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On uniqueness of graphs with constant mean curvature*

By

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Abstract

A result due to Serrin assures that a graph with constant mean curvature $H \neq 0$ in Euclidean space $\mathbb{R}^3$ cannot keep away a distance $1/|H|$ from its boundary. When the distance is exactly $1/|H|$, then the surface is a hemisphere. Following ideas due to Meeks, in this note we treat the aspect of the equality in the Serrin's estimate as well as generalizations in other situations and ambient spaces.

1. Mise in scène

Let Ω be a bounded domain of the xy-plane $\mathbb{R}^2$. Consider a non-parametric solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of the equation of constant mean curvature in Euclidean space $\mathbb{R}^3$:

$$(1+u_y^2)u_{xx} - 2u_xu_yu_{xy} + (1+u_x^2)u_{yy} = 2H(1+u_x^2+u_y^2)^{3/2} \tag{1.1}$$

where H is a nonzero real number. The graph S of u, $z = u(x,y)$, is a surface of $\mathbb{R}^3$ with mean curvature H with respect to the orientation

$$N = \frac{(-u_x, -u_y, 1)}{\sqrt{1+u_x^2+u_y^2}}. \tag{1.2}$$

Serrin proved in [14, Th. 4] the following height estimate of S: assume that $H > 0$. Set

$$m = \min_{\partial\Omega} u \qquad M = \max_{\partial\Omega} u.$$

Then

$$m - \frac{1}{H} \leq u(x,y) < M \qquad (x,y) \in \Omega. \tag{1.3}$$

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Equality is attained on the left if and only if u describes a hemisphere of radius $1/H$. The proof involves a comparison between the parallel surface to S at a distance $1/H$ with a hemisphere of radius $1/H$ at the point of minimum ordinate, together a Taylor expansion of both surfaces at that point.

Remark 1. The prescribed sign of H is not a restriction in (1.3). In the case that $H < 0$, we obtain the analogous inequality:

$$m < u \leq M - \frac{1}{H}.$$

Remark 2. As it was pointed out by Serrin in [14], if $H \to 0$, the left hand inequality in (1.3) goes to $-\infty$. However the maximum principle for the minimal surface equation gives $m \leq u \leq M$ in Ω.

Meeks showed the left inequality in (1.3) using the maximum principle for superharmonic functions for an appropriate function that involves information of the geometry of the graph [12]. The same technique has been used for a number of authors in other situations and ambient spaces. We refer to the reader to the recent book [4] as an introduction in the theory of surfaces of constant mean curvature.

However, the aspect of equality has not been clearly treated in the literature. This note aims to consider the case of the equality in each setting, as well as, to generalize the estimates when the boundary values are arbitrary. Much of the results obtained in this work can be generalized in the n-dimensional case. According to [12], the steps to follow in the proof are:

1. The fact that S is a graph means that a certain function $\psi = \psi(N)$, depending on the Gauss map N, has sign on S. Let η be the height function of S with respect to the xy-domain.

2. Let us define an appropriate function φ on the surface S that involves the functions ψ and η. The constancy of the mean curvature implies that φ is superharmonic, that is, $\Delta\varphi \leq 0$.

3. The maximum principle for φ leads to $\varphi \geq \min_{\partial S} \varphi$, which gives the optimal estimate of η.

4. In the case that the graph attains the equality in this estimate, the function φ is constant and this means that S is umbilical.

Let us recall the proof of Meeks. Let $S = \{(x, y, u(x, y)); (x, y) \in \Omega\}$ be the graph of u and let us consider the inclusion $\mathbf{x} : S \to \mathbb{R}^3$ with the orientation given by N. Assume that the mean curvature H is a positive constant. Set $\vec{a} = (0, 0, 1)$. By (1.2), the function $\psi = \langle N, \vec{a}\rangle$ defined on S is positive. On the other hand, the height of S is measured by the function $\eta = \langle \mathbf{x}, \vec{a}\rangle$, with $\eta(p) = u(\pi(p))$, where π is the projection $\pi(x, y, z) = (x, y)$. Denote Δ the Laplacian-Beltrami operator on S induced by $\mathbf{x}$. Because the mean curvature H is constant, we have the known identity:

$$\Delta\langle N, \vec{a}\rangle = -|\sigma|^2\langle N, \vec{a}\rangle, \tag{1.4}$$

where σ is the second fundamental form of the immersion. Recall that $|\sigma|^2 = 4H^2 - 2K \geq 2H^2$, where K is the Gaussian curvature of S and that the equality holds if and only if S is an umbilical surface. On the other hand, the height function $\langle \mathbf{x}, \vec{a}\rangle$ satisfies (independently of the constancy of H)

$$\Delta\langle \mathbf{x}, \vec{a}\rangle = 2H\langle N, \vec{a}\rangle \tag{1.5}$$

Let us define the function

$$\varphi = H\langle \mathbf{x}, \vec{a}\rangle + \langle N, \vec{a}\rangle.$$

By combining (1.4) and (1.5), we have

$$\Delta\varphi = (2H^2 - |\sigma|^2)\langle N, \vec{a}\rangle \leq 0 \tag{1.6}$$

and thus φ is superharmonic. By the maximum principle, the minimum of φ is attained at some boundary point. Since $0 < \langle N, \vec{a}\rangle \leq 1$, we have

$$H\langle \mathbf{x}, \vec{a}\rangle + 1 \geq \varphi \geq H \min_{\partial S}\langle \mathbf{x}, \vec{a}\rangle = Hm, \tag{1.7}$$

and so,

$$\langle \mathbf{x}, \vec{a}\rangle \geq m - \frac{1}{H},$$

obtaining the left hand inequality of (1.3).

The right hand follows from (1.5): since $\Delta\langle \mathbf{x}, \vec{a}\rangle > 0$, the strong maximum principle leads to $\langle \mathbf{x}, \vec{a}\rangle < \max_{\partial S}\langle \mathbf{x}, \vec{a}\rangle = M$.

Now, we study the equality in the left hand inequality (1.3). In such case, for some point $p \in S$, $m - 1/H = \langle p, \vec{a}\rangle$. In particular, $p \notin \partial S$: on the contrary, $\langle p, \vec{a}\rangle \geq m$. As conclusion,

$$\varphi(p) = Hm - 1 + \langle N(p), \vec{a}\rangle \leq Hm.$$

From (1.7), we deduce that $\varphi(p) = Hm$. This implies that at the (interior) point p, φ attains its minimum and, hence, φ is a constant function. The inequality (1.6) is now an identity, and then, $2H^2 = |\sigma|^2$ on S, that is, S is an umbilical surface. Since S is a graph, S must be included in a hemisphere of radius $1/H$. As in a hemisphere, $M - u \leq 1/H$, at the point p, we have

$$m - \frac{1}{H} = u(p) \geq M - \frac{1}{H},$$

that is, $m = M$. This means that ∂S is a circle and that S is a hemisphere of radius $1/H$. This concludes the proof of the equality in (1.3).

2. Radial graphs with constant mean curvature in Euclidean space

In Section 1, we have studied graphs on planar domains. We consider now graphs on the other kind of umbilical surfaces of Euclidean space, that is, spheres. Let $\mathbb{S}^2$ be the unit sphere of $\mathbb{R}^3$ with center at the origin. By a *radial graph* we mean a surface S in Euclidean space $\mathbb{R}^3$ with injective central projection over $\mathbb{S}^2$. Assume that $S = \{u(q)q; q \in \Omega\}$, where Ω is a domain of $\mathbb{S}^2$ and $0 < u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. If $\mathbf{x}$ is the immersion, the concept of radial graph means that the support function $\psi = \langle N, \mathbf{x}\rangle$ does not vanish on S, where N is the Gauss map of S. Our interest is to estimate the function $\eta = |\mathbf{x}| = u$. Let

$$r = \min_{\partial S} |\mathbf{x}| = \min_{\partial\Omega} u, \qquad R = \max_{\partial S} |\mathbf{x}| = \max_{\partial\Omega} u.$$

We assume that $\langle N, \mathbf{x}\rangle > 0$ and $H > 0$. With the above notation, we have

Theorem 2.1 ([10, 11])**.** *The distance* $|\mathbf{x}|$ *from the origin satisfies*

$$\frac{-1 + \sqrt{1 + H^2 r^2}}{H} \leq |\mathbf{x}| < R. \tag{2.1}$$

Proof. The proof follows the same steps as in (1.3) by defining

$$\varphi = \frac{H}{2}|\mathbf{x}|^2 + \langle N, \mathbf{x}\rangle$$

and using the identities

$$\Delta|\mathbf{x}|^2 = 4 + 4H\langle N, \mathbf{x}\rangle, \qquad \Delta\langle N, \mathbf{x}\rangle = -2H - |\sigma|^2\langle N, \mathbf{x}\rangle. \tag{2.2}$$

Let us remark again that the first equation holds for an arbitrary immersion $\mathbf{x}$ but that the second one holds if the mean curvature H is constant. By combining the two equations of (2.2), we have

$$\Delta\varphi = (2H^2 - |\sigma|^2)\langle N, \mathbf{x}\rangle \leq 0. \tag{2.3}$$

The the maximum principle says that $\varphi \geq \min_{\partial S} \varphi$. Since $|\mathbf{x}| \geq \langle N, \mathbf{x}\rangle > 0$, we have

$$\frac{H}{2}|\mathbf{x}|^2 + |\mathbf{x}| \geq \varphi \geq \min_{\partial S} \varphi \geq \frac{H}{2} r^2, \tag{2.4}$$

and from this, it follows the left side in the inequality (2.1). For the other one, we use in (2.2) that $\langle N, \mathbf{x}\rangle > 0$ to get $\Delta|\mathbf{x}|^2 > 0$. Then $|\mathbf{x}|^2 < \max_{\partial S} |\mathbf{x}|^2 = R^2$. ∎

Theorem 2.2. *The equality in the left side of* (2.1) *holds if and only if* S *describes a spherical cap of radius* $1/H$ *intersecting tangentially the cone determined by the origin and the circle* ∂S*.*

Proof. Let $p \in S$ such that

$$|p| = \frac{1}{H}\left(-1 + \sqrt{1 + H^2 r^2}\right). \tag{2.5}$$

Again, p is not a boundary point because in such case, $|p| \geq r$, in contradiction with (2.5). Thus φ is a constant function. Then $\Delta\varphi = 0$ and (2.3) yields that S is umbilical, that is, S is an open set of a sphere of radius $1/H$.

At the point p, $N(p) = p/|p|$, and $\varphi(p) = Hr^2/2$. Because along the boundary $\varphi \geq H|\mathbf{x}|/2 \geq Hr^2/2$, we conclude that $\langle N, \mathbf{x}\rangle = 0$ along ∂S and so, $r = R$. This means that ∂S is a circle and that S is a spherical cap. Thus, S intersects tangentially the cone determined by ∂S and the origin. ∎

Remark 3. In the case that for the orientation N with $\langle N, \mathbf{x}\rangle > 0$, the sign of H is negative, then we obtain

$$|\mathbf{x}| \leq \frac{1+\sqrt{1+R^2H^2}}{-H}.$$

However, in this situation it is not possible to obtain an estimate of $|\mathbf{x}|$ from below independent on H as in (2.1): let us consider C a great circle of $\mathbb{S}^2$. Let S_H be the family of spherical caps whose boundary is C, included in the open ball determined by $\mathbb{S}^2$ and parametrized each surface by its mean curvature H. Then for the orientation N such that $\langle N, \mathbf{x}\rangle > 0$, the mean curvature H is negative. Here $r = R = 1$, but letting $H \to 0$, $\min_S |\mathbf{x}| \to 0$.

Remark 4. If we add the hypothesis that Ω is included in an open hemisphere, it is possible to have more information on the geometry of the graph (for example, some results of existence, [14, 7]). Assume the mean curvature of a radial graph S is a constant $H < 0$ for $\langle N, \mathbf{x}\rangle > 0$ and that $\Omega \subset \mathbb{S}^2 \cap \{x_3 > 0\}$. The maximum principle comparying S with planes $\{x_3 = t\}$ shows that $\min_S x_3 = \min_{\partial S} x_3 = m > 0$. Thus $|\mathbf{x}| \geq m$ ($r \geq m$).

3. Graphs with constant mean curvature in hyperbolic space

We consider graphs on umbilical surfaces of hyperbolic space $\mathbb{H}^3$. In this ambient space, the types of umbilical surfaces P are: geodesic planes, equidistant surfaces, horospheres and spheres. As in Euclidean ambient, by a graph we mean the following: let Ω be a bounded domain of P, and ξ a fix orientation on P. For each point $q \in \Omega$, there exists a unique unit speed geodesic $\gamma_q : \mathbb{R} \to \mathbb{H}^3$ with $\gamma_q(0) = q$ and $\gamma_q'(0) = \xi(q)$. If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, we define the graph of u as the set of all points $\gamma_q(u(q))$, with $q \in \Omega$.

We consider the Minkowski model for $\mathbb{H}^3$: let $\mathbb{L}^4$ be the vector space $\mathbb{R}^4 = \{x = (x_0, \ldots, x_3); x_i \in \mathbb{R}\}$ equipped with the Lorentzian metric $\langle , \rangle = -dx_0^2 + dx_1^2 + \cdots + dx_3^2$. Then

$$\mathbb{H}^3 = \{x \in \mathbb{L}^4; \langle x, x\rangle = -1, x_0 > 0\}.$$

In this setting, the umbilical surfaces in $\mathbb{H}^3$ are given as the subsets $U_{\vec{a},\tau} = \{q \in \mathbb{H}^3; \langle q, \vec{a}\rangle = \tau\}$. Here $\tau \in \mathbb{R}$ and $\vec{a} \in \mathbb{L}^4$ is a vector such that $\langle \vec{a}, \vec{a}\rangle = \epsilon$, with $\epsilon = -1, 0, 1$. The classification of the umbilical surfaces of $\mathbb{H}^3$ is the following: (i) geodesic planes for $\epsilon = 1$ and $\tau = 0$; (ii) equidistant surfaces for $\epsilon = 1$ and $\tau \neq 0$; (iii) horospheres for $\epsilon = 0$ and $\tau \neq 0$ and (iv) spheres for $\epsilon = -1$ and $|\tau| > 1$.

A unit normal vector field ξ on $U_{\vec{a},\tau}$ is defined by

$$\xi(q) = -\lambda(\vec{a} + \tau q), \qquad \lambda = \frac{1}{\sqrt{\tau^2 + \epsilon}}.$$

With this orientation, the mean curvature h of $U_{\vec{a},\tau}$ is $h = \lambda\tau$. By reversing the vector $\vec{a}$ by $-\vec{a}$ if it is necessary, we assume throughout this section that $\tau \leq 0$.

Now let Ω be a bounded domain in $U_{\vec{a},\tau}$ and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. The graph of u is given by $S = \{\cosh(u(q))q + \sinh(u(q))\xi(q); q \in \Omega\}$. Let us remark that in the case of a graph on a sphere $U_{\vec{a},\tau}$, the geodesic $\cosh(t)q + \sinh(t)\xi(q)$ starts from the center of the sphere, namely, the point a and then, $t > -\mathrm{arccosh}(-\tau)$. Taking in account the orientation ξ, we conclude that the value of the function $\eta = \langle \mathbf{x}, \vec{a}\rangle$ is

$$\langle \mathbf{x}, \vec{a}\rangle = \tau \cosh u - \sqrt{\tau^2 + \epsilon}\sinh u. \tag{3.1}$$

Because S is a graph, then function $\psi = \langle N, \vec{a}\rangle$ has sign on S, N is the Gauss map of S. Assume that $\langle N, \vec{a}\rangle > 0$. Then

$$\langle N, \vec{a}\rangle = \frac{(\cosh u - h\sinh u)^2}{\lambda\sqrt{(\cosh u - h\sinh u)^2 + |\nabla u|^2}}. \tag{3.2}$$

As in Euclidean setting, let $m = \min_{\partial\Omega} u$ and $M = \max_{\partial\Omega} u$. On the other hand, for an immersion $\mathbf{x} : S \to \mathbb{H}^3$ with constant mean curvature H, we have ([13])

$$\Delta\langle \mathbf{x}, \vec{a}\rangle = 2\langle \mathbf{x}, \vec{a}\rangle + 2H\langle N, \vec{a}\rangle, \tag{3.3}$$

$$\Delta\langle N, \vec{a}\rangle = -2H\langle \mathbf{x}, \vec{a}\rangle - |\sigma|^2\langle N, \vec{a}\rangle. \tag{3.4}$$

We define

$$\varphi = H\langle \mathbf{x}, \vec{a}\rangle + \langle N, \vec{a}\rangle.$$

Then (3.3) and (3.4) imply $\Delta\varphi = (2H^2 - |\sigma|^2)\langle N, \vec{a}\rangle \leq 0$ and so,

$$H\langle \mathbf{x}, \vec{a}\rangle + \langle N, \vec{a}\rangle \geq \min_{\partial S}\left(H\langle \mathbf{x}, \vec{a}\rangle\right). \tag{3.5}$$

For each point p of S, we decompose the vector $\vec{a}$ as

$$\vec{a} = \langle \vec{a}, \vec{a}^T\rangle\vec{a}^T + \langle N(p), \vec{a}\rangle N(p) - \langle \vec{a}, p\rangle p,$$

where a^T is the (unit) orthogonal projection of $\vec{a}$ in the tangent plane of S at p. Then

$$\epsilon = \langle \vec{a}, \vec{a}\rangle \geq \langle N(p), \vec{a}\rangle^2 - \langle p, \vec{a}\rangle^2,$$

and so,

$$\langle N(p), \vec{a}\rangle \leq \sqrt{\epsilon + \langle p, \vec{a}\rangle^2}. \tag{3.6}$$

This bound is equivalent to consider in (3.2) the inequality $\langle N(p), \vec{a}\rangle \leq |\cosh u - h\sinh u|/\lambda$.

Next we distinguish each one of the cases of graphs depending on which umbilical surface is defined.

3.1. Graphs on geodesic planes and equidistant surfaces

Theorem 3.1. *Let $U_{\vec{a},\tau}$ be a geodesic plane or an equidistant surface of $\mathbb{H}^3$ with $\tau \leq 0$. Let $S \subset \mathbb{H}^3$ be the graph of a smooth function u defined in a domain $\Omega \subset U_{\vec{a},\tau}$. If $1 < H < \infty$,*

$$m < u \leq \log \frac{\sqrt{H^2\alpha^2 + H^2 - 1} - H\alpha}{(H-1)(\sqrt{\tau^2+1} - \tau)}, \tag{3.7}$$

where $\alpha = \tau \cosh M - \sqrt{\tau^2+1}\sinh M$. The equality in the right side holds if and only if S is a (hyperbolic) hemisphere that tangentially meets $U_{\vec{a},\tau}$ along $\partial\Omega$.

Proof. From (3.5) and (3.6), we obtain,

$$H\langle \mathbf{x}, \vec{a}\rangle + \sqrt{1 + \langle \mathbf{x}, \vec{a}\rangle^2} \geq H\alpha.$$

Now $\sqrt{1+\langle \mathbf{x}, \vec{a}\rangle^2} = \sqrt{\tau^2+1}\cosh u - \tau \sinh u$. Then (3.1) leads to

$$(H-1)(\tau - \sqrt{\tau^2+1})e^{2u} - 2H\alpha e^u + (H+1)(\tau + \sqrt{\tau^2+1}) \geq 0$$

which implies the left inequality in (3.7).

For the proof the right hand of (3.7), we consider two possibilities.

1. Assume that there exist points of S with $\langle \mathbf{x}, \vec{a}\rangle > 0$. If the function $\langle \mathbf{x}, \vec{a}\rangle$ achieves its maximum at some interior point $q \in S$, $\Delta\langle \mathbf{x}, \vec{a}\rangle(q) \leq 0$, which it is a contradicition with (3.3). Thus this maximum is attained at some boundary point $p \in \partial S$. Then $\langle \mathbf{x}, \vec{a}\rangle \leq \langle p, \vec{a}\rangle \leq \max_{\partial S}\langle \mathbf{x}, \vec{a}\rangle$.

2. The function $\langle \mathbf{x}, \vec{a}\rangle$ is non-positive. If the maximum is attained at some interior point $q \in S$, then $\langle N(q), \vec{a}\rangle = \sqrt{1 + \langle q, \vec{a}\rangle^2}$. Because $H > 1$, we have

$$\Delta\langle \mathbf{x}, \vec{a}\rangle(q) = 2\left(\langle q, \vec{a}\rangle + H\sqrt{1+\langle q, \vec{a}\rangle^2}\right) > 0,$$

because $H > 1$. This gets a contradiction since $\Delta\langle \mathbf{x}, \vec{a}\rangle(q) \leq 0$. As conclusion, the maximum of the function $\langle \mathbf{x}, \vec{a}\rangle$ occurs at some boundary point $p \in \partial S$. Thus $\langle \mathbf{x}, \vec{a}\rangle < \max_{\partial S}\langle \mathbf{x}, \vec{a}\rangle$.

In both situations, $\langle \mathbf{x}, \vec{a}\rangle < \max_{\partial S}\langle \mathbf{x}, \vec{a}\rangle$. This implies the left hand inequality in (3.7).

Assume now that u attains the upper bound (3.7) at some point $p \in S$. Then $H\alpha = H\langle p, a\rangle + \langle N(p), \vec{a}\rangle$. This implies that p is an interior point, since $\langle p, a\rangle \geq \alpha$. In particular, φ is a constant function and then $0 = \Delta\varphi = (2H^2 - |\sigma|^2)\langle N, \vec{a}\rangle$. This mean that S is an umbilical surface, that is, an open set of a (hyperbolic) sphere. Moreover, if $q \in \partial S$, then

$$H\alpha = \varphi(p) = \varphi(q) = H\langle q, \vec{a}\rangle + \langle N(q), \vec{a}\rangle \geq H\langle q, a\rangle \geq H\alpha.$$

Thus $\langle q, \vec{a}\rangle = \langle q, \xi(q)\rangle = 0$. Then ∂S is a circle and S is a spherical cap. Moreover, $\langle N(q), \vec{a}\rangle = 0$. Hence, $\langle N(q), \xi(q)\rangle = 0$, that is, S is a hemisphere orthogonal to $U_{\vec{a},\tau}$ along $\partial\Omega$. $\square$

If we reverse the sign of H, and if we assume that $H < -1$, an analogous reasoning gives

$$\log \frac{\sqrt{H^2\beta^2 + H^2 - 1} + H\beta}{(H-1)(\tau - \sqrt{\tau^2+1})} \leq u < M,$$

with $\beta = \tau \cosh m - \sqrt{\tau^2+1} \sinh m$.

Remark 5. Assume that $U_{\vec{a},\tau}$ is a geodesic plane and $u = 0$ along $\partial\Omega$. The right inequality in (3.7) generalizes the obtained one in [13], namely $0 < u \leq \operatorname{arcsinh}\left(1/\sqrt{H^2-1}\right)$.

Remark 6. In the range $-1 \leq H \leq 1$, and fixed H, there are examples of graphs with arbitrary height. See Figure 1.

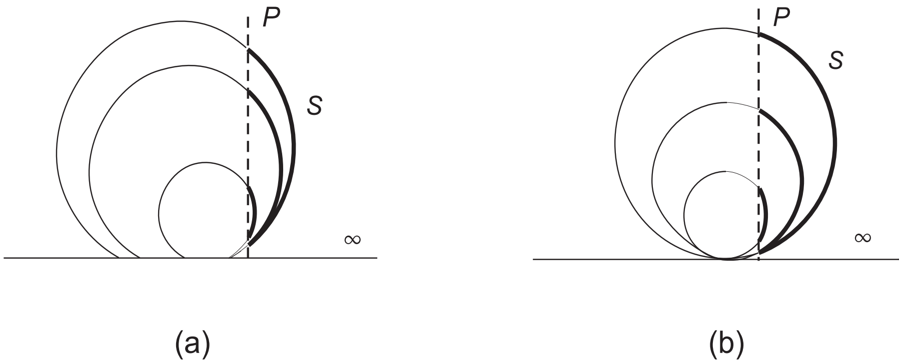

Figure 1. P is a geodesic plane. Case (a): pieces of equidistant surfaces with the same mean curvature; case (b) pieces of horospheres.

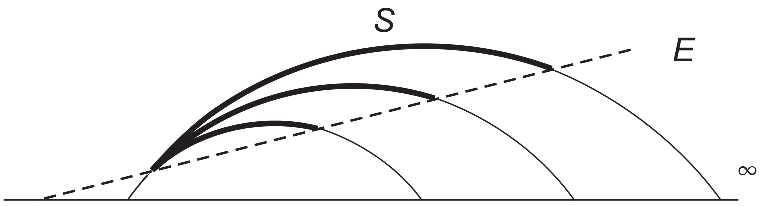

Figure 2. E is an equidistant surface and S are equidistant surfaces with the same mean curvature.

3.2. Graphs on horospheres

Theorem 3.2. *Let $U_{\vec{a},\tau}$ be a horosphere of $\mathbb{H}^3$ with $\tau < 0$. Let $S \subset \mathbb{H}^3$ be the graph of a smooth function u defined in a domain $\Omega \subset U_{\vec{a},\tau}$.*

1. *Assume* $1 < H < \infty$. *Then*

$$m < u \leq M + \log \frac{H}{H-1}. \tag{3.8}$$

2. *If* $-\infty < H < 0$, *then*

$$m + \log \frac{H}{H-1} \leq u < M. \tag{3.9}$$

In both expressions (3.8) *and* (3.9), *the equality holds if and only if* S *is an umbilical surface whose boundary is a circle and that orthogonally meets* $U_{\vec{a},\tau}$ *along* $\partial\Omega$.

The estimates (3.8) and (3.9) were obtained in [9] for some particular cases.

Proof. Here, $\langle \mathbf{x}, \vec{a} \rangle = \tau e^u$ and $\langle N, \vec{a} \rangle \leq -\tau e^u$. We assume $H > 1$. From (3.5),

$$(H-1)\tau e^u \geq H \min_{\partial\Omega} \tau e^u = H\tau e^M,$$

giving the right inequality (3.8). Now we prove $u > m$. From (3.3), if $p \in S$ is a interior critical point of the function $\langle \mathbf{x}, \vec{a} \rangle$, then

$$\Delta\langle \mathbf{x}, \vec{a} \rangle(p) = 2(\langle p, \vec{a} \rangle - H\langle p, \vec{a} \rangle) = 2(1-H)\langle p, \vec{a} \rangle > 0.$$

This means that the maximum of $\langle \mathbf{x}, \vec{a} \rangle$ is achieved at some boundary point. As a consequence, $\langle \mathbf{x}, \vec{a} \rangle < \max_{\partial S} \langle \mathbf{x}, \vec{a} \rangle$ which leads to $u > m$.

If $H < 0$, then (3.5) means $H\langle \mathbf{x}, \vec{a} \rangle + \langle N, \vec{a} \rangle \geq H \max_{\partial S} \langle \mathbf{x}, a \rangle$, that is,

$$(H-1)\tau e^u \geq H \max_{\partial\Omega} \tau e^u = H\tau e^m,$$

which leads the left side in (3.9). On the other hand, $\Delta\langle \mathbf{x}, \vec{a} \rangle < 0$, and the maximum principles gives $\langle \mathbf{x}, \vec{a} \rangle \geq \min_{\partial S} \langle \mathbf{x}, \vec{a} \rangle$, that is, $u < M$.

Assume now that the right inequality (3.8) is attained at some point $p \in S$. Then $e^{u(p)} = H/(H-1)e^M$ and so, p is an interior point of S. Thus φ is a constant function and so, $\Delta\varphi = 0$. This means that $2H^2 = |\sigma|^2$ and S is an umbilical surface. Moreover, for each $q \in \partial S$,

$$H\tau e^M = (H-1)\langle p, \vec{a} \rangle = \varphi(q) = H\tau e^{u(q)} + \langle N(q), \vec{a} \rangle \geq H\tau e^{u(q)} \geq H\tau e^M.$$

Therefore $\langle q, \vec{a} \rangle = \alpha$ and $\langle N, \vec{a} \rangle = 0$ along ∂S. In particular, ∂S is a circle. A similar reasoning holds in the case of equality in (3.9). $\square$

Remark 7. It is possible to obtain analogous estimates as in the minimal case as it occurs in Euclidean space, see Remark 2. Exactly: *Let* P *be either a geodesic plane, an equidistant surface or a horosphere, and let* $\Omega \subset P$ *be a bounded domain. Assume that* h *is the mean curvature of* P. *If* S *is a graph of a function* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *with constant mean curvature* $H = h$, *then* $m \leq u \leq M$. The proof uses the maximum principle together the fact that $\mathbb{H}^3$ can be foliated by a family of umbilical surfaces that contains P (c.f. [6, Th. 2.2]).

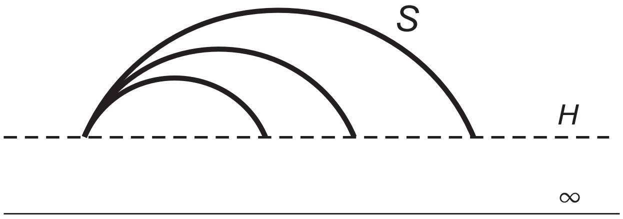

Figure 3. H is a horosphere. The graphs S are equidistant surfaces with the same mean curvature.

Remark 8. As in the above subsection, in the range $0 \leq H \leq 1$, and fixed H, there are examples of graphs with arbitrary height. See Figure 3.

3.3. Graphs on spheres

Theorem 3.3. *Let $U_{\vec{a},\tau}$ be a sphere of $\mathbb{H}^3$ with $\tau < 0$. Let $S \subset \mathbb{H}^3$ be the graph of a smooth function u defined in a domain $\Omega \subset U_{\vec{a},\tau}$. Define*

$$\alpha = \frac{\tau - \sqrt{\tau^2 - 1}}{2} e^{M} + \frac{\tau + \sqrt{\tau^2 - 1}}{2} e^{-m},$$
$$\beta = \frac{\tau - \sqrt{\tau^2 - 1}}{2} e^{m} + \frac{\tau + \sqrt{\tau^2 - 1}}{2} e^{-M}.$$

1. *If $1 < H < \infty$,*

$$\log \frac{H\alpha + \sqrt{H^2\alpha^2 + 1 - H^2}}{(H-1)(\tau - \sqrt{\tau^2 - 1})} \leq u \leq \log \frac{H\alpha - \sqrt{H^2\alpha^2 + 1 - H^2}}{(H-1)(\tau - \sqrt{\tau^2 - 1})}. \tag{3.10}$$

2. *If $H = 1$,*

$$\log \frac{\tau + \sqrt{\tau^2 - 1}}{\alpha} \leq u. \tag{3.11}$$

3. *If $0 < H < 1$,*

$$\log \frac{H\alpha + \sqrt{H^2\alpha^2 + 1 - H^2}}{(H-1)(\tau - \sqrt{\tau^2 - 1})} \leq u < \log \frac{\alpha - \sqrt{\alpha^2 - 1}}{\tau - \sqrt{\tau^2 - 1}}. \tag{3.12}$$

4. *If $-\infty < H < 0$,*

$$\log \frac{H\beta + \sqrt{H^2\beta^2 + 1 - H^2}}{(H-1)(\tau - \sqrt{\tau^2 - 1})} \leq u < \log \frac{\alpha - \sqrt{\alpha^2 - 1}}{\tau - \sqrt{\tau^2 - 1}}. \tag{3.13}$$

Proof. From (3.6), $\langle \mathbf{x}, \vec{a}\rangle^2 - 1 \geq 0$ and so, $\langle \mathbf{x}, \vec{a}\rangle \leq -1$ or $\langle \mathbf{x}, \vec{a}\rangle \geq 1$. But $\langle \mathbf{x}, \vec{a}\rangle = \tau \cosh u - \sqrt{\tau^2 - 1} \sinh u < 0$. Thus $\langle \mathbf{x}, \vec{a}\rangle \leq -1$. Assume $H < 1$. If p is a critical (interior) point of $\langle \mathbf{x}, \vec{a}\rangle$, then (3.3) and (3.6) give

$$\Delta\langle \mathbf{x}, \vec{a}\rangle(p) = 2(\langle p, \vec{a}\rangle + H\sqrt{\langle p, \vec{a}\rangle^2 - 1}),$$

that is, $\Delta\langle \mathbf{x}, \vec{a}\rangle(p) < 0$. Thus the minimum of the function $\langle \mathbf{x}, \vec{a}\rangle$ must be achieved at some boundary point. In particular, $\langle \mathbf{x}, \vec{a}\rangle \geq \min_{\partial S}\langle \mathbf{x}, \vec{a}\rangle = \alpha$ which implies

$$\frac{\alpha+\sqrt{\alpha^2-1}}{\tau-\sqrt{\tau^2-1}} < e^u < \frac{\alpha-\sqrt{\alpha^2-1}}{\tau-\sqrt{\tau^2-1}}. \tag{3.14}$$

This gives the right sides in the inequalities (3.12) and (3.13).

Consider $H \neq 0$. The inequalities (3.5) and (3.6) imply

$$H\langle \mathbf{x}, \vec{a}\rangle + \sqrt{\langle \mathbf{x}, \vec{a}\rangle^2 - 1} \geq \min_{\partial S}(H\langle \mathbf{x}, \vec{a}\rangle) := C = \begin{cases} H\alpha & H > 0 \\ H\beta & H < 0 \end{cases} \tag{3.15}$$

On the other hand, recall that $u > -\text{arccosh}(-\tau)$. This means that

$$\sqrt{\langle \mathbf{x}, \vec{a}\rangle^2 - 1} = \sqrt{\tau^2-1}\cosh u - \tau \sinh u.$$

Substituing in (3.15),

$$P := (H-1)(\tau-\sqrt{\tau^2-1})e^{2u} - 2Ce^u + (H+1)(\tau+\sqrt{\tau^2-1}) \geq 0. \tag{3.16}$$

Assume $1 < H < \infty$. The polynomial P of second order on e^u of (3.16) has two positive roots. This gives (3.10). If $H = 1$, from (3.16), $-\alpha e^u + \tau + \sqrt{\tau^2-1} \geq 0$. This implies (3.11).

If $H^2 < 1$, then P has one negative and one positive root. This gives

$$e^u \geq \frac{H\alpha + \sqrt{H^2\alpha^2 + 1 - H^2}}{(H-1)(\tau-\sqrt{\tau^2-1})} \qquad 0 < H < 1. \tag{3.17}$$

$$e^u \geq \frac{H\beta + \sqrt{H^2\beta^2 + 1 - H^2}}{(H-1)(\tau-\sqrt{\tau^2-1})} \qquad -1 < H < 0. \tag{3.18}$$

Now we do compare each one of these estimates with the obtained one in the left side of (3.14). If $0 < H < 1$, the right side in (3.17) is decreasing on H. By letting $H \to 1$, we prove that the limit of (3.17) is bigger that the left side of (3.14). If $-1 < H < 0$, the lower bound in (3.18) is decreasing on β. Since $\beta \leq -1$, we put $\beta = -1$ in (3.18) and we get a bound bigger than the left side in (3.14). In any both cases, we show the lower bounds in (3.12) and (3.13) for $-1 < H < 1$.

If $H < -1$, P has two positive roots. Then

$$e^u \geq \frac{H\beta + \sqrt{H^2\beta^2 + 1 - H^2}}{(H-1)(\tau-\sqrt{\tau^2-1})} \quad \text{or} \quad e^u \leq \frac{H\beta - \sqrt{H^2\beta^2 + 1 - H^2}}{(H-1)(\tau-\sqrt{\tau^2-1})}. \tag{3.19}$$

If the second possibility holds, and since this expression is decreasing on β, we change the value of β by α. We can combine this with the left side in (3.14) to obtain

$$(H-1)\left(\alpha+\sqrt{\alpha^2-1}\right) \leq H\alpha - \sqrt{H^2\alpha^2+1-H^2},$$

which it is a contradiction. Thus, the only possibility is the first case in (3.19). If $H=-1$, P has 0 and a positive number as roots. This yields (3.13) again for this case of H. A similar reasoning as in the case $-1<H<0$ shows that this bound improves that the left side in (3.14). $\square$

Theorem 3.4. *Under the same hypothesis as in Theorem* 3.3*, the equalities hold in* (3.10)*,* (3.11)*,* (3.12) *and* (3.13) *if and only if S is an umbilical surface whose boundary is a circle and that orthogonally meets $U_{\vec{a},\tau}$ along $\partial\Omega$.*

Proof. First, we consider the equality in (3.10). Then there exists a point $p\in S$ such that $H\langle p,\vec{a}\rangle+\sqrt{\langle p,\vec{a}\rangle^2-1}=H\alpha$. In particular, p is not a boundary point, and the maximum principle says that φ is constant, with $\varphi=H\alpha$. Then $\Delta\varphi=0$ and S is an umbilical surface. For each boundary point q, we have $H\alpha=\varphi(q)=\langle q,\vec{a}\rangle+\langle N(q),\vec{a}\rangle\geq H\alpha$. Then $\langle\mathbf{x},\vec{a}\rangle=\alpha$ along ∂S, this is, ∂S is a circle. Moreover, $\langle N,\vec{a}\rangle=0$ along ∂S and so, S meets orthogonally $U_{\vec{a},\tau}$ along $\partial\Omega$. In the remaining cases, the reasoning is analogous. $\square$

4. Graphs with constant mean curvature in Euclidean sphere

Let $\mathbb{S}^3=\{x=(x_1,x_2,x_3,x_4)\in\mathbb{R}^4;|x|^2=1\}$ be the unit sphere in Euclidean space $\mathbb{R}^4$. Let $P=\{x_4=0\}$ and $\Sigma=\mathbb{S}^3\cap P$ be a geodesic sphere. Let us fix $\mathbf{e}_4=(0,0,0,1)$ an orientation in Σ. Given Ω a domain of Σ and $f\in C^2(\Omega)\cap C^0(\overline{\Omega})$, we define the graph of f as the set $S=\{\gamma_q(f(q));q\in\Omega\}$, where for each point $q\in\Omega$, $\gamma_q:\mathbb{R}\to\mathbb{S}^3$ is the unit speed geodesic with $\gamma_q(\pi/2)=q$ and $\gamma_q'(\pi/2)=\mathbf{e}_4$. Here $\gamma_q(t)=\sin(t)q-\cos(t)\mathbf{e}_4$, $t\in(0,\pi)$. By the stereographic projection Π from the point $\mathbf{e}_4$, $\Pi:\mathbb{S}^3\setminus\{\mathbf{e}_4\}:\to\{x\in\mathbb{R}^4;x_4=0\}$, the surface S is viewed as a radial graph on P from the origin of $\mathbb{R}^4$, exactly, $S=\{u(q)q;q\in\Omega\}$, with

$$0<u=\frac{\sin(f)}{1+\cos(f)}=\tan(f/2).$$

Recall that $\Pi_{|P}=\mathrm{id}$ and so, $\Pi(\Sigma)=\Sigma$. Denote $\mathbf{x}:S\to P$ the immersion of S. We want to estimate the function $\eta=u$. The fact that S is a graph means that the function $\psi=\langle N,\mathbf{e}_4\rangle$ has sign on S, where N is an orientation on S. As in Section 2, let

$$r=\min_{\partial S}|\mathbf{x}|=\min_{\partial\Omega}u,\qquad R=\max_{\partial S}|\mathbf{x}|=\max_{\partial\Omega}u.$$

We assume that the mean curvature H of $\mathbf{x}$ is a constant positive with respect to the orientation N that satisfies $\langle N,\mathbf{e}_4\rangle>0$.

Theorem 4.1. *The distance $u=|\mathbf{x}|$ from the origin satisfies*

$$\frac{-(1+r^2)+\sqrt{(1+r^2)^2+4H^2r^2}}{2H}\leq u. \tag{4.1}$$

The equality holds if and only if S describes a spherical cap of radius $1/H$ intersecting tangentially the cone determined by the origin and the circle ∂S.

Proof. In this setting, we have ([13])

$$\Delta\langle \mathbf{x}, \mathbf{e}_4\rangle = -2\langle \mathbf{x}, \mathbf{e}_4\rangle + 2H\langle N, \mathbf{e}_4\rangle. \tag{4.2}$$

$$\Delta\langle N, \mathbf{e}_4\rangle = 2H\langle \mathbf{x}, \mathbf{e}_4\rangle - |\sigma|^2\langle N, \mathbf{e}_4\rangle. \tag{4.3}$$

Then $\varphi = H\langle \mathbf{x}, \mathbf{e}_4\rangle + \langle N, \mathbf{e}_4\rangle$ satisfies

$$\Delta\varphi = (2H^2 - |\sigma|^2)\langle N, \mathbf{e}_4\rangle \leq 0. \tag{4.4}$$

The maximum principle yields

$$H\langle \mathbf{x}, \mathbf{e}_4\rangle + \langle N, \mathbf{e}_4\rangle \geq H \min_{\partial S}\langle \mathbf{x}, \mathbf{e}_4\rangle := H\alpha.$$

Here $\langle \mathbf{x}, \mathbf{e}_4\rangle = (u^2-1)/(1+u^2)$ and $\alpha = (r^2-1)/(r^2+1)$. Because $1 = \langle \mathbf{e}_4, \mathbf{e}_4\rangle \geq \langle \mathbf{x}, \mathbf{e}_4\rangle^2 + \langle N, \mathbf{e}_4\rangle^2$, we conclude

$$H\langle \mathbf{x}, \mathbf{e}_4\rangle + \sqrt{1 - \langle \mathbf{x}, \mathbf{e}_4\rangle^2} \geq H\alpha,$$

that is,

$$H\frac{u^2-1}{1+u^2} + \frac{2u}{1+u^2} \geq H\alpha.$$

Then

$$H(1-\alpha)u^2 + 2u - H(1+\alpha) \geq 0,$$

which leads to

$$u \geq \frac{-1 + \sqrt{1 + H^2(1-\alpha^2)}}{H(1-\alpha)}.$$

This implies (4.1). If we have equality in (4.1) for some point $p \in S$, then $p \notin \partial S$ since the left side in (4.1) is strictly less than r. Thus p is an interior point and so, φ is a constant function. Then (4.4) implies that $2H^2 = |\sigma|^2$ on S, that is, S is an open set of a sphere. Moreover, for each point $q \in \partial\Omega$,

$$H\alpha = \varphi(p) = \varphi(q) = H\langle q, \mathbf{e}_4\rangle + \langle N(q), \mathbf{e}_4\rangle \geq H\alpha,$$

that is, $\langle N, \mathbf{e}_4\rangle = 0$ and $\langle \mathbf{x}, \mathbf{e}_4\rangle = \alpha$ along $\partial\Omega$. Then ∂S is included in the hyperplane $\{x_4 = \alpha\}$. This means that ∂S is a circle of radius $\sqrt{1-\alpha^2}$ and that S is a spherical cap that tangentially meets the cone determined by the origin of P and ∂S. □

In the case that $H < 0$ for the orientation N such that $\langle N, \mathbf{e}_4\rangle > 0$, we obtain

$$u \leq \frac{1 + R^2 + \sqrt{(1+R^2)^2 + 4H^2R^2}}{-2H}.$$

Remark 9. When $\partial S = \partial\Omega$, then $r = 1$ and the inequality (4.1) writes as $-\frac{1}{\sqrt{1+H^2}} \leq \langle \mathbf{x}, \mathbf{e}_4 \rangle$. This was proved in [1]. On the other hand, when Ω is included in a hemisphere of Σ, then $u < R$ on Ω: it is a consequence of the maximum principle by considering an appropriate family of spheres. See [1], Remark 4 in Section 2 and Remark 7 for a similar argument in the hyperbolic context.

5. Graphs with constant Gaussian curvature in Euclidean space

We end by considering graphs S with positive constant Gaussian curvature K in Euclidean space. Although in this situation the equation of constant Gaussian curvature is of Monge-Ampère type, it is possible to obtain a priori height estimates with a technique similar as for Equation (1.1). Also, we can consider the cases of vertical and radial graphs. We use the notation of Section 1. In the first setting, we assume that the graphs are oriented in such way that the principal curvatures κ_1 and κ_2 are positive and that $\langle N, \vec{a} \rangle > 0$.

Theorem 5.1. *Let S be a graph in $\mathbb{R}^3$ of a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ defined on a planar domain Ω with positive constant Gauss curvature K. Then*

$$m - \frac{1}{\sqrt{K}} \leq u < M. \tag{5.1}$$

The inequality in the left side was proved in [13], [2] in the case that the boundary is a planar curve.

Proof. Because the Gauss curvature K is positive, the second fundamental form

$$\sigma_p(u, v) = -\langle dN_p(u), v \rangle, \qquad u, v \in T_pS$$

defines a Riemannian metric on S. If Δ^σ is the Laplacian operator in S with the metric σ, then one can compute [3]:

$$\Delta^\sigma \langle \mathbf{x}, \vec{a} \rangle = 2\langle N, \vec{a} \rangle. \tag{5.2}$$

$$\Delta^\sigma \langle N, \vec{a} \rangle = -2H\langle N, \vec{a} \rangle. \tag{5.3}$$

Now the proof works as in Serrin's theorem with the function $\varphi = \sqrt{K}\langle \mathbf{x}, \vec{a} \rangle + \langle N, \vec{a} \rangle$ since

$$\Delta\varphi = 2(\sqrt{K} - H)\langle N, \vec{a} \rangle \leq 0. \tag{5.4}$$

For the right side in (5.1) and from (5.2), we have $\Delta\langle \mathbf{x}, \vec{a} \rangle > 0$, and then, $\langle \mathbf{x}, \vec{a} \rangle \leq \max_{\partial S} = M$. $\square$

Theorem 5.2. *The equality on the left side of* (5.1) *occurs if and only if S is a hemisphere of radius $1/\sqrt{K}$.*

Proof. Assume that for $p \in S$, $\langle p, \vec{a} \rangle = m - 1/\sqrt{K}$. Then $p \notin \partial S$. It follows that φ is a constant function and $\varphi = \varphi(p) = \sqrt{K}m$. Now (5.4) implies $\sqrt{K} = H$ on S, that is, S is an open set of a sphere of radius $1/\sqrt{K}$. The same argument that in the case of constant mean curvature shows that $m = M$ and that S is a hemisphere. $\square$

In the case that for positive principal curvature, the orientation N corresponds with $\langle N, \vec{a} \rangle < 0$, a similar argument as above proves that $m < u \leq M - 1/\sqrt{K}$.

Finally, we study radial graphs with constant Gaussian curvature. Let Ω be a domain of the unit sphere $\mathbb{S}^2$ and let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a positive function. We know now that the function $\psi = \langle N, \mathbf{x} \rangle$ does not vanishes on S. Assume that κ_i are positive with the orientation N that does $\langle N, \mathbf{x} \rangle < 0$.

Theorem 5.3 ([8]). *Let S be the radial graph of u. Assume that S has positive constant Gauss curvature K. Then the distance $|x|$ from the origin satisfies*

$$|\mathbf{x}| \leq \frac{1 + \sqrt{1 + KR^2}}{\sqrt{K}} \tag{5.5}$$

Proof. Now we use the equations

$$\Delta^\sigma |\mathbf{x}|^2 = 4\langle N, \mathbf{x} \rangle + \frac{4H}{K}, \tag{5.6}$$

and

$$\Delta^\sigma \langle N, \mathbf{x} \rangle = -2 - 2H\langle N, \mathbf{x} \rangle. \tag{5.7}$$

Let us define $\varphi = \frac{\sqrt{K}}{2}|\mathbf{x}|^2 + \langle N, \mathbf{x} \rangle$. Then

$$\Delta\varphi = 2(\sqrt{K} - H)\langle N, \mathbf{x} \rangle + \frac{2}{\sqrt{K}}(H - \sqrt{K}) \geq 0. \tag{5.8}$$

Then φ is a subharmonic function. The maximum principle and the fact that $\langle N, \mathbf{x} \rangle \geq -|\mathbf{x}|$ imply

$$\frac{\sqrt{K}}{2}|\mathbf{x}|^2 - |\mathbf{x}| \leq \varphi \leq \frac{\sqrt{K}}{2}R^2. \tag{5.9}$$

This proves (5.5) $\square$

Remark 10. Under the hypothesis of Theorem 5.3, it is not possible to get estimates for $|\mathbf{x}|$ from below independent on K as it shows the examples of Remark 3.

Theorem 5.4. *The equality holds in* (5.5) *if and only if S is a spherical cap of radius $1/\sqrt{K}$ intersecting tangentially the cone determined by the origin and the circle ∂S.*

Proof. Let $p \in S$ such that $|p| = \frac{1+\sqrt{1+R^2K}}{\sqrt{K}}$. Then p is not a boundary point because in such case, $|p| \leq R$ in contradiction with the value of $|p|$. Thus, p is an interior point of S and φ is constant. Since the function $|\mathbf{x}|$ achieves a maximum at p, $N(p) = -p/|p|$. Thus,

$$\varphi(p) = \frac{\sqrt{K}}{2}|p|^2 - |p| = \frac{\sqrt{K}}{2}R^2.$$

From (5.8), $\sqrt{K} - H$ on S, that is, S is an umbilical surface. In particular, S is included in a sphere of radius $1/\sqrt{K}$. The proof finishes as in Theorem 2.2. □

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