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3-graded decompositions of exceptional Lie algebras $\mathfrak{g}$ and group realizations of $\mathfrak{g}_{ev}, \mathfrak{g}_0$ and $\mathfrak{g}_{ed}$ Part II, $G = E_7$, Case 1

By

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The 3-graded decompositions of simple Lie algebras $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3, \quad [\mathfrak{g}_k, \mathfrak{g}_l] \subset \mathfrak{g}_{k+l}$$

are classified and the types of subalgebras $\mathfrak{g}_{ev} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$, $\mathfrak{g}_0$ and $\mathfrak{g}_{ed} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_3$ of $\mathfrak{g}$ are determined. The following table is the results of $\mathfrak{g}_{ev}, \mathfrak{g}_0$ and $\mathfrak{g}_{ed}$ for the exceptional Lie algebras of type E_7 ([1]).

Case 1	$\mathfrak{g}$	$\mathfrak{g}_{ev}$ / $\mathfrak{g}_{ed}$	$\mathfrak{g}_0$ / $\dim\mathfrak{g}_1, \dim\mathfrak{g}_2, \dim\mathfrak{g}_3$
	${\mathfrak{e}_7}^C$	$\mathfrak{sl}(2,C) \oplus \mathfrak{so}(12,C)$	$\mathfrak{sl}(2,C) \oplus C \oplus \mathfrak{sl}(6,C)$
		$\mathfrak{sl}(3,C) \oplus \mathfrak{sl}(6,C)$	$30, 15, 2$
	$\mathfrak{e}_{7(7)}$	$\mathfrak{sl}(2,\boldsymbol{R}) \oplus \mathfrak{so}(6,6)$	$\mathfrak{sl}(2,\boldsymbol{R}) \oplus \boldsymbol{R} \oplus \mathfrak{sl}(6,\boldsymbol{R})$
		$\mathfrak{sl}(3,\boldsymbol{R}) \oplus \mathfrak{sl}(6,\boldsymbol{R})$	$30, 15, 2$
	$\mathfrak{e}_{7(-5)}$	$\mathfrak{sl}(2,\boldsymbol{R}) \oplus \mathfrak{so}^*(12)$	$\mathfrak{sl}(2,\boldsymbol{R}) \oplus \boldsymbol{R} \oplus \mathfrak{su}^*(6)$
		$\mathfrak{sl}(3,\boldsymbol{R}) \oplus \mathfrak{su}^*(6)$	$30, 15, 2$

Case 2	$\mathfrak{g}$	$\mathfrak{g}_{ev}$ / $\mathfrak{g}_{ed}$	$\mathfrak{g}_0$ / $\dim\mathfrak{g}_1, \dim\mathfrak{g}_2, \dim\mathfrak{g}_3$
	${\mathfrak{e}_7}^C$	$\mathfrak{sl}(2,C) \oplus \mathfrak{so}(12,C)$	$C \oplus C \oplus \mathfrak{sl}(6,C)$
		$C \oplus \mathfrak{sl}(7,C)$	$26, 16, 6$
	$\mathfrak{e}_{7(7)}$	$\mathfrak{sl}(2,\boldsymbol{R}) \oplus \mathfrak{so}(6,6)$	$\boldsymbol{R} \oplus \boldsymbol{R} \oplus \mathfrak{sl}(6,\boldsymbol{R})$
		$\boldsymbol{R} \oplus \mathfrak{sl}(7,\boldsymbol{R})$	$26, 16, 6$

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Case 3	$\mathfrak{g}$	$\mathfrak{g}_{ev}$ $\mathfrak{g}_{ed}$	$\mathfrak{g}_0$ $\dim \mathfrak{g}_1, \dim \mathfrak{g}_2, \dim \mathfrak{g}_3$
	${\mathfrak{e}_7}^C$	$C \oplus {\mathfrak{e}_6}^C$ $C \oplus \mathfrak{so}(12, C)$	$C \oplus C \oplus \mathfrak{so}(10, C)$ $17, 16, 10$
	$\mathfrak{e}_{7(7)}$	$\boldsymbol{R} \oplus \mathfrak{e}_{6(6)}$ $\boldsymbol{R} \oplus \mathfrak{so}(6, 6)$	$\boldsymbol{R} \oplus \boldsymbol{R} \oplus \mathfrak{so}(5, 5)$ $17, 16, 10$
	$\mathfrak{e}_{7(-25)}$	$\boldsymbol{R} \oplus \mathfrak{e}_{6(-26)}$ $\boldsymbol{R} \oplus \mathfrak{so}(2, 10)$	$\boldsymbol{R} \oplus \boldsymbol{R} \oplus \mathfrak{so}(1, 9)$ $17, 16, 10$

Case 4	$\mathfrak{g}$	$\mathfrak{g}_{ev}$ $\mathfrak{g}_{ed}$	$\mathfrak{g}_0$ $\dim \mathfrak{g}_1, \dim \mathfrak{g}_2, \dim \mathfrak{g}_3$
	${\mathfrak{e}_7}^C$	$C \oplus {\mathfrak{e}_6}^C$ $\mathfrak{sl}(2, C) \oplus C \oplus \mathfrak{so}(10, C)$	$C \oplus C \oplus \mathfrak{so}(10, C)$ $26, 16, 1$
	$\mathfrak{e}_{7(7)}$	$\boldsymbol{R} \oplus \mathfrak{e}_{6(6)}$ $\mathfrak{sl}(2, \boldsymbol{R}) \oplus \boldsymbol{R} \oplus \mathfrak{so}(5, 5)$	$\boldsymbol{R} \oplus \boldsymbol{R} \oplus \mathfrak{so}(5, 5)$ $26, 16, 1$
	$\mathfrak{e}_{7(-25)}$	$\boldsymbol{R} \oplus \mathfrak{e}_{6(-26)}$ $\mathfrak{sl}(2, \boldsymbol{R}) \oplus \boldsymbol{R} \oplus \mathfrak{so}(5, 5)$	$\boldsymbol{R} \oplus \boldsymbol{R} \oplus \mathfrak{so}(1, 9)$ $26, 16, 1$

Case 5	$\mathfrak{g}$	$\mathfrak{g}_{ev}$ $\mathfrak{g}_{ed}$	$\mathfrak{g}_0$ $\dim \mathfrak{g}_1, \dim \mathfrak{g}_2, \dim \mathfrak{g}_3$
	${\mathfrak{e}_7}^C$	$\mathfrak{sl}(8, C)$ $\mathfrak{sl}(3, C) \oplus \mathfrak{sl}(6, C)$	$\mathfrak{sl}(3, C) \oplus C \oplus \mathfrak{sl}(5, C)$ $30, 15, 5$
	$\mathfrak{e}_{7(7)}$	$\mathfrak{sl}(8, \boldsymbol{R})$ $\mathfrak{sl}(3, \boldsymbol{R}) \oplus \mathfrak{sl}(6, \boldsymbol{R})$	$\mathfrak{sl}(3, \boldsymbol{R}) \oplus \boldsymbol{R} \oplus \mathfrak{sl}(5, \boldsymbol{R})$ $30, 15, 5$

In the previous paper [7], we gave the group realizations of $\mathfrak{g}_{ev}, \mathfrak{g}_0$ and $\mathfrak{g}_{ed}$ for the exceptional universal linear Lie groups G of type G_2, F_4 and E_6. In the present paper, for exceptional universal linear Lie groups G of type E_7, we realize the subgroups G_{ev}, G_0 and G_{ed} of G corresponding to $\mathfrak{g}_{ev}, \mathfrak{g}_0$ and $\mathfrak{g}_{ed}$ of $\mathfrak{g} = \mathrm{Lie}\, G$. The result of the present paper is only shown about the case 1, so we continue to report the remaining cases. Our results of the case 1 are as follows:

G	G_{ev} G_{ed}	G_0
${E_7}^C$	$(SL(2, C) \times Spin(12, C))/\boldsymbol{Z}_2$ $(SL(3, C) \times SL(6, C))/\boldsymbol{Z}_3$	$(SL(2, C) \times C^* \times SL(6, C))/(\boldsymbol{Z}_6 \times \boldsymbol{Z}_2)$
$E_{7(7)}$	$(SL(2, \boldsymbol{R}) \times spin(6, 6))/\boldsymbol{Z}_2 \times 2$ $SL(3, \boldsymbol{R}) \times SL(6, \boldsymbol{R})$	$(SL(2, \boldsymbol{R}) \times \boldsymbol{R}^+ \times SL(6, \boldsymbol{R}))/\boldsymbol{Z}_2 \times 2^2$
$E_{7(-5)}$	$(SL(2, \boldsymbol{R}) \times spin^*(12))/\boldsymbol{Z}_2 \times 2$ $SL(3, \boldsymbol{R}) \times SU^*(6)$	$(SL(2, \boldsymbol{R}) \times \boldsymbol{R}^+ \times SU^*(6))/\boldsymbol{Z}_2 \times 2^2$

This paper is a continuation of [7], so the numbering of sections and theorems start from 4. We use the same notations as that in [7].

4. Group E_7

Let $\mathfrak{J}^C$ (resp. $\mathfrak{J}'$) be the exceptional C-Jordan algebra (resp. the split exceptional $\boldsymbol{R}$-Jordan algebra) and we define the C-vector space $\mathfrak{P}^C$ (resp. the $\boldsymbol{R}$-vector space $\mathfrak{P}'$) by

$$\mathfrak{P}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C \oplus C \quad (\text{resp. } \mathfrak{P}' = \mathfrak{J}' \oplus \mathfrak{J}' \oplus \boldsymbol{R} \oplus \boldsymbol{R}),$$

with inner products

$$(P, Q) = (X, Z) + (Y, W) + \xi\zeta + \eta\omega, \quad \{P, Q\} = (X, W) - (Y, Z) + \xi\omega - \eta\zeta,$$

where $P = (X, Y, \xi, \eta)$, $Q = (Z, W, \zeta, \omega) \in \mathfrak{P}^C$ (resp. $\mathfrak{P}'$).

For $\phi \in {\mathfrak{e}_6}^C, A, B \in \mathfrak{J}^C$ and $\nu \in C$, we define the C-linear mapping $\Phi(\phi, A, B, \nu)$ of $\mathfrak{P}^C$ by

$$\Phi(\phi, A, B, \nu)\begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \phi X - \dfrac{1}{3}\nu X + 2B \times Y + \eta A \\ 2A \times X - {}^t\phi Y + \dfrac{1}{3}\nu Y + \xi B \\ (A, Y) + \nu\xi \\ (B, X) - \nu\eta \end{pmatrix}.$$

For $P = (X, Y, \xi, \eta), Q = (Z, W, \zeta, \omega) \in \mathfrak{P}^C$, we define the C-linear mapping $P \times Q$ of $\mathfrak{P}^C$ by

$$P \times Q = \Phi(\phi, A, B, \nu), \quad \begin{cases} \phi = -\dfrac{1}{2}(X \vee W + Z \vee Y) \\ A = -\dfrac{1}{4}(2Y \times W - \xi Z - \zeta X) \\ B = \dfrac{1}{4}(2X \times Z - \eta W - \omega Y) \\ \nu = \dfrac{1}{8}((X, W) + (Z, Y) - 3(\xi\omega + \zeta\eta)), \end{cases}$$

where $X \vee W \in {\mathfrak{e}_6}^C$ is defined by $X \vee W = [\widetilde{X}, \widetilde{W}] + \Big(X \circ W - \dfrac{1}{3}(X, W)E\Big)^{\sim}$, here $\widetilde{X} : \mathfrak{J}^C \to \mathfrak{J}^C$ is defined by $\widetilde{X}Z = X \circ Z$, $Z \in \mathfrak{J}^C$.

We arrange C-linear transformations of $\mathfrak{P}^C$ used later. By using the mapping $\varphi_2 : Sp(1, \boldsymbol{H}^C) \times Sp(1, \boldsymbol{H}^C) \to {G_2}^C$,

$$\varphi_2(p, q)(a + be_4) = qa\overline{q} + (pb\overline{q})e_4, \quad a + be_4 \in \boldsymbol{H}^C \oplus \boldsymbol{H}^C e_4 = \mathfrak{C}^C,$$

we define C-linear transformations $\gamma, \gamma', \gamma_1, \varepsilon_1, \varepsilon_2, w_3$ and δ_4 of $\mathfrak{C}^C$ by

$$\gamma = \varphi_2(1, -1), \quad \gamma' = \varphi_2(e_1, e_1), \quad \gamma_1 = \varphi_2(e_2, e_2),$$
$$\varepsilon_1 = \varphi_2(e_1, 1), \quad \varepsilon_2 = \varphi_2(e_2, 1), \quad w_3 = \varphi_2(1, \overline{w}_1), \quad \delta_4 = \varphi_2(1, -e_1),$$

where $w_1 = e^{2\pi e_1/3} \in \mathfrak{C} \subset \mathfrak{C}^C$. These C-linear transformations of $\mathfrak{C}^C$ are naturally extended to C-linear transformations $\gamma, \gamma', \gamma_1, \varepsilon_1, \varepsilon_2, w_3$ and δ_4 of $\mathfrak{P}^C$ as

$$\gamma(X, Y, \xi, \eta) = (\gamma X, \gamma Y, \xi, \eta)$$

etc. Then $\gamma, \gamma', \gamma_1, \varepsilon_1, \varepsilon_2, w_3, \delta_4 \in ({G_2}^C)^\tau \subset {G_2}^C \subset {F_4}^C \subset {E_6}^C \subset {E_7}^C$ and $\gamma^2 = {\gamma'}^2 = {\gamma_1}^2 = 1, {w_3}^3 = 1, {\varepsilon_1}^4 = {\varepsilon_2}^4 = {\delta_4}^4 = 1$.

The connected universal linear Lie groups ${E_7}^C, E_{7(7)}$ and $E_{7(-5)}$ are given by

$$\begin{aligned}
{E_7}^C &= \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C) \,|\, \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q\},\\
E_{7(7)} &= \{\alpha \in \mathrm{Iso}_R(\mathfrak{P}') \,|\, \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q\},\\
E_{7(-5)} &= \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C) \,|\, \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q\rangle_\gamma = \langle P, Q\rangle_\gamma\},
\end{aligned}$$

where $\langle P, Q\rangle_\gamma = (\tau\gamma P, Q),\ P, Q \in \mathfrak{P}^C$.

The Lie algebra ${\mathfrak{e}_7}^C$ of the group ${E_7}^C$ is given by

$${\mathfrak{e}_7}^C = \{\Phi(\phi, A, B, \nu) \,|\, \phi \in {\mathfrak{e}_6}^C,\ A, B \in \mathfrak{J}^C, \nu \in C\}.$$

The Lie bracket $[\Phi_1, \Phi_2]$ of ${\mathfrak{e}_7}^C$ is given by

$$[\Phi(\phi_1, A_1, B_1, \nu_1), \Phi(\phi_2, A_2, B_2, \nu_2)] = \Phi(\phi, A, B, \nu),$$

$$\begin{cases}
\phi = [\phi_1, \phi_2] + 2A_1 \vee B_2 - 2A_2 \vee B_1\\
A = \Big(\phi_1 + \dfrac{2}{3}\nu_1\Big)A_2 - \Big(\phi_2 + \dfrac{2}{3}\nu_2\Big)A_1\\
B = -\Big({}^t\phi_1 + \dfrac{2}{3}\nu_1\Big)B_2 + \Big({}^t\phi_2 + \dfrac{2}{3}\nu_2\Big)B_1\\
\nu = (A_1, B_2) - (B_1, A_2).
\end{cases}$$

In the Lie algebra ${\mathfrak{e}_7}^C$, we use the same notations G_{ij}, $\widetilde{A}_k(e_i)$, $(E_k - E_l)^\sim$, $\widetilde{F}_k(e_i)$, $\check{E}_k$, $\hat{E}_k$, $\check{F}_k(e_i)$, $\hat{F}_k(e_i)$, $\mathbf{1}$ as that used in the preceding papers [5], [6], or [7].

4.1. Subgroups of type ${A_1}^C \oplus {D_6}^C, {A_1}^C \oplus C \oplus {A_5}^C$ and ${A_2}^C \oplus {A_5}^C$ of ${E_7}^C$

Since γ and γ_1 are conjugate in $({G_2}^C)^\tau \subset {G_2}^C \subset {E_7}^C$ ([3], [4]), we have

$$E_{7(7)} = ({E_7}^C)^{\tau\gamma} \cong ({E_7}^C)^{\tau\gamma_1}.$$

In the Lie algebra ${\mathfrak{e}_7}^C$, let

$$Z = \Phi(i(-2G_{23} + G_{45} + G_{67}), 0, 0, 0).$$

Theorem 4.1. *The 3-graded decomposition of the Lie algebra* $\mathfrak{e}_{7(7)} = ({\mathfrak{e}_7}^C)^{\tau\gamma_1}$ *(or* ${\mathfrak{e}_7}^C$*),*

$$\mathfrak{e}_{7(7)} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

with respect to $\mathrm{ad}Z, Z = \Phi(i(-2G_{23} + G_{45} + G_{67}), 0, 0, 0)$*, is given by*

$$\mathfrak{g}_0 = \left\{ \begin{array}{l} iG_{01}, iG_{23}, iG_{45}, iG_{67}, G_{46} + G_{57}, i(G_{47} - G_{56}), \\ \widetilde{A}_k(1), i\widetilde{A}_k(e_1), (E_1 - E_2)^\sim, (E_2 - E_3)^\sim, \widetilde{F}_k(1), i\widetilde{F}_k(e_1) \\ \check{E}_k, \hat{E}_k, \check{F}_k(1), i\check{F}_k(e_1), \hat{F}_k(1), i\hat{F}_k(e_1), k = 1,2,3, \ \mathbf{1} \end{array} \right\} \ 39$$

$$\mathfrak{g}_{-1} = \left\{ \begin{array}{l} G_{04} + iG_{05}, G_{06} + iG_{07}, iG_{14} - G_{15}, iG_{16} - G_{17}, \\ (2G_{15} + G_{26} - G_{37}) - i(2G_{14} + G_{27} + G_{36}), \\ (2G_{17} - G_{24} + G_{35}) - i(2G_{16} - G_{25} - G_{34}), \\ \widetilde{A}_k(e_4 + ie_5), \widetilde{A}_k(e_6 + ie_7), \widetilde{F}_k(e_4 + ie_5), \widetilde{F}_k(e_6 + ie_7), \\ \check{F}_k(e_4 + ie_5), \check{F}_k(e_6 + ie_7), \hat{F}_k(e_4 + ie_5), \hat{F}_k(e_6 + ie_7), k = 1,2,3 \end{array} \right\} \ 30$$

$$\mathfrak{g}_{-2} = \left\{ \begin{array}{l} G_{02} - iG_{03}, iG_{12} + G_{13}, \\ (-2G_{13} + G_{46} - G_{57}) - i(2G_{12} - G_{47} - G_{56}), \\ \widetilde{A}_k(e_2 - ie_3), \widetilde{F}_k(e_2 - ie_3), \check{F}_k(e_2 - ie_3), \hat{F}_k(e_2 - ie_3), k = 1,2,3 \end{array} \right\} \ 15$$

$$\mathfrak{g}_{-3} = \left\{ (G_{24} + G_{35}) + i(G_{25} - G_{34}), (G_{26} + G_{37}) + i(G_{27} - G_{36}) \right\} \ 2$$

$$\mathfrak{g}_1 = \tau(\mathfrak{g}_{-1})\tau, \ \mathfrak{g}_2 = \tau(\mathfrak{g}_{-2})\tau, \ \mathfrak{g}_3 = \tau(\mathfrak{g}_{-3})\tau$$

Proof. We can prove this theorem in a way similar to [6, Theorem 4.5], by using [6, Lemma 4.4]. □

As is shown in ${G_2}^C, {F_4}^C$ or ${E_6}^C$ ([7]), we have

$$z_2 = \exp\frac{2\pi i}{2}Z = \gamma, \quad z_4 = \exp\frac{2\pi i}{4}Z = \delta_4, \quad z_3 = \exp\frac{2\pi i}{3}Z = w_3.$$

Now, since $({\mathfrak{e}_7}^C)_{ev} = ({\mathfrak{e}_7}^C)^{z_2} = ({\mathfrak{e}_7}^C)^\gamma, ({\mathfrak{e}_7}^C)_0 = ({\mathfrak{e}_7}^C)^{z_4} = ({\mathfrak{e}_7}^C)^{\delta_4}, ({\mathfrak{e}_7}^C)_{ed} = ({\mathfrak{e}_7}^C)^{z_3} = ({\mathfrak{e}_7}^C)^{w_3}$, we shall determine the structures of groups

$$({E_7}^C)_{ev} = ({E_7}^C)^{z_2} = ({E_7}^C)^\gamma, \quad ({E_7}^C)_0 = ({E_7}^C)^{z_4} = ({E_7}^C)^{\delta_4},$$
$$({E_7}^C)_{ed} = ({E_7}^C)^{z_3} = ({E_7}^C)^{w_3}.$$

4.1.1. Involution γ and subgroup $(SL(2,C) \times Spin(12,C))/\boldsymbol{Z}_2$ of ${E_7}^C$

Let $({E_7}^C)^\gamma = \{\alpha \in {E_7}^C \,|\, \gamma\alpha = \alpha\gamma\}$ and we will show

$$({E_7}^C)^\gamma \cong (SL(2,C) \times Spin(12,C))/\boldsymbol{Z}_2$$

(Theorem 4.1.1). For this end, we have to find subgroups which are isomorphic to $SL(2,C)$ and $Spin(12,C)$ in the group $({E_7}^C)^\gamma$. As for $SL(2,C)$, by using the mapping $\varphi_2 : Sp(1, \boldsymbol{H}^C) \times Sp(1, \boldsymbol{H}^C) \to {G_2}^C$, we may prefer

$$Sp(1, \boldsymbol{H}^C) = \{\varphi_2(p, 1) \,|\, p \in Sp(1, \boldsymbol{H}^C)\},$$

which is isomorphic to $SL(2,C)$. As for $Spin(12,C)$, we prefer

$$({E_7}^C)^{\varepsilon_1,\varepsilon_2} = \{\alpha \in {E_7}^C \,|\, \varepsilon_1\alpha = \alpha\varepsilon_1,\ \varepsilon_2\alpha = \alpha\varepsilon_2\}.$$

Since ${\varepsilon_1}^2 = \gamma$, if $\alpha \in {E_7}^C$ satisfies $\varepsilon_1\alpha = \alpha\varepsilon_1$, then $\gamma\alpha = \alpha\gamma$ is automatically satisfied, so $({E_7}^C)^{\varepsilon_1,\varepsilon_2}$ is a subgroup of $({E_7}^C)^\gamma$. $({E_7}^C)^{\varepsilon_1,\varepsilon_2} \cong Spin(12, C)$ will be proved in Proposition 1.1.7. In oder to show the connectedness of the group $({E_7}^C)^{\varepsilon_1,\varepsilon_2}$, we consider a series of subgroups

$$({E_7}^C)^{\varepsilon_1,\varepsilon_2} \supset (({E_7}^C)^{\varepsilon_1,\varepsilon_2})_{\dot{1}} \supset ({E_6}^C)^{\varepsilon_1,\varepsilon_2},$$

and we will show the connectedness of these groups in the order. We will start on a study of the group $({E_6}^C)^{\varepsilon_1,\varepsilon_2}$.

Proposition 1.1.1. $({E_6}^C)^{\varepsilon_1,\varepsilon_2} \cong SU^*(6, \boldsymbol{C}^C)$.
In particular, the group $({E_6}^C)^{\varepsilon_1,\varepsilon_2}$ *is connected.*

Proof. Let $SU^*(6, \boldsymbol{C}^C) = \{A \in M(6, \boldsymbol{C}^C) \,|\, JA = \overline{A}J, \det A = 1\}$, $J = \mathrm{diag}(J,\ J, J)$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The mapping $\varphi_6 : Sp(1, \boldsymbol{H}^C) \times SU^*(6, \boldsymbol{C}^C) \to ({E_6}^C)^\gamma$ is defined by

$$\varphi_6(p, A)(M + \boldsymbol{n}) = (k^{-1}A)M(k^{-1}A)^* + p\boldsymbol{n}(k^{-1}A)^{-1},$$
$$M + \boldsymbol{n} \in \mathfrak{J}(3, \boldsymbol{H}^C) \oplus (\boldsymbol{H}^C)^3 = \mathfrak{J}^C,$$

where $k : M(3, \boldsymbol{H}^C) \to M(6, \boldsymbol{C}^C)$ is defined by $k\Big((a+be_2)\Big) = \Big(\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}\Big)$, a, $b \in \boldsymbol{C}^C$. Then φ_6 induces the isomorphism $({E_6}^C)^\gamma \cong (Sp(1, \boldsymbol{H}^C) \times SU^*(6, \boldsymbol{C}^C))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{(1, E), (-1, -E)\}$ (see [5], [7] for details). Now, we define a mapping $\varphi_{6,r} : SU^*(6, \boldsymbol{C}^C) \to ({E_6}^C)^{\varepsilon_1,\varepsilon_2}$ by

$$\varphi_{6,r}(A) = \varphi_6(1, A)$$

as the restriction mapping of $\varphi_6 : Sp(1, \boldsymbol{H}^C) \times SU^*(6, \boldsymbol{C}^C) \to {E_6}^C$. It is easily verified that $\varphi_{6,r}$ is well-defined and a homomorphism. We shall show that $\varphi_{6,r}$ is onto. For $\alpha \in ({E_6}^C)^{\varepsilon_1,\varepsilon_2} \subset ({E_6}^C)^\gamma$, there exist $p \in Sp(1, \boldsymbol{H}^C)$ and $A \in SU^*(6, \boldsymbol{C}^C)$ such that $\alpha = \varphi_6(p, A)$. From the condition $\varepsilon_k\alpha = \alpha\varepsilon_k, k = 1, 2$, we see that $\alpha = \varphi_6(1, A)$ or $\alpha = \varphi_6(-1, A)$. In the latter case, $\alpha = \varphi_6(-1, A) = \varphi_6(1, -A) = \varphi_{6,r}(-A)$. Hence $\varphi_{6,r}$ is onto. It is easily obtained that $\mathrm{Ker}\,\varphi_{6,r} = \{E\}$. Thus we have the required isomorphism $SU^*(6, \boldsymbol{C}^C) \cong ({E_6}^C)^{\varepsilon_1,\varepsilon_2}$. □

Lemma 1.1.2. (1) *The Lie algebra* $({\mathfrak{e}_7}^C)^{\varepsilon_1,\varepsilon_2}$ *of the Lie group* $({E_7}^C)^{\varepsilon_1,\varepsilon_2}$ *is given by*

$$({\mathfrak{e}_7}^C)^{\varepsilon_1,\varepsilon_2} = \left\{ \Phi(D+\widetilde{S}+\widetilde{T},A,B,\nu) \in {\mathfrak{e}_7}^C \;\middle|\; \right.$$

$$D = \begin{pmatrix} 0 & d_{01} & d_{02} & d_{03} & 0 & 0 & 0 & 0 \\ -d_{01} & 0 & d_{12} & d_{13} & 0 & 0 & 0 & 0 \\ -d_{02} & -d_{12} & 0 & d_{23} & 0 & 0 & 0 & 0 \\ -d_{03} & -d_{13} & -d_{23} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_1 & d_2 & d_3 \\ 0 & 0 & 0 & 0 & -d_1 & 0 & d_3 & -d_2 \\ 0 & 0 & 0 & 0 & -d_2 & -d_3 & 0 & d_1 \\ 0 & 0 & 0 & 0 & -d_3 & d_2 & -d_1 & 0 \end{pmatrix} \in \mathfrak{so}(8,C),$$

$$\left. S = \begin{pmatrix} 0 & s_3 & -\overline{s}_2 \\ -\overline{s}_3 & 0 & s_1 \\ s_2 & -\overline{s}_1 & 0 \end{pmatrix}, s_k \in \boldsymbol{H}^C, T \in \mathfrak{J}(3,\boldsymbol{H}^C)_0, A,B \in \mathfrak{J}(3,\boldsymbol{H}^C), \nu \in C \right\}.$$

In particular, $\dim_C(({\mathfrak{e}_7}^C)^{\varepsilon_1,\varepsilon_2}) = 9+12+14+15\times 2+1 = 66$.

(2) *The Lie algebra* $(({\mathfrak{e}_7}^C)^{\varepsilon_1,\varepsilon_2})_{\rm i}$ *of the Lie group* $(({E_7}^C)^{\varepsilon_1,\varepsilon_2})_{\rm i}$ *is given by*

$$(({\mathfrak{e}_7}^C)^{\varepsilon_1,\varepsilon_2})_{\rm i} = \{\Phi(D+\widetilde{S}+\widetilde{T},A,B,\nu) \in ({\mathfrak{e}_7}^C)^{\varepsilon_1,\varepsilon_2} \,|B=0,\nu=0\}.$$

The Freudenthal manifold $\mathfrak{M}^C$ is defined by

$$\begin{aligned} \mathfrak{M}^C &= \{P \in \mathfrak{P}^C \,|\, P\times P = 0,\ P \neq 0\} \\ &= \left\{ P=(X,Y,\xi,\eta) \in \mathfrak{P}^C \;\middle|\; \begin{array}{l} X\vee Y = 0, X\times X = \eta Y, Y\times Y = \xi X, \\ (X,Y) = 3\xi\eta,\ P\neq 0 \end{array} \right\}. \end{aligned}$$

We define a submanifold $(\mathfrak{M}^C)_{\varepsilon_1,{\rm i}}$ of $\mathfrak{M}^C$ by

$$\begin{aligned} (\mathfrak{M}^C)_{\varepsilon_1,{\rm i}} &= \{P \in \mathfrak{P}^C \,|\, P\times P = 0,\ \varepsilon_1 P = P,\ \{\dot{1},P\} = 1\} \\ &= \left\{ P=(X,Y,\xi,\eta) \in \mathfrak{P}^C \;\middle|\; \begin{array}{l} X\vee Y = 0, X\times X = \eta Y, Y\times Y = \xi X, \\ (X,Y) = 3\xi\eta,\ X,Y \in \mathfrak{J}(3,\boldsymbol{H}^C), \\ \{\dot{1},P\} = 1 \end{array} \right\} \\ &= \{\,(X, X\times X, \det X, 1) \,|\, X \in \mathfrak{J}(3,\boldsymbol{H}^C), X\vee(X\times X) = 0\}. \end{aligned}$$

Proposition 1.1.3. $(({E_7}^C)^{\varepsilon_1,\varepsilon_2})_{\rm i}/({E_6}^C)^{\varepsilon_1,\varepsilon_2} \simeq (\mathfrak{M}^C)_{\varepsilon_1,{\rm i}}$.
In particular, the group $(({E_7}^C)^{\varepsilon_1,\varepsilon_2})_{\rm i}$ *is connected.*

Proof. The group $(({E_7}^C)^{\varepsilon_1,\varepsilon_2})_{\rm i}$ acts on $(\mathfrak{M}^C)_{\varepsilon_1,{\rm i}}$. We shall show that this action is transitive. To prove this, it is sufficient to show that any element $P \in (\mathfrak{M}^C)_{\varepsilon_1,{\rm i}}$ can be transformed to $\underset{\cdot}{1} = (0,0,0,1) \in (\mathfrak{M}^C)_{\varepsilon_1,{\rm i}}$ by some $\alpha \in (({E_7}^C)^{\varepsilon_1,\varepsilon_2})_{\rm i}$, moreover by some $\alpha \in ((({E_7}^C)^{\varepsilon_1,\varepsilon_2})_{\rm i})^0$ (which is the connected component subgroup of $(({E_7}^C)^{\varepsilon_1,\varepsilon_2})_{\rm i}$). Now, for a given $P = (X, X\times X, \det X, 1) \in (\mathfrak{M}^C)_{\varepsilon_1,{\rm i}}$, we see that $\Phi(0,X,0,0) \in (({\mathfrak{e}_7}^C)^{\varepsilon_1,\varepsilon_2})_{\rm i}$ (Lemma 1.1.2. (2)). Hence $\alpha(X) = \exp(\Phi(0,X,0,0)) \in ((({E_7}^C)^{\varepsilon_1,\varepsilon_2})_{\rm i})^0$. Operate $\alpha(X)$ on

$\dot{1}$, then we have $\alpha(X)\dot{1} = (X, X \times X, \det X, 1)$. This shows the transitivity of $((E_7{}^C)^{\varepsilon_1,\varepsilon_2})_{\rm i}$. Since $(\mathfrak{M}^C)_{\varepsilon_1,{\rm i}} = (((E_7{}^C)^{\varepsilon_1,\varepsilon_2})_{\rm i})^0\dot{1}$, $(\mathfrak{M}^C)_{\varepsilon_1,{\rm i}}$ is connected. The isotropy subgroup of $((E_7{}^C)^{\varepsilon_1,\varepsilon_2})_{\rm i}$ at $\dot{1}$ is $(E_6{}^C)^{\varepsilon_1,\varepsilon_2}$. Thus we have the required homeomorphism $((E_7{}^C)^{\varepsilon_1,\varepsilon_2})_{\rm i}/(E_6{}^C)^{\varepsilon_1,\varepsilon_2} \simeq (\mathfrak{M}^C)_{\varepsilon_1,{\rm i}}$. $(\mathfrak{M}^C)_{\varepsilon_1,{\rm i}}$ and $(E_6{}^C)^{\varepsilon_1,\varepsilon_2}$ are connected (Proposition 1.1.1), so $((E_7{}^C)^{\varepsilon_1,\varepsilon_2})_{\rm i}$ is also connected. □

Lemma 1.1.4. *For $\Phi(0,0,B,\nu), B \in \mathfrak{J}(3,\boldsymbol{H}^C), \nu \in C$, there exist $Y \in \mathfrak{J}(3,\boldsymbol{H}^C)$ and $\xi \in C, \xi \neq 0$ such that*

$$(\exp(\Phi(0,0,B,\nu)))\dot{1} = \Big(\frac{1}{\xi}(Y \times Y), Y, \xi, \frac{1}{\xi^2}(\det Y)\Big).$$

Conversely, for $\Big(\frac{1}{\xi}(Y \times Y), Y, \xi, \frac{1}{\xi^2}(\det Y)\Big) \in \mathfrak{P}^C$, there exist $B \in \mathfrak{J}(3,\boldsymbol{H}^C)$ and $\nu \in C$ such that $\Big(\frac{1}{\xi}(Y \times Y), Y, \xi, \frac{1}{\xi^2}(\det Y)\Big) = (\exp(\Phi(0,0,B,\nu)))\dot{1}$.

Proof.

$$(\exp(\Phi(0,0,B,\nu)))\dot{1} = \begin{pmatrix} (e^\nu - 2e^{\frac{\nu}{3}} + e^{-\frac{\nu}{3}})\dfrac{9}{4\nu^2}(B \times B) \\ (e^\nu - e^{\frac{\nu}{3}})\dfrac{3}{2\nu}B \\ e^\nu \\ ((e^\nu - e^{-\nu}) - 3(e^{\frac{\nu}{3}} - e^{-\frac{\nu}{3}}))\dfrac{27\det B}{8\nu^3} \end{pmatrix} \in \mathfrak{P}^C,$$

(in the case of $\nu = 0$, the parts of $\nu = 0$ need to replace by $\lim\limits_{\nu\to 0}$). Now, put $Y = (e^\nu - e^{\frac{\nu}{3}})\dfrac{3}{2\nu}B,\ \xi = e^\nu$ $(*)$, then we have

$$(\exp(\Phi(0,0,B,\nu)))\dot{1} = \Big(\frac{1}{\xi}\,(Y \times Y), Y, \xi, \frac{1}{\xi^2}\,(\det Y)\Big).$$

Conversely, for $P = \Big(\frac{1}{\xi}\,(Y \times Y), Y, \xi, \frac{1}{\xi^2}\,(\det Y)\Big)$, we can choose $B \in \mathfrak{J}(3,\boldsymbol{H}^C)$ and $\nu \in C$ satisfying the condition $(*)$ above. Then we obtain $(\exp(\Phi(0,0,B,\nu)))$ $\dot{1} = P$. □

We define a submanifold $(\mathfrak{M}^C)_{\varepsilon_1}$ of the Freudenthal manifold $\mathfrak{M}^C$ by

$$\begin{aligned}(\mathfrak{M}^C)_{\varepsilon_1} &= \{P \in \mathfrak{P}^C \mid P \times P = 0,\ \varepsilon_1 P = P,\ P \neq 0\} \\ &= \left\{ P = (X, Y, \xi, \eta) \in \mathfrak{P}^C \;\middle|\; \begin{array}{l} X \vee Y = 0, X \times X = \eta Y, Y \times Y = \xi X, \\ (X, Y) = 3\xi\eta,\ X, Y \in \mathfrak{J}(3,\boldsymbol{H}^C),\ P \neq 0 \end{array} \right\}.\end{aligned}$$

Proposition 1.1.5. $(E_7{}^C)^{\varepsilon_1,\varepsilon_2}/((E_7{}^C)^{\varepsilon_1,\varepsilon_2})_{\rm i} \simeq (\mathfrak{M}^C)_{\varepsilon_1}$.
In particular, the group $(E_7{}^C)^{\varepsilon_1,\varepsilon_2}$ is connected.

Proof. The group $({E_7}^C)^{\varepsilon_1,\varepsilon_2}$ acts on $(\mathfrak{M}^C)_{\varepsilon_1}$. We shall show that this action is transitive. To prove this, it is sufficient to show that any element $P \in (\mathfrak{M}^C)_{\varepsilon_1}$ can be transformed to $\dot{1} = (0,0,1,0) \in (\mathfrak{M}^C)_{\varepsilon_1}$ by some $\alpha \in ({E_7}^C)^{\varepsilon_1,\varepsilon_2}$, moreover by $\alpha \in (({E_7}^C)^{\varepsilon_1,\varepsilon_2})^0$ (which is the connected component subgroup of $({E_7}^C)^{\varepsilon_1,\varepsilon_2}$).

Case (1) $P = (X,Y,\xi,\eta), \xi \neq 0$. From the condition of $(\mathfrak{M}^C)_{\varepsilon_1}$, we see

$$X = \frac{1}{\xi}(Y \times Y), \quad \eta = \frac{1}{\xi^2}(\det Y).$$

For these Y and ξ, choose $B \in \mathfrak{J}(3, \boldsymbol{H}^C)$ and $\nu \in C$ of Lemma 1.1.4, then we see $\Phi(0,0,B,\nu) \in ({\mathfrak{e}_7}^C)^{\varepsilon_1,\varepsilon_2}$ (Lemma 1.1.2.(1)). Hence $\alpha = \exp(\Phi(0,0,B,\nu)) \in (({E_7}^C)^{\varepsilon_1,\varepsilon_2})^0$ and we have $\alpha\dot{1} = P$.

Case (2) $P = (X,Y,0,\eta),\ Y \neq 0$. For a given P, we see $\Phi(0,\tau Y,0,0) \in ({\mathfrak{e}_7}^C)^{\varepsilon_1,\varepsilon_2}$ (Lemma 1.1.2.(1)). Hence $\exp(\Phi(0,\tau Y,0,0)) \in (({E_7}^C)^{\varepsilon_1,\varepsilon_2})^0$. We have

$$(\exp(\Phi(0,\tau Y,0,0)))(X,Y,0,\eta) = (X + \eta(\tau Y), Y + 2\tau Y \times X, (\tau Y, Y), \eta).$$

If $Y \neq 0$, then $(\tau Y, Y) \neq 0$. Hence this case is reduced to the case (1).

Case (3) $P = (X,0,0,\eta),\ X \neq 0$. $\exp(\Phi(0,E,0,0)) \in (({E_7}^C)^{\varepsilon_1,\varepsilon_2})^0$ and we have

$$(\exp(\Phi(0,E,0,0)))(X,0,0,\eta) = (X + \eta E, (\mathrm{tr}(X)+\eta)E - X, \mathrm{tr}(X)+\eta, 0)\ (*).$$

If $(\mathrm{tr}(X)+\eta)E - X \neq 0$, then this is reduced to the case (2). In the case of $(\mathrm{tr}(X)+\eta)E - X = 0$, we see that $(*)$ is equal to $-\dfrac{1}{3}\mathrm{tr}(X)(E,0,-1,0)$, so this case is reduced to the case (1).

Case (4) $P = (0,0,0,\eta),\ \eta \neq 0$. $\exp(\Phi(0,E_1,0,0)) \in (({E_7}^C)^{\varepsilon_1,\varepsilon_2})^0$ and we have

$$(\exp(\Phi(0,E_1,0,0)))(0,0,0,\eta) = (\eta E_1, 0, 0, \eta).$$

Hence this case is reduced to the case (3).
Thus the proof of the transitivity of $(({E_7}^C)^{\varepsilon_1,\varepsilon_2})^0$ on $(\mathfrak{M}^C)_{\varepsilon_1}$ is completed.

Now, the group $({E_7}^C)^{\varepsilon_1,\varepsilon_2}$ acts on $(\mathfrak{M}^C)_{\varepsilon_1}$ transitively and the isotoropy subgroup of the group $({E_7}^C)^{\varepsilon_1,\varepsilon_2}$ at $\dot{1}$ is $(({E_7}^C)^{\varepsilon_1,\varepsilon_2})_{\dot{1}}$. Hence we have the homeomorphism $({E_7}^C)^{\varepsilon_1,\varepsilon_2}/(({E_7}^C)^{\varepsilon_1,\varepsilon_2})_{\dot{1}} \simeq (\mathfrak{M}^C)_{\varepsilon_1}$. Since $(\mathfrak{M}^C)_{\varepsilon_1} = (({E_7}^C)^{\varepsilon_1,\varepsilon_2})^0\dot{1}$, $(\mathfrak{M}^C)_{\varepsilon_1}$ is connected, and $(({E_7}^C)^{\varepsilon_1,\varepsilon_2})_{\dot{1}}$ is connected (Propositions 1.1.3), hence $({E_7}^C)^{\varepsilon_1,\varepsilon_2}$ is also connected. □

To prove the following proposition, we use the following two mappings $\phi_1(\theta), \lambda$. For $\theta \in C^*$, we define the C-linear transformation $\phi_1(\theta)$ of $\mathfrak{P}^C$ by

$$\phi_1(\theta)(X,Y,\xi,\eta) = (\theta^{-1}X, \theta Y, \theta^3\xi, \theta^{-3}\eta), \quad (X,Y,\xi,\eta) \in \mathfrak{P}^C,$$

and we define the C-linear transformation λ of $\mathfrak{P}^C$ by

$$\lambda(X, Y, \xi, \eta) = (Y, -X, \eta, -\xi), \quad (X, Y, \xi, \eta) \in \mathfrak{P}^C.$$

Then $\phi(\theta), \lambda \in ({E_7}^C)^{\varepsilon_1,\varepsilon_2}$.

Proposition 1.1.6. *The center $z(({E_7}^C)^{\varepsilon_1,\varepsilon_2})$ of the group $({E_7}^C)^{\varepsilon_1,\varepsilon_2}$ is isomorphic to the direct product group of two cyclic groups of order* 2:

$$z(({E_7}^C)^{\varepsilon_1,\varepsilon_2}) = \{1, \gamma\} \times \{1, -\gamma\} = \boldsymbol{Z}_2 \times \boldsymbol{Z}_2.$$

Proof. Let $\alpha \in z(({E_7}^C)^{\varepsilon_1,\varepsilon_2})$. For $\beta \in ({E_6}^C)^{\varepsilon_1,\varepsilon_2} \subset ({E_7}^C)^{\varepsilon_1,\varepsilon_2}$, we see

$$\beta\alpha\dot{1} = \alpha\beta\dot{1} = \alpha\dot{1}.$$

Denote $\alpha\dot{1} = (X, Y, \xi, \eta)$. From $(\beta X, {}^t\beta^{-1}Y, \xi, \eta) = (X, Y, \xi, \eta)$, we have

$$\beta X = X, \quad {}^t\beta^{-1}Y = Y \quad \text{for all } \beta.$$

Choose $\omega 1 \in ({E_6}^C)^{\varepsilon_1,\varepsilon_2}$ $(\omega \in C, \omega^3 = 1)$ as β, then we have $X = Y = 0$. Hence $\alpha\dot{1}$ is of the form

$$\alpha\dot{1} = (0, 0, \xi, \eta), \quad \xi, \eta \in C.$$

From $\alpha\dot{1} \in \mathfrak{M}^C$, we have $\xi\eta = 0$. Suppose $\xi = 0$, then we see $\alpha\dot{1} = \underset{\cdot}{\eta}$, $\eta \neq 0$. Since α commutes with $\phi_1(\theta) \in ({E_7}^C)^{\varepsilon_1,\varepsilon_2}$, we have

$$\theta^{-3}\underset{\cdot}{\eta} = \phi_1(\theta)\underset{\cdot}{\eta} = \phi_1(\theta)\alpha\dot{1} = \alpha\phi_1(\theta)\dot{1} = \alpha(\theta^3\dot{1}) = \theta^3\underset{\cdot}{\eta}$$

for any $\theta \in C^*$, so that we have $\eta = 0$. This is a contradiction. Hence $\xi \neq 0$, that is, $\eta = 0$. Thus $\alpha\dot{1} = \dot{\xi}$. By a similar argument as above, we have $\alpha\underset{\cdot}{1} = \underset{\cdot}{\zeta}$. Since $\xi\zeta = \{\dot{\xi}, \underset{\cdot}{\zeta}\} = \{\alpha\dot{1}, \alpha\underset{\cdot}{1}\} = \{\dot{1}, \underset{\cdot}{1}\} = 1$, that is, $\xi\zeta = 1$, we have $\alpha\dot{1} = \dot{\xi}, \alpha\underset{\cdot}{1} = \underset{\cdot}{\xi^{-1}}$. Moreover, α commutes with $\lambda \in ({E_7}^C)^{\varepsilon_1,\varepsilon_2}$, so we have

$$-\underset{\cdot}{\xi} = \lambda\dot{\xi} = \lambda\alpha\dot{1} = \alpha\lambda\dot{1} = \alpha(-\underset{\cdot}{1}) = -\underset{\cdot}{\xi^{-1}}.$$

Hence $\xi = \xi^{-1}$, so $\xi = 1$ or $\xi = -1$.

(i) Case $\xi = 1$. Since $\alpha\dot{1} = \dot{1}$ and $\alpha\underset{\cdot}{1} = \underset{\cdot}{1}$, we see $\alpha \in {E_6}^C$. Hence $\alpha \in z(({E_6}^C)^{\varepsilon_1,\varepsilon_2})$. Since $({E_6}^C)^{\varepsilon_1,\varepsilon_2} \cong SU^*(6, \boldsymbol{C}^C)$ (Proposition 1.1.1), we have

$$\begin{aligned} z(({E_6}^C)^{\varepsilon_1,\varepsilon_2}) &= z(\varphi_{6,r}(SU^*(6, \boldsymbol{C}^C)) \\ &= \{\varphi_{6,r}(cE) \,|\, c = 1, \omega, \omega^2, -1, -\omega, -\omega^2\}, \end{aligned}$$

where $\omega = e^{2\pi i/3}$. However $\varphi_{6,r}(cE) \notin z(({E_7}^C)^{\varepsilon_1,\varepsilon_2})$ for $c = \pm\omega, \pm\omega^2$. Hence we see $\alpha = \varphi_{6,r}(E) = 1$ or $\alpha = \varphi_{6,r}(-E) = \gamma$.

(ii) Case $\xi = -1$. By a similar argument as (i), we have $-\alpha \in z(({E_6}^C)^{\varepsilon_1, \varepsilon_2})$. Hence we have $-\alpha = 1$ or $-\alpha = \gamma$, that is, $\alpha = -1$ or $\alpha = -\gamma$.

Therefore we get $z(({E_7}^C)^{\varepsilon_1, \varepsilon_2}) \subset \{1, \gamma, -1, -\gamma\}$. The converse inclusion is trivial. Thus we have $z(({E_7}^C)^{\varepsilon_1, \varepsilon_2}) = \{1, \gamma, -1, -\gamma\}$. □

We define a 12-demensional C-vector space $(V^C)^{12}$ by

$$
\begin{aligned}
(V^C)^{12} &= \{P \in \mathfrak{P}^C \,|\, \varepsilon_1 P = -iP\} \\
&= \left\{ \left(\begin{pmatrix} 0 & x_3 & \overline{x}_2 \\ \overline{x}_3 & 0 & x_1 \\ x_2 & \overline{x}_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & y_3 & \overline{y}_2 \\ \overline{y}_3 & 0 & y_1 \\ y_2 & \overline{y}_1 & 0 \end{pmatrix}, 0, 0 \right) \,\middle|\, x_k,\, y_k \in (\boldsymbol{H}^C e_4)_{\varepsilon_1} \right\},
\end{aligned}
$$

with a norm

$$(P, P)_{\varepsilon_2} = \frac{1}{8}\{P, \varepsilon_2 P\},$$

where $(\boldsymbol{H}^C e_4)_{\varepsilon_1} = \{x \in \mathfrak{C}^C \,|\, x = p(e_4 + ie_5) + q(e_6 - ie_7),\, p, q \in C\}$. For $P \in (V^C)^{12}, x_k = p_k(e_4 + ie_5) + q_k(e_6 - ie_7), y_k = s_k(e_4 + ie_5) + t_k(e_6 - ie_7), k = 1, 2, 3$, the explicit form of $(P, P)_{\varepsilon_2}$ is given by

$$(P, P)_{\varepsilon_2} = (p_1 t_1 - q_1 s_1) + (p_2 t_2 - q_2 s_2) + (p_3 t_3 - q_3 s_3).$$

Proposition 1.1.7. $({E_7}^C)^{\varepsilon_1, \varepsilon_2} \cong Spin(12, C)$.

Proof. The group $({E_7}^C)^{\varepsilon_1, \varepsilon_2}$ acts on $(V^C)^{12}$ and any element $\alpha \in ({E_7}^C)^{\varepsilon_1, \varepsilon_2}$ leaves invariant the norm $(P, P)_{\varepsilon_2}$ of $(V^C)^{12}$. Furthermore, the group $({E_7}^C)^{\varepsilon_1, \varepsilon_2}$ is connected (Proposition1.1.5), so we can define a homomorphism $\pi : ({E_7}^C)^{\varepsilon_1, \varepsilon_2} \to SO((V^C)^{12}) = SO(12, C)$ by $\pi(\alpha) = \alpha|(V^C)^{12}$. We shall find $\operatorname{Ker} \pi$. For this end, we will show that the kernel of the differential mapping $\pi_* : ({\mathfrak{e}_7}^C)^{\varepsilon_1, \varepsilon_2} \to \mathfrak{so}((V^C)^{12})$ of π is trivial, that is, $\operatorname{Ker} \pi_* = 0$ (which is easily obtained). Hence $\operatorname{Ker} \pi$ is a discrete group. Moreover, the group $({E_7}^C)^{\varepsilon_1, \varepsilon_2}$ is connected, so we have

$$\operatorname{Ker} \pi \subset z(({E_7}^C)^{\varepsilon_1, \varepsilon_2}) = \{1, \gamma, -1, -\gamma\}$$

(Proposition 1.1.6). However $-1, \gamma \notin \operatorname{Ker} \pi$. Hence we get $\operatorname{Ker} \pi = \{1, -\gamma\} = \boldsymbol{Z}_2$. Since $\dim_C(({\mathfrak{e}_7}^C)^{\varepsilon_1, \varepsilon_2}) = 66$ (Lemma 1.1.2.(1)) $= \dim_C(\mathfrak{so}(12, C))$, π is onto. Thus we have $({E_7}^C)^{\varepsilon_1, \varepsilon_2}/\boldsymbol{Z}_2 \cong SO(12, C)$. Therefore $({E_7}^C)^{\varepsilon_1, \varepsilon_2}$ is $Spin(12, C)$ as a double covering group of $SO(12, C)$. □

By using the mapping $\varphi_2 : Sp(1, \boldsymbol{H}^C) \times Sp(1, \boldsymbol{H}^C) \to {G_2}^C$, we define a mapping $\varphi_{2,l} : Sp(1, \boldsymbol{H}^C) \to {G_2}^C$ by

$$\varphi_{2,l}(p) = \varphi_2(p, 1).$$

Then $\varphi_{2,l}(p) \in {G_2}^C \subset {F_4}^C \subset {E_6}^C \subset {E_7}^C$.

Lemma 1.1.8. *In the Lie algebra* ${\mathfrak e_7}^C$*, the Lie algebra* $\mathfrak{sp}(1, \boldsymbol{H}^C)$ *of the Lie group* $Sp(1, \boldsymbol{H}^C) = \varphi_{2,l}(Sp(1, \boldsymbol{H}^C))$ *is given by*

$$\mathfrak{sp}(1, \boldsymbol{H}^C) = \left\{ \varPhi(D, 0, 0, 0) \in {\mathfrak e_7}^C \;\middle|\; D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_1 J & 0 & 0 \\ 0 & 0 & d_2 J & 0 \\ 0 & 0 & 0 & d_3 J \end{pmatrix} \in \mathfrak{so}(8, C), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

Proposition 1.1.9. *The subgroups* $Sp(1, \boldsymbol{H}^C)$ *and* $Spin(12, C)$ *of* ${E_7}^C$ *are commutative elementwisely.*

Proof. Any $\varPhi_1 \in \mathfrak{sp}(1, \boldsymbol{H}^C) = {\varphi_{2,l}}_*(\mathfrak{sp}(1, \boldsymbol{H}^C))$ (${\varphi_{2,l}}_*$ is the defferential mapping of $\varphi_{2,l}$) commutes with any $\varPhi_2 \in \mathfrak{spin}(12, C) = ({\mathfrak e_7}^C)^{\varepsilon_1, \varepsilon_2} : [\varPhi_1, \varPhi_2] = 0$ (Lemma 1.1.8 and Lemma 1.1.2), furthermore the groups $Sp(1, \boldsymbol{H}^C)$ and $Spin(12, C) = ({E_7}^C)^{\varepsilon_1, \varepsilon_2}$ are connected. Hence, any $\varphi_{2,l}(p), p \in Sp(1, \boldsymbol{H}^C)$ commutes with $\beta \in Spin(12, C) : \varphi_{2,l}(p)\beta = \beta\varphi_{2,l}(p)$. □

Now, we will prove the main theorem of this section by using the preparations above.

Theorem 4.1.1. $({E_7}^C)_{ev} \cong (SL(2, C) \times Spin(12, C))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{(E, 1), (-E, \gamma)\}$.

Proof. We define a mapping $\varphi_\gamma : Sp(1, \boldsymbol{H}^C) \times Spin(12, C) \to ({E_7}^C)^\gamma = ({E_7}^C)_{ev}$ by

$$\varphi_\gamma(p, \beta) = \varphi_{2,l}(p)\beta.$$

$\varphi_{2,l}(p) \in ({E_7}^C)^\gamma$ is clear and $Spin(12, C) = ({E_7}^C)^{\varepsilon_1, \varepsilon_2}$ (Proposition 1.1.7) $\subset ({E_7}^C)^\gamma$, so φ_γ is well-defined. Since $\varphi_{2,l}(p)$ commutes with $\beta : \varphi_{2,l}(p)\beta = \beta\varphi_{2,l}(p)$ (Proposition 1.1.9), φ_γ is a homomorphism. $\mathrm{Ker}\,\varphi_\gamma = \{(1, 1), (-1, \gamma)\} = \boldsymbol{Z}_2$. The group $({E_7}^C)^\gamma$ is connected and $\dim_C(\mathfrak{sp}(1, \boldsymbol{H}^C) \oplus \mathfrak{spin}(12, C)) = 3 + 66 = 69 = 39 + 15 \times 2 = \dim_C(({\mathfrak e_7}^C)_{ev})$ (Theorem 4.1) $= \dim_C(({\mathfrak e_7}^C)^\gamma)$, hence φ_γ is onto. Thus we have the required isomorphism $({E_7}^C)_{ev} \cong (Sp(1, \boldsymbol{H}^C) \times Spin(12, C))/\boldsymbol{Z}_2$ ($\boldsymbol{Z}_2 = \{(1, 1), (-1, \gamma)\}$) $\cong (SL(2, C) \times Spin(12, C))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{(E, 1), (-E, \gamma)\}$. □

4.1.2. Automorphism δ_4 of order 4 and subgroup $(SL(2, C) \times C^* \times SL(6, C))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_6)$ of ${E_7}^C$

Let $({E_7}^C)^{\delta_4} = \{\alpha \in {E_7}^C \,|\, \delta_4\alpha = \alpha\delta_4\}$ and we will show

$$({E_7}^C)^{\delta_4} \cong (SL(2, C) \times C^* \times SL(6, C))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_6).$$

(Theorem 4.1.2). As for $SL(2,C)$, we may prefer $Sp(1,\boldsymbol{H}^C)$ as in the case of $({E_7}^C)_{ev}$. Before we consider the remainder part $C^* \times SL(6,C)$, we find a subgroup of $({E_7}^C)^{\delta_4}$ of type $GL(6,C)$, that is, consider the subgroup $({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}$ $= \{\alpha \in {E_7}^C \,|\, \varepsilon_1\alpha = \alpha\varepsilon_1, \varepsilon_2\alpha = \alpha\varepsilon_2,\ \delta_4\alpha = \alpha\delta_4\}$ of the group $({E_7}^C)^{\delta_4}$ and we will show

$$({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}/\boldsymbol{Z}_2 \cong GL(6,C)$$

(Proposition 1.2.4). And then we decompose $GL(6,C)$ into $C^* \times SL(6,C)$. For this end, we need to find subgroups C^* and $SL(6,C)$ in the group $({E_7}^C)^{\delta_4}$. We will start on a study of the group $({F_4}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}$.

Proposition 1.2.1. $({F_4}^C)^{\varepsilon_1,\varepsilon_2,\delta_4} \cong U(3,\boldsymbol{C}^C)$.

Proof. The mapping $\varphi_4 : Sp(1,\boldsymbol{H}^C) \times Sp(3,\boldsymbol{H}^C) \to ({F_4}^C)^\gamma$ is defined by

$$\varphi_4(p,A)(M+\boldsymbol{n}) = AMA^* + p\boldsymbol{n}A^*, \quad M+\boldsymbol{n} \in \mathfrak{J}(3,\boldsymbol{H}^C) \oplus (\boldsymbol{H}^C)^3 = \mathfrak{J}^C.$$

Then φ_4 induces the isomorphism $({F_4}^C)^\gamma \cong (Sp(1,\boldsymbol{H}^C) \times Sp(3,\boldsymbol{H}^C))/\boldsymbol{Z}_2, \boldsymbol{Z}_2$ $= \{(1,E),(-1,-E)\}$ (see [5] for details). Now, we define a mapping $\varphi_{4,r}$: $U(3,\boldsymbol{C}^C) \to ({F_4}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}$ by

$$\varphi_{4,r}(A) = \varphi_4(1,A),$$

as the restriction mapping of $\varphi_4 : Sp(1,\boldsymbol{H}^C) \times Sp(3,\boldsymbol{H}^C) \to ({F_4}^C)^\gamma$. It is easy to verify that $\varphi_{4,r}$ is well-defined and a homomorphism. We shall show that $\varphi_{4,r}$ is onto. For $\alpha \in ({F_4}^C)^{\varepsilon_1,\varepsilon_2,\delta_4} \subset ({F_4}^C)^\gamma$, there exist $p \in Sp(1,\boldsymbol{H}^C)$ and $A \in Sp(3,\boldsymbol{H}^C)$ such that $\alpha = \varphi_4(p,A)$. From the conditions $\varepsilon_k\alpha = \alpha\varepsilon_k, k = 1,2$ and $\delta_4\alpha = \alpha\delta_4$, we have $\alpha = \varphi_4(1,A) = \varphi_{4,r}(A)$ or $\alpha = \varphi_4(-1,A) = \varphi_{4,r}(-A), A \in U(3,\boldsymbol{C}^C)$. Hence $\varphi_{4,r}$ is onto. It is easily obtained that $\mathrm{Ker}\,\varphi_{4,r} = \{E\}$. Thus we have the required isomorphism $U(3,\boldsymbol{C}^C) \cong ({F_4}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}$. □

Lemma 1.2.2. *The Lie algebra* $({\mathfrak{e}_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}$ *of the Lie group* $({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}$ *is given by*

$$({\mathfrak{e}_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4} = \left\{ \Phi(D+\widetilde{S}+\widetilde{T},A,B,\nu) \in {\mathfrak{e}_7}^C \;\middle|\; \begin{array}{l} D = \begin{pmatrix} d_1J & 0 & 0 & 0 \\ 0 & d_2J & 0 & 0 \\ 0 & 0 & d_3J & 0 \\ 0 & 0 & 0 & d_3J \end{pmatrix} \in \mathfrak{so}(8,C), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ S = \begin{pmatrix} 0 & s_3 & -\overline{s}_2 \\ -\overline{s}_3 & 0 & s_1 \\ s_2 & -\overline{s}_1 & 0 \end{pmatrix}, s_k \in \boldsymbol{C}^C, T \in \mathfrak{J}(3,\boldsymbol{C}^C)_0, A,B \in \mathfrak{J}(3,\boldsymbol{C}^C), \nu \in C \end{array} \right\}.$$

In particular, $\dim_C(({\mathfrak{e}_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}) = 3+6+8+9\times 2+1 = 36$.

Proposition 1.2.3. *The center $z(({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4})$ of the group $({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}$ is given by*

$$z(({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}) = \{\alpha,\ -\alpha \,|\, \alpha = \varphi_{4,r}(cE),\ c \in U(1, \boldsymbol{C}^C)\}.$$

Proof. Let $\alpha \in z(({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4})$. Note that $\phi_1(\theta), \lambda \in ({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}$. Then by the same argument as in Proposition 1.1.6, we have

$$\alpha\dot{1} = (0, 0, \xi, 0), \quad \xi = 1 \quad \text{or} \quad \xi = -1.$$

(i) Case $\xi = 1$. Since $\alpha\dot{1} = \dot{1}$ and $\alpha\underset{\cdot}{1} = \underset{\cdot}{1}$, we see $\alpha \in {E_6}^C$. Hence

$$\alpha \in z(({E_6}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}).$$

For $\beta \in ({F_4}^C)^{\varepsilon_1,\varepsilon_2,\delta_4} \subset ({E_6}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}$, we see

$$\beta\alpha E = \alpha\beta E = \alpha E.$$

Putting $\alpha E = Y = Y(\eta, y) \in \mathfrak{J}^C$, we have

$$\beta Y = Y \quad \text{for all } \beta \in ({F_4}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}.$$

For $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, we define mappings $\delta : \mathfrak{J}^C \to \mathfrak{J}^C$ by $\delta X = TXT^{-1}$, then $\delta \in ({F_4}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}$. From the condition of $\delta Y = Y$, we have

$$y_1 = y_2 = y_3 = 0, \quad \eta_1 = \eta_2 = \eta_3 (= \omega).$$

Hence $\alpha E = \omega E,\ \omega \in C$. Moreover $\omega^3 = \det(\omega E) = \det\alpha E = \det E = 1$. So we see $\omega^{-1}\alpha E = E$, hence $\omega^{-1}\alpha \in ({F_4}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}$. Thus

$$\omega^{-1}\alpha \in z(({F_4}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}).$$

Since $({F_4}^C)^{\varepsilon_1,\varepsilon_2,\delta_4} = U(3, \boldsymbol{C}^C)$ (Proposition 1.2.1) and $z(U(3, \boldsymbol{C}^C)) = \{cE \,|\, c \in U(1, \boldsymbol{C}^C)\}$, we see

$$z(({F_4}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}) = z(\varphi_{4,r}(U(3, \boldsymbol{C}^C))) = \{\varphi_{4,r}(cE) \,|\, c \in U(1, \boldsymbol{C}^C)\}.$$

Hence there exists $c \in U(1, \boldsymbol{C}^C)$ such that $\omega^{-1}\alpha = \varphi_{4,r}(cE)$, that is,

$$\alpha = \omega\varphi_{4,r}(cE), \quad \omega \in C, \quad \omega^3 = 1, \quad c \in U(1, \boldsymbol{C}^C).$$

The condition $\alpha \in z(({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4})$ implies that α commutes with all elements $\Phi(\phi, A, B, \nu) \in ({\mathfrak{e}_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}$, that is,

$$\omega\varphi_{4,r}(cE)\Phi(\phi, A, B, \nu) = \Phi(\phi, A, B, \nu)\omega\varphi_{4,r}(cE).$$

Hence, for all $\phi \in ({\mathfrak{e}_6}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}$, $A, B \in \mathfrak{J}(3, \boldsymbol{C}^C)$ (Lemma 1.2.2), we have

$$\begin{cases} \varphi_{4,r}(cE)\phi\varphi_{4,r}(\overline{c}E) = \phi & \cdots\cdots \ (1) \\ \omega\varphi_{4,r}(cE)A = A & \cdots\cdots \ (2) \\ \omega^{-1}\varphi_{4,r}(cE)B = B & \cdots\cdots \ (3). \end{cases}$$

Since $\omega\varphi_{4,r}(cE)A = \omega(cE)A(cE)^* = \omega(c\overline{c})A = \omega A$, from the condition (2), we have $\omega = 1$, thereby we see the condition (3). The condition (1) is always valid. Thus we see that α is of the form $\varphi_{4,r}(cE)$.

(ii) Case $\xi = -1$. By a similar argument as (i), there exists $c \in U(1, \boldsymbol{C}^C)$ such that $-\alpha = \varphi_{4,r}(cE)$. Hence α is of the form $-\varphi_{4,r}(cE)$.
Thus this proposition is completely proved. □

We define a C-vector subspace $(V^C)^6$ of the C-vector space $(V^C)^{12}$ by

$$\begin{aligned}(V^C)^6 &= \{P \in (V^C)^{12} \,|\, \delta_4 P = -iP\} \\ &= \left\{ \left(\begin{pmatrix} 0 & x_3 & \overline{x}_2 \\ \overline{x}_3 & 0 & x_1 \\ x_2 & \overline{x}_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & y_3 & \overline{y}_2 \\ \overline{y}_3 & 0 & y_1 \\ y_2 & \overline{y}_1 & 0 \end{pmatrix}, 0, 0 \right) \,\middle|\, x_k, \, y_k \in (\boldsymbol{H}^C e_4)_{\varepsilon_1,\delta_4} \right\},\end{aligned}$$

where $(\boldsymbol{H}^C e_4)_{\varepsilon_1,\delta_4} = \{x \in \mathfrak{C}^C \,|\, x = p(e_4 + ie_5), \, p \in C\}$.

Proposition 1.2.4. $({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}/\boldsymbol{Z}_2 \cong GL(6, C)$, $\boldsymbol{Z}_2 = \{1, -\gamma\}$.

Proof. Let $GL(6, C) = \mathrm{Iso}_C((V^C)^6)$. Any element $\alpha \in ({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}$ leaves invariant the space $(V^C)^6$, so α induces an element of $GL(6, C)$. Hence we can define a mapping $g : ({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4} \to GL(6, C)$ by

$$g(\alpha) = \alpha|(V^C)^6, \quad \alpha \in ({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}.$$

It is clear that g is a homomorphism. We shall calculate $\mathrm{Ker}\, g$. For this end, first, we show that the kernel of the differential mapping $g_* : ({\mathfrak{e}_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4} \to \mathfrak{gl}(6, C)$ of g is trivial : $\mathrm{Ker}\, g_* = \{0\}$ (which is easily obtained). Hence $\mathrm{Ker}\, g$ is a discrete group. Moreover since the group $({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}$ is connected (because $({E_7}^C)^{\varepsilon_1,\varepsilon_2}$ is simply connected (Proposition 1.1.7)), we have

$$\mathrm{Ker}\, g \subset z(({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}).$$

Let $\alpha \in \mathrm{Ker}\, g$. Then α is of the form $\alpha = \varphi_{4,r}(cE)$ or $\alpha = -\varphi_{4,r}(cE)$ for some $c \in U(1, \boldsymbol{C}^C)$ (Proposition 1.2.3). $\varphi_{4,r}(cE)$ is nothing but $\varphi_{2,r}(c) = \varphi_2(1, c)$. Since $\varphi_{2,r}(c)(e_4 + ie_5) = \overline{c}(e_4 + ie_5)$, from the condition $\varphi_{2,r}(c)(e_4 + ie_5) = e_4 + ie_5$, we see $c = 1$, that is, $\alpha = 1$. In the case of $\alpha = \varphi_{4,r}(cE)$, by a similar way above, we see $\alpha = -\varphi_{4,r}(-E) = -\gamma$. Hence $\mathrm{Ker}\, g = \{1, -\gamma\} = \boldsymbol{Z}_2$. Furthermore $\dim_C(({\mathfrak{e}_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}) = 36$ (Lemma 1.2.2) $= \dim_C(\mathfrak{gl}(6, C))$ and $({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}$ is connected, hence g is onto. Thus we have the required isomorphism $({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}/\boldsymbol{Z}_2 \cong GL(6, C)$. □

Proposition 1.2.5. $({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4} \cong (C^* \times SL(6,C))/\boldsymbol{Z}_6$, $\boldsymbol{Z}_6 = \{(a, a^{-1}E) \mid a \in C, a^6 = 1\}$.

Proof. The general linear group $GL_1(6,C) = \{A \in M(6,C) \mid \det A \neq 0\}$ is decomposable as

$$GL_1(6,C) = {C_1}^* SL_1(6,C), \quad {C_1}^* \cap SL_1(6,C) = \{aE \mid a \in C, a^6 = 1\},$$

where ${C_1}^* = \{aE \mid a \in C^*\}$ which is the connected component subgroup of the center of $GL_1(6,C)$ and $SL_1(6,C) = \{A \in GL_1(6,C) \mid \det A = 1\}$. On the other hand, the connected component subgroup of $({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}$ is $\{\varphi_{4,r}(cE) \mid c \in U(1, \boldsymbol{C}^C)\}$ (Proposition 1.2.3). If we give the isomorphism $h : C^* \to U(1, \boldsymbol{C}^C)$ by

$$h(a) = \frac{a + a^{-1}}{2} + \frac{a - a^{-1}}{2} ie_1 = \iota a + \bar{\iota} a^{-1}, \quad \iota = \frac{1 + ie_1}{2},$$

we have

$$\varphi_{4,r}(h(a)E)F_k(x) = F_k(ax), \quad x \in (\boldsymbol{H}^C e_4)_{\varepsilon_1,\delta_4}, \quad k = 1,2,3.$$

Hereafter, for $a \in C^*$, we denote $\varphi_{4,r}(h(a)E)$ by $\zeta(a)$:

$$\zeta(a) = \varphi_{4,r}(h(a)E).$$

Then the restriction mapping of $\zeta(a)$ to $(V^C)^6$ is given by

$$\zeta(a)(X,Y,0,0) = (aX, aY, 0, 0).$$

Hence we see $g(\zeta(a)) = aE$ for $a \in C^*$ (as for g, see Proposition 1.2.4), so g induces an isomorphism $g : C^* \to {C_1}^*$. Next we will find a subgroup $SL(6,C)$ of $({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}$ which is isomorphic to the group $SL_1(6,C)$ under g. Consider the subgroup $\widetilde{SL} = g^{-1}(SL_1(6,C))$ of $({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}$. Then $\widetilde{SL}/\boldsymbol{Z}_2 \cong SL_1(6,C)$. Since $SL_1(6,C)$ is simply connected, $\widetilde{SL}$ is never connected. Let $SL(6,C)$ be the connected component subgroup of $\widetilde{SL}$, then $SL(6,C)$ is the requiered one. Then we have the following diagram

$$\begin{array}{ccc} C^* \times SL(6,C) & \xrightarrow{\mu} & ({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4} \\ g \ \downarrow \ g & & \downarrow g \\ {C_1}^* \times SL_1(6,C) & \xrightarrow{\mu_1} & GL_1(6,C), \end{array}$$

where μ, μ_1 are multiplication mappings in the groups, respectively. Obviously μ is a surjective homomorphism. We shall find the kernel of μ. Let $(\zeta(a), \beta) \in \operatorname{Ker}\mu$. From the diagram above, we have $g(\zeta(a))g(\beta) = g((\zeta(a)\beta)) = g(1) = E$. Hence we obtain $\operatorname{Ker}\mu = \{(\zeta(a), \zeta(a^{-1}) \mid a \in C, a^6 = 1\} = \boldsymbol{Z}_6$. Since $g : C^* \to {C_1}^*$ is isomorphic, $\operatorname{Ker}\mu$ is denoted by $\{(a, a^{-1}E) \mid a \in C, a^6 = 1\}$. Thus we have the required isomorphism $({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4} \cong (C^* \times SL(6,C))/\boldsymbol{Z}_6$. $\square$

Hereafter, we identify two groups $SL(6,C), SL_1(6,C)$ and $C^*, {C_1}^*$, respectively.

Now, we will prove the main theorem of this section by using the preparations above.

Theorem 4.1.2. $({E_7}^C)_0 \cong (SL(2,C)\times C^*\times SL(6,C))/(\boldsymbol{Z}_2\times\boldsymbol{Z}_6)$, $\boldsymbol{Z}_2 = \{(E,1,E),(-E,1,-E)\}$, $\boldsymbol{Z}_6 = \{(E,a,a^{-1}E)\,|\, a = 1,\omega,\omega^2,-1,-\omega,-\omega^2\}$, *where* $\omega = e^{2\pi i/3}$.

Proof. We define a mapping $\varphi_{\delta_4} : Sp(1,\boldsymbol{H}^C)\times C^*\times SL(6,C) \to Sp(1,\boldsymbol{H}^C) \times({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4} \to ({E_7}^C)^{\delta_4} = ({E_7}^C)_0$ by

$$\varphi_{\delta_4}(p,a,\beta) = \varphi_{2,l}(p)\zeta(a)\beta.$$

φ_{δ_4} is well-defined because $\varphi_{2,l}(p)\in({E_7}^C)^{\delta_4}$ and $\zeta(a),\beta\in({E_7}^C)^{\varepsilon_1,\varepsilon_2,\delta_4}$ (Proposition 1.2.5) $\subset ({E_7}^C)^{\delta_4}$. Since $\varphi_{2,l}(p)$ commutes with $\zeta(a)\beta$: $\varphi_{2,l}(p)\zeta(a)\beta = \zeta(a)\beta\varphi_{2,l}(p)$ (Proposition 1.1.9), φ_{δ_4} is a homomorphism. It is easily obtained that $\mathrm{Ker}\,\varphi_{\delta_4} = \{(1,1,1),(-1,1,\gamma)\}\times\{(1,\zeta(a),\zeta(a^{-1}))\,|\, a = 1,\omega,\omega^2, -1,-\omega,-\omega^2\} = \boldsymbol{Z}_2\times\boldsymbol{Z}_6$. Moreover $\dim_C(\mathfrak{sp}(1,\boldsymbol{H}^C)\oplus C\oplus\mathfrak{sl}(6,C)) = 3+1+35 = 39 = \dim_C(({\mathfrak{e}_7}^C)_0)$ (Theorem 4.1) $= \dim_C(({\mathfrak{e}_7}^C)^{\delta_4})$, hence φ_{δ_4} is onto. Thus we have the isomorphism $({E_7}^C)^{\delta_4} \cong (Sp(1,\boldsymbol{H}^C)\times C^*\times SL(6,C))/(\boldsymbol{Z}_2\times \boldsymbol{Z}_6)\,(\boldsymbol{Z}_2 = \{(1,1,1),(-1,1,\gamma)\}) \cong (SL(2,C)\times C^*\times SL(6,C))/(\boldsymbol{Z}_2\times\boldsymbol{Z}_6)$, $\boldsymbol{Z}_2 = \{(E,1,E),(-E,1,-E)\}$. $\square$

4.1.3. Automorphism w_3 of order 3 and subgroup $(SL(3,C)\times SL(6,C))/\boldsymbol{Z}_3$ of ${E_7}^C$

Let $({E_7}^C)^{w_3} = \{\alpha\in{E_7}^C\,|\,w_3\alpha = \alpha w_3\}$ and we will show

$$({E_7}^C)^{w_3} \cong (SL(3,C)\times SL(6,C))/\boldsymbol{Z}_3$$

(Theorem 4.1.3). For this end, we have to find subgroups which are isomorphic to $SL(3,C)$ and $SL(6,C)$ in the group $({E_7}^C)^{w_3}$. As for $SL(3,C)$, we use the embedding $\varphi_{3,l} : SU(3,\boldsymbol{C}^C) \to {G_2}^C$. As for $SL(6,C)$, we prefer $({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3} = \{\alpha\in{E_7}^C\,|\,w_3\alpha = \alpha w_3, \varepsilon_1\alpha = \alpha\varepsilon_1,\ \varepsilon_2\alpha = \alpha\varepsilon_2,\ \gamma_3\alpha = \alpha\gamma_3\}$ (Proposition 1.3.7).

The mapping $\varphi_{3,l} : SU(3,\boldsymbol{C}^C) \to {G_2}^C$ is defined by

$$\varphi_{3,l}(A)(a+\boldsymbol{m}) = a + A\boldsymbol{m}, \quad a+\boldsymbol{m}\in\boldsymbol{C}^C\oplus(\boldsymbol{C}^C)^3 = \mathfrak{C}^C.$$

Then $\varphi_{3,l}$ induces the isomorphism $SU(3,\boldsymbol{C}^C)\cong({G_2}^C)^{w_3}$ (see [2] for details). By using this mapping $\varphi_{3,l} : SU(3,\boldsymbol{C}^C)\to{G_2}^C$, we define a C-linear transformation γ_3 of $\mathfrak{C}^C$ by

$$\gamma_3 = \varphi_{3,l}\left(\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right).$$

Then $\gamma_3 \in {G_2}^C \subset {F_4}^C \subset {E_6}^C \subset {E_7}^C$ and ${\gamma_3}^3 = 1$. Note that using the mapping $\varphi_{3,l}$, the C-linear transformations ε_1 and w_3 are expressed as follows:

$$\varepsilon_1 = \varphi_{3,l}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & -e_1 \end{pmatrix}\right), \quad w_3 = \varphi_{3,l}\left(\begin{pmatrix} w_1 & 0 & 0 \\ 0 & w_1 & 0 \\ 0 & 0 & w_1 \end{pmatrix}\right).$$

We use the mappings h and φ_3 of [2], so we will review of these mappings. First, the mapping $h : M(3, \boldsymbol{C}^C) \times M(3, \boldsymbol{C}^C) \to M(3, \boldsymbol{C}^C)$ is defined by

$$h(A, B) = \frac{A+B}{2} + \frac{A-B}{2} ie_1 = \iota A + \overline{\iota} B, \quad \iota = \frac{1}{2}(1 + ie_1).$$

The mapping $\varphi_3 : SU(3, \boldsymbol{C}^C) \times SU(3, \boldsymbol{C}^C) \times SU(3, \boldsymbol{C}^C) \to ({E_6}^C)^{w_3}$ is defined by

$$\varphi_3(P, A, B)(X + M) = h(A, B)Xh(A, B)^* + PM\tau h(A, B)^*,$$
$$X + M \in \mathfrak{J}(3, \boldsymbol{C}^C) \oplus M(3, \boldsymbol{C}^C) = \mathfrak{J}^C.$$

Then φ_3 induces the isomorphism $({E_6}^C)^{w_3} \cong (SU(3, \boldsymbol{C}^C) \times SU(3, \boldsymbol{C}^C) \times SU(3, \boldsymbol{C}^C))/\boldsymbol{Z}_3$, $\boldsymbol{Z}_3 = \{ (E, E, E), (w_1E, w_1E, w_1E), ({w_1}^2E, {w_1}^2E, {w_1}^2E)\}$. The mapping φ_3 is an extension of the mapping $\varphi_{3,l} : SU(3, \boldsymbol{C}^C) \to {G_2}^C$, that is, the following holds.

$$\varphi_{3,l}(A) = \varphi_3(A, E, E), \quad A \in SU(3, \boldsymbol{C}^C).$$

Now, we will begin on a study of the group $({E_6}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}$.

Proposition 1.3.1. $({E_6}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3} \cong SU(3, \boldsymbol{C}^C) \times SU(3, \boldsymbol{C}^C)$. *In particular, the group* $({E_6}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}$ *is connected.*

Proof. We define a mapping $\varphi_{3,r} : SU(3, \boldsymbol{C}^C) \times SU(3, \boldsymbol{C}^C) \to ({E_6}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}$ by

$$\varphi_{3,r}(A, B) = \varphi_3(E, A, B),$$

as the restriction mapping of $\varphi_3 : SU(3, \boldsymbol{C}^C) \times SU(3, \boldsymbol{C}^C) \times SU(3, \boldsymbol{C}^C) \to ({E_6}^C)^{w_3}$. It is not difficult to verify that $\varphi_{3,r}$ is well-defined and a homomorphism. We shall show that $\varphi_{3,r}$ is onto. For $\alpha \in ({E_6}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3} \subset ({E_6}^C)^{w_3}$, there exist $P, A, B \in SU(3, \boldsymbol{C}^C)$ such that $\alpha = \varphi_3(P, A, B)$. From the conditions of $\varepsilon_k\alpha = \alpha\varepsilon_k, k = 1, 2$ and $\gamma_3\alpha = \alpha\gamma_3$, we have $\alpha = \varphi_3(E, A, B) = \varphi_{3,r}(A, B)$. Hence $\varphi_{6,r}$ is onto. It is easily obtained that $\operatorname{Ker}\varphi_{3,r} = \{(E, E)\}$. Therefore we have the required isomorphism $SU(3, \boldsymbol{C}^C) \times SU(3, \boldsymbol{C}^C) \cong ({E_6}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}$. □

Lemma 1.3.2. (1) *The Lie algebra* $({\mathfrak{e}_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}$ *of the Lie group*

$({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}$ *is given by*

$$({\mathfrak{e}_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3} = \left\{ \varPhi(D+\widetilde{S}+\widetilde{T},A,B,\nu) \in {\mathfrak{e}_7}^C \;\middle|\; \begin{array}{l} D = \begin{pmatrix} d_1J & 0 & 0 & 0 \\ 0 & d_2J & 0 & 0 \\ 0 & 0 & d_2J & 0 \\ 0 & 0 & 0 & d_2J \end{pmatrix} \in \mathfrak{so}(8,C), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ S = \begin{pmatrix} 0 & s_3 & -\overline{s}_2 \\ -\overline{s}_3 & 0 & s_1 \\ s_2 & -\overline{s}_1 & 0 \end{pmatrix}, s_k \in \boldsymbol{C}^C, T \in \mathfrak{J}(3,\boldsymbol{C}^C)_0, A,B \in \mathfrak{J}(3,\boldsymbol{C}^C), \nu \in C \end{array} \right\}.$$

In particular, $\dim_C(({\mathfrak{e}_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}) = 2+6+8+9\times 2+1 = 35.$

(2) *The Lie algebra* $(({\mathfrak{e}_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3})_{\dot{1}}$ *of the Lie group* $(({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3})_{\dot{1}}$ *is given by*

$$(({\mathfrak{e}_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3})_{\dot{1}} = \{\varPhi(D+\widetilde{S}+\widetilde{T},A,B,\nu) \in ({\mathfrak{e}_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3} \mid B=0, \nu=0\}.$$

We define a submanifold $(\mathfrak{M}^C)_{w_3,\varepsilon_1,\dot{1}}$ of the Freudenthal manifold $\mathfrak{M}^C$ by

$$\begin{aligned} (\mathfrak{M}^C)_{w_3,\varepsilon_1,\dot{1}} &= \{P \in \mathfrak{P}^C \mid P\times P = 0,\ w_3P = P,\ \varepsilon_1 P = P,\ \{\dot{1},P\} = 1\} \\ &= \left\{ P = (X,Y,\xi,\eta) \in \mathfrak{P}^C \;\middle|\; \begin{array}{l} X \vee Y = 0,\ X\times X = \eta Y, \\ Y\times Y = \xi X,\ (X,Y) = 3\xi\eta, \\ X,Y \in \mathfrak{J}(3,\boldsymbol{C}^C),\ \{\dot{1},P\} = 1 \end{array} \right\} \\ &= \{\,(X, X\times X, \det X, 1) \mid X \in \mathfrak{J}(3,\boldsymbol{C}^C),\ X \vee (X\times X) = 0\}. \end{aligned}$$

Proposition 1.3.3. $(({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3})_{\dot{1}}/({E_6}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3} \simeq (\mathfrak{M}^C)_{w_3,\varepsilon_1,\dot{1}}$. *In particular, the group* $(({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3})_{\dot{1}}$ *is connected.*

Proof. We can prove this proposition in a way similar to Proposition 1.1.3 by replacing $\boldsymbol{H}^C$ by $\boldsymbol{C}^C$. □

We define a submanifold $(\mathfrak{M}^C)_{w_3,\varepsilon_1}$ of the Freudenthal manifold $\mathfrak{M}^C$ by

$$\begin{aligned} (\mathfrak{M}^C)_{w_3,\varepsilon_1} &= \{P \in \mathfrak{P}^C \mid P\times P = 0,\ w_3P = P,\ \varepsilon_1 P = P,\ P \neq 0\} \\ &= \left\{ P = (X,Y,\xi,\eta) \in \mathfrak{P}^C \;\middle|\; \begin{array}{l} X \vee Y = 0, X\times X = \eta Y, Y\times Y = \xi X, \\ (X,Y) = 3\xi\eta, X,Y \in \mathfrak{J}(3,\boldsymbol{C}^C), P \neq 0 \end{array} \right\}. \end{aligned}$$

Proposition 1.3.4. $({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}/(({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3})_{\dot{1}} \simeq (\mathfrak{M}^C)_{w_3,\varepsilon_1}$. *In particular, the group* $({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}$ *is connected.*

Proof. We can prove this proposition in a way similar to Proposition 1.1.5 by replacing $\boldsymbol{H}^C$ by $\boldsymbol{C}^C$. □

Proposition 1.3.5. *The center of* $z(({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3})$ *of the group* $({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}$ *is the cyclic group of order* 6:

$$z(({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}) = \{1, w_3, {w_3}^2, -1, -w_3, -{w_3}^2\} = \boldsymbol{Z}_6.$$

Proof. Let $\alpha \in z(({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3})$. Note that $\phi_1(\theta), \lambda \in ({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}$. Then by the same argument as in Proposition 1.1.6, we have

$$\alpha\dot{1} = (0, 0, \xi, 0), \quad \xi = 1 \quad \text{or} \quad \xi = -1.$$

(i) Case $\xi = 1$. Since $\alpha\dot{1} = \dot{1}$ and $\alpha\underset{.}{1} = \underset{.}{1}$, we see $\alpha \in {E_6}^C$. Hence

$$\alpha \in z(({E_6}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}).$$

Since $({E_6}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3} = SU(3, \boldsymbol{C}^C) \times SU(3, \boldsymbol{C}^C)$ (Proposition 1.3.1), we have

$$\begin{aligned} z(({E_6}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}) &= z(\varphi_{3,r}(SU(3, \boldsymbol{C}^C) \times SU(3, \boldsymbol{C}^C))) \\ &= \{\varphi_{3,r}(wE, w'E) \,|\, w, w' \in \boldsymbol{C}, w^3 = {w'}^3 = 1\}. \end{aligned}$$

We shall find the the condition $\varphi_{3,r}(wE, w'E) \in z(({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}$. For this end, we shall find the condition that $\varphi_{3,r}(wE, w'E)$ commutes with all elements $\varPhi(\phi, A, B, \nu) \in ({\mathfrak{e}_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}$, that is,

$$\varphi_{3,r}(wE, w'E)\varPhi(\phi, A, B, \nu) = \varPhi(\phi, A, B, \nu)\varphi_{3,r}(wE, w'E).$$

Hence, for all $\phi \in ({\mathfrak{e}_6}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}, A, B \in \mathfrak{J}(3, \boldsymbol{C}^C)$ (Lemma 1.3.2.(1)), we have

$$\begin{cases} \varphi_{3,r}(wE, w'E)\phi\,\varphi_{3,r}(wE, w'E)^{-1} = \phi & \cdots\cdots \ (1) \\ \varphi_{3,r}(wE, w'E)A = A & \cdots\cdots \ (2) \\ \tau\varphi_{3,r}(wE, w'E)\tau B = B & \cdots\cdots \ (3). \end{cases}$$

From the condition (2),

$$\begin{aligned} \varphi_{3,r}(wE, w'E)A &= h(wE, w'E)Ah(wE, w'E)^* \\ &= h(wE, w'E)Ah(\overline{w}'E, \overline{w}E) = (\iota(w\overline{w}') + \overline{\iota}(w'\overline{w}))A, \end{aligned}$$

we have $\iota(w\overline{w}') + \overline{\iota}(w'\overline{w}) = 1$. This relation implies $w\overline{w}' = 1$, that is, $w = w' = 1, w_1$ or ${w_1}^2$. We get the same result from the condition (2). Furthermore, from $\varphi_{3,r}(wE, w'E) = \varphi_{3,r}(w_1E, w_1E) = {w_3}^2$, the condition (1) is clear. Thus we see that an element of $z(({E_7}^C)^{\varepsilon_1,\varepsilon_2,\gamma_3,w_3})$ is either one of the following

$$\varphi_{3,r}(E, E) = 1, \quad \varphi_{3,r}(w_1E, w_1E) = {w_3}^2, \quad \varphi_{3,r}({w_1}^2E, {w_1}^2E) = w_3.$$

(ii) Case $\xi = -1$. By a similar argument as (i), we see that an element of $z(({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3})$ is either one of the following

$$-\varphi_{3,r}(E, E) = -1, \ \ -\varphi_{3,r}(w_1E, w_1E) = -{w_3}^2, \ \ -\varphi_{3,r}({w_1}^2E, {w_1}^2E) = -w_3.$$

Thus we have $z(({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}) \subset \{1, w_3, {w_3}^2, -1, -w_3, -{w_3}^2\}$. The converse inclusion is trivial. Therefore we have $z(({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}) = \{1, w_3, {w_3}^2, -1, -w_3, -{w_3}^2\} = \boldsymbol{Z}_6$. □

Proposition 1.3.6. $({\mathfrak{e}_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3} \cong \mathfrak{su}(6, \boldsymbol{C}^C)$.

Proof. The mappings $\phi_C : \mathfrak{su}(3, \boldsymbol{C}^C) \oplus \mathfrak{su}(3, \boldsymbol{C}^C) \to {\mathfrak{e}_6}^C$ and $h_C : M(3, \boldsymbol{C}^C) \to \mathfrak{J}(3, \boldsymbol{C}^C)$ are defined by

$$\phi_C(A, B)X = h(A, B)X + Xh(A, B)^*, \quad X \in \mathfrak{J}(3, \boldsymbol{C}^C),$$
$$h_C(L) = \frac{L^* + L}{2} + ie_1\frac{L^* - L}{2} = \iota L^* + \bar{\iota}L, \quad \iota = \frac{1 + ie_1}{2},$$

respectively. Now, the mapping $\varphi_* : \mathfrak{su}(6, \boldsymbol{C}^C) \to ({\mathfrak{e}_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}$

$$\varphi_*\left(\begin{pmatrix} B & L \\ -L^* & C \end{pmatrix} + \frac{\nu}{3}\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}\right) = \varPhi(\phi_C(B, C), h_C(L), -\tau h_C(L), -ie_1\nu),$$

where $B, C \in \mathfrak{su}(3, \boldsymbol{C}^C), L \in M(3, \boldsymbol{C}^C), \nu \in e_1C$, gives the isomorphism (see [2] for details). □

Proposition 1.3.7. $({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3} \cong SU(6, \boldsymbol{C}^C)$.

Proof. The group $({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}$ is connected (Proposition 1.3.4). Hence, from Proposition 1.3.6, the group $({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}$ is isomorphic to either one of the following groups

$$SU(6, \boldsymbol{C}^C), \quad SU(6, \boldsymbol{C}^C)/\boldsymbol{Z}_2, \quad SU(6, \boldsymbol{C}^C)/\boldsymbol{Z}_3 \quad \text{or} \quad SU(6, \boldsymbol{C}^C)/\boldsymbol{Z}_6.$$

Since $z(({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}) = \boldsymbol{Z}_6$ (Proposition 1.3.5), it cannot but become that $({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3} \cong SU(6, \boldsymbol{C}^C)$. □

Lemma 1.3.8. *In the Lie algebra ${\mathfrak{e}_7}^C$, the Lie algebra $\mathfrak{su}(3, \boldsymbol{C}^C)$ of the Lie group $SU(3, \boldsymbol{C}^C) = \varphi_{3,l}(SU(3, \boldsymbol{C}^C))$ is given by*

$$\mathfrak{su}(3, \boldsymbol{C}^C) = \Big\{\varPhi(D, 0, 0, 0) \in {\mathfrak{e}_7}^C \;\Big|$$
$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{23} & d_{24} & d_{25} & d_{26} & d_{27} \\ 0 & 0 & -d_{23} & 0 & -d_{25} & d_{24} & -d_{27} & d_{26} \\ 0 & 0 & -d_{24} & d_{25} & 0 & d_{45} & d_{46} & d_{47} \\ 0 & 0 & -d_{25} & -d_{24} & -d_{45} & 0 & -d_{47} & d_{46} \\ 0 & 0 & -d_{26} & d_{27} & -d_{46} & d_{47} & 0 & d_{67} \\ 0 & 0 & -d_{27} & -d_{26} & -d_{47} & -d_{46} & -d_{67} & 0 \end{pmatrix} \in \mathfrak{so}(8, C),$$
$$d_{ij} \in C, d_{23} + d_{45} + d_{67} = 0\Big\}.$$

Now we will prove the main theorem of this section by using the preparations above.

Theorem 4.1.3. $({E_7}^C)_{ed} \cong (SL(3, C) \times SL(6, C))/\boldsymbol{Z}_3, \boldsymbol{Z}_3 = \{(E, E), (\omega E, \omega E), (\omega^2 E, \omega^2 E)\}$*, where* $\omega = e^{2\pi i/3}$.

Proof. We define a mapping $\varphi_{w_3} : SU(3, \boldsymbol{C}^C) \times SU(6, \boldsymbol{C}^C) \to ({E_7}^C)^{w_3} = ({E_7}^C)_{ed}$ by

$$\varphi_{w_3}(A, \beta) = \varphi_{3,l}(A)\beta.$$

φ_{w_3} is well-defined because $\varphi_{3,l}(A) \in ({E_7}^C)^{w_3}$ and $\beta \in SU(6, \boldsymbol{C}^C) \cong ({E_7}^C)^{w_3, \varepsilon_1, \varepsilon_2, \gamma_3}$ (Proposition 1.3.7) $\subset ({E_7}^C)^{w_3}$. Any $\Phi_1 \in \mathfrak{su}(3, \boldsymbol{C}^C)$ commutes with any $\Phi_2 \in \mathfrak{su}(6, \boldsymbol{C}^C)$: $[\Phi_1, \Phi_2] = 0$ (Lemma 1.3.2.(1) and Lemma 1.3.8) and groups $SU(3, \boldsymbol{C}^C)$ and $SU(6, \boldsymbol{C}^C)$ are connected, so $\varphi_{3,l}(A)$ commutes with β : $\varphi_{3,l}(A)\beta = \beta\varphi_{3,l}(A)$. Hence φ_{w_3} is a homomorphism. It is obtained that $\operatorname{Ker}\varphi_{w_3} = \{(E, 1), (w_1 E, \varphi_{3,l}({w_1}^2 E)), ({w_1}^2 E, \varphi_{3,l}(w_1 E))\}$ (in $SU(3, \boldsymbol{C}^C) \times ({E_7}^C)^{w_3, \varepsilon_1, \varepsilon_2, \gamma_3}$) $\cong \{(E, E), (w_1 E, w_1 E), ({w_1}^2 E, {w_1}^2 E)\}$ (in $SU(3, \boldsymbol{C}^C) \times SU(6, \boldsymbol{C}^C)$) $= \boldsymbol{Z}_3$. Moreover $\dim_C(\mathfrak{su}(3, \boldsymbol{C}^C) \oplus \mathfrak{su}(6, \boldsymbol{C}^C)) = 8 + 35 = 43 = 39 + 2 \times 2 = \dim_C(({\mathfrak{e}_7}^C)_{ed})$ (Theorem 4.1) $= \dim_C(({\mathfrak{e}_7}^C)^{w_3})$, hence φ_{w_3} is onto. Thus we have the required isomorphism

$$({E_7}^C)^{w_3} \cong (SU(3, \boldsymbol{C}^C) \times SU(6, \boldsymbol{C}^C))/\boldsymbol{Z}_3,$$
$$\boldsymbol{Z}_3 = \{(E, E), (w_1 E, w_1 E), ({w_1}^2 E, {w_1}^2 E)\}.$$

Since the group $SU(6, \boldsymbol{C}^C)$ is isomorphic to $SL(6, C)$ under the mapping

$$f : SL(6, C) \to SU(6, \boldsymbol{C}^C), \quad f(A) = \iota A + \bar{\iota}\, {}^t A^{-1}, \quad \iota = \frac{1 + ie_1}{2},$$

we have the isomorphism $({E_7}^C)^{w_3} \cong (SL(3, C) \times SL(6, C))/\boldsymbol{Z}_3$, $\boldsymbol{Z}_3 = \{(E, E), (\omega E, \omega E), (\omega^2 E, \omega^2 E)\}$. Note that ωE is transformed to $w_1 E$ under the isomorphism f. $\square$

4.2. Subgroups of type $A_{1(1)} \oplus D_{6(6)}, A_{1(1)} \oplus \boldsymbol{R} \oplus A_{5(5)}$ and $A_{2(2)} \oplus A_{5(5)}$ of $E_{7(7)}$

Since $(\mathfrak{e}_{7(7)})_{ev} = ({\mathfrak{e}_7}^C)_{ev} \cap ({\mathfrak{e}_7}^C)^{\tau\gamma_1} = ({\mathfrak{e}_7}^C)^{\gamma} \cap ({\mathfrak{e}_7}^C)^{\tau\gamma_1}$, $(\mathfrak{e}_{7(7)})_0 = ({\mathfrak{e}_7}^C)_0 \cap ({\mathfrak{e}_7}^C)^{\tau\gamma_1} = ({\mathfrak{e}_7}^C)^{\delta_4} \cap ({\mathfrak{e}_7}^C)^{\tau\gamma_1}$, $(\mathfrak{e}_{7(7)})_{ed} = ({\mathfrak{e}_7}^C)_{ed} \cap ({\mathfrak{e}_7}^C)^{\tau\gamma_1} = ({\mathfrak{e}_7}^C)^{w_3} \cap ({\mathfrak{e}_7}^C)^{\tau\gamma_1}$, we shall determine the structures of groups

$$(E_{7(7)})_{ev} = ({E_7}^C)_{ev} \cap ({E_7}^C)^{\tau\gamma_1} = ({E_7}^C)^{\gamma} \cap ({\mathfrak{e}_7}^C)^{\tau\gamma_1},$$
$$(E_{7(7)})_0 = ({E_7}^C)_0 \cap ({E_7}^C)^{\tau\gamma_1} = ({E_7}^C)^{\delta_4} \cap ({E_7}^C)^{\tau\gamma_1},$$
$$(E_{7(7)})_{ed} = ({E_7}^C)_{ed} \cap ({E_7}^C)^{\tau\gamma_1} = ({E_7}^C)^{w_3} \cap ({E_7}^C)^{\tau\gamma_1}.$$

Theorem 4.2. (1) $(E_{7(7)})_{ev} \cong (SL(2, \boldsymbol{R}) \times spin(6, 6))/\boldsymbol{Z}_2 \times \{1, \gamma\}$, $\boldsymbol{Z}_2 = \{(E,\, 1), (-E,\, \gamma)\}$.

(2) $(E_{7(7)})_0 \cong (SL(2, \boldsymbol{R}) \times \boldsymbol{R}^+ \times SL(6, \boldsymbol{R}))/\boldsymbol{Z}_2 \times \{1, \gamma, \gamma', \gamma\gamma'\}$, $\boldsymbol{Z}_2 = \{(E, 1,\, E), (-E, 1, -E)\}$.

(3) $(E_{7(7)})_{ed} \cong SL(3, \boldsymbol{R}) \times SL(6, \boldsymbol{R})$.

Proof. (1) For $\alpha \in (E_{7(7)})_{ev} \subset ({E_7}^C)_{ev} = ({E_7}^C)^\gamma$, there exist $p \in Sp(1, \boldsymbol{H}^C)$ and $\beta \in Spin(12, C)$ such that $\alpha = \varphi_\gamma(p, \beta) = \varphi_{2,l}(p)\beta$ (Theorem 4.1.1). From the condition $\tau\gamma_1\alpha\gamma_1\tau = \alpha$, that is, $\tau\gamma_1\varphi_{2,l}(p)\beta\gamma_1\tau = \varphi_{2,l}(p)\beta$, we have $\varphi_{2,l}(\tau\gamma_1 p)\tau\gamma_1\beta\gamma_1\tau = \varphi_{2,l}(p)\beta$. Hence

$$\begin{cases} \tau\gamma_1 p = p \\ \tau\gamma_1\beta\gamma_1\tau = \beta \end{cases} \quad \text{or} \quad \begin{cases} \tau\gamma_1 p = -p \\ \tau\gamma_1\beta\gamma_1\tau = \gamma\beta. \end{cases}$$

In the former case, from $\tau\gamma_1 p = p$, we have $p \in Sp(1, \boldsymbol{H}')$. The group $\{\beta \in Spin(12, C) \,|\, \tau\gamma_1\beta\gamma_1\tau = \beta\} = Spin(12, C)^{\tau\gamma_1} = ({E_7}^C)^{\varepsilon_1, \varepsilon_2, \tau\gamma_1}$ acts on the $\boldsymbol{R}$-vector space

$$\begin{aligned} V^{6,6} &= \{P \in \mathfrak{P}^C \,|\, \varepsilon_1 P = -iP, \tau\gamma_1 P = P\} \\ &= \left\{ \left(\begin{pmatrix} 0 & x_3 & \overline{x}_2 \\ \overline{x}_3 & 0 & x_1 \\ x_2 & \overline{x}_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & y_3 & \overline{y}_2 \\ \overline{y}_3 & 0 & y_1 \\ y_2 & \overline{y}_1 & 0 \end{pmatrix}, 0, 0 \right) \middle|\; x_k, y_k \in (\boldsymbol{H}^C e_4)_{\varepsilon_1, \tau\gamma_1} \right\}, \end{aligned}$$

with the norm

$$(P, P)_{\varepsilon_2} = \frac{1}{8}\{P, \varepsilon_2 P\},$$

where $(\boldsymbol{H}^C e_4)_{\varepsilon_1, \tau\gamma_1} = \{\, x \in \mathfrak{C}^C \,|\, x = p(e_4 + ie_5) + q(e_6 - ie_7),\ p, q \in \boldsymbol{R}\}$. Since $Spin(12, C)$ is simply connected, $Spin(12, C)^{\tau\gamma_1}$ is connected, so we can define a homomorphism $\pi : Spin(12, C)^{\tau\gamma_1} \to O(V^{6,6})^0 = O(6,6)^0$ (which is the connected component subgroup of $O(6,6)$) by $\pi(\alpha) = \alpha | V^{6,6}$. It is easily obtained that $\operatorname{Ker}\pi = \{1, -\gamma\}$. Moreover $\dim(\mathfrak{spin}(12, C)^{\tau\gamma_1}) = \dim((\mathfrak{e}_{7(7)})_{ev}) - \dim(\mathfrak{sp}(1, \boldsymbol{H}')) = 39 + 15 \times 2 - 3$ (Theorem 4.1) $= 66 = \dim(\mathfrak{o}(6,6))$, hence π is onto. Therefore $Spin(12, C)^{\tau\gamma_1}$ is denoted by $spin(6,6)$ as a covering group of $O(6,6)^0$. Therefore the group of the former case is $(Sp(1, \boldsymbol{H}') \times spin(6,6))/\boldsymbol{Z}_2$ $(\boldsymbol{Z}_2 = \{(1,1), (-1, \gamma)\}) \cong (SL(2, \boldsymbol{R}) \times spin(6,6))/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{(E,1), (-E, \gamma)\}$. In the latter case, $p = e_1, \beta = \varepsilon_1$ satisfy the conditions, and $\varphi_\gamma(e_1, \varepsilon_1) = \varphi_2(e_1 e_1, 1) = \varphi_2(-1, 1) = \gamma$. Thus we have the required isomorphim $(E_{7(7)})_{ev} \cong (SL(2, \boldsymbol{R}) \times spin(6,6))/\boldsymbol{Z}_2 \times \{1, \gamma\}$, $\boldsymbol{Z}_2 = \{(E,1), (-E, \gamma)\}$.

(2) For $\alpha \in (E_{7(7)})_0 \subset ({E_7}^C)_0 = ({E_7}^C)^{\delta_4}$, there exist $p \in Sp(1, \boldsymbol{H}^C), a \in C^*$ and $\beta \in SL(6, C)$ such that $\alpha = \varphi_{\delta_4}(p, a, \beta) = \varphi_{2,l}(p)\zeta(p)\beta$ (Theorem 4.1.2). Note that $\tau\gamma_1\varphi_{2,l}(p)\gamma_1\tau = \varphi_{2,l}(\tau\gamma_1 p)$ and $\tau\gamma_1\zeta(a)\gamma_1\tau = \tau\gamma_1\varphi_{4,r}(h(a)E)$ $\gamma_1\tau = \varphi_{4,r}(h(\tau\gamma_1 a\gamma_1\tau)E) = \varphi_{4,r}(h(\tau a)E) = \zeta(\tau a)$. Now, from the condition $\tau\gamma_1\alpha\gamma_1\tau = \alpha$, that is, $\tau\gamma_1\varphi_{2,l}(p)\zeta(a)\beta\gamma_1\tau = \varphi_{2,l}\zeta(a)\beta$, we have $\varphi_{2,l}(\tau\gamma_1 p)\zeta(\tau a)$ $\tau\gamma_1\beta\gamma_1\tau = \varphi_{2,l}(p)\zeta(a)\beta$. Hence

$$\text{(i)}\begin{cases}\tau\gamma_1 p = p\\ \zeta(\tau a) = \zeta(a)\\ \tau\gamma_1\beta\gamma_1\tau = \beta,\end{cases}\qquad \text{(ii)}\begin{cases}\tau\gamma_1 p = -p\\ \zeta(\tau a) = \zeta(a)\\ \tau\gamma_1\beta\gamma_1\tau = \gamma\beta,\end{cases}$$

$$\text{(iii)}\begin{cases}\tau\gamma_1 p = p\\ \zeta(\tau a) = \gamma\zeta(a)\\ \tau\gamma_1\beta\gamma_1\tau = \gamma\beta,\end{cases}\qquad \text{(iv)}\begin{cases}\tau\gamma_1 p = -p\\ \zeta(\tau a) = \gamma\zeta(a)\\ \tau\gamma_1\beta\gamma_1\tau = \beta,\end{cases}$$

$$\text{(v)}\begin{cases}\tau\gamma_1 p = p\\ \zeta(\tau a) = \zeta(\pm\omega^k)\zeta(a)\\ \tau\gamma_1\beta\gamma_1\tau = \zeta(\pm\omega^{-k})\beta\\ \qquad\qquad k = 1, 2,\end{cases}\qquad \text{(vi)}\begin{cases}\tau\gamma_1 p = -p\\ \zeta(\tau a) = \zeta(\pm\omega^k)\zeta(a)\\ \tau\gamma_1\beta\gamma_1\tau = \zeta(\pm\omega^{-k})\gamma\beta\\ \qquad\qquad k = 1, 2.\end{cases}$$

Case (i) From $\tau\gamma_1 p = p$, we have $p \in Sp(1, \boldsymbol{H}')$, and from $\zeta(\tau a) = \zeta(a)$, that is, $\tau a = a$, hence we have $a \in \boldsymbol{R}^*$. To determine the structure of the group $\{\beta \in SL(6, C) \,|\, \tau\gamma_1\beta\gamma_1\tau = \beta\} = SL(6, C)^{\tau\gamma_1}$, consider an $\boldsymbol{R}$-vector space

$$V^6 = \{P \in (V^C)^6 \,|\, \tau\gamma_1 P = P\}$$

and let $GL(6, \boldsymbol{R}) = \mathrm{Iso}_R(V^6)$. Then by the correspondence

$$\alpha \in \mathrm{Iso}_C((V^6)^C)^{\tau\gamma_1} \to \alpha \,|\, V^6 \in \mathrm{Iso}_R(V^6),$$

we have the isomorphism $\mathrm{Iso}_C((V^6)^C)^{\tau\gamma_1} \cong \mathrm{Iso}_R(V^6)$, so that $GL(6, C)^{\tau\gamma_1} \cong GL(6, \boldsymbol{R})$ and so we have $SL(6, C)^{\tau\gamma_1} \cong SL(6, \boldsymbol{R})$. Hence for $\alpha \in (E_{7(7)})_0$, there exist $p \in Sp(1, \boldsymbol{H}'), a \in \boldsymbol{R}^*$ and $\beta \in SL(6, \boldsymbol{R})$ such that $\alpha = \varphi_{\delta_4}(p, a, \beta)$. Denote the group of (i) by $G_{\text{(i)}}$. The mapping $\varphi_{\delta_4} : Sp(1, \boldsymbol{H}') \times \boldsymbol{R}^* \times SL(6, \boldsymbol{R}) \to G_{\text{(i)}}$ is a surjective homomorphism and $\mathrm{Ker}\,\varphi_{\delta_4} = \{(1, 1, 1), (-1, -1, 1)\} \times \{(1, 1, 1), (-1, 1, \gamma)\} = \boldsymbol{Z}_2 \times \boldsymbol{Z}_2$. Therefore we have the isomorphism $G_{\text{(i)}} \cong (Sp(1, \boldsymbol{H}') \times \boldsymbol{R}^* \times SL(6, \boldsymbol{R}))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2) \cong (Sp(1, \boldsymbol{H}') \times \boldsymbol{R}^+ \times SL(6, \boldsymbol{R}))/\boldsymbol{Z}_2$ $(\boldsymbol{Z}_2 = \{(1, 1, 1), (-1, 1, \gamma)\}) \cong (SL(2, \boldsymbol{R}) \times \boldsymbol{R}^+ \times SL(6, \boldsymbol{R}))/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{(E, 1, E), (-E, 1, -E)\}$.

Case (ii) $\varphi_{\delta_4}(e_1, 1, \delta_4) = \varphi_2(e_1, 1)\varphi_2(1, -e_1) = \varphi_2(e_1, -e_1) = \gamma\gamma'$.

Case (iii) $\varphi_{\delta_4}(1, i, \delta_4) = \varphi_2(1, -e_1)\varphi_2(1, -e_1) = \varphi_2(1, -1) = \gamma$.

Case (iv) $\varphi_{\delta_4}(e_1, -i, 1) = \varphi_2(e_1, 1)\varphi_2(1, e_1) = \varphi_2(e_1, e_1) = \gamma'$.

Cases (v) and (vi) are impossible. Because there exists no element $a \in C^*$ satisfying the condition $\tau a = (\pm\omega^k)a$ for $k = 1, 2$.

Thus we have the required isomorphism $(E_{7(7)})_0 \cong (SL(2, \boldsymbol{R}) \times \boldsymbol{R}^+ \times SL(6, \boldsymbol{R}))/\boldsymbol{Z}_2 \times \{1, \gamma, \gamma', \gamma\gamma'\}$, $\boldsymbol{Z}_2 = \{(E, 1, 1), (-E, 1, -E)\}$.

(3) For $\alpha \in (E_{7(7)})_{ed} \subset ({E_7}^C)_{ed} = ({E_7}^C)^{w_3}$, there exist $A \in SU(3, \boldsymbol{C}^C)$ and $\beta \in SU(6, \boldsymbol{C}^C)$ such that $\alpha = \varphi_{w_3}(A, \beta) = \varphi_{3,l}(A)\beta$ (Theorem 4.1.3).

From $\tau\gamma_1\alpha\gamma_1\tau = \alpha$, that is, $\tau\gamma_1\varphi_{3,l}(A)\beta\gamma_1\tau = \varphi_{3,l}(A)\beta$, we have $\varphi_{3,l}(\tau\gamma_1 A)\tau\gamma_1 \beta\gamma_1\tau = \varphi_{3,l}(A)\beta$. Hence

$$\text{(i)} \begin{cases} \tau\gamma_1 A = A \\ \tau\gamma_1\beta\gamma_1\tau = \beta, \end{cases} \qquad \text{(ii)} \begin{cases} \tau\gamma_1 A = w_1 A \\ \tau\gamma_1\beta\gamma_1\tau = w_1\beta, \end{cases}$$

$$\text{(iii)} \begin{cases} \tau\gamma_1 A = {w_1}^2 A \\ \tau\gamma_1\beta\gamma_1\tau = {w_1}^2\beta. \end{cases}$$

Case (i) From $\tau\gamma_1 A = A$, we have $A \in SU(3, \boldsymbol{C}')$. To determine the structure of the group $\{\beta \in SU(6, \boldsymbol{C}^C) \,|\, \tau\gamma_1\beta\gamma_1\tau = \beta\} = SU(6, \boldsymbol{C}^C)^{\tau\gamma_1}$, we use the following fact.

$$({\mathfrak{e}_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3,\tau\gamma_1} \cong \mathfrak{su}(6, \boldsymbol{C}').$$

In fact, since the C-Lie isomorphism $\varphi_* : \mathfrak{su}(6, \boldsymbol{C}^C) \to ({\mathfrak{e}_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}$ of Proposition 1.3.6 satisfies

$$\tau\gamma_1\varphi_*\left(\begin{pmatrix} B & L \\ -L^* & C \end{pmatrix} + \frac{\nu}{3}\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}\right)\gamma_1\tau = \varphi_*\left(\tau\gamma_1\left(\begin{pmatrix} B & L \\ -L^* & C \end{pmatrix} + \frac{\nu}{3}\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}\right)\right),$$

φ_* induces a Lie isomorphism ${\varphi'}_* : \mathfrak{su}(6, \boldsymbol{C}') \to ({\mathfrak{e}_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3,\tau\gamma_1}$,

$${\varphi'}_*\left(\begin{pmatrix} B & L \\ -L^* & C \end{pmatrix} + \frac{\nu}{3}\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}\right) = \varPhi(\phi_C(B, C), h_C(L), -\tau h_C(L), -ie_1\nu),$$

$B, C \in \mathfrak{su}(3, \boldsymbol{C}'), L \in M(3, \boldsymbol{C}'), \nu \in e_1(i\boldsymbol{R})$. Therefore we have the required isomorphism $\mathfrak{su}(6, \boldsymbol{C}') \cong ({\mathfrak{e}_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3,\tau\gamma_1}$.

Now, the group $({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3,\tau\gamma_1}$ is connected (because $({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3} \cong SU(6, \boldsymbol{C}^C)$ (Proposition 1.3.7) $\cong SL(6, C)$ is simply connected). Furthermore $({\mathfrak{e}_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3,\tau\gamma_1} = \mathfrak{su}(6, \boldsymbol{C}')$, hence the group $({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3,\tau\gamma_1}$ is isomorphic to either one of the groups

$$SU(6, \boldsymbol{C}') \quad \text{or} \quad SU(6, \boldsymbol{C}')/\boldsymbol{Z}_2.$$

Moreover, since it is easily obtained that $z(({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3,\tau\gamma_1}) \supset \{1, -1\} \cong \boldsymbol{Z}_2$, we have

$$({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3,\tau\gamma_1} \cong SU(6, \boldsymbol{C}').$$

Therefore the group of case (i) is $SU(3, \boldsymbol{C}') \times SL(6, \boldsymbol{C}')$.

Case (ii) $\varphi_{w_3}(w_1 E, w_1 E) = 1$.

Case (iii) $\varphi_{w_3}({w_1}^2 E, {w_1}^2 E) = 1$.

Thus we have the required isomorphism $(E_{7(7)})_{ed} \cong SU(3, \boldsymbol{C}') \times SU(6, \boldsymbol{C}') \cong SL(3, \boldsymbol{R}) \times SL(6, \boldsymbol{R})$. □

4.3. Subgroups of type $A_{1(1)}\oplus D_{6(-6)}, A_{1(1)}\oplus \boldsymbol{R}\oplus A_{5(-7)}$ and $A_{2(2)}\oplus A_{5(-7)}$ of $E_{7(-5)}$

Since γ and γ_1 are conjugate in $({G_2}^C)^\tau \subset ({E_7}^C)^{\tau\lambda} \subset {E_7}^C$, we have

$$E_{7(-5)} = ({E_7}^C)^{\tau\lambda\gamma} \cong ({E_7}^C)^{\tau\lambda\gamma_1}.$$

Theorem 4.3. *The 3-graded decomposition of the Lie algebra* $\mathfrak{e}_{7(-5)} = ({\mathfrak{e}_7}^C)^{\tau\lambda\gamma_1}$ *(or* ${\mathfrak{e}_7}^C$*),*

$$\mathfrak{e}_{7(-5)} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

with respect to $\mathrm{ad}Z, Z = \varPhi(i(-2G_{23} + G_{45} + G_{67}), 0, 0, 0)$*, is given by*

$$\mathfrak{g}_0 = \left\{ \begin{array}{l} iG_{01}, iG_{23}, iG_{45}, G_{46}, iG_{47}, iG_{67} \\ \widetilde{A}_k(1), i\widetilde{A}_k(e_i), i(E_1 - E_2)^\sim, i(E_2 - E_3)^\sim, i\widetilde{F}_k(1), \widetilde{F}_k(e_1) \\ \check{E}_k - \hat{E}_k, i(\check{E}_k + \hat{E}_k), \check{F}_k(1) - \hat{F}_k(1), i(\check{F}_k(1) + \hat{F}_k(1)), \\ \check{F}_k(e_1) + \hat{F}_k(e_1), i(\check{F}_k(e_1) - \hat{F}_k(e_1)), k = 1, 2, 3,\ i\mathbf{1} \end{array} \right\} \quad 39$$

$$\mathfrak{g}_{-1} = \left\{ \begin{array}{l} G_{04} + iG_{05}, G_{06} + iG_{07}, iG_{14} - G_{15}, iG_{16} - G_{17}, \\ (G_{24} - G_{35}) - i(G_{25} + G_{34}), (G_{26} - G_{37}) - i(G_{27} + G_{36}), \\ \widetilde{A}_k(e_4 + ie_5), \widetilde{A}_k(e_6 + ie_7), i\widetilde{F}_k(e_4 + ie_5), i\widetilde{F}_k(e_6 + ie_7), \\ \check{F}_k(e_4 + ie_5) - \hat{F}_k(e_4 + ie_5), i(\check{F}_k(e_4 + ie_5) + \hat{F}_k(e_4 + ie_5)), \\ \check{F}_k(e_6 + ie_7) - \hat{F}_k(e_6 + ie_7), i(\check{F}_k(e_6 + ie_7) + \hat{F}_k(e_6 + ie_7)), \\ \hfill k = 1, 2, 3 \end{array} \right\} \quad 30$$

$$\mathfrak{g}_{-2} = \left\{ \begin{array}{l} G_{02} - iG_{03}, iG_{02} + G_{13}, (G_{46} - G_{57}) + i(G_{47} + G_{56}), \\ \widetilde{A}_k(e_2 - ie_3), \check{F}_k(e_2 - ie_3) - \hat{F}_k(e_2 - ie_3), \\ \widetilde{F}_k(e_2 - ie_3), i(\check{F}_k(e_2 - ie_3) + \hat{F}_k(e_2 - ie_3)), k = 1, 2, 3 \end{array} \right\} \quad 15$$

$$\mathfrak{g}_{-3} = \left\{\ (G_{24} + G_{35}) + i(G_{25} - G_{34}), (G_{26} + G_{37}) + i(G_{27} - G_{36})\ \right\} \ 2$$

$$\mathfrak{g}_1 = \tau(\mathfrak{g}_{-1})\tau,\ \mathfrak{g}_2 = \tau(\mathfrak{g}_{-2})\tau,\ \mathfrak{g}_3 = \tau(\mathfrak{g}_{-3})\tau$$

Proof. We can prove this theorem in a way similar to [6, Theorem 4.13], using [6, Lemma 4.12]. □

Since $(\mathfrak{e}_{7(-5)})_{ev} = ({\mathfrak{e}_7}^C)_{ev} \cap ({\mathfrak{e}_7}^C)^{\tau\lambda\gamma_1} = ({\mathfrak{e}_7}^C)^{\gamma} \cap ({\mathfrak{e}_7}^C)^{\tau\lambda\gamma_1}, (\mathfrak{e}_{7(-5)})_0 = ({\mathfrak{e}_7}^C)_0 \cap ({\mathfrak{e}_7}^C)^{\tau\lambda\gamma_1} = ({\mathfrak{e}_7}^C)^{\delta_4} \cap ({\mathfrak{e}_7}^C)^{\tau\lambda\gamma_1}, (\mathfrak{e}_{7(-5)})_{ed} = ({\mathfrak{e}_7}^C)_{ed} \cap ({\mathfrak{e}_7}^C)^{\tau\lambda\gamma_1} = ({\mathfrak{e}_7}^C)^{w_3} \cap ({\mathfrak{e}_7}^C)^{\tau\lambda\gamma_1}$, we shall determine the structures of groups

$$(E_{7(-5)})_{ev} = ({E_7}^C)_{ev} \cap ({E_7}^C)^{\tau\lambda\gamma_1} = ({E_7}^C)^{\gamma} \cap ({E_7}^C)^{\tau\lambda\gamma_1},$$
$$(E_{7(-5)})_0 = ({E_7}^C)_0 \cap ({E_7}^C)^{\tau\lambda\gamma_1} = ({E_7}^C)^{\delta_4} \cap ({E_7}^C)^{\tau\lambda\gamma_1},$$
$$(E_{7(-5)})_{ed} = ({E_7}^C)_{ed} \cap ({E_7}^C)^{\tau\lambda\gamma_1} = ({E_7}^C)^{w_3} \cap ({E_7}^C)^{\tau\lambda\gamma_1}.$$

Theorem 4.3.1. (1) $(E_{7(-5)})_{ev} \cong (SL(2, \boldsymbol{R}) \times spin^*(12))/\boldsymbol{Z}_2 \times \{1, \gamma\gamma'\}$, $\boldsymbol{Z}_2 = \{(E, 1), (-E, \gamma)\}$.

(2) $(E_{7(-5)})_0 \cong (SL(2,\boldsymbol{R}) \times \boldsymbol{R}^+ \times SU^*(6))/\boldsymbol{Z}_2 \times \{1, \gamma, \gamma', \gamma\gamma'\}$, $\boldsymbol{Z}_2 = \{(E, 1, E), (-E,\ 1, -E)\}$.

(3) $(E_{7(-5)})_{ed} \cong SL(3,\boldsymbol{R}) \times SU^*(6)$.

Proof. (1) For $\alpha \in (E_{7(-5)})_{ev} \subset ({E_7}^C)_{ev} = ({E_7}^C)^\gamma$, there exist $p \in Sp(1, \boldsymbol{H}^C)$ and $\beta \in Spin(12, C)$ such that $\alpha = \varphi_\gamma(p, \beta) = \varphi_{2,l}(p)\beta$ (Theorem 4.1.1). From the condition $\tau\lambda\gamma_1\alpha\gamma_1\lambda^{-1}\tau = \alpha$, that is, $\tau\lambda\gamma_1\varphi_{2,l}(p)\beta\gamma_1\lambda^{-1}\tau = \varphi_{2,l}(p)\beta$, we have $\varphi_{2,l}(\tau\gamma_1 p)\tau\lambda\gamma_1\beta\gamma_1\lambda^{-1}\tau = \varphi_{2,l}(p)\beta$. Hence

$$\begin{cases} \tau\gamma_1 p = p \\ \tau\lambda\gamma_1\beta\gamma_1\lambda^{-1}\tau = \beta \end{cases} \quad \text{or} \quad \begin{cases} \tau\gamma_1 p = -p \\ \tau\lambda\gamma_1\beta\gamma_1\lambda^{-1}\tau = \gamma\beta. \end{cases}$$

In the former case, from $\tau\gamma_1 p = p$, we have $p \in Sp(1, \boldsymbol{H}')$. In order to determine the structure of the group $\{\beta \in Spin(12, C) \,|\, \tau\lambda\gamma_1\beta\gamma_1\lambda^{-1}\tau = \beta\} = Spin(12, C)^{\tau\lambda\gamma_1} = ({E_7}^C)^{\varepsilon_1, \varepsilon_2, \tau\lambda\gamma_1}$, we consider a C-vector space

$$\begin{aligned} (V^C)^{12} &= \{P \in \mathfrak{P}^C \,|\, \varepsilon_1 P = -iP\} \\ &= \left\{ P = \left(\begin{pmatrix} 0 & x_3 & \overline{x}_2 \\ \overline{x}_3 & 0 & x_1 \\ x_2 & \overline{x}_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & y_3 & \overline{y}_2 \\ \overline{y}_3 & 0 & y_1 \\ y_2 & \overline{y}_1 & 0 \end{pmatrix}, 0, 0 \right) \,\middle|\, x_k, y_k \in (\boldsymbol{H}e_4)_{\varepsilon_1} \right\}, \end{aligned}$$

with norms

$$(P, P)_{\varepsilon_2} = \frac{1}{8}\{P, \varepsilon_2 P\}, \quad (P, P)_{\tau\lambda\gamma_1, \varepsilon_2} = \frac{1}{4}\{\tau\lambda\gamma_1 P, \varepsilon_2 P\},$$

where $(\boldsymbol{H}e_4)_{\varepsilon_1} = \{x \in \mathfrak{C}^C \,|\, x = p(e_4 + ie_5) + q(e_6 - ie_7),\ p, q \in C\}$. The explicit forms of $(P, P)_{\varepsilon_2}$ and $(P, P)_{\tau\lambda\gamma_1, \varepsilon_2}$ for $P \in (V^C)^{12}, x_k = p_k(e_4 + ie_5) + q_k(e_6 - ie_7)$, $y_k = s_k(e_4 + ie_5) + t_k(e_6 - ie_7), k = 1, 2, 3$ are given by

$$\begin{aligned} (P, P)_{\varepsilon_2} &= (p_1 t_1 - q_1 s_1) + (p_2 t_2 - q_2 s_2) + (p_3 t_3 - q_3 s_3), \\ (P, P)_{\tau\lambda\gamma_1, \varepsilon_2} &= \frac{1}{2}\big((\tau p_1)q_1 - (\tau q_1)p_1 + (\tau p_2)q_2 - (\tau q_2)p_2 + (\tau p_3)q_3 - (\tau q_3)p_3 \\ &\quad + (\tau s_1)t_1 - (\tau t_1)s_1 + (\tau s_2)t_2 - (\tau t_2)s_2 + (\tau s_3)t_3 - (\tau t_3)s_3\big), \end{aligned}$$

respectively. By the following coordinate transformation ($m_k \in C$)

$$\begin{cases} p_1 = m_1 + im_2, & t_1 = m_1 - im_2, \\ q_1 = m_3 + im_4, & s_1 = -m_3 + im_4, \end{cases}$$

$$\begin{cases} p_2 = m_5 + im_6, & t_2 = m_5 - im_6, \\ q_2 = m_7 + im_8, & s_2 = -m_7 + im_8, \end{cases}$$

$$\begin{cases} p_3 = m_9 + im_{10}, & t_3 = m_9 - im_{10}, \\ q_3 = m_{11} + im_{12}, & s_3 = -m_{11} + im_{12}, \end{cases}$$

we have

$$\begin{aligned}(P,P)_{\varepsilon_2} &= {m_1}^2+{m_2}^2+\cdots+{m_{11}}^2+{m_{12}}^2={}^t\boldsymbol{m}\,\boldsymbol{m},\\ (P,P)_{\tau\lambda\gamma_1,\varepsilon_2} &= (\tau m_1)m_3-(\tau m_3)m_1+(\tau m_2)m_4-(\tau m_4)m_2+\cdots\\ &\quad+(\tau m_9)m_{11}-(\tau m_{11})m_9+(\tau m_{10})m_{12}-(\tau m_{12})m_{10}\\ &= {}^t(\tau\boldsymbol{m})J'\,\boldsymbol{m},\end{aligned}$$

where $\boldsymbol{m} = {}^t(m_1, m_2, \ldots, m_{12})$, $J' = \begin{pmatrix} Q & 0 & 0\\ 0 & Q & 0\\ 0 & 0 & Q\end{pmatrix}$, $Q = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix}$.
This shows that we have an isomorphism

$$\begin{aligned}\{\alpha\in \mathrm{Iso}_C(({V^C})^{12})\,|\,(\alpha P,\alpha P)_{\varepsilon_2}=(P,P)_{\varepsilon_2},(\alpha P,\alpha P)_{\tau\lambda\gamma_1,\varepsilon_2}=(P,P)_{\tau\lambda\gamma_1,\varepsilon_2}\}\\ \cong\{A\in M(12,C)\,|\,{}^tAA=E,J'A=(\tau A)J'\}.\end{aligned}$$

Since

$$\begin{aligned}O^*(12)=O^*(({V^C})^{12})&=\{A\in M(12,C)\,|\,{}^tAA=E,JA=(\tau A)J\}\\ &\cong\{A\in M(12,C)\,|\,{}^tAA=E,J'A=(\tau A)J'\},\end{aligned}$$

we have

$$\begin{aligned}&O^*(12)\cong\\ &\quad\{\alpha\in \mathrm{Iso}_C(({V^C})^{12})\,|\,(\alpha P,\alpha P)_{\varepsilon_2}=(P,P)_{\varepsilon_2},(\alpha P,\alpha P)_{\tau\lambda\gamma_1,\varepsilon_2}=(P,P)_{\tau\lambda\gamma_1,\varepsilon_2}\}.\end{aligned}$$

Now, since the group $Spin(12,C)^{\tau\lambda\gamma_1}$ is connected, we can define a homomorphism $\pi : Spin(12,C)^{\tau\lambda\gamma_1} \to O^*(12)^0$ (which is the connected component subgroup of $O^*(12)$) by $\pi(\alpha) = \alpha|({V^C})^{12}$. $\dim(\mathfrak{spin}(12,C)^{\tau\lambda\gamma_1}) = \dim((\mathfrak{e}_{7(-5)})_{ev}) - \dim(\mathfrak{sp}(1,\boldsymbol{H}')) = 39 + 15\times 2 - 3$ (Theorem 4.3) $= 66 = \dim(\mathfrak{o}^*(12))$ and $\mathrm{Ker}\,\pi = \{1,-\gamma\}$. Hence

$$Spin(12,C)^{\tau\lambda\gamma_1}/\boldsymbol{Z}_2 \cong O^*(12)^0.$$

Thus $Spin(12,C)^{\tau\lambda\gamma_1}$ is denoted by $spin^*(12)$ as a double covering group of $O^*(12)^0$, that is, $Spin(12,C)^{\tau\lambda\gamma_1} \cong spin^*(12)$. Hence the group of the former case is $(Sp(1,\boldsymbol{H}')\times spin^*(12))/\boldsymbol{Z}_2$ ($\boldsymbol{Z}_2 = \{(1,1),(1,-\gamma)\}$) $\cong (SL(2,\boldsymbol{R})\times spin^*(12))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{(E,1),(-E,\gamma)\}$. In the latter case, $p = e_1, \beta = \delta_4$ satisfy these conditions and $\varphi_\gamma(e_1,\delta_4) = \gamma\gamma'$. Therefore $(E_{7(-5)})_{ev} \cong (SL(2,\boldsymbol{R})\times spin^*(12))/\boldsymbol{Z}_2\times\{1,\gamma\gamma'\}$, $\boldsymbol{Z}_2 = \{(E,1),(-E,\gamma)\}$.

(2) For $\alpha\in(E_{7(-5)})_0\subset({E_7}^C)_0 = ({E_7}^C)^{\delta_4}$, there exist $p\in Sp(1,\boldsymbol{H}^C)$, $a\in C^*$ and $\beta\in SL(6,C)$ such that $\alpha = \varphi_{\delta_4}(p,a,\beta) = \varphi_{2,l}(p)\zeta(a)\beta$ (Theorem 4.1.2). From the condition $\tau\lambda\gamma_1\alpha\gamma_1\lambda^{-1}\tau = \alpha$, that is, $\tau\lambda\gamma_1\varphi_{2,l}(p)\zeta(a)\beta\gamma_1\lambda^{-1}\tau$

$= \varphi_{2,l}(p)\zeta(a)\beta$, we have $\varphi_{2,l}(\tau\gamma_1 p)\zeta(\tau a)\tau\lambda\gamma_1\beta\gamma_1\lambda^{-1}\tau = \varphi_{2,l}(p)\zeta(a)\beta$. Hence

$$\text{(i)} \begin{cases} \tau\gamma_1 p = p \\ \zeta(\tau a) = \zeta(a) \\ \tau\gamma_1\lambda\beta\gamma_1\lambda^{-1}\tau = \beta, \end{cases} \qquad \text{(ii)} \begin{cases} \tau\gamma_1 p = -p \\ \zeta(\tau a) = \zeta(a) \\ \tau\gamma_1\lambda\beta\gamma_1\lambda^{-1}\tau = \gamma\beta, \end{cases}$$

$$\text{(iii)} \begin{cases} \tau\gamma_1 p = p \\ \zeta(\tau a) = \gamma\zeta(a) \\ \tau\gamma_1\lambda\beta\lambda^{-1}\gamma_1\tau = \gamma\beta, \end{cases} \qquad \text{(iv)} \begin{cases} \tau\gamma_1 p = -p \\ \zeta(\tau a) = \gamma\zeta(a) \\ \tau\gamma_1\lambda\beta\gamma_1\lambda^{-1}\tau = \beta, \end{cases}$$

$$\text{(v)} \begin{cases} \tau\gamma_1 p = p \\ \zeta(\tau a) = \zeta(\pm\omega^k)\zeta(a) \\ \tau\gamma_1\lambda\beta\gamma_1\lambda^{-1}\tau = \zeta(\pm\omega^{-k})\beta \\ \qquad k = 1, 2, \end{cases} \qquad \text{(vi)} \begin{cases} \tau\gamma_1 p = -p \\ \zeta(\tau a) = \zeta(\pm\omega^k)\zeta(a) \\ \tau\gamma_1\lambda\beta\gamma_1\lambda^{-1}\tau = \zeta(\pm\omega^{-k})\gamma\beta \\ \qquad k = 1, 2. \end{cases}$$

Case (i) From $\tau\gamma_1 p = p$, we have $p \in Sp(1, \boldsymbol{H}')$, and from $\zeta(\tau a) = \zeta(a)$, we have $a \in \boldsymbol{R}^*$. We will determine the structure of the group $\{\beta \in SL(6, C) \,|\, \tau\lambda\gamma_1\beta\gamma_1\lambda^{-1}\tau = \beta\} = SL(6, C)^{\tau\lambda\gamma_1}$. For this end, consider the correspondence

$$(V^C)^6 \ni \left(\begin{pmatrix} 0 & p_3(e_4 + ie_5) & -p_2(e_4 + ie_5) \\ -p_3(e_4 + ie_5) & 0 & p_1(e_4 + ie_5) \\ p_2(e_4 + ie_5) & -p_1(e_4 + ie_5) & 0 \end{pmatrix}, \begin{pmatrix} 0 & s_3(e_4 + ie_5) & -s_2(e_4 + ie_5) \\ -s_3(e_4 + ie_5) & 0 & s_1(e_4 + ie_5) \\ s_2(e_4 + ie_5) & -s_1(e_4 + ie_5) & 0 \end{pmatrix}, 0, 0 \right) \to \begin{pmatrix} p_1 \\ s_1 \\ p_2 \\ s_2 \\ p_3 \\ s_3 \end{pmatrix} \in C^6.$$

Under this correspondence, the actions λ and $\tau\gamma_1$ on $(V^C)^6$ correspond to the actions J and τ on C^6, respectively. Let $B \in M(6, C)$ be the matrix corresponds to $\beta \in SL(6, C)$, then we see that the condition $\lambda\tau\gamma_1\beta\tau\gamma_1\lambda^{-1} = \beta$ corresponds to the condition $J(\tau B)J^{-1} = B$, that is, $JB = (\tau B)J$. Hence we have

$$\beta \in SL(6, C)^{\tau\lambda\gamma_1} \cong SU^*(6) = \{B \in M(6, C) \,|\, JB = (\tau B)J, \det B = 1\}.$$

Thus we see that for $\alpha \in (E_{7(-5)})_0$, there exist $p \in Sp(1, \boldsymbol{H}'), a \in \boldsymbol{R}^*$ and $\beta \in SU^*(6)$ such that $\alpha = \varphi_{\delta_4}(p, a, \beta)$. As similar to Theorem 4.2.(2), the group of (i) is isomorphic to $(Sp(1, \boldsymbol{H}') \times \boldsymbol{R}^* \times SU^*(6))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2)$ $(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2 = \{(1,1,1), (-1,-1,1)\} \times \{(1,1,1), (-1,1,\gamma)\}) \cong (Sp(1, \boldsymbol{H}') \times \boldsymbol{R}^+ \times SU^*(6))/\boldsymbol{Z}_2$ $(\boldsymbol{Z}_2 = \{(1,1,1), (-1,1,\gamma)\}) \cong (Sp(1, \boldsymbol{H}') \times \boldsymbol{R}^+ \times SU^*(6))/\boldsymbol{Z}_2$ $(\boldsymbol{Z}_2 = \{(1,1,1), (-1,1,\gamma)\}) \cong (SL(2, \boldsymbol{R}) \times \boldsymbol{R}^+ \times SU^*(6))/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{(E, 1, E), (-E, 1, -E)\}$.

Case (ii) $\varphi_{\delta_4}(e_1, 1, \delta_4) = \gamma\gamma'$.

Case (iii) $\varphi_{\delta_4}(1, i, \delta_4) = \gamma$.

Case (iv) $\varphi_{\delta_4}(e_1, -i, 1) = \gamma'$.

Cases (v) and (vi) are impossible. Because there exists no $a \in C^*$ satisfying the condition $\tau a = (\pm\omega^k)a$ for $k = 1, 2$.

Thus we have the required isomorphism $(E_{7(-5)})_0 \cong ((SL(2, \boldsymbol{R}) \times \boldsymbol{R}^+ \times SU^*(6)) /\boldsymbol{Z}_2 \times \{1, \gamma, \gamma', \gamma\gamma'\}$, $\boldsymbol{Z}_2 = \{(E, 1, E), (-E, 1, -E)\}$.

(3) For $\alpha \in (E_{7(-5)})_{ed} \subset ({E_7}^C)_{ed} = ({E_7}^C)^{w_3}$, there exist $A \in SU(3, \boldsymbol{C}^C)$ and $\beta \in SU(6, \boldsymbol{C}^C)$ such that $\alpha = \varphi_{w_3}(A, \beta) = \varphi_{3,l}(A)\beta$ (Theorem 4.1.3). From the condition $\tau\lambda\gamma_1\alpha\gamma_1\lambda^{-1}\tau = \alpha$, that is, $\tau\lambda\gamma_1\varphi_{3,l}(A)\beta\gamma_1\lambda^{-1}\tau = \varphi_{3,l}(A)\beta$, we have $\varphi_{3,l}(\tau\gamma_1 A)\tau\lambda\gamma_1\beta\gamma_1\lambda^{-1}\tau = \varphi_{3,l}(A)\beta$. Hence

$$\text{(i)} \begin{cases} \tau\gamma_1 A = A \\ \tau\lambda\gamma_1\beta\gamma_1\lambda^{-1}\tau = \beta, \end{cases} \qquad \text{(ii)} \begin{cases} \tau\gamma_1 A = w_1 A \\ \tau\lambda\gamma_1\beta\gamma_1\lambda^{-1}\tau = w_1\beta, \end{cases}$$

$$\text{(iii)} \begin{cases} \tau\gamma_1 A = {w_1}^2 A \\ \tau\lambda\gamma_1\beta\gamma_1\lambda^{-1}\tau = {w_1}^2\beta. \end{cases}$$

Case (i) From $\tau\gamma_1 A = A$, we have $A \in SU(3, \boldsymbol{C}')$. From the proof of (2) above, we see $\beta \in SU^*(6)$. Therefore the group of (i) is $SU(3, \boldsymbol{C}') \times SU^*(6) \cong SL(3, \boldsymbol{R}) \times SU^*(6)$.

Case (ii) $\varphi_{w_3}(w_1 E, w_1 E) = 1$.

Case (iii) $\varphi_{w_3}({w_1}^2 E, {w_1}^2 E) = 1$.

Thus we have the required isomorphism $(E_{7(-5)})_{ed} \cong SU(3, \boldsymbol{C}') \times SU^*(6) \cong SL(3, \boldsymbol{R}) \times SU^*(6)$. □

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