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A 2-knot connected-sum and 4-dimensional diffeomorphism

By

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Abstract

We consider a sufficient condition that a knot self-concordance surgery introduced in [3] gives rise to the diffeomorphic manifolds. The main theorem in the present paper is a certain generalization of Akbulut's result in [3] and author's result in [6].

1. Introduction

Let C be a cusp neighborhood, which is an elliptic surface over D^2 having only one singular fiber as a cusp. R. Fintushel and R. Stern in [6] introduced a knot surgery of C to construct infinitely many 4-manifolds which are homeomorphic but non-diffeomorphic to C [4]. Let K be a classical knot in S^3. Their surgeries are defined by removing a neighborhood of a general fiber T from C and gluing $(S^3 - \nu(K)) \times S^1$ to the boundary via a certain gluing map. Through this paper, $\nu(M)$ or $\nu(f)$ stands for an open tubular neighborhood either of a submanifold M or a submanifold embedded by a map f. We denote the resulting manifold by C_K. For a 4-manifold X that contains C, we construct X_K from X by replacing C with C_K. We call this operation $X \to X_K$ *Fintushel-Stern knot surgery.* They showed that for some 4-manifold X, for example K3 surface and so on, X_K is non-diffeomorphic to X if the Alexander polynomial Δ_K of K is not 1, by computing Seiberg-Witten invariants of X and X_K.

Let $\tilde{\varphi}$ be any knot self-concordance $S^1 \times I \hookrightarrow S^3 \times I$, which is defined by a knot concordance such that for $i \in \{0,1\} = \partial I$ each image $\tilde{\varphi}(S^1 \times \{i\})$ represents the same knot K. We can construct a torus-embedding $\varphi : S^1 \times S^1 \hookrightarrow S^3 \times S^1$ from $\tilde{\varphi}$ by identifying $S^3 \times \{0\}$ with $S^3 \times \{1\}$. We define $\mathcal{T}$ as the set of all isotopy classes of torus-embeddings in $S^3 \times S^1$ such that the embedding is obtained by using such an identification for a knot K. For a 4-manifold X that contains C, removing an open neighborhood of a general fiber T from $C \subset X$ and gluing $S^3 \times S^1 - \nu(\varphi)$ by a self-diffeomorphism f of the 3-dimensional torus T^3 for any $\varphi \in \mathcal{T}$ we construct a 4-manifold.

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It is known that diffeomorphisms over T^3 are classified by the induced homomorphisms on the fundamental group as elements in $SL(3,\mathbb{Z})$. The generators of $\pi_1(\partial[S^3 \times S^1 - \nu(\varphi)])$ are induced from a longitude l, a meridian m of K, and a loop n induced from $\varphi(\mathrm{pt} \times S^1)$. We identify $\nu(T)$ with $T \times D^2$, where D^2 is a 2-disk. The vanishing cycles $l_1,\ l_2$ around the cusp fiber generate $\pi_1(T)$. We fix a gluing map $f : \partial[S^3 \times S^1 - \nu(\varphi)] \to \partial(T \times D^2)$ that satisfies the following:

$$\begin{aligned} f(l) &= \mathrm{pt} \times \partial D \\ f(m) &= \quad l_1 \\ f(n) &= \quad l_2. \end{aligned}$$

We denote the resulting manifold $[X - \nu(T)] \cup_f [S^3 \times S^1 - \nu(\varphi)]$ by X_φ. After Selman Akbulut's work [3] we call this operation $X \to X_\varphi$ *Akbulut surgery for* φ. Fintushel-Stern knot surgery is the case where $\varphi(S^1 \times t) = K$ for any t, that is, $X_\varphi = X_K$.

Akbulut surgery might give rise to smooth structures other than the Fintushel and Stern's example, but for the present the author do not know whether it occurs. The purpose of this paper is to give a sufficient condition for certain two Akbulut surgeries to give the same 4-manifold. Here we say that a 2-knot $S \subset S^4$ is *a ribbon* 2*-knot in a double position* if (S^4, S) is a double of a ribbon disk (D^4, D), i.e., which is represented by $(S^4, D \cup D^{mr})$, where D^{mr} is a mirror image of D. The main theorem in this paper is as follows:

Theorem 1.1. *For any $\varphi_1 \in \mathcal{T}$, we denote the image of φ_1 by T_1. We consider a connected-sum $T_2 := T_1 \# S$ or $T_1 \# \overline{S}$, where S is a ribbon 2-knot in a double position. We denote by $\varphi_2 \in \mathcal{T}$ an embedding whose image is T_2. Then for any 4-manifold X that contains C, X_{φ_2} is diffeomorphic to X_{φ_1} for both the connected-sums.*

Akbulut's Theorem 2.3 of [3] is the case where T_1 is a torus obtained from a trivial concordance of an unknot and S is spun trefoil 2-knot. The main theorem in [6] is the case where T_1 is the same torus as Akbulut's case and S is any spun 2-knot.

Remark 1.1. In the 3-dimensional case $(S^2 \times S^1, S_0) \# (S^3, K)$ is always isotopic to $(S^2 \times S^1, S_0)$ for any classical knot K where $S_0 = \mathrm{pt} \times S^1$. On the other hand in our case $(S^3 \times S^1, T_1) \neq (S^3 \times S^1, T_2)$ in general.

In Section 2 we will describe an exterior of any ribbon 2-knot in a double position by Kirby calculus, and in Section 3 prove the main theorem. We will also prove more generalized theorem by means of the main theorem's proof.

2. The Kirby diagram of the exterior of any ribbon 2-knot in a double position

A method to describe any surface exterior in any 4-manifold by a Kirby diagram have been studied in Chapter 6 in [5]. By using this method, *level* *picture description*, we describe the exterior of any ribbon 2-knot in a double position concretely. We describe the 0-framed 2-handle corresponding to a 1-handle of 2-knot by an arc as from Fig. 3 to Fig. 4. The way how the 2-handle winds around two 1-handles is expressed by two arrows as in Fig. 5. We call such an arc with two arrows *a simplified 2-handle*. When the 2-handle has k half twists as the right in Fig. 6, we put the integer k to a side of the simplified 2-handle as the left in Fig. 6. We call this integer *the twisting* of the simplified 2-handle. For example the level picture Fig. 1 corresponds to the diagram Fig. 4.

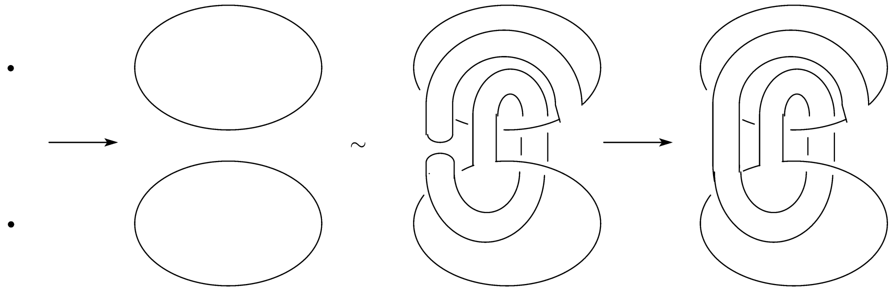

Figure 1. Level picture.

Any ribbon 2-knot in a double position has a symmetric level picture, in fact, the upper half is the copy of the lower half with the level upside down. In the lower half part of the 2-knot, there is no 2-handle of the 2-knot. Thus, in a handle decomposition of the exterior of such a 2-knot 1-handles $\{\alpha_i\}_{i\in I}$ and 3-handles $\{\overline{\alpha}_i\}_{i\in I}$ appear in pairs and 2-handles appear in pairs of simplified 2-handles $\{\beta_j\}_{j\in J}$ and its meridian $\{\overline{\beta}_j\}_{j\in J}$ with 0-framing. Such pairs are derived from gluing $D^4 - \nu(D) = 0\text{-handle} \cup_i \alpha_i \cup_j \beta_j$ and the symmetric copy $\cup_j \overline{\beta}_j \cup_i \overline{\alpha}_i \cup 4\text{-handle}$. Each $\overline{\alpha}_i$ (or $\overline{\beta}_j$) is called *a dual 3-handle* (or *dual 2-handle*) *to the handle* α_i (or β_j respectively).

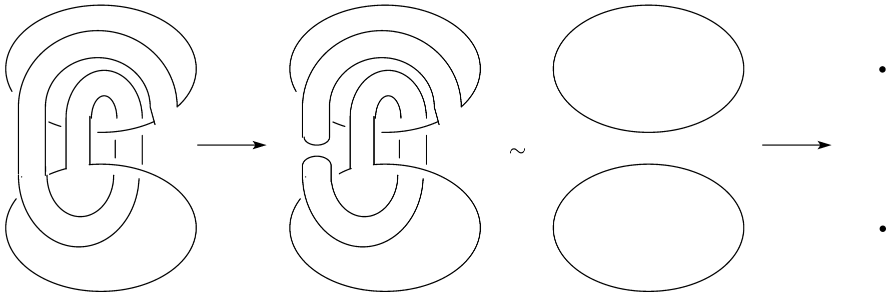

Figure 2. Double level picture.

For example the combination of Fig. 1 and Fig. 2 is a level picture of a

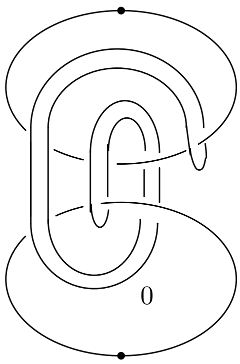

Figure 3.

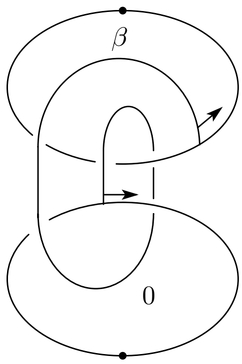

Figure 4.

ribbon 2-knot in a double position. The Kirby diagram of the surface exterior corresponding to this combined level picture is Fig. 7, where two dual 3-handles and one 4-handle are abbreviated.

The Kirby diagram of the exterior of any ribbon 2-knot in a double position can be characterized as follows.

Proposition 2.1. *The twistings of all simplified* 2*-handles in the Kirby diagram of the exterior of any ribbon* 2*-knot in a double position depend only on parities of the twistings. Namely the Kirby diagram of such a ribbon* 2*-knot exterior can be described by using* 1*-handles, simplified* 2*-handles with arrows, their dual* 2*-handles, and dual* 3*-handles.*

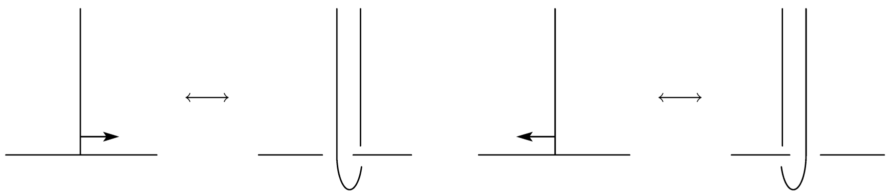

Figure 5. The definition of arrows.

Proof. Since any ribbon 2-knot in a double position has a symmetric level picture, the exterior of the 2-knot can be described by using several 1-handles, simplified 2-handles, and the dual 2- and 3-handles according to the level picture. Sliding any simplified 2-handle, after changing it back to the ordinary 2-handle, over the corresponding dual 2-handle as in Fig. 8, we can change the twisting by ± 2. By applying this operation to all simplified 2-handles finitely

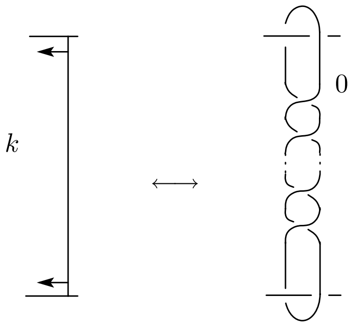

Figure 6. Twisting.

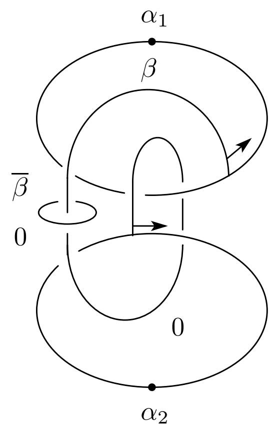

Figure 7.

many times we can reduce their twistings to 0 or 1. If the twisting is 1 it can be changed into 0 by suitably turning one of the two arrows associated with the simplified 2-handle to the counter side and sliding if necessary. Hence the twistings of simplified 2-handles can be represented as the direction of the arrows. □

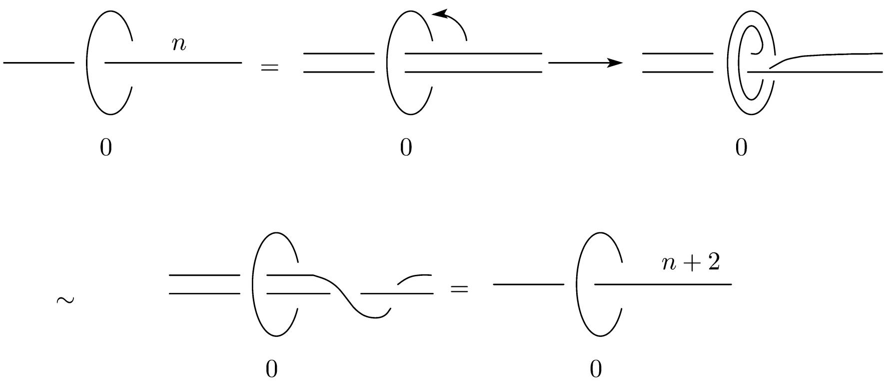

Figure 8.

From now on, in the Kirby diagrams of such ribbon 2-knots, we adjust the twistings of the simplified 2-handles into 0, and we omit writing 0 near the components.

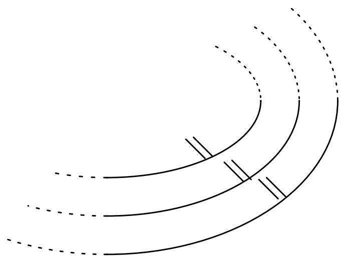

Figure 9. 3-handles

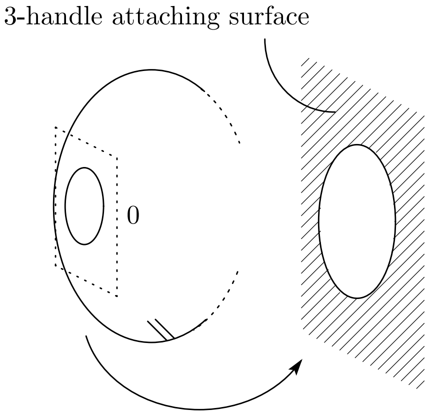

Figure 10. Canceling 2-, 3-handle.

3. Proofs of the main theorem and a generalized theorem derived from it

In this paper we use Kirby diagrams and Kirby calculus that contain 3-handles or 4-handles. We describe a 3-handle as a surface attaching the 3-handle in the boundary of 0-handle and in ordinary calculus only with 1- and 2-handles. In order to distinguish 3-handle from 1-handles and 2-handles we describe two back slashes per a 3-handle, see Fig. 9. The two back slashes are described on the boundary of the surface projected on the plane. For example Fig. 10 is a diagram where a 3-handle runs through a 0-framed 2-handle once, thus the pair of two handles can be canceled. The right picture in Fig. 10 is the closeup illustration. Any 4-handle is described by drawing a hatched region on a place attaching the 3-sphere on the boundary of 0-handle.

We consider the connected-sum of two separated surfaces S_1 and S_2 embedded in a 4-manifold X. Here the separateness means that there exist a 4-disk D^4 in X such that either $S_1 \subset D^4$ and $S_2 \cap D^4 = \emptyset$ or $S_2 \subset D^4$ and $S_1 \cap D^4 = \emptyset$ is satisfied. We describe $X - \nu(S_1) - \nu(S_2)$ from level picture description. Since S_1, and S_2 are connected, we can move 1-handles and 3-handles as the top picture in Fig. 11 by an isotopy of the Kirby diagram. The choices of such two 1- and 3-handles are arbitrary as long as these handles are attached by the level pictures. As a result connected-sum operation of embedded two surfaces can be described as follows.

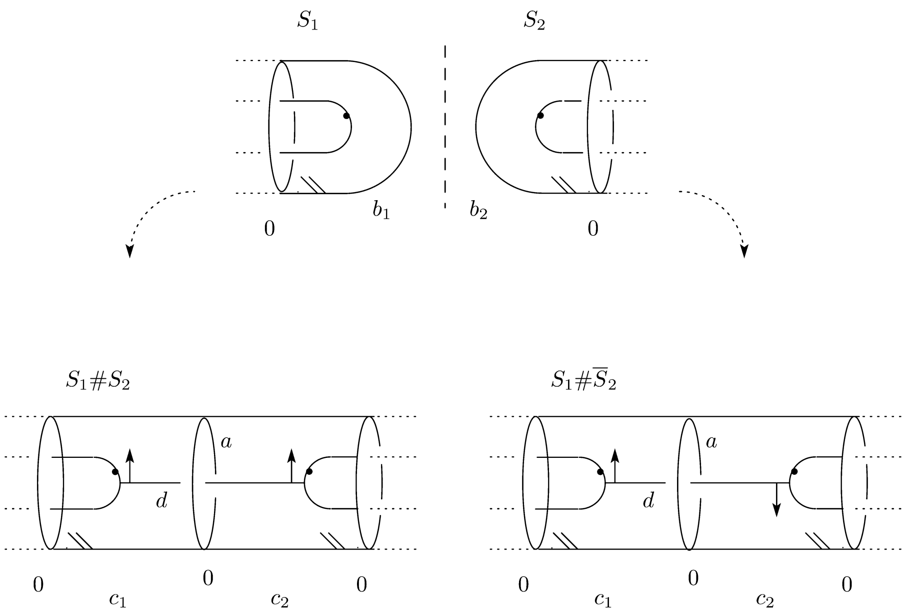

Figure 11. The construction of exteriors of connected-sums $S_1 \# S_2$ and $S_1 \# \overline{S}_2$.

Lemma 3.1. *Kirby diagram of the exterior of a connected-sum of two separated surfaces S_1 and S_2 in a 4-manifolds X is obtained as follows. First, one move the Kirby diagram of $X - \nu(S_1) - \nu(S_2)$ in the form of Fig.* 11. *Secondly one attaches* 0*-framed* 2*-handle a, whose attaching circle is unknotted and separated from other attaching spheres. Next, one exchanges each* 3*-handle b_1 and b_2 in S_1 and S_2 for new* 3*-handles c_1 and c_2 that run through a once. Finally one attach a simplified* 2*-handle d according to the orientations of connected-sum operation. Namely that connected-sum operation is as in Fig.* 11.

Proof. From the level picture description of the connected-sum of S_1 and S_2 the attachment of a, d and reattachment of c_1, c_2 in place of b_1, b_2 is obvious. The simplified 2-handle d is connected to each 1-handle in S_i $(i = 1, 2)$. The attachment of d has an ambiguity of the twisting and arrows. Since a is the dual 2-handle of d, the ways to attach d are two kinds by the operation of Fig. 8. Therefore if the bottom-left of Fig. 11 is $S_1 \# S_2$, then the bottom-right is $S_1 \# \overline{S}_2$, where $\overline{S}_2$ is S_2 with the reversed-orientation. □

Here we prove the main theorem by using Lemma 3.1.

Proof of the main theorem. In the connected-sum operation there exist two choices as above up to the move in Fig. 8, at the direction of the arrow of the simplified 2-handle. We choose one of connected-sum of T_1 and S arbitrarily.

Applying the above construction to T_1 in C and S in S^4, the diagram of C_{φ_2} is obtained by attaching two -1-framed 2-handles along the positions m, n to the exterior. We can move the 2-handle γ which is attached along m, to a meridian of any 1-handle of the S-part by finitely many handle slides as in Fig. 12. Here S-part indicates all 1-, 2-, and 3-handles induced from the exterior of S according to the level picture of S. Then we slide a simplified 2-handle in the S-part to γ as in Fig. 13 (a) and slide γ to the dual 2-handle as in Fig. 13 (b) to obtain Fig. 13 (c). We also reduce the twistings of the simplified 2-handles by the move in Fig. 8. By these slides we can change any crossing between the simplified 2-handles and the 1-handles in the S-part. In order to remove all the crossings we perform the procedures finitely many times. Thus we can change the S-part to a linearly-arranged diagram as the upper part in Fig. 14. Note that we do not have to consider all arrows associated to any simplified 2-handle since the S-part can be rotated in the direction of the arrow in Fig. 14. The dual 2-handles in the S-part are canceled with the dual 3-handles, and the simplified 2-handles are canceled with 1-handles. Therefore the resulting diagram is given as the lower part in Fig. 14. This diagram is that of C_{φ_1}. By carrying out these operations in X, $X_{\varphi_1} = X_{\varphi_2}$ is obtained. □

We prove the same theorem as the main theorem in slightly general condition in the rest of this section.

Definition 3.1. Let $S \subset X$ be an embedded sphere in any 4-manifold X and φ a self-diffeomorphism of $S^2 \times S^1$. We denote $[X - \nu(S)] \cup_\varphi S^2 \times D^2$ by $X_{S,\varphi}$. We call $X_{S,\varphi}$ a Gluck-surgery of X with respect to (S, φ).

In particular if φ is non-isotopic to the identity, then the regluing is non-trivial, thus this construction is called a non-trivial Gluck-surgery.

Some of non-trivial Gluck-surgeries can construct exotic 4-manifolds, for example as in [1]. The non-trivial Gluck-surgeries of S^4 give rise to homotopy S^4. But for the present the author does not know whether the resulting manifold is exotic or not in general.

Let S an embedded sphere in S^4 (which is not necessarily in a double position). Gluck-surgery along S is to attach a 2-handle γ along a meridian of any 1-handle of $S^4 - \nu(S)$ and a 4-handle. Note that γ is isotopic to a meridian of any 1-handle of $S^4 - \nu(S)$ by Fig. 12 and that the framing of γ depends only on the parity because of sliding γ over all 1-handles (this slidings is also written in [2]). The framing of trivial Gluck-surgery is 0, since the framings

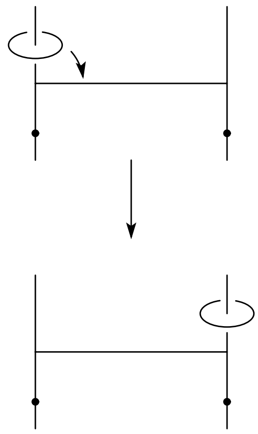

Figure 12.

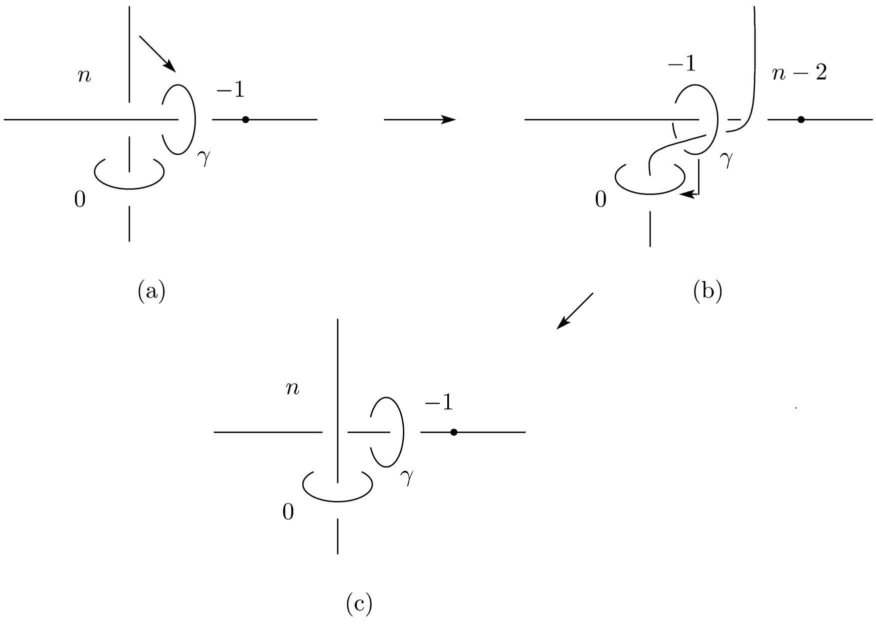

Figure 13.

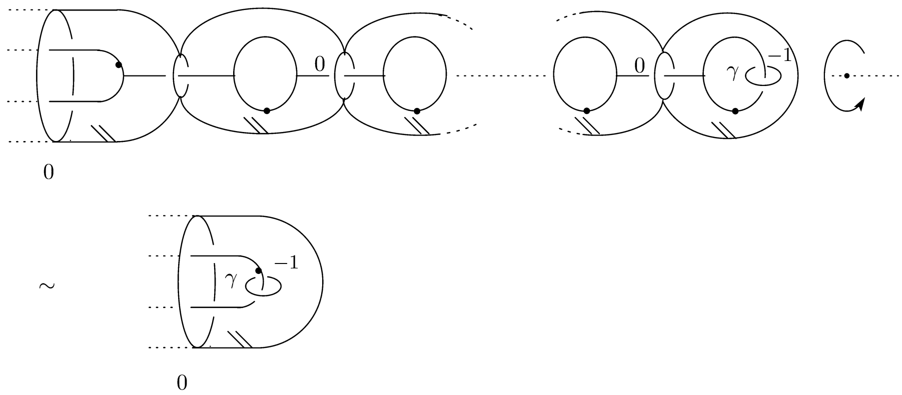

Figure 14.

of 2-handles, including the original 2-handles of simplified 2-handles, are all 0. We summarize it as follows:

Lemma 3.2. *Suppose that a Kirby diagram of the exterior of an embedded 2-sphere S in S^4 is obtained from the level picture of S by the Kirby's method. The non-trivial Gluck-surgery along S is to attach a -1-framed 2-handle along the meridian of any 1-handle and a 4-handle.*

Here we generalize the main theorem as follows.

Theorem 3.1. *Let S be a 2-knot in S^4 whose non-trivial Gluck-surgery is diffeomorphic to the standard 4-sphere. For any $\varphi_1 \in \mathcal{T}$ we define φ_2 as an embedding corresponding to a connected-sum of the image of φ_1 and S. Then for any 4-manifold X that contains C, X_{φ_2} is diffeomorphic to X_{φ_1}.*

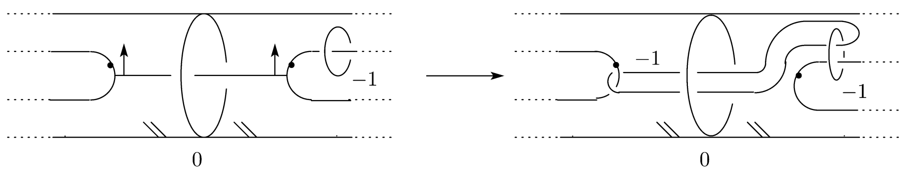

Figure 15.

Proof. C_{φ_1} is constructed by attaching two -1-framed 2-handles along the positions m, n on $\partial[S^3 \times S^1 - \nu(\varphi_2)]$. The -1-framed 2-handle γ attached along m can be moved to a meridian of a 1-handle of the S-part as in the proof of Theorem 1.1. We can describe the -1-framed 2-handle as the right picture in Fig. 15. Thus we slide the simplified 2-handle over the -1-framed 2-handle as in

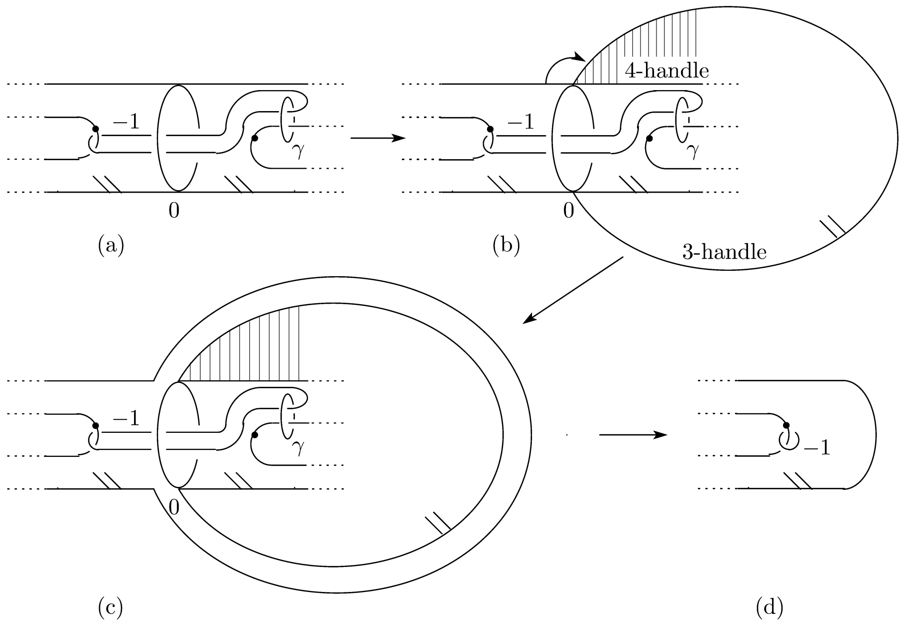

Figure 16.

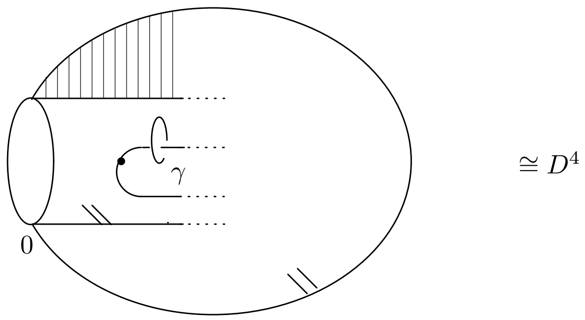

Figure 17.

Fig. 15. We attach the pair of canceling 3-, 4-handle as in Fig. 16 (b), and slide the left 3-handle to the right 3-handle to obtain Fig. 16 (c). On the other hand the manifold in the left picture Fig. 17 contained in the diagram of Fig. 16 (c) is diffeomorphic to $(S^4 - \nu(S)) \cup$ 2-handle. From the assumption, this manifold is diffeomorphic to D^4. Exchanging this part for the empty diagram (it is also D^4) without changing the other parts of the diagram, we obtain the diagram in Fig. 16 (d). Since the diffeomorphism of S^3 can be extended to D^4 uniquely, the manifolds which represent (c) and (d) are diffeomorphic. Therefore C_{φ_2} is diffeomorphic to C_{φ_1}, as a consequence X_{φ_2} is diffeomorphic to X_{φ_1}. □

From Theorem 3.1, if 4-dimensional smooth Poincaré conjecture holds, Theorem 1.1 is true in the case where S is any 2-knot.

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References

[1] S. Akbulut, *Constructing a fake 4-manifold by Gluck construction to a standard 4-manifold*, Topology **27**-2 (1988), 239–243.

[2] ———, *Scharlemann's manifold is standard*, Ann. Math. **149** (1999), 497–510.

[3] ———, *Variations on Fintushel-Stern knot surgery on 4-manifolds*, Turkish J. Math. **26**-1 (2002), 81–92.

[4] R. Fintushel and R. Stern, *Knots, links, and 4-manifolds*, Invent. Math. **134**-2 (1998), 363–400.

[5] R. Gompf and A. Stipsicz, *4-manifolds and Kirby calculus*, Grad. Stud. Math. **20** (1999), American Mathematical Society.

[6] M. Tange, *On the diffeomorphisms for Akbulut's knot concordance surgery*, J. Knot Theory Ramifications **14**-5 (2005), 539–563.