in \mathbb{R}$, $\lambda_2 < 0$, $u_+ \in \mathcal{D}$, and u_1 and B be defined by* (1.6) *and* (1.7)–(1.9), *respectively. Then there exist a constant $C > 0$, a polynomial $P(\|u_+\|_{\mathcal{D}})$ of $\|u_+\|_{\mathcal{D}}$ and a positive integer k such that for $t \geq 3$,*

$$\|\partial_x^j u_1(t)\|_{L_x^\infty} \leq C\|u_+\|_{\mathcal{D}} t^{-1/2}, \quad j = 0, 1, 2, \tag{4.8}$$

$$\|\mathcal{L}u_1(t)\|_{L_x^2} \leq C\|u_+\|_{\mathcal{D}} t^{-2}, \tag{4.9}$$

$$\left\| \frac{ix}{t} u_1(t) + \partial_x^3 u_1(t) \right\|_{L_x^2} \leq C\|u_+\|_{\mathcal{D}} t^{-1}, \tag{4.10}$$

$$\|\partial_x^j B(t)\|_{L_x^2} \leq P(\|u_+\|_{\mathcal{D}})(\log t)^k, \quad j = 0, 1, 2, 3, 4, \tag{4.11}$$

$$\|\partial_x^j B(t)\|_{L_x^\infty} \leq P(\|u_+\|_{\mathcal{D}})(\log t)^k, \quad j = 0, 1. \tag{4.12}$$

Proof. Putting $\phi(x) = \hat{u}_+(\sqrt[3]{x})/|\sqrt[3]{x}|$ and using the Leibniz rule, Hölder's inequality, the relations $\|\psi(x/t)\|_{L_x^\infty} = \|\psi\|_{L_x^\infty}$ and $\|\psi(x/t)\|_{L_x^2} = t^{1/2}\|\psi\|_{L_x^2}$ for $t \in \mathbb{R} \setminus \{0\}$, Lemmas 2.3 and 2.4, and the Sobolev embedding theorem $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, we can show the estimates (4.8), (4.11) and (4.12) by a direct calculation. Hereafter we describe the proof of the estimates (4.9) and (4.10).

Let

$$\phi(x) = \frac{1}{|\sqrt[3]{x}|}\hat{u}_+(\sqrt[3]{x}), \quad q(t,x) = \frac{3}{4}x\sqrt[3]{\frac{x}{t}}.$$

Then

$$u_1(t,x) = \frac{1}{(3it)^{1/2}}\phi\left(\frac{x}{t}\right)e^{iq(t,x)}.$$

We note that

$$\partial_x e^{iq} = i\partial_x q e^{iq}, \quad \partial_x^2 e^{iq} = (i\partial_x^2 q - (\partial_x q)^2)e^{iq}, \tag{4.13}$$

$$\partial_x^3 e^{iq} = (i\partial_x^3 q - 3(\partial_x q)(\partial_x^2 q) - i(\partial_x q)^3)e^{iq}, \tag{4.14}$$

$$\partial_x^4 e^{iq} = (i\partial_x^4 q - 4(\partial_x q)(\partial_x^3 q) - 3(\partial_x^2 q)^2 - 6i(\partial_x q)^2(\partial_x^2 q) + (\partial_x q)^4)e^{iq} \tag{4.15}$$

and

$$\partial_x q(t,x) = \left(\frac{x}{t}\right)^{1/3}, \quad \partial_x^2 q(t,x) = \frac{1}{3}\frac{1}{t}\left(\frac{x}{t}\right)^{-2/3}, \tag{4.16}$$

$$\partial_x^3 q(t,x) = -\frac{2}{9}\frac{1}{t^2}\left(\frac{x}{t}\right)^{-5/3}, \quad \partial_x^4 q(t,x) = \frac{10}{27}\frac{1}{t^3}\left(\frac{x}{t}\right)^{-8/3}. \tag{4.17}$$

We show the estimate (4.9). By the Leibniz rule, we have

$$\begin{aligned}\frac{1}{4}\partial_x^4 u_1 = &\frac{1}{(3it)^{1/2}}\left\{\frac{1}{4t^4}(\partial^4\phi)\left(\frac{x}{t}\right)e^{iq}\right.\\ &+\frac{1}{t^3}(\partial^3\phi)\left(\frac{x}{t}\right)\partial_x e^{iq} + \frac{3}{2t^2}(\partial^2\phi)\left(\frac{x}{t}\right)\partial_x^2 e^{iq}\\ &\left.+\frac{1}{t}(\partial\phi)\left(\frac{x}{t}\right)\partial_x^3 e^{iq} + \frac{1}{4}\phi\left(\frac{x}{t}\right)\partial_x^4 e^{iq}\right\}.\end{aligned} \tag{4.18}$$

By the identities (4.13)–(4.17) and (4.18), Hölder's inequality and Lemmas 2.3 and 2.4, we can write $(1/4)\partial_x^4 u_1$ as

$$\begin{aligned}\frac{1}{4}\partial_x^4 u_1 =&\frac{1}{(3it)^{1/2}}\frac{1}{4}\phi\left(\frac{x}{t}\right)(\partial_x q)^4 e^{iq}\\ &-\frac{1}{(3it)^{1/2}}\frac{3i}{2}\phi\left(\frac{x}{t}\right)(\partial_x q)^2(\partial_x^2 q)e^{iq}\\ &-\frac{1}{(3it)^{1/2}}\frac{i}{t}(\partial\phi)\left(\frac{x}{t}\right)(\partial_x q)^3 e^{iq}-\frac{1}{t^{5/2}}Y\left(\frac{x}{t}\right)e^{iq}\\ =&\frac{1}{4}\frac{1}{(3it)^{1/2}}\phi\left(\frac{x}{t}\right)\left(\frac{x}{t}\right)^{4/3}e^{iq}-\frac{i}{2(3i)^{1/2}}\frac{1}{t^{3/2}}\phi\left(\frac{x}{t}\right)e^{iq}\\ &-\frac{i}{(3i)^{1/2}}\frac{1}{t^{3/2}}(\partial\phi)\left(\frac{x}{t}\right)\frac{x}{t}e^{iq}-\frac{1}{t^{5/2}}Y\left(\frac{x}{t}\right)e^{iq},\end{aligned}\tag{4.19}$$

where Y is some complex valued function on $\mathbb{R}$ satisfying

$$\frac{1}{t^{1/2}}\left\|Y\left(\frac{x}{t}\right)\right\|_{L_x^2}=\|Y\|_{L_x^2}\le C\|u_+\|_{\mathcal{D}}\tag{4.20}$$

with some constant $C>0$. On the other hand, by a simple calculation, we have

$$\begin{aligned}i\partial_t u_1 =&\frac{1}{4}\frac{1}{(3it)^{1/2}}\phi\left(\frac{x}{t}\right)\left(\frac{x}{t}\right)^{4/3}e^{iq}-\frac{i}{2(3i)^{1/2}}\frac{1}{t^{3/2}}\phi\left(\frac{x}{t}\right)e^{iq}\\ &-\frac{i}{(3i)^{1/2}}\frac{1}{t^{3/2}}(\partial\phi)\left(\frac{x}{t}\right)\frac{x}{t}e^{iq}.\end{aligned}\tag{4.21}$$

From the identities (4.19) and (4.21), we obtain

$$\mathcal{L}u_1=\frac{1}{t^{5/2}}Y\left(\frac{x}{t}\right)e^{iq}.\tag{4.22}$$

The equality (4.22) and the estimate (4.20) imply the estimate (4.9).

Finally we prove the estimate (4.10). By the Leibniz rule, we have

$$\begin{aligned}\partial_x^3 u_1 =&\frac{1}{(3it)^{1/2}}\left\{\frac{1}{t^3}(\partial^3\phi)\left(\frac{x}{t}\right)e^{iq}+\frac{3}{t^2}(\partial^2\phi)\left(\frac{x}{t}\right)\partial_x e^{iq}\right.\\ &\left.+\frac{3}{t}(\partial\phi)\left(\frac{x}{t}\right)\partial_x^2 e^{iq}+\phi\left(\frac{x}{t}\right)\partial_x^3 e^{iq}\right\}.\end{aligned}\tag{4.23}$$

By the identities (4.13)–(4.14), (4.16)–(4.17) and (4.23), Hölder's inequality and Lemmas 2.3 and 2.4, we can write $\partial_x^3 u_1$ as

$$\begin{aligned}\partial_x^3 u_1 =&\frac{1}{(3it)^{1/2}}\phi\left(\frac{x}{t}\right)(-i(\partial_x q)^3)e^{iq}+\frac{1}{t^{3/2}}Z\left(\frac{x}{t}\right)e^{iq},\\ =&-\frac{ix}{t}\frac{1}{(3it)^{1/2}}\phi\left(\frac{x}{t}\right)e^{iq}+\frac{1}{t^{3/2}}Z\left(\frac{x}{t}\right)e^{iq},\\ =&-\frac{ix}{t}u_1+\frac{1}{t^{3/2}}Z\left(\frac{x}{t}\right)e^{iq},\end{aligned}\tag{4.24}$$

where Z is some complex valued function on $\mathbb{R}$ satisfying

$$\frac{1}{t^{1/2}}\left\|Z\left(\frac{x}{t}\right)\right\|_{L_x^2}=\|Z\|_{L_x^2}\le C\|u_+\|_{\mathcal{D}} \tag{4.25}$$

with some constant $C>0$. The estimate (4.10) follows from the identity (4.24) and the inequality (4.25). This lemma is proved. □

Remark 6. According to the estimate (4.9), $\mathcal{L}u_1(t)$ decays in L_x^2 with respect to t, though $u_1(t)$ does not decay. (Note that $\|u_1(t)\|_{L^2}=\|\hat{u}_+\|_{L^2}$.) This is because u_1 is an approximate solution of the free equation (1.10).

We prove Proposition 4.1.

Proof of Proposition 4.1. Let $t\ge 3$. By the definition of u_a, we see that

$$\begin{aligned}
|u_a(t,x)| &= \left|u_1(t,x)W\left(t,\frac{x}{t}\right)\right| \\
&= \frac{1}{(3t)^{1/2}}\left|\frac{1}{\sqrt[3]{x/t}}\hat{u}_+\left(\sqrt[3]{\frac{x}{t}}\right)\right| \\
&\quad\times\left(1+\frac{2|\lambda_2|}{3}\frac{1}{(\sqrt[3]{x/t})^2}\left|\hat{u}_+\left(\sqrt[3]{\frac{x}{t}}\right)\right|^2\log t\right)^{-1/2}.
\end{aligned}$$

The above equality and the fact $\sup_{y\in\mathbb{R}}(|y|(1+ay^2)^{-1/2})=a^{-1/2}$ for $a>0$ imply the estimate (4.1).

We show the estimate (4.2). By the identity (4.7), Hölder's inequality, the relations $\|\psi(x/t)\|_{L_x^\infty}=\|\psi\|_{L_x^\infty}$ and $\|\psi(x/t)\|_{L_x^2}=t^{1/2}\|\psi\|_{L_x^2}$ for $t\in\mathbb{R}\setminus\{0\}$ and Lemma 4.1, we see that

$$\begin{aligned}
&\|\mathcal{L}u_a(t)-f(u_a(t))\|_{L_x^2} \\
&\le\|\mathcal{L}u_1(t)\|_{L_x^2}\|B(t)\|_{L_x^\infty}+\frac{1}{t}\left\|\frac{ix}{t}u_1(t)+\partial_x^3u_1(t)\right\|_{L_x^2}\|\partial_xB(t)\|_{L_x^\infty} \\
&\quad+\frac{3}{2}\frac{1}{t^2}\|\partial_x^2u_1(t)\|_{L_x^\infty}t^{1/2}\|\partial_x^2B(t)\|_{L_x^2} \\
&\quad+\frac{1}{t^3}\|\partial_xu_1(t)\|_{L_x^\infty}t^{1/2}\|\partial_x^3B(t)\|_{L_x^2} \\
&\quad+\frac{1}{4}\frac{1}{t^4}\|u_1(t)\|_{L_x^\infty}t^{1/2}\|\partial_x^4B(t)\|_{L_x^2} \\
&\le P(\|u_+\|_{\mathcal{D}})t^{-2}(\log t)^k
\end{aligned}$$

with some polynomial $P(\|u_+\|_{\mathcal{D}})$ of $\|u_+\|_{\mathcal{D}}$ and some positive integer k. This yields the estimate (4.2). □

Now we prove Theorem 1.1.

Proof of Theorem 1.1. Suppose that all the assumptions in Theorem 1.1 are satisfied. Namely, assume that $u_+\in\mathcal{D}$, $\lambda_1\in\mathbb{R}$ and $\lambda_2<0$. Let u_a

be defined by (1.5)–(1.9). Then by Proposition 4.1, we see that the asymptotic profile u_a satisfies the the assumptions in Proposition 3.1. Theorem 1.1 immediately follows from Proposition 3.1. □

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