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Asymptotics of Green functions and the limiting absorption principle for elliptic operators with periodic coefficients

By

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Abstract

We give the asymptotics of Green functions $G_{\lambda\pm i0}(x,y)$ as $|x-y|\to\infty$ for an elliptic operator with periodic coefficients on $\mathbf{R}^d$ in the case where $d\geq 2$ and the spectral parameter λ is close to and greater than the bottom of the spectrum of the operator. The main tools are the Bloch representation of the resolvent and the stationary phase method. As a by-product, we also show directly the limiting absorption principle. In the one dimensional case, we show that Green functions are written as products of exponential functions and periodic functions for any λ in the interior of the spectrum or the resolvent set.

1. Introduction

The scattering and spectral theory for periodic Schrödinger operators L on $\mathbf{R}^d$ has been developed to some extent; in its study the limiting absorption principle (i.e., the existence of the limits $(L-\lambda\mp i0)^{-1}$ in a certain topology) has played a crucial role (cf. [T], [Be], [Si], [G], [GN1,2], [Su], [RS], [BY], [FS]).

The main purpose of this paper is to give the asymptotics of the integral kernel $G_{\lambda\pm i0}(x,y)$ as $|x-y|\to\infty$ of the operator $(L-\lambda\mp i0)^{-1}$ in the case where $d\geq 2$ and the parameter λ is greater than and close to λ_0, the bottom of the spectrum of L. In the subcritical case (i.e., either $\lambda<\lambda_0$ or $\lambda=\lambda_0$ and $d\geq 3$) we gave the asymptotics of the resolvent kernel and used it to determine the Martin boundary in [MT] (see also [Ba], [Se]). This paper is a continuation of [MT]; and the basic tool used in both papers is the Bloch representation of the resolvent. The secondary purpose is to give a direct and elementary proof of the limiting absorption principle by the method employed in establishing the asymptotics of Green functions. The last purpose is to describe precisely Green functions for any $\lambda\in\mathbf{R}$ in the one dimensional case.

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We consider a second order elliptic operator on $\mathbf{R}^d$ with periodic coefficients

$$L = -\sum_{j,k=1}^{d} \frac{\partial}{\partial x_k}\left(a_{jk}(x)\frac{\partial}{\partial x_j}\right) + c(x) = -\nabla \cdot a(x)\nabla + c(x),$$

where $d \geq 2$, $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_d)$, and $a(x) = (a_{jk}(x))_{j,k=1}^d$. We assume that the coefficients are real-valued measurable functions on $\mathbf{R}^d$ which are $\mathbf{Z}^d$-periodic, i.e., $a_{jk}(x+z) = a_{jk}(x)$ and $c(x+z) = c(x)$ for any $x \in \mathbf{R}^d$ and $z \in \mathbf{Z}^d$. We further assume that a is a symmetric matrix-valued function satisfying

$$\mu|\xi|^2 \leq \sum_{j,k=1}^{d} a_{jk}(x)\xi_j\xi_k \leq \mu^{-1}|\xi|^2, \quad x, \xi \in \mathbf{R}^d,$$

for some $\mu > 0$, and that $c \in L^p_{loc}(\mathbf{R}^d)$ for some $p > d/2$. We regard L as the selfadjoint operator on $L^2(\mathbf{R}^d)$ with the domain $D(L) = \{u \in H^1(\mathbf{R}^d);\, Lu \in L^2(\mathbf{R}^d)\}$, where $H^1(\mathbf{R}^d)$ is the Sobolev space of order one.

We recall some results to state our theorem. For each $\zeta \in \mathbf{C}^d$, let $L(\zeta)$ be the operator on the d-dimensional torus $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$ defined by

$$L(\zeta) = e^{-i\zeta\cdot x} L e^{i\zeta\cdot x} = -(\nabla + i\zeta)\cdot a(x)(\nabla + i\zeta) + c(x),$$

where $i = \sqrt{-1}$ is the imaginary unit. We regard $L(\zeta)$ as a closed operator with compact resolvent on $L^2(\mathbf{T}^d)$ with the domain

$$D(L(\zeta)) = \{u \in H^1(\mathbf{T}^d);\, L(\zeta)u \in L^2(\mathbf{T}^d)\}.$$

$\{L(\zeta)\}$ is a holomorphic family of type (B) (cf. [Ka]). Similarly, we regard the formal adjoint $L(\zeta)^*$ of $L(\zeta)$ as a closed operator in $L^2(\mathbf{T}^d)$. By the Krein-Rutman theorem, for each $\xi \in \mathbf{R}^d$, $L(i\xi)$ has an eigenvalue $\Lambda(i\xi) \in \mathbf{R}$ of multiplicity one such that the corresponding eigenspace is generated by a positive function, which is Hölder continuous by the elliptic regularity; furthermore, $\Lambda(i\xi)$ is also an eigenvalue of $L(i\xi)^*$ of multiplicity one such that the eigenspace is generated by a positive Hölder continuous function on $\mathbf{T}^d$ (cf. [A1], [P2], [Mu], [St]). We call

$$E(\xi) := \Lambda(i\xi)$$

the principal eigenvalue of $L(i\xi)$. The following results are known (cf. [KP, Lemma 12], [Ku, Theorem 4.6.7], [A2], [P1], [LP]).

Fact AP. *The function $E(\xi)$ is real analytic and strictly concave. Its Hessian* Hess $E(\xi)$ *is negative definite for any $\xi \in \mathbf{R}^d$. The supremum* $\sup_\xi$ *$E(\xi)$ is attained only at $\xi = 0$, and $\nabla_\xi E(\xi) = 0$ if and only if $\xi = 0$. $E(\xi)$ is nondegenerate* (*algeblaically simple*).

We denote by $\Lambda(\zeta)$ the analytic continuation of $\Lambda(i\xi)$ in some neighborhood of $i\mathbf{R}^d$ in $\mathbf{C}^d$. Since $\Lambda(i\xi)$ is the nondegenerate eigenvalue of $L(i\xi)$, the analytic perturbation theory implies that $\Lambda(\zeta)$ is a nondegenerate eigenvalue of $L(\zeta)$ for ζ near $i\mathbf{R}^d$. If $\xi \in \mathbf{R}^d$, since $L(\xi)$ is selfadjoint, $\Lambda(\xi)$ is real-valued. Since the Hessian of $E(\xi)$ is negative definite, $\operatorname{Hess}\Lambda(0)$ is positive definite. Hence, there exists a sufficiently small positive number δ such that for any λ with $\Lambda(0) < \lambda < \Lambda(0) + \delta$, $\{\xi \in \mathbf{R}^d;\ \Lambda(\xi) \le \lambda\}$ is a compact and strictly convex set; furthermore, $\nabla\Lambda(\xi) \neq 0$ on X_λ for any $\lambda \in (\Lambda(0), \Lambda(0)+\delta)$, where

$$X_\lambda := \{\xi \in \mathbf{R}^d;\ \Lambda(\xi) = \lambda\}.$$

Hence, for each s in the unit sphere $\mathbf{S}^{d-1}$ there exists a unique $\xi_s \in X_\lambda$ such that $s = \nabla\Lambda(\xi_s)/|\nabla\Lambda(\xi_s)|$. Regarding X_λ as the hypersurface oriented by $N(\xi) = -\nabla\Lambda(\xi)/|\nabla\Lambda(\xi)|$, $\xi \in X_\lambda$, we denote by $K_\lambda(\xi)$ the Gauss-Kronecker curvature of X_λ at ξ. For $\xi \in X_\lambda$, let u_ξ be an eigenfunction to $L(\xi)u = \Lambda(\xi)u$. For $u \in L^2(\mathbf{T}^d)$, put $\|u\|^2 = \int_{\mathbf{T}^d} |u(x)|^2 dx$. The symbol $O(|x-y|^{-N})$ stands for a function $f(x,y)$ on $\mathbf{R}^{2d}$ satisfying $|f(x,y)| \le C|x-y|^{-N}$ on $\{|x-y| > R\}$ for some positive constants C and R independent of x, y. Let $R(z) = (L-z)^{-1}$ be the resolvent of L for z in the resolvent set. Our main theorem is the following.

Theorem 1.1. *There exists $\delta > 0$ such that for any $\Lambda(0) < \lambda < \Lambda(0) + \delta$, the limit $R(\lambda \pm i0)f(x) := \lim_{\varepsilon \downarrow 0} R(\lambda \pm i\varepsilon)f(x)$ in $L^2_{loc}(\mathbf{R}^d)$ exists for $f \in L^2(\mathbf{R}^d)$ with compact support, and the integral kernels $G_{\lambda \pm i0}(x,y)$ of $R(\lambda \pm i0)$ admit the following asymptotics as $|x-y| \to \infty$:*

(1.1)
$$G_{\lambda+i0}(x,y) = \frac{e^{i\pi(3-d)/4}}{|\nabla\Lambda(\xi_s)|\sqrt{K_\lambda(\xi_s)}} \frac{e^{i(x-y)\cdot\xi_s}}{(2\pi|x-y|)^{(d-1)/2}} \frac{u_{\xi_s}(x)\overline{u_{\xi_s}(y)}}{\|u_{\xi_s}\|^2}(1 + O(|x-y|^{-1})),$$

(1.2)
$$G_{\lambda-i0}(x,y) = \frac{e^{-i\pi(3-d)/4}}{|\nabla\Lambda(\xi_s)|\sqrt{K_\lambda(\xi_s)}} \frac{e^{-i(x-y)\cdot\xi_s}}{(2\pi|x-y|)^{(d-1)/2}} \frac{\overline{u_{\xi_s}(x)}u_{\xi_s}(y)}{\|u_{\xi_s}\|^2}(1 + O(|x-y|^{-1})),$$

where $s = (x-y)/|x-y|$.

The rest of this paper is organized as follows. In Section 2, we shall give and prove a precise version of Theorem 1.1 (Theorem 2.3). Theorem 2.3 specifies precisely the interval in which the spectral parameter λ can be contained. In Section 3, we give the asymptotics at infinity of the m-th derivative of $G_{\lambda \pm i0}(x,y)$ with respect to λ, $m = 1, 2, \cdots$ (see Theorem 3.1). In Section 4, as a by-product of the proof of the theorems, we give a direct and elementary proof of the limiting absorption principle. Finally, in Section 5, in the case $d = 1$ we calculate $G_{\lambda \pm i0}(x,y)$ and show that the limiting absorption principle holds for any λ in the interior of the spectrum. We study also the case that λ is in the resolvent set.

2. Proof of Theorem 1.1

For each $\xi \in \mathbf{R}^d$, $L(\xi)$ is a selfadjoint operator with compact resolvent, so it has discrete spectrum $\lambda_1(\xi) \leq \lambda_2(\xi) \leq \cdots$ counted with multiplicities. Each $\lambda_n(\xi)$ is continuous and $(2\pi\mathbf{Z})^d$-periodic, and called the n-th band function (see e.g. [Ku, p.161] or [Sk]). Put $\lambda_{2,min} = \min_{\xi\in\mathbf{R}^d} \lambda_2(\xi)$ and let

$$\begin{aligned} W := &\text{ the connected component of } \{\xi \in \mathbf{R}^d;\, \lambda_1(\xi) < \lambda_{2,min}\} \\ &\text{ containing } \xi = 0. \end{aligned}$$

Lemma 2.1. (i) $\sup_{\xi\in\mathbf{R}^d} \Lambda(i\xi) = \Lambda(0) = \inf_{\xi\in\mathbf{R}^d} \lambda_1(\xi) = \lambda_1(0) < \lambda_1(\xi)$ *for* $\xi \in [-\pi, \pi]^d \setminus \{0\}$.

(ii) W *is not empty. Furthermore,* $\lambda_1(\xi)$ *is a nondegenerate eigenvalue of* $L(\xi)$*,* $\xi \in W$; *real analytic on* W; *and the analytic continuation of* $\Lambda(\xi)$*.*

Proof. (i) The first equality has been seen in Fact AP. The second follows from [MT, Proposition 2.1]. The third equality and the last inequality is known for the case $a_{ij}(x) = \delta_{ij}$ in [KS, Theorem 2.1]. The proof works out similarly for our general case. (ii) By (i) W is not empty. Since $L(\xi)$ is selfadjoint, $\lambda_1(\xi)$, $\xi \in W$, is nondegenerate. The other statements follows from the nondegeneracy and the continuity of λ_1 by the analytic perturbation theory. □

Taking account of Lemma 2.1(ii), we denote $\lambda_1(\xi)$ by $\Lambda(\xi)$ for $\xi \in W$. We have $\Lambda(\xi) = \Lambda(-\xi)$ for $\xi \in W$. For $\Lambda(0) < \lambda < \lambda_{2,min}$, let

$$W_\lambda := \text{ the connected component of } \{\xi \in W; \Lambda(\xi) < \lambda\} \text{ containing } \xi = 0.$$

Put

$$\begin{aligned} I := \big\{\lambda' \in (\Lambda(0), \lambda_{2,min});\ &\text{for any } \lambda \in (\Lambda(0), \lambda'), \\ &\text{(i) } \operatorname{Hess}\Lambda(\xi) \text{ is positive definite on } W_\lambda; \\ &\text{(ii) } \cup_{m\in\mathbf{Z}^d} (W_\lambda + 2\pi m) = \{\xi \in \mathbf{R}^d; \lambda_1(\xi) < \lambda\}\big\} \end{aligned}$$

and

$$\lambda_{conv} := \sup I.$$

Lemma 2.2. *The set* I *is not empty and* $\Lambda(0) < \lambda_{conv}$*. Furthermore, for any* $\lambda \in (\Lambda(0), \lambda_{conv})$*,* $\overline{W_\lambda}$ *is compact and strictly convex.*

Proof. From the positive definiteness of $\operatorname{Hess}\Lambda(0)$, the periodicity of the function λ_1, and Lemma 2.1(i), it follows that the former part of the lemma holds. We show the latter part. First we claim that if $\overline{W_\lambda}$ is strictly convex for $\Lambda(0) < \lambda < \lambda_{conv}$, then W_λ is bounded. Suppose that there exists a sequence $\{\xi_n\} \subset W_\lambda$ such that $|\xi_n| > n$. We may assume that $\xi_n/|\xi_n|$ converges to some $\eta \in \mathbf{S}^{d-1}$. Since $0 \in W_\lambda$, the convexity of W_λ implies that $t\xi_n/|\xi_n| \in W_\lambda$ for $0 \leq t \leq |\xi_n|$. So we have $t\eta \in \overline{W_\lambda}$ for any $t \geq 0$. $\overline{W_\lambda}$ contains the ball with the center at the origin and radius $r > 0$ for some r. Hence the segment connecting $t\eta$ with each point in the ball belongs to $\overline{W_\lambda}$. Taking $t \to \infty$, we obtain that $\overline{W_\lambda}$ contains the tubular neighborhood with rudius r of the half line with the direction η. The tubular neighborhood contains some $2\pi m \neq 0, m \in \mathbf{Z}^d$. Then by the periodicity of Λ, we have the contradiction

$$0 = m \cdot (\nabla\Lambda(2\pi m) - \nabla\Lambda(0)) = 2\pi \int_0^1 m \cdot \mathrm{Hess}\Lambda(\theta 2\pi m) m d\theta > 0.$$

Next we show that $\overline{W_\lambda}$ is strictly convex for $\Lambda(0) < \lambda < \lambda_{conv}$. Since $\mathrm{Hess}\,\Lambda(0)$ is positive definite, there exists $\delta > 0$ such that for $\Lambda(0) < \lambda < \Lambda(0) + \delta$, $\overline{W_\lambda}$ is strictly convex. We claim that if $\overline{W_\lambda}$ is strictly convex for any $\Lambda(0) < \lambda < \lambda_0$ with some $\lambda_0 < \lambda_{conv}$, so is $\overline{W_{\lambda_0}}$. In fact, for any $\xi, \xi' \in \overline{W_{\lambda_0}}$, we can take $\{\xi_n\}, \{\xi'_n\} \subset W_{\lambda_0}$ so that $\xi_n \to \xi$, $\xi'_n \to \xi'$. Then the segment connecting ξ_n with ξ'_n belongs to W_{λ_0}. So the segment connecting ξ with ξ' belongs to $\overline{W_{\lambda_0}}$. Since $\mathrm{Hess}\,\Lambda$ is positive definite on $\overline{W_{\lambda_0}}$, it follows that for $0 < t < 1$, $\Lambda(t\xi + (1-t)\xi') < t\Lambda(\xi) + (1-t)\Lambda(\xi') \leq \lambda_0$. This yields that $t\xi + (1-t)\xi' \in W_{\lambda_0}$ and hence $\overline{W_{\lambda_0}}$ is strictly convex. Furthermore, we claim that if $\overline{W_{\lambda_0}}$ is strictly convex, then there exists $\lambda' < \lambda_{conv}$ such that $\overline{W_\lambda}$ is strictly convex for $\lambda_0 \leq \lambda < \lambda'$. In fact, since $\overline{W_{\lambda_0}}$ is strictly convex, there exists a compact, strictly convex neighborhood $K \subset W$ of $\overline{W_{\lambda_0}}$ such that $\mathrm{Hess}\,\Lambda$ is positive definite on K. The function Λ is strictly convex on K. Put $\lambda' = \min_{\xi \in \partial K} \Lambda(\xi)$. Then $\overline{W_\lambda}$ is strictly convex for $\lambda_0 \leq \lambda < \lambda'$. The arguments above imply that $\sup\{\lambda < \lambda_{conv}; \overline{W_\lambda} \text{ is strictly convex}\} = \lambda_{conv}$. Thus we have proved the lemma. □

Let $X_\lambda := \{\xi \in W; \Lambda(\xi) = \lambda\}$. For $s \in \mathbf{S}^{d-1}$, let ξ_s be the point such that $s = \nabla\Lambda(\xi_s)/|\nabla\Lambda(\xi_s)|$, and choose $\{e_{s,j}\}_{j=1}^{d-1} \subset \mathbf{R}^d$ such that $\{e_{s,1}, \ldots, e_{s,d-1}, s\}$ is an orthonormal basis of $\mathbf{R}^d$. Theorem 1.1 is a consequence of the following theorem.

Theorem 2.3. *Let $\Lambda(0) < \lambda < \lambda_{conv}$. Then the limit $R(\lambda \pm i0)f(x)$ in $L^2_{loc}(\mathbf{R}^d)$ exists for $f \in L^2(\mathbf{R}^d)$ with compact support, and the convergence is locally uniform in $(\Lambda(0), \lambda_{conv})$. For any $\omega \in \mathbf{S}^{d-1}$, there exist a conic neighborhood V_ω of ω and a constant $C_\omega > 0$ such that $G_{\lambda+i0}(x,y)$ admits the asymptotics*

$$\begin{aligned} G_{\lambda+i0}(x,y) = {} & \frac{|\nabla\Lambda(\xi_s)|^{(d-3)/2}}{(\det(e_{s,j} \cdot \mathrm{Hess}\,\Lambda(\xi_s) e_{s,k})_{jk})^{1/2}} \frac{e^{i\pi(3-d)/4} e^{i(x-y)\cdot\xi_s}}{(2\pi|x-y|)^{(d-1)/2}} \frac{u_{\xi_s}(x)\overline{u_{\xi_s}(y)}}{\|u_{\xi_s}\|^2} \\ & \times (1 + O(|x-y|^{-1})), \end{aligned}$$

where $s = (x-y)/|x-y|$ and $|O(|x-y|^{-1})| \leq C_\omega |x-y|^{-1}$ for any $x - y \in V_\omega$.

Proof of Theorem 1.1. Let $N(\xi) = -\nabla\Lambda(\xi)/|\nabla\Lambda(\xi)|$, $\xi \in X_\lambda$. Then we have

$$K_\lambda(\xi) = \det(-((e_j \cdot \nabla)(N(\xi) \cdot e_k))_{jk}) = \frac{\det((e_j \cdot \mathrm{Hess}\,\Lambda(\xi) e_k)_{jk})}{|\nabla\Lambda(\xi)|^{d-1}},$$

where $\{e_1, \ldots, e_{d-1}\}$ is an orthonormal basis of the tangent plane of K_λ at ξ (cf. [Th]). From this Theorem 2.3 implies (1.1). (1.2) follows from (1.1) and the relation $G_{\lambda-i0}(x, y) = \overline{G_{\lambda+i0}(y, x)}$. □

In the rest of this section we shall prove Theorem 2.3.

We denote by $\sigma(T)$ the spectrum of an operator T. Let

$$\Gamma := \{(\xi, z) \in \mathbf{R}^d \times \mathbf{C}; z \notin \sigma(L(\xi))\},$$
$$R(\xi, z) := (L(\xi) - z)^{-1} \text{ for } (\xi, z) \in \Gamma.$$

Assume $\Lambda(0) < \lambda < \lambda_{conv}$. Let $f \in L^2(\mathbf{R}^d)$ with compact support. In the same way as Proposition 2.3 of [MT] we obtain that for any $\varepsilon > 0$

$$\text{(2.1)} \quad R(\lambda + i\varepsilon)f(x) = \mathcal{U}^{-1}R(\xi, \lambda + i\varepsilon)\mathcal{U}f(x) = \int_{(-\pi,\pi]^d} F_{\lambda+i\varepsilon}(\xi, x)\,\frac{d\xi}{(2\pi)^d},$$

where

$$\mathcal{U}f(\xi, x) := \sum_{m\in\mathbf{Z}^d} f(x-m)e^{-i(x-m)\cdot\xi}, \quad f \in L^2(\mathbf{R}^d)$$
$$\mathcal{U}^{-1}g(x) := \int_{(-\pi,\pi]^d} e^{ix\cdot\xi}g(\xi, x)\,\frac{d\xi}{(2\pi)^d}, \quad g \in L^2((-\pi, \pi]^d; L^2(\mathbf{T}^d))$$

$$\text{(2.2)} \quad F_{\lambda+i\varepsilon}(\xi, x) := e^{ix\cdot\xi}R(\xi, \lambda + i\varepsilon)\mathcal{U}f(\xi, x).$$

Here we regard $R(\xi, \lambda + i\varepsilon)\mathcal{U}f(\xi, x)$ as a periodic function of x. Note that $F_{\lambda+i\varepsilon}(\xi, x)$ is the $(2\pi\mathbf{Z})^d$-periodic function of ξ (see [MT, Lemma 2.4]). Since $\lambda < \lambda_{2,min}$, there exists $\delta > 0$ such that $\Lambda(0) < \lambda - 3\delta < \lambda + 3\delta < \lambda_{conv}$. Put

$$D_\delta := \{\xi \in W; \lambda - \delta < \Lambda(\xi) < \lambda + \delta\}.$$

For $\xi \in D_\delta$,

$$\sigma(L(\xi)) \cap \{z \in \mathbf{C}; |z - \lambda| < 2\delta\} = \{\Lambda(\xi)\}.$$

We have for $(\xi, z) \in D_\delta \times \{z \in \mathbf{C}; |z - \lambda| < \delta\}$, $z \neq \Lambda(\xi)$,

$$\text{(2.3)} \quad R(\xi, z) = (\Lambda(\xi) - z)^{-1}P(\xi) + Q_z(\xi),$$

where

(2.4)
$$P(\xi) := \frac{-1}{2\pi i}\oint_{|z'-\lambda|=2\delta} R(\xi, z')dz', \quad Q_z(\xi) := \frac{1}{2\pi i}\oint_{|z'-\lambda|=2\delta} \frac{R(\xi, z')}{z' - z}dz'.$$

$P(\xi)$ and $Q_z(\xi)$ are defined for $\xi \in D_\delta$ and for $(\xi, z) \in D_\delta \times \{z \in \mathbf{C}; |z-\lambda| < \delta\}$, respectively. Note that $(\xi, z) \in \Gamma$ if and only if $(\xi + 2\pi m, z) \in \Gamma$, $m \in \mathbf{Z}^d$. So we extend $P(\xi)$ and $Q_z(\xi)$ to functions on $\cup_{m\in\mathbf{Z}^d}(D_\delta + 2\pi m)$ and $\cup_{m\in\mathbf{Z}^d}(D_\delta + 2\pi m) \times \{z \in \mathbf{C}; |z - \lambda| < \delta\}$ by (2.4).

Fix $\tilde{\xi} \in D_\delta$ and let $u_{\tilde{\xi}}$ be an eigenfunction of $L(\tilde{\xi})u = \Lambda(\tilde{\xi})u$. Then $\varphi_\xi := P(\xi)u_{\tilde{\xi}}$ is a Hölder continuous solution to $L(\xi)u = \Lambda(\xi)u$ and a $H^1(\mathbf{T}^d)$-valued analytic function in ξ (see [MT, Lemma 3.3]). Furthermore, we have $\varphi_\xi \neq 0$ for $\xi \in B(r, \tilde{\xi})$, the ball of some radius $r = r(\tilde{\xi}) > 0$ with the center $\tilde{\xi}$. Since $P(\xi)$ is the orthogonal projection onto the eigenspace, the integral kernel $p(\xi; x, y)$ of $P(\xi)$ is written as

$$p(\xi; x, y) = \varphi_\xi(x)\overline{\varphi_\xi(y)}/\|\varphi_\xi\|^2, \quad \xi \in B(r, \tilde{\xi}), \quad x, y \in \mathbf{T}^d. \tag{2.5}$$

By Lemma 3.3 (iii) of [MT], for any multi-index β

$$\sup_{\xi \in B(r,\tilde{\xi})} \|\partial_\xi^\beta p(\xi; \cdot, \cdot)\|_{C(\mathbf{T}^d \times \mathbf{T}^d)} \leq C_\beta \tag{2.6}$$

for some $C_\beta > 0$. By covering D_δ with finite balls $\{B(r(\tilde{\xi}_j), \tilde{\xi}_j); \tilde{\xi}_j \in D_\delta, 1 \leq j \leq J\}$, we have the estimate (2.6) with $\sup_{\xi \in B(r,\tilde{\xi})}$ replaced by $\sup_{\xi \in D_\delta}$.

For $\omega \in \mathbf{S}^{d-1}$ let $\xi_\omega \in X_\lambda$ be the point such that $\omega = \nabla\Lambda(\xi_\omega)/|\nabla\Lambda(\xi_\omega)|$. Note that $-\xi_\omega \in X_\lambda$ since $\Lambda(\xi) = \Lambda(-\xi)$. We take functions $\psi_j(\xi) \in C_0^\infty(D_\delta)$, $j = 1, 2, 3$, such that

(i) $\psi_1 = 1$ near ξ_ω, and $\omega \cdot \nabla\Lambda > 0$ on the support of ψ_1;
(ii) $\psi_2 = 1$ near $-\xi_\omega$, and $\omega \cdot \nabla\Lambda < 0$ on the support of ψ_2;
(iii) $\psi_3 = 1$ near $\{\xi \in X_\lambda; \omega \cdot \nabla\Lambda(\xi) = 0\}$;
(iv) $\psi(\xi) := \sum_{j=1}^3 \psi_j(\xi) = 1$ near X_λ.

Let $r_0 > 0$ be a number such that $\psi_1(\xi) = 1$ for ξ, $|\xi - \xi_\omega| < r_0$, and let V_ω be the conic neighborhood of ω:

(2.7)
$$V_\omega := \{x \in \mathbf{R}^d \setminus \{0\}; x = t\nabla\Lambda(\xi) \text{ for some } t > 0 \text{ and some } \xi,\ |\xi - \xi_\omega| < r_0/2\}.$$

We claim that for any m, $m' \in \mathbf{Z}^d$, $m \neq m'$, and any $\lambda < \lambda_{conv}$,

$$(W_\lambda + 2\pi m) \cap (W_\lambda + 2\pi m') = \emptyset.$$

In fact, suppose that $\xi \in W_\lambda \cap (W_\lambda + 2\pi m)$, $m \neq 0$. Then ξ, $\xi - 2\pi m \in W_\lambda$. By the periodicity of Λ and the convexity of W_λ, we have the contradiction

$$0 = m \cdot (\nabla\Lambda(\xi) - \nabla\Lambda(\xi - 2\pi m)) = 2\pi \int_0^1 m \cdot \mathrm{Hess}\Lambda(\theta 2\pi m + (\xi - 2\pi m)) m d\theta > 0.$$

By this claim, we extend $\psi \in C_0^\infty(D_\delta)$ to a periodic C^∞-function ψ on $\mathbf{R}^d$ by putting $\psi(\xi + 2\pi m) := \psi(\xi)$ for $\xi \in D_\delta$ and $m \in \mathbf{Z}^d$, and $\psi(\xi) = 0$ otherwise. Put

$$E_\delta := \left[\mathbf{R}^d \setminus \bigcup_{m \in \mathbf{Z}^d} (D_\delta + 2\pi m)\right] \cap (-\pi, \pi]^d.$$

By the periodicity of $F_{\lambda+i\varepsilon}$ we have

$$\begin{aligned}\int_{(-\pi,\pi]^d} F_{\lambda+i\varepsilon}(\xi,x)\,d\xi &= \int_{E_\delta} + \int_{(-\pi,\pi]^d\setminus E_\delta} F_{\lambda+i\varepsilon}(\xi,x)\,d\xi \\ &= \int_{E_\delta} + \int_{\cup_m[(-\pi,\pi]^d\cap(D_\delta+2\pi m)]} F_{\lambda+i\varepsilon}(\xi,x)\,d\xi \\ &= \int_{E_\delta} + \int_{\cup_m[((-\pi,\pi]^d-2\pi m)\cap D_\delta]} F_{\lambda+i\varepsilon}(\xi,x)\,d\xi \\ &= \int_{E_\delta} + \int_{D_\delta} F_{\lambda+i\varepsilon}(\xi,x)\,d\xi.\end{aligned}$$

Thus by (2.1), (2.2), and (2.3) we have for $\varepsilon>0$ small enough

$$R(\lambda+i\varepsilon)f(x) = \sum_{j=1}^{4} I_{j,\lambda+i\varepsilon}f(x),$$

where

$$\begin{aligned}I_{j,\lambda+i\varepsilon}f(x) &= \int_{D_\delta} \frac{\psi_j(\xi)e^{ix\cdot\xi}P(\xi)\mathcal{U}f(\xi,x)}{\Lambda(\xi)-\lambda-i\varepsilon}\frac{d\xi}{(2\pi)^d}, \quad j=1,2,3, \\ I_{4,\lambda+i\varepsilon}f(x) &= \int_{E_\delta} F_{\lambda+i\varepsilon}(\xi,x)\frac{d\xi}{(2\pi)^d} + \int_{D_\delta}(1-\psi(\xi))F_{\lambda+i\varepsilon}(\xi,x)\frac{d\xi}{(2\pi)^d} \\ &\quad + \int_{D_\delta}\psi(\xi)e^{ix\cdot\xi}Q_{\lambda+i\varepsilon}(\xi)\mathcal{U}f(\xi,x)\frac{d\xi}{(2\pi)^d}.\end{aligned}$$

We denote by $I_{j,\lambda+i\varepsilon}(x,y)$ the integral kernel of each $I_{j,\lambda+i\varepsilon}$.

First we treat $I_{1,\lambda+i\varepsilon}$. We have

$$I_{1,\lambda+i\varepsilon}(x,y) = \int_{D_\delta}\frac{e^{i(x-y)\cdot\xi}p_1(\xi;x,y)}{\Lambda(\xi)-\lambda-i\varepsilon}\frac{d\xi}{(2\pi)^d},$$

where $p_1(\xi;x,y) := \psi_1(\xi)p(\xi;x,y)$ is regarded as a periodic function of $x,y\in\mathbf{R}^d$. We choose $\{e_2',\dots,e_d'\}\subset\mathbf{R}^d$ such that $\{\omega,e_2',\dots,e_d'\}$ is an orthonormal basis of $\mathbf{R}^d$, and use the coordinates $(\xi_1,\xi')=(\xi_1,\dots,\xi_d)$ and $((x-y)_1,(x-y)')=((x-y)_1,\dots,(x-y)_d)$ such that

$$\xi=\xi_1\omega+\xi'\cdot e'=\xi_1\omega+\sum_{j=2}^{d}\xi_je_j' \text{ and } x-y=(x-y)_1\omega+(x-y)'\cdot e'. \tag{2.8}$$

Changing the integral variables from (ξ_1,ξ') to $\zeta=(\zeta_1,\zeta')$ such that $\zeta_1=\Lambda(\xi)-\lambda$, $\zeta'=\xi'$, we have

$$I_{1,\lambda+i\varepsilon}(x,y) = \int_{\mathbf{R}^d} e^{i[(x-y)_1\xi_1(\zeta)+(x-y)'\cdot\zeta']}\frac{p_1(\xi_1(\zeta)\omega+\zeta'\cdot e';x,y)}{\zeta_1-i\varepsilon}\left|\frac{\partial\xi_1}{\partial\zeta_1}\right|\frac{d\zeta}{(2\pi)^d}.$$

Since $\omega \cdot \nabla\Lambda > 0$ on the support of ψ_1, we have $0 < \partial\xi_1/\partial\zeta_1 = (\omega \cdot \nabla\Lambda(\xi_1(\zeta)\omega + \zeta' \cdot e'))^{-1} < \infty$. Note that for $\varphi \in C_0^1(\mathbf{R})$ and $\varepsilon > 0$,

$$\left| \int_{\mathbf{R}} \left[\frac{1}{x - i\varepsilon} - \left(\mathrm{p.v.}\frac{1}{x} + i\pi\delta(x) \right) \right] \varphi(x)dx \right| \leq C\varepsilon \|\varphi\|_{C^1}.$$

Taking the limit $\varepsilon \downarrow 0$, we obtain that

(2.9)

$$\begin{aligned} &I_{1,\lambda+i0}(x,y) := \lim_{\varepsilon\downarrow 0} I_{1,\lambda+i\varepsilon}(x,y) \\ &= \int_{\mathbf{R}^d} e^{i[(x-y)_1\xi_1(\zeta)+(x-y)'\cdot\zeta']} \frac{p_1(\xi_1(\zeta)\omega + \zeta' \cdot e'; x, y)}{\omega \cdot \nabla\Lambda(\xi_1(\zeta)\omega + \zeta' \cdot e')} \left(\mathrm{p.v.}\,\frac{1}{\zeta_1} + i\pi\delta(\zeta_1) \right) \frac{d\zeta}{(2\pi)^d}. \end{aligned}$$

Here the convergence is uniform with respect to (x, y) and locally uniform with respect to λ. Hence $I_{1,\lambda+i\varepsilon}f(x) \to I_{1,\lambda+i0}f(x) := \int I_{1,\lambda+i0}(x,y)f(y)dy$ in $L^2_{loc}(\mathbf{R}^d)$ locally uniformly with respect to λ. We prepare a lemma to estimate the integral, which plays a crucial role in proving Theorem 2.3.

Lemma 2.4. *Let $b(x) \in C_0^\infty(\mathbf{R})$, and $\varphi(x)$ be real-valued $C^\infty(\mathbf{R})$- function. Assume that $\varphi'(x) > 0$ on $\operatorname{supp} b$. Then for any positive integer N,*

$$\int_{-\infty}^{\infty} e^{i\nu\varphi(x)} b(x)\mathrm{p.v.}\frac{1}{x}\,dx = \pm i\pi e^{i\nu\varphi(0)} b(0) + O(|\nu|^{-N})$$

as $\nu \to \pm\infty$, where $O(|\nu|^{-N})$ satisfies the estimate

$$|O(|\nu|^{-N})| \leq C_N |\operatorname{supp} b| \|b\|_{C^{2N+1}} \|\varphi\|_{C^{2N+3}} |\nu|^{-N}$$

with some constant C_N. Here $|\operatorname{supp} b|$ is the Lebesgue measure of $\operatorname{supp} b$.

Proof. First we shall show the following: for any positive integer N,

$$\int_{-\infty}^{\infty} e^{i\nu x} b(x)\mathrm{p.v.}\frac{1}{x}\,dx = \pm i\pi b(0) + O(|\nu|^{-N}) \tag{2.10}$$

as $\nu \to \pm\infty$, where $|O(|\nu|^{-N})| \leq C_N |\nu|^{-N} \int |b^{(N+1)}(x)|dx$. In fact, the left-hand side equals

$$\begin{aligned} &\left(\mathrm{p.v.}\frac{1}{x}\right)^{\wedge} * \hat{b}(-\nu) = -i\sqrt{\pi/2}\,(\operatorname{sgn}\,\nu * \hat{b})(-\nu) \\ &\quad = -i\sqrt{\pi/2}\left(-\int_{-\infty}^{\infty} \hat{b}(\nu')d\nu' + 2\int_{-\infty}^{-\nu} \hat{b}(\nu')d\nu' \right) = i\pi b(0) + O(\nu^{-N}) \end{aligned}$$

as $\nu \to \infty$, where $\hat{b}(\nu) = \frac{1}{\sqrt{2\pi}} \int e^{-i\nu x} b(x)dx$ is the Fourier transform of b. The asymptotics for the case $\nu \to -\infty$ can be obtained similarly. Next we show the estimate in the lemma. Take $\chi(x) \in C_0^\infty((-1,1))$ such that $\chi = 1$ near the origin. We divide the integral in question into two parts

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{i\nu\varphi(x)} b(x) \mathrm{p.v.}\frac{1}{x}\, dx \\
&= \int_{-\infty}^{\infty} e^{i\nu\varphi(x)} b(x) \mathrm{p.v.}\frac{1}{x}\chi(|\nu|^{1/2}x)\, dx + \int_{-\infty}^{\infty} e^{i\nu\varphi(x)} \frac{b(x)}{x}(1-\chi(|\nu|^{1/2}x))\, dx \\
&=: J_1(\nu) + J_2(\nu).
\end{aligned}$$

Write $\varphi(x)$ as

$$\varphi(x) = \varphi(0) + x\varphi'(0) + r(x), \quad r(x) = x^2 \int_0^1 (1-\theta)\varphi''(\theta x) d\theta,$$

and change the integral variables to $y = |\nu|^{1/2}x$ to obtain

$$J_1(\nu) = e^{i\nu\varphi(0)} \int_{-\infty}^{\infty} e^{i\nu\varphi'(0)y/|\nu|^{1/2}} b_\nu(y) \mathrm{p.v.}\frac{1}{y}\, dy,$$

where $b_\nu(y) = e^{i\nu r(y/|\nu|^{1/2})} b(y/|\nu|^{1/2})\chi(y)$. Note that

$$\int_{-\infty}^{\infty} |b_\nu^{(N+1)}(y)| dy \le C_N |\mathrm{supp}\, b| \|b\|_{C^{N+1}} \|\varphi\|_{C^{N+3}}$$

with some constant $C_N > 0$ independent of $|\nu| > 1$. Thus by (2.10) we obtain

$$J_1(\nu) = e^{i\nu\varphi(0)}(\pm i\pi b(0) + O(|\nu|^{-N/2})) \text{ as } \nu \to \pm\infty,$$

with $|O(|\nu|^{-N/2})| \le C_N |\mathrm{supp}\, b| \|b\|_{C^{N+1}} \|\varphi\|_{C^{N+3}} |\nu|^{-N/2}$. Next we estimate $J_2(\nu)$. Since $\varphi' \ne 0$ on $\mathrm{supp}\, b$, it follows that

$$J_2(\nu) = (-i\nu)^{-N} \int_{-\infty}^{\infty} e^{i\nu\varphi(x)} \Phi^N \left(\frac{b(x)}{x}(1-\chi(|\nu|^{1/2}x)) \right) dx,$$

where Φ is the differential operator given by $\Phi u(x) = (u(x)/\varphi'(x))'$. Using that

$$\sup_x |\Phi^N(b(x)(1-\chi(|\nu|^{1/2}x))/x)| \le C_N \|b\|_{C^N} \|\varphi\|_{C^{N+1}} |\nu|^{(N+1)/2},$$

we have $|J_2(\nu)| \le C_N |\mathrm{supp}\, b| \|b\|_{C^N} \|\varphi\|_{C^{N+1}} |\nu|^{(-N+1)/2}$. $\square$

Note that if $x - y \in V_\omega$ (see (2.7) for the definition of V_ω) then $(x-y)_1 \ge c|(x-y)'|$ for some $c > 0$. Since $\partial\xi_1/\partial\zeta_1 > 0$ on $\mathrm{supp}\, \psi_1$, Lemma 2.4 implies that for any positive integer N and $x - y \in V_\omega$

(2.11)

$$\begin{aligned}
&I_{1,\lambda+i0}(x,y) \\
&\quad = i \int_{\mathbf{R}^{d-1}} \left(e^{i[(x-y)_1\xi_1(\zeta)+(x-y)'\cdot\zeta']} \frac{p_1(\xi_1(\zeta)\omega + \zeta'\cdot e'; x, y)}{\omega\cdot\nabla\Lambda(\xi_1(\zeta)\omega + \zeta'\cdot e')} \right)\Bigg|_{\zeta_1=0} \frac{d\zeta'}{(2\pi)^{d-1}} \\
&\qquad + O((x-y)_1^{-N}) \\
&\quad = i \int_{\mathbf{R}^{d-1}} e^{i|x-y|[s_1\xi_1(0,\zeta')+s'\cdot\zeta']} \frac{p_1(\xi_1(0,\zeta')\omega + \zeta'\cdot e'; x, y)}{\omega\cdot\nabla\Lambda(\xi_1(0,\zeta')\omega + \zeta'\cdot e')} \frac{d\zeta'}{(2\pi)^{d-1}} \\
&\qquad + O((|x-y|^{-N}),
\end{aligned}$$

where $s = (x-y)/|x-y| = s_1\omega + s' \cdot e'$.

Since $\Lambda(\xi_1(0,\zeta')\omega + \zeta' \cdot e') = \lambda$, ζ' satisfies the equation $\partial_{\zeta'}(s_1\xi_1(0,\zeta') + s' \cdot \zeta') = 0$ if and only if s is the direction of $\nabla\Lambda(\xi_1(0,\zeta')\omega + \zeta' \cdot e')$. So the equation has a unique solution $\zeta' = \zeta'_*$, and we have $\xi_s = \xi_1(0,\zeta'_*)\omega + \zeta'_* \cdot e'$. We apply the stationary phase method (cf. [H, Theorem 7.7.5]) to the integral in (2.11) to obtain that

(2.12)

$$\begin{aligned}
&I_{1,\lambda+i0}(x,y) \\
&\quad = \frac{i}{(2\pi)^{d-1}}\left(\frac{2\pi}{|x-y|}\right)^{(d-1)/2} \frac{e^{i[(x-y)_1\xi_1(0,\zeta'_*)+(x-y)'\cdot\zeta'_*]}e^{i\pi\operatorname{sgn}\operatorname{Hess}\xi_1(0,\zeta'_*)/4}}{s_1^{(d-1)/2}|\det\operatorname{Hess}\xi_1(0,\zeta'_*)|^{1/2}} \\
&\qquad \times \left(\frac{p(\xi_s;x,y)}{\omega\cdot\nabla\Lambda(\xi_s)} + O(|x-y|^{-1})\right) + O(|x-y|^{-N}).
\end{aligned}$$

Since

$$\begin{aligned}
(\operatorname{Hess}\xi_1(0,\zeta'_*))_{jk} &:= \partial_{\zeta_j}\partial_{\zeta_k}\xi_1(0,\zeta'_*) \\
&= -(\omega\cdot\nabla\Lambda(\xi_s))^{-1}(\partial_{\zeta_j}\xi_1\omega + e'_j)\cdot\operatorname{Hess}\Lambda(\xi_s)(\partial_{\zeta_k}\xi_1\omega + e'_k),
\end{aligned} \tag{2.13}$$

the Hessian is negative definite, and so $\operatorname{sgn}\operatorname{Hess}\xi_1(0,\zeta'_*) = 1-d$.

We shall show that

$$\begin{aligned}
&\omega\cdot\nabla\Lambda(\xi_s)s_1^{(d-1)/2}|\det\operatorname{Hess}\xi_1(0,\zeta'_*)|^{1/2} \\
&\qquad = |\nabla\Lambda(\xi_s)|^{-(d-3)/2}|\det(e_{s,j}\cdot\operatorname{Hess}\Lambda(\xi_s)e_{s,k})_{jk}|^{1/2},
\end{aligned} \tag{2.14}$$

where $\{e_{s,1},\dots,e_{s,d-1},s\}$ is an orthonormal basis of $\mathbf{R}^d$. By using (2.13) and $s_1 = \omega\cdot\nabla\Lambda(\xi_s)/|\nabla\Lambda(\xi_s)|$, we have that the left-hand side of (2.14) equals

$$s_1|\nabla\Lambda(\xi_s)|^{-(d-3)/2}|\det[(\partial_{\zeta_j}\xi_1\omega + e'_j)\cdot\operatorname{Hess}\Lambda(\xi_s)(\partial_{\zeta_k}\xi_1\omega + e'_k)]_{jk}|^{1/2}.$$

It suffices to show that

$$|\det[(\partial_{\zeta_j}\xi_1\omega + e'_j)\cdot\operatorname{Hess}\Lambda(\xi_s)(\partial_{\zeta_k}\xi_1\omega + e'_k)]_{jk}| = s_1^{-2}|\det(e_{s,j}\cdot\operatorname{Hess}\Lambda(\xi_s)e_{s,k})_{jk}|.$$

Since $\{\partial_{\zeta_j}\xi_1\omega + e'_j\}_{j=2}^d$ and $\{e_{s,j}\}_{j=1}^{d-1}$ are basis of the tangent space of X_λ at ξ_s, we can write $\partial_{\zeta_j}\xi_1\omega + e'_j = \sum_{k=1}^{d-1} b_{jk}e_{s,k}$, $j = 2,\dots,d$, for some $B = (b_{jk})_{j=2,\dots,d,k=1,\dots,d-1}$. Then

$$\begin{aligned}
&|\det[(\partial_{\zeta_j}\xi_1\omega + e'_j)\cdot\operatorname{Hess}\Lambda(\xi_s)(\partial_{\zeta_k}\xi_1\omega + e'_k)]_{jk}| \\
&\qquad = (\det B)^2|\det(e_{s,j}\cdot\operatorname{Hess}\Lambda(\xi_s)e_{s,k})_{jk}|.
\end{aligned}$$

On the other hand, since $\{e_{s,1},\dots,e_{s,d-1},s\}$ is an orthonormal basis, we have

$$|\det B| = |\det(s, \partial_{\zeta_2}\xi_1\omega + e'_2,\dots,\partial_{\zeta_d}\xi_1\omega + e'_d))| = 1/s_1,$$

where we used $\partial_{\zeta_j}\xi_1 = -s_j/s_1$ in the last equality. We have thus shown (2.14). Combining (2.12) with (2.14), we obtain

Lemma 2.5. *For $x-y \in V_\omega$,*

$$I_{1,\lambda+i0}(x,y) = \frac{e^{i\pi(3-d)/4}e^{i(x-y)\cdot\xi_s}}{(2\pi|x-y|)^{(d-1)/2}} \frac{|\nabla\Lambda(\xi_s)|^{(d-3)/2}p(\xi_s;x,y)}{(\det(e_{s,j}\cdot \mathrm{Hess}\,\Lambda(\xi_s)e_{s,k})_{jk})^{1/2}}(1+O(|x-y|^{-1})).$$

By (2.5) we get the main term of the asymptotics.

In the same way as above, we obtain that $I_{2,\lambda+i0}(x,y) := \lim_{\varepsilon\downarrow 0} I_{2,\lambda+i\varepsilon}(x,y)$ is equal to the right-hand side of (2.9) with $p_1(\xi;x,y)$ replaced by $p_2(\xi;x,y) := \psi_2(\xi)p(\xi;x,y)$. Here the convergence is uniform with respect to (x,y) and locally uniform with respect to λ. Hence $I_{2,\lambda+i\varepsilon}f(x) \to I_{2,\lambda+i0}f(x) := \int I_{2,\lambda+i0}(x,y)f(y)dy$ in $L^2_{loc}(\mathbf{R}^d)$ locally uniformly with respect to λ. Since $\partial\xi_1/\partial\zeta_1 = (\omega\cdot\nabla\Lambda(\xi_1(\zeta)\omega+\zeta'\cdot e'))^{-1} < 0$ on $\mathrm{supp}\,\psi_2$, Lemma 2.4 implies that the term with the factor $i\pi\delta(\zeta_1)$ cancels the one with the factor p.v.ζ_1^{-1} modulo the remainder $O((x-y)_1^{-N}) = O(|x-y|^{-N})$ for $x-y\in V_\omega$ and any positive integer N. Thus we obtain

Lemma 2.6. *For any positive integer N, $I_{2,\lambda+i0}(x,y) = O(|x-y|^{-N})$ for $x-y\in V_\omega$.*

Next we treat $I_{3,\lambda+i\varepsilon}$.

Lemma 2.7. *For any positive integer N,*

$$I_{3,\lambda+i0}(x,y) := \lim_{\varepsilon\downarrow 0} I_{3,\lambda+i\varepsilon}(x,y) = O(|x-y|^{-N})$$

for $x-y\in V_\omega$.

Proof. We have

$$I_{3,\lambda+i\varepsilon}(x,y) = \int_{D_\delta} \frac{\psi_3(\xi)e^{i(x-y)\cdot\xi}p(\xi;x,y)}{\Lambda(\xi)-\lambda-i\varepsilon}\frac{d\xi}{(2\pi)^d}.$$

By using a partition of unity for $\mathrm{supp}\,\psi_3$, it suffices to consider integrals restricted on sufficiently small integral domains, i.e., let $\chi \in C_0^\infty$ be a cutoff function such that $\chi = 1$ near a point $\xi_0 \in X_\lambda \cap \mathrm{supp}\,\psi_3$ and consider the integral

$$I_{3,\lambda+i\varepsilon,\chi}(x,y) := \int_{D_\delta} \frac{\chi(\xi)\psi_3(\xi)e^{i(x-y)\cdot\xi}p(\xi;x,y)}{\Lambda(\xi)-\lambda-i\varepsilon}\frac{d\xi}{(2\pi)^d}.$$

Let n be the outward unit normal vector to X_λ at ξ_0, i.e., $n = \nabla\Lambda(\xi_0)/|\nabla\Lambda(\xi_0)|$. Take $\{\tilde{e}_3,\dots,\tilde{e}_d\}$ such that $\{n,\omega,\tilde{e}_3,\dots,\tilde{e}_d\}$ is a basis of $\mathbf{R}^d$. Using the coordinates $(\eta_1,\eta_2,\tilde{\eta}) = (\eta_1,\eta_2,\dots,\eta_d)$, we change the integral variables such that $\xi = \eta_1 n + \eta_2\omega + \tilde{\eta}\cdot\tilde{e} = \eta_1 n + \eta_2\omega + \sum_{j=3}^d \eta_j\tilde{e}_j$. Then putting $p_3(\xi;x,y) := \chi(\xi)\psi_3(\xi)p(\xi;x,y)$, we have

$$I_{3,\lambda+i\varepsilon,\chi}(x,y) = \frac{D}{(2\pi)^d}\int_{\mathbf{R}^d} e^{i(x-y)\cdot(\eta_1 n+\eta_2\omega+\tilde{\eta}\cdot\tilde{e})}\frac{p_3(\eta_1 n+\eta_2\omega+\tilde{\eta}\cdot\tilde{e};x,y)}{\Lambda(\eta_1 n+\eta_2\omega+\tilde{\eta}\cdot\tilde{e})-\lambda-i\varepsilon}d\eta_1 d\eta_2 d\tilde{\eta},$$

where $D = |\det(n, \omega, \tilde{e}_3, \dots, \tilde{e}_d)|$. We change the integral variables from $(\eta_1, \eta_2, \tilde{\eta})$ to $\zeta = (\zeta_1, \zeta_2, \tilde{\zeta})$ such that $\zeta_1 = \Lambda(\eta_1 n + \eta_2\omega + \tilde{\eta}\cdot\tilde{e}) - \lambda$ and $(\zeta_2, \tilde{\zeta}) = (\eta_2, \tilde{\eta})$. Then

$$I_{3,\lambda+i\varepsilon,\chi}(x,y) = \int_{\mathbf{R}^d} e^{i|x-y|s\cdot(\eta_1(\zeta)n+\zeta_2\omega+\tilde{\zeta}\cdot\tilde{e})}\frac{r(\zeta)}{\zeta_1 - i\varepsilon}d\zeta_1 d\zeta_2 d\tilde{\zeta},$$

where

$$r(\zeta) := \frac{D}{(2\pi)^d}p_3(\eta_1(\zeta)n + \zeta_2\omega + \tilde{\zeta}\cdot\tilde{e}; x, y)\left|\frac{\partial\eta_1}{\partial\zeta_1}\right|. \tag{2.15}$$

We may assume that $n\cdot\nabla\Lambda > 0$ on $\operatorname{supp}\chi$ without loss of generality. Then $\partial_{\zeta_1}\eta_1 = (n\cdot\nabla\Lambda)^{-1} < \infty$. Since $\zeta_1 = \Lambda(\eta_1(\zeta)n + \zeta_2\omega + \tilde{\zeta}\cdot\tilde{e}) - \lambda$, we have $\partial_{\zeta_2}\eta_1 = -\omega\cdot\nabla\Lambda/n\cdot\nabla\Lambda$. Thus

$$\partial_{\zeta_2}(s\cdot(\eta_1(\zeta)n + \zeta_2\omega)) = -\frac{\omega\cdot\nabla\Lambda}{n\cdot\nabla\Lambda}s\cdot n + s\cdot\omega =: t(\zeta).$$

If $s \in V_\omega$ and $\eta_1(\zeta)n+\zeta_2\omega+\tilde{\zeta}\cdot\tilde{e} \in \operatorname{supp}\chi\psi_3$, then $t(\zeta) > 0$ since $V_\omega\cap\{\nabla\Lambda(\xi); \xi \in \operatorname{supp}\chi\psi_3\} = \emptyset$. Then for any positive integer N,

$$I_{3,\lambda+i\varepsilon,\chi}(x,y) = (-i|x-y|)^{-N}\int_{\mathbf{R}^d} e^{i|x-y|s\cdot(\eta_1(\zeta)n+\zeta_2\omega+\tilde{\zeta}\cdot\tilde{e})}\frac{T^N r(\zeta)}{\zeta_1 - i\varepsilon}d\zeta_1 d\zeta_2 d\tilde{\zeta},$$

where $Tr(\zeta) = \partial_{\zeta_2}(r(\zeta)/t(\zeta))$. Thus we have

$$\begin{aligned}(-i|x-y|)^N I_{3,\lambda+i0,\chi}(x,y) &:= \lim_{\varepsilon\downarrow 0}(-i|x-y|)^N I_{3,\lambda+i\varepsilon,\chi}(x,y)\\ &= \int_{\mathbf{R}^d} e^{i|x-y|s\cdot(\eta_1(\zeta)n+\zeta_2\omega+\tilde{\zeta}\cdot\tilde{e})}T^N r(\zeta)\left(\text{p.v.}\frac{1}{\zeta_1} + i\pi\delta(\zeta_1)\right)d\zeta_1 d\zeta_2 d\tilde{\zeta},\end{aligned}$$

where the convergence is uniform with respect to (x, y) and locally uniform with respect to λ. This implies that $I_{3,\lambda+i\varepsilon,\chi}f(x) \to I_{3,\lambda+i0,\chi}f(x)$ $:= \int I_{3,\lambda+i0,\chi}(x,y)f(y)dy$ in $L^2(\mathbf{R}^d)$ locally uniformly with respect to λ. Hence it follows that $I_{3,\lambda+i0,\chi}(x,y) = O(|x-y|^{-N})$. We have thus proved the lemma. □

Next we treat $I_{4,\lambda+i\varepsilon}$. Since the functions of ξ, $(1-\psi(\xi))F_{\lambda+i\varepsilon}(\xi,x)$ and

$$\psi(\xi)e^{ix\cdot\xi}Q_{\lambda+i\varepsilon}(\xi)\mathcal{U}f(\xi,x) = \frac{1}{2\pi i}\oint_{|z'-\lambda|=2\delta}\frac{\psi(\xi)F_{z'}(\xi,x)}{z'-\lambda-i\varepsilon}dz'$$

have $(2\pi\mathbf{Z})^d$-periodicity, we have

$$\begin{aligned}I_{4,\lambda+i\varepsilon}f(x) = &\int_{(-\pi,\pi]^d}(1-\psi(\xi))F_{\lambda+i\varepsilon}(\xi,x)\frac{d\xi}{(2\pi)^d}\\ &+\int_{(-\pi,\pi]^d}\psi(\xi)e^{ix\cdot\xi}Q_{\lambda+i\varepsilon}(\xi)\mathcal{U}f(\xi,x)\frac{d\xi}{(2\pi)^d}.\end{aligned}$$

Since $(1-\psi(\xi))(R(\xi,\lambda+i\varepsilon)-R(\xi,\lambda))\to 0$ in the norm of $B(L^2(\mathbf{T}^d))$ as $\varepsilon\downarrow 0$ uniformly on $\operatorname{supp}(1-\psi(\xi))$,

$$\begin{aligned}&\int_{(-\pi,\pi]^d}(1-\psi(\xi))F_{\lambda+i\varepsilon}(\xi,x)\frac{d\xi}{(2\pi)^d}=\mathcal{U}^{-1}(1-\psi(\xi))R(\xi,\lambda+i\varepsilon)\mathcal{U}f(x)\\ &\to I'_{4,\lambda+i0}f(x):=\mathcal{U}^{-1}(1-\psi(\xi))R(\xi,\lambda)\mathcal{U}f(x)\\ &\qquad\qquad=\int_{(-\pi,\pi]^d}(1-\psi(\xi))e^{ix\cdot\xi}R(\xi,\lambda)\mathcal{U}f(\xi,x)\frac{d\xi}{(2\pi)^d},\end{aligned}$$

in $L^2(\mathbf{R}^d)$ as $\varepsilon\downarrow 0$. Similarly, since $\psi(\xi)(Q_{\lambda+i\varepsilon}(\xi)-Q_\lambda(\xi))\to 0$ in the norm of $B(L^2(\mathbf{T}^d))$ as $\varepsilon\downarrow 0$ uniformly on $\operatorname{supp}\psi(\xi)$, we have

$$\begin{aligned}&\int_{(-\pi,\pi]^d}\psi(\xi)e^{ix\cdot\xi}Q_{\lambda+i\varepsilon}(\xi)\mathcal{U}f(\xi,x)\frac{d\xi}{(2\pi)^d}=\mathcal{U}^{-1}\psi(\xi)Q_{\lambda+i\varepsilon}(\xi)\mathcal{U}f(x)\\ &\to I''_{4,\lambda+i0}f(x):=\mathcal{U}^{-1}\psi(\xi)Q_\lambda(\xi)\mathcal{U}f(x)\\ &\qquad\qquad=\int_{(-\pi,\pi]^d}\psi(\xi)e^{ix\cdot\xi}Q_\lambda(\xi)\mathcal{U}f(\xi,x)\frac{d\xi}{(2\pi)^d},\end{aligned}$$

in $L^2(\mathbf{R}^d)$ as $\varepsilon\downarrow 0$. Thus $I_{4,\lambda+i0}f(x):=\lim_{\varepsilon\downarrow 0}I_{4,\lambda+i\varepsilon}f(x)=I'_{4,\lambda+i0}f(x)+I''_{4,\lambda+i0}f(x)$, where the convergence is locally uniform with respect to λ.

We claim that

$$(L-\lambda)I_{4,\lambda+i0}f(x)=f(x)-\int_{D_\delta}\psi(\xi)e^{ix\cdot\xi}P(\xi)\mathcal{U}f(\xi,x)\frac{d\xi}{(2\pi)^d}. \tag{2.16}$$

In fact, we have

$$\begin{aligned}(L-\lambda)I'_{4,\lambda+i0}f(x)&=\mathcal{U}^{-1}(L(\xi)-\lambda)(1-\psi(\xi))R(\xi,\lambda)\mathcal{U}f(x)\\ &=\mathcal{U}^{-1}(1-\psi(\xi))\mathcal{U}f(x),\end{aligned}$$

and

$$\begin{aligned}(L-\lambda)I''_{4,\lambda+i0}f(x)&=\mathcal{U}^{-1}(L(\xi)-\lambda)\psi(\xi))Q_\lambda(\xi)\mathcal{U}f(x)\\ &=\mathcal{U}^{-1}\frac{\psi(\xi)}{2\pi i}\oint_{|z'-\lambda|=2\delta}\frac{(L(\xi)-z'+z'-\lambda)R(\xi,z')\mathcal{U}f(x)}{z'-\lambda}dz'\\ &=\mathcal{U}^{-1}\psi(\xi)\mathcal{U}f(x)-\mathcal{U}^{-1}\psi(\xi)P(\xi)\mathcal{U}f(x).\end{aligned}$$

Hence we have (2.16).

Denote the integral kernel of the resolvent $R(\xi,z)$ by $R(\xi,z;x,y)$ for $(\xi,z)\in\Gamma$.

Lemma 2.8. *The function $R(\xi,z;x,y)$ is a measurable function on $\Gamma\times\mathbf{T}^d\times\mathbf{T}^d$. For each fixed $y\in\mathbf{T}^d$, $R(\xi,z;x,y)$ is the $W^{1,1}(\mathbf{T}^d)$-valued analytic function on Γ; furthermore, $R(\xi,z;x,y)\in H^1_{loc}(\mathbf{T}^d\setminus\{y\})$. For any compact set*

K in Γ and any multi-index α, there exists a constant $C_{K,\alpha} > 0$ such that for $(\xi, z) \in K$

$$|\partial_\xi^\alpha R(\xi, z; x, y)| \le \begin{cases} C_{K,\alpha}(1 + |\log|x-y||), & d = 2, \\ C_{K,\alpha}|x-y|^{2-d}, & d \ge 3. \end{cases}$$

Proof. See [GW] or [Mi], and use Cauchy's integral formula. □

Let $Q_\lambda(\xi; x, y)$, $\xi \in \cup_{m\in\mathbf{Z}^d}(D_\delta + 2\pi m)$, be the integral kernel of $Q_\lambda(\xi)$:

$$Q_\lambda(\xi; x, y) = \frac{1}{2\pi i}\oint_{|z'-\lambda|=2\delta} \frac{R(\xi, z'; x, y)}{z' - \lambda} dz'.$$

Corollary 2.9. *For each fixed $y \in \mathbf{T}^d$, $Q_\lambda(\xi; x, y)$ is the $W^{1,1}(\mathbf{T}^d)$-valued analytic function on $\cup_{m\in\mathbf{Z}^d}(D_\delta + 2\pi m)$; furthermore, $Q_\lambda(\xi; x, y) \in H^1_{loc}(\mathbf{T}^d \setminus \{y\})$. For any multi-index α, there exists a constant C_α such that for $\xi \in D_\delta$*

$$|\partial_\xi^\alpha Q_\lambda(\xi; x, y)| \le \begin{cases} C_\alpha(1 + |\log|x-y||), & d = 2, \\ C_\alpha|x-y|^{2-d}, & d \ge 3. \end{cases}$$

The integral kernel $I_{4,\lambda+i0}(x, y)$ of $I_{4,\lambda+i0}$ is written as $I_{4,\lambda+i0}(x, y) = I'_{4,\lambda+i0}(x, y) + I''_{4,\lambda+i0}(x, y)$, where

$$I'_{4,\lambda+i0}(x, y) = \int_{(-\pi,\pi]^d} \frac{d\xi}{(2\pi)^d}(1 - \psi(\xi))R(\xi, \lambda; x, y)e^{i(x-y)\cdot\xi},$$
$$I''_{4,\lambda+i0}(x, y) = \int_{(-\pi,\pi]^d} \frac{d\xi}{(2\pi)^d}\psi(\xi)Q_\lambda(\xi; x, y)e^{i(x-y)\cdot\xi}.$$

Lemma 2.10. *For any positive integer N, $I_{4,\lambda+i0}(x, y) = O(|x-y|^{-N})$.*

Combining Lemmas 2.5, 2.6, 2.7, and the lemma, we complete the proof of Theorem 2.3.

Proof. We suppose $d \ge 3$. Since the case $d = 2$ is similarly shown, we omit the proof. Let $\rho > 0$ and $y \in \mathbf{R}^d$. Put $\chi_{\rho,y}(x) := |B(\rho, y)|^{-1}\chi_{B(\rho,y)}(x)$, where $\chi_{B(\rho,y)}(x)$ is the characteristic function of the ball $B(\rho, y)$ and $|B(\rho, y)|$ is the volume. We shall show that for any positive integer N there exists C_N independent of $\rho > 0$ such that

$$|I_{4,\lambda+i0}\chi_{\rho,y}(x)| \le C_N|x-y|^{-N}. \tag{2.17}$$

We have

$$I'_{4,\lambda+i0}\chi_{\rho,y}(x)$$
$$= \int_{(-\pi,\pi]^d}(1 - \psi(\xi))R(\xi, \lambda)\left(\sum_{m\in\mathbf{Z}^d}\chi_{\rho,y}(\cdot - m)e^{i(x-\cdot+m)\cdot\xi}\right)(x)\frac{d\xi}{(2\pi)^d}.$$

Note that for any multi-index α and $m \in \mathbf{Z}^d$

$$\partial_\xi^\alpha R(\xi,\lambda) = e^{i2\pi m\cdot x}\partial_\xi^\alpha R(\xi+2\pi m,\lambda)e^{-i2\pi m\cdot y}. \tag{2.18}$$

Write $\tilde{s} = (x - \cdot + m)/|x - \cdot + m|$. By using $(i|x - \cdot + m|)^{-1}\tilde{s}\cdot\nabla_\xi e^{i(x-\cdot+m)\cdot\xi} = e^{i(x-\cdot+m)\cdot\xi}$ and the periodicity (2.18), we make integration by parts $(N+N')$-times to obtain that

$$\begin{aligned} I'_{4,\lambda+i0}\chi_{\rho,y}(x) = &\sum_{\substack{|\alpha|=N+N'\\ \alpha_1+\alpha_2=\alpha}} C_{\alpha_1,\alpha_2}\int_{(-\pi,\pi]^d}\frac{d\xi}{(2\pi)^d}\\ &\times\partial_\xi^{\alpha_1}(1-\psi(\xi))\partial_\xi^{\alpha_2}R(\xi,\lambda)\left(\sum_{m\in\mathbf{Z}^d}\chi_{\rho,y}(\cdot - m)\frac{\tilde{s}^\alpha e^{i(x-\cdot+m)\cdot\xi}}{|x-\cdot+m|^{N+N'}}\right), \end{aligned}$$

with some constants C_{α_1,α_2}. Here N' is chosen sufficiently large later. Hence, by Lemma 2.8 there exists $C_{N+N'} > 0$ independent $\rho > 0$ such that

$$|I'_{4,\lambda+i0}\chi_{\rho,y}(x)| \le C_{N+N'}|x-y|^{-N-N'}|[x]-[y]|^{2-d}, \tag{2.19}$$

where $[x] \in \mathbf{T}^d$ denotes the equivalence class of x. Similarly, by Corollary 2.9 we have (2.19) with $I'_{4,\lambda+i0}\chi_{\rho,y}(x)$ replaced by $I''_{4,\lambda+i0}\chi_{\rho,y}(x)$. These imply that for $l \in \mathbf{Z}^d$, $|l| \gg 1$, $z \in (-1/2, 1/2]^d$,

$$|I_{4,\lambda+i0}\chi_{\rho,y}(y+l+z)| \le C_{N+N'}|l|^{-N-N'}|z|^{2-d}.$$

If l and z satisfy $|l|^{-N'}|z|^{2-d} \le 1$, i.e., $|z| \ge |l|^{-N'/(d-2)}$, then

$$|I_{4,\lambda+i0}\chi_{\rho,y}(y+l+z)| \le C_N|l|^{-N}.$$

Next we consider the case $|z| < r := |l|^{-N'/(d-2)}$. Since $I_{4,\lambda+i0}\chi_{\rho,y}$ belongs to $H^1_{loc}(\mathbf{R}^d)$, and satisfies (2.16) with f replaced by $\chi_{\rho,y}$, the Hölder continuity of solutions (cf. [St, Théorème 7.2] or [GT, Theorem 8.22]) implies that there exist α, $K > 0$ independent of $y \in \mathbf{R}^d$, $l \in \mathbf{Z}^d$, $|l| \gg 1$, $\rho > 0$, such that for z, $z' \in (-1/2, 1/2]^d$

$$\begin{aligned} &|I_{4,\lambda+i0}\chi_{\rho,y}(y+l+z) - I_{4,\lambda+i0}\chi_{\rho,y}(y+l+z')|\\ &\quad\le K|z-z'|^\alpha\left(\sup_{z\in(-1/2,1/2]^d}|I_{4,\lambda+i0}\chi_{\rho,y}(y+l+z)| + M\right), \end{aligned} \tag{2.20}$$

where M is the constant

$$\begin{aligned} M &:= C\sup_{\xi\in D_\delta}\|p(\xi;\cdot,\cdot)\|_{C(\mathbf{T}^d\times\mathbf{T}^d)}\\ &\ge \left\|\int_{D_\delta}\psi(\xi)e^{-i(y+l+z)\cdot\xi}P(\xi)\mathcal{U}\chi_{\rho,y}(\xi,[y+z])\frac{d\xi}{(2\pi)^d}\right\|_{L^p((-1/2,1/2]^d)}, \quad p > d/2, \end{aligned}$$

where the L^p-norm is taken with respect to z-variable. Take z' such that $|z'| = r$ in (2.20). Then

$$\begin{aligned}
&\sup_{|z|<r} |I_{4,\lambda+i0}\chi_{\rho,y}(y+l+z)| \\
&\le C|l|^{-N} + K(2r)^\alpha \left(\sup_{z\in(-1/2,1/2]^d} |I_{4,\lambda+i0}\chi_{\rho,y}(y+l+z)| + M \right) \\
&\le C|l|^{-N} + K(2r)^\alpha \max\left(\sup_{|z|<r} |I_{4,\lambda+i0}\chi_{\rho,y}(y+l+z)|, C|l|^{-N} \right) + MK(2r)^\alpha.
\end{aligned}$$

Choose N' so large that $-\alpha N'/(d-2) \le -N$, and take $|l|$ so large that $K(2r)^\alpha < 1/2$. Then we have

$$\sup_{|z|<r} |I_{4,\lambda+i0}\chi_{\rho,y}(y+l+z)| \le C_N |l|^{-N}.$$

We thus obtain (2.17). Since

$$I_{4,\lambda+i0}\chi_{\rho,y}(x) - I_{4,\lambda+i0}(x,y) = \int (I_{4,\lambda+i0}(x,y+z) - I_{4,\lambda+i0}(x,y))\chi_{\rho,0}(z)\,dz,$$

Lebesgue's theorem implies that for fixed x, $I_{4,\lambda+i0}\chi_{\rho,y}(x) \to I_{4,\lambda+i0}(x,y)$ as a function of y in L^1_{loc} as $\rho \downarrow 0$. We have thus shown the lemma. □

Remark 2.11. We can show that $G_{\lambda+i0}$ admits the following asymptotic expansion as $|x-y| \to \infty$: There exist functions $g_j(x,y), j = 1,2,\cdots$, such that for any natural number n

$$\begin{aligned}
G_{\lambda+i0}(x,y) = &\frac{e^{i\pi(3-d)/4}}{|\nabla\Lambda(\xi_s)|\sqrt{K_\lambda(\xi_s)}} \frac{e^{i(x-y)\cdot\xi_s}}{(2\pi|x-y|)^{(d-1)/2}} \\
&\times \left(p(\xi_s;x,y) + \sum_{j=1}^n \frac{g_j(x,y)}{|x-y|^j} + O(|x-y|^{-n-1}) \right).
\end{aligned}$$

In order to prove this expansion, we have only to apply to the integral in (2.11) the stationary phase method which gives the asymptotic expansion with the higher order terms; and note Lemmas 2.6, 2.7, and Lemma 2.10. In principle, we can explicitly calculate the functions $g_j(x,y)$, which are written by using the derivatives of $\Lambda(\xi)$ and $p(\xi;x,y)$ at $\xi = \xi_s$.

3. Asymptotics of derivatives of the Green functions

Let $G^{(m)}_{\lambda\pm i0}(x,y)$ be the integral kernel of $\lim_{\varepsilon\downarrow 0} \big(\frac{d}{d\lambda}\big)^m R(\lambda \pm i\varepsilon)$. Our aim of this section is to prove the following.

Theorem 3.1. *Let $\Lambda(0) < \lambda < \lambda_{conv}$. Then $G^{(m)}_{\lambda+i0}(x,y)$, $m \ge 1$, ad-*

mits the following asymptotics as $|x-y| \to \infty$:

$$G^{(m)}_{\lambda+i0}(x,y) \tag{3.1}$$
$$= \left(\frac{i|x-y|}{|\nabla\Lambda(\xi_s)|}\right)^m \frac{e^{i\pi(3-d)/4}}{|\nabla\Lambda(\xi_s)|\sqrt{K_\lambda(\xi_s)}} \frac{e^{i(x-y)\cdot\xi_s}}{(2\pi|x-y|)^{(d-1)/2}} \frac{u_{\xi_s}(x)\overline{u_{\xi_s}(y)}}{\|u_{\xi_s}\|^2}$$
$$\times (1+O(|x-y|^{-1})),$$

where $s=(x-y)/|x-y|$.

We prove the following theorem, which clearly implies Theorem 3.1.

Theorem 3.2. *Let* $\Lambda(0)<\lambda<\lambda_{conv}$. *For any* $\omega \in \mathbf{S}^{d-1}$, *there exist a conic neighborhood* V_ω *of* ω *and a constant* $C_\omega>0$ *such that* $G^{(m)}_{\lambda+i0}(x,y)$ *satisfies the asymptotics* (3.1), *where* $|O(|x-y|^{-1})| \le C_\omega|x-y|^{-1}$ *for any* $x-y \in V_\omega$.

As will be clearly seen in the proof of the theorem, for $f \in L^2$ with compact support, the convergence of $\lim_{\varepsilon\downarrow 0}\big(\frac{d}{d\lambda}\big)^m R(\lambda\pm i\varepsilon)f$ in L^2_{loc} is locally uniform with respect to λ. Hence $\lim_{\varepsilon\downarrow 0}\big(\frac{d}{d\lambda}\big)^m R(\lambda\pm i\varepsilon)f=\big(\frac{d}{d\lambda}\big)^m R(\lambda\pm i0)f$. Let

$$Q^{(m)}_z(\xi) := \left(\frac{d}{dz}\right)^m Q_z(\xi) = \frac{m!}{2\pi i}\oint_{|z'-\lambda|=2\delta} \frac{R(\xi,z')}{(z'-z)^{m+1}}dz',$$

for $(\xi,z) \in \cup_{m\in\mathbf{Z}^d}(D_\delta+2\pi m)\times\{z\in\mathbf{C};|z-\lambda|<\delta\}$, and denote its integral kernel by $Q^{(m)}_z(\xi;x,y)$. We need a lemma which follows from Lemma 2.8 immediately. Set $(x)_+=\max\{x,0\}$ for $x\in\mathbf{R}$.

Lemma 3.3. *For any compact set* K *in* Γ, *any multi-index* α, *and* $m \ge 0$, *there exists a constant* $C_{K,\alpha,m}>0$ *such that for* $(\xi,z)\in K$

$$|\partial^\alpha_\xi \partial^m_z R(\xi,z;x,y)|$$
$$\le \begin{cases} C_{K,\alpha,m}(1+|\log|x-y||)^{(1-m)_+}, & d=2,\\ C_{K,\alpha,m}|x-y|^{-(d-2-2m)_+}, & d\ge 3,\ d\ne 2+2m,\\ C_{K,\alpha,m}(1+|\log|x-y||), & d\ge 3,\ d=2+2m.\end{cases}$$

Furthermore, for any multi-index α *and* $m\ge 0$, *there exists a constant* $C_{\alpha,m}$ *such that for* $\xi\in D_\delta$

$$|\partial^\alpha_\xi Q^{(m)}_\lambda(\xi;x,y)| \le \begin{cases} C_{\alpha,m}(1+|\log|x-y||)^{(1-m)_+}, & d=2,\\ C_{\alpha,m}|x-y|^{-(d-2-2m)_+}, & d\ge 3,\ d\ne 2+2m,\\ C_{\alpha,m}(1+|\log|x-y||), & d\ge 3,\ d=2+2m.\end{cases}$$

Proof of Theorem 3.2. We use the same notations as in Section 2. Since $R(\xi,z)$ is a bounded operator-valued holomorphic function on $\{(\xi,z)\in\mathbf{C}^d\times\mathbf{C}; z\notin\sigma(L(\xi))\}$, by (2.1) and (2.2) we have

$$\left(\frac{d}{d\lambda}\right)^m R(\lambda+i\varepsilon)f(x) = \int_{(-\pi,\pi]^d} e^{ix\cdot\xi}\left(\frac{d}{d\lambda}\right)^m R(\xi,\lambda+i\varepsilon)\mathcal{U}f(\xi,x)\frac{d\xi}{(2\pi)^d}.$$

By (2.3) we have

$$\left(\frac{d}{d\lambda}\right)^m R(\xi,\lambda+i\varepsilon)=\frac{m!P(\xi)}{(\Lambda(\xi)-\lambda-i\varepsilon)^{m+1}}+Q^{(m)}_{\lambda+i\varepsilon}(\xi),\quad \xi\in D_\delta.$$

The same argument as in Section 2 shows that for $f\in L^2(\mathbf{R}^d)$ with compact support,

$$\left(\frac{d}{d\lambda}\right)^m R(\lambda+i\varepsilon)f(x)=\sum_{j=1}^{4}I^{(m)}_{j,\lambda+i\varepsilon}f(x),$$

where

$$\begin{aligned}
I^{(m)}_{j,\lambda+i\varepsilon}f(x)&:=m!\int_{D_\delta}\frac{\psi_j(\xi)e^{ix\cdot\xi}P(\xi)\mathcal{U}f(\xi,x)}{(\Lambda(\xi)-\lambda-i\varepsilon)^{m+1}}\frac{d\xi}{(2\pi)^d},\quad j=1,2,3,\\
I^{(m)}_{4,\lambda+i\varepsilon}f(x)&:=m!\int_{(-\pi,\pi]^d}(1-\psi(\xi))e^{ix\cdot\xi}R(\xi,\lambda+i\varepsilon)^{m+1}\mathcal{U}f(\xi,x)\frac{d\xi}{(2\pi)^d}\\
&\qquad+\int_{D_\delta}\psi(\xi)e^{ix\cdot\xi}Q^{(m)}_{\lambda+i\varepsilon}(\xi)\mathcal{U}f(\xi,x)\frac{d\xi}{(2\pi)^d}.
\end{aligned}$$

Here ψ_j, $j=1,2,3$, and $\psi=\sum_{j=1}^3\psi_j$ are the cutoff functions given below (2.6).

First we treat $I^{(m)}_{1,\lambda+i\varepsilon}$. By the same calculation as stated in the case $I_{1,\lambda+i\varepsilon}$ we obtain that the integral kernel $I^{(m)}_{1,\lambda+i\varepsilon}(x,y)$ of $I^{(m)}_{1,\lambda+i\varepsilon}$ equals

$$\begin{aligned}
&I^{(m)}_{1,\lambda+i\varepsilon}(x,y)\\
&=m!\int_{\mathbf{R}^d}e^{i[(x-y)_1\xi_1(\zeta)+(x-y)'\cdot\zeta']}\frac{p_1(\xi_1(\zeta)\omega+\zeta'\cdot e';x,y)}{\omega\cdot\nabla\Lambda(\xi_1(\zeta)s+\zeta'\cdot e')}(\zeta_1-i\varepsilon)^{-m-1}\frac{d\zeta}{(2\pi)^d}\\
&=\int_{\mathbf{R}^d}\partial^m_{\zeta_1}\left(e^{i[(x-y)_1\xi_1(\zeta)+(x-y)'\cdot\zeta']}\frac{p_1(\xi_1(\zeta)\omega+\zeta'\cdot e';x,y)}{\omega\cdot\nabla\Lambda(\xi_1(\zeta)\omega+\zeta'\cdot e')}\right)(\zeta_1-i\varepsilon)^{-1}\frac{d\zeta}{(2\pi)^d},
\end{aligned}$$

where $p_1(\xi;x,y):=\psi_1(\xi)p(\xi;x,y)$. Taking the limit $\varepsilon\downarrow 0$, we have

(3.2)

$$\begin{aligned}
&I^{(m)}_{1,\lambda+i0}(x,y):=\lim_{\varepsilon\downarrow 0}I^{(m)}_{1,\lambda+i\varepsilon}(x,y)\\
&=\int_{\mathbf{R}^d}\partial^m_{\zeta_1}\left(e^{i[(x-y)_1\xi_1(\zeta)+(x-y)'\cdot\zeta']}\frac{p_1(\xi_1(\zeta)\omega+\zeta'\cdot e';x,y)}{\omega\cdot\nabla\Lambda(\xi_1(\zeta)\omega+\zeta'\cdot e')}\right)\\
&\times\left(\text{p.v.}\frac{1}{\zeta_1}+i\pi\delta(\zeta_1)\right)\frac{d\zeta}{(2\pi)^d}\\
&=\int_{\mathbf{R}^d}\frac{d\zeta}{(2\pi)^d}e^{i[(x-y)_1\xi_1(\zeta)+(x-y)'\cdot\zeta']}\left(\text{p.v.}\frac{1}{\zeta_1}+i\pi\delta(\zeta_1)\right)\\
&\times\left((i(x-y)_1\partial_{\zeta_1}\xi_1)^m\frac{p_1(\xi_1(\zeta)\omega+\zeta'\cdot e';x,y)}{\omega\cdot\nabla\Lambda(\xi_1(\zeta)\omega+\zeta'\cdot e')}+\sum_{j=0}^{m-1}(x-y)_1^j a_j^{(m)}(\zeta;x,y)\right),
\end{aligned}$$

where $a_j^{(m)}(\zeta;x,y)$ are $C(\mathbf{T}^d\times\mathbf{T}^d)$-valued smooth functions. Since $\partial_{\zeta_1}\xi_1 = (\omega\cdot\nabla\Lambda(\xi_1(\zeta)\omega+\zeta'\cdot e'))^{-1}>0$ on $\operatorname{supp}\psi_1$, Lemma 2.4 implies that for $x-y\in V_\omega$ and any positive N,

$$\begin{aligned} I^{(m)}_{1,\lambda+i0}(x,y) = i\int_{\mathbf{R}^{d-1}} \frac{d\zeta'}{(2\pi)^{d-1}} &\Bigg[e^{i[(x-y)_1\xi_1(\zeta)+(x-y)'\cdot\zeta']} \\ &\times\Bigg((i(x-y)_1)^m \frac{p_1(\xi_1(\zeta)s+\zeta'\cdot e';x,y)}{(\omega\cdot\nabla\Lambda(\xi_1(\zeta)\omega+\zeta'\cdot e'))^{m+1}} \\ &+\sum_{j=0}^{m-1}(x-y)_1^j a_j^{(m)}(\zeta;x,y)\Bigg)\Bigg]\Bigg|_{\zeta_1=0} + O(|x-y|^{-N}). \end{aligned} \tag{3.3}$$

Applying the stationary phase method, we have for $x-y\in V_\omega$

$$\begin{aligned} &I^{(m)}_{1,\lambda+i0}(x,y) \\ &\quad= \left(\frac{i|x-y|}{|\nabla\Lambda(\xi_s)|}\right)^m \frac{e^{i\pi(3-d)/4}e^{i(x-y)\cdot\xi_s}}{(2\pi|x-y|)^{(d-1)/2}} \frac{|\nabla\Lambda(\xi_s)|^{(d-3)/2}}{(\det(e_{s,j}\cdot\operatorname{Hess}\Lambda(\xi_s)e_{s,k})_{jk})^{1/2}} \\ &\quad\times \frac{u_{\xi_s}(x)\overline{u_{\xi_s}(y)}}{\|u_{\xi_s}\|^2}(1+O(|x-y|^{-1})), \end{aligned}$$

which gives the main term. Here we have used $(x-y)_1/\omega\cdot\nabla\Lambda(\xi_s) = |x-y|/|\nabla\Lambda(\xi_s)|$.

In the same way as above, we obtain that $I^{(m)}_{2,\lambda+i0}(x,y) := \lim_{\varepsilon\downarrow 0} I^{(m)}_{2,\lambda+i\varepsilon}(x,y)$ is equal to the right-hand side of (3.2) with p_1 replaced by p_2. Since

$$\partial_{\zeta_1}\xi_1 = (\omega\cdot\nabla\Lambda(\xi_1(\zeta)\omega+\zeta'\cdot e'))^{-1}<0 \text{ on } \operatorname{supp}\psi_2,$$

Lemma 2.4 implies that the term with the factor $i\pi\delta(\zeta_1)$ cancels the one with the factor $\mathrm{p.v.}\zeta_1^{-1}$ modulo the remainder $O(|x-y|^{-N})$ for $x-y\in V_\omega$. Thus we obtain that for any positive integer N,

$$I^{(m)}_{2,\lambda+i0}(x,y) = O(|x-y|^{-N}) \quad \text{for } x-y\in V_\omega. \tag{3.4}$$

Next we treat $I^{(m)}_{3,\lambda+i\varepsilon}$. Let χ be the cutoff function appeared in the proof of Lemma 2.7. By the same calculation as stated in the proof of Lemma 2.7, it suffices to estimate the quantity

$$I^{(m)}_{3,\lambda+i\varepsilon,\chi}(x,y) := m!\int_{\mathbf{R}^d} e^{i|x-y|s\cdot(\eta_1(\zeta)n+\zeta_2\omega+\tilde\zeta\cdot\tilde e)} \frac{r(\zeta)}{(\zeta_1-i\varepsilon)^{m+1}}d\zeta_1 d\zeta_2 d\tilde\zeta,$$

where $r(\zeta)$ is given in (2.15). For any positive integer N and $x-y\in V_\omega$,

$$\begin{aligned} &I^{(m)}_{3,\lambda+i\varepsilon,\chi}(x,y) \\ &\quad= m!(-i|x-y|)^{-N}\int_{\mathbf{R}^d} e^{i|x-y|s\cdot(\eta_1(\zeta)n+\zeta_2\omega+\tilde\zeta\cdot\tilde e)} \frac{T^N r(\zeta)}{(\zeta_1-i\varepsilon)^{m+1}}d\zeta_1 d\zeta_2 d\tilde\zeta \\ &\quad= (-i|x-y|)^{-N}\int_{\mathbf{R}^d} \frac{\partial^m_{\zeta_1}[e^{i|x-y|s\cdot(\eta_1(\zeta)n+\zeta_2\omega+\tilde\zeta\cdot\tilde e)}T^N r(\zeta)]}{\zeta_1-i\varepsilon}d\zeta_1 d\zeta_2 d\tilde\zeta, \end{aligned}$$

where T is the operator given in the proof of Lemma 2.7. Thus we have $\lim_{\varepsilon\downarrow 0} I^{(m)}_{3,\lambda+i\varepsilon,\chi}(x,y) = O(|x-y|^{-N+m})$ for $x-y\in V_\omega$. Hence for any positive N,

$$I^{(m)}_{3,\lambda+i0}(x,y) := \lim_{\varepsilon\downarrow 0} I^{(m)}_{3,\lambda+i\varepsilon}(x,y) = O(|x-y|^{-N}), \quad x-y\in V_\omega. \tag{3.5}$$

Next we treat $I^{(m)}_{4,\lambda+i\varepsilon}$. In the same way as stated in the case of $I_{4,\lambda+i\varepsilon}$ we have

$$I^{(m)}_{4,\lambda+i0}f(x) := \lim_{\varepsilon\downarrow 0} I^{(m)}_{4,\lambda+i\varepsilon}f(x) = I'^{(m)}_{4,\lambda+i0}f(x) + I''^{(m)}_{4,\lambda+i0}f(x),$$

where

$$\begin{aligned} I'^{(m)}_{4,\lambda+i0}f(x) &:= m!\,\mathcal{U}^{-1}(1-\psi(\xi))R(\xi,\lambda)^{m+1}\mathcal{U}f(x), \\ I''^{(m)}_{4,\lambda+i0}f(x) &:= \mathcal{U}^{-1}\psi(\xi)Q^{(m)}_\lambda(\xi)\mathcal{U}f(x). \end{aligned}$$

We claim that for $m\geq 1$

$$(L-\lambda)I^{(m)}_{4,\lambda+i0}f(x) = mI^{(m-1)}_{4,\lambda+i0}f(x). \tag{3.6}$$

In fact

$$\begin{aligned} (L-\lambda)I'^{(m)}_{4,\lambda+i0}f(x) &= m!\,\mathcal{U}^{-1}(L(\xi)-\lambda)(1-\psi(\xi))R(\xi,\lambda)^{m+1}\mathcal{U}f(x) \\ &= m!\mathcal{U}^{-1}(1-\psi(\xi))R(\xi,\lambda)^{m}\mathcal{U}f(x) = mI'^{(m-1)}_{4,\lambda+i0}f(x), \\ (L-\lambda)I''^{(m)}_{4,\lambda+i0}f(x) &= \mathcal{U}^{-1}\psi(\xi)(L(\xi)-\lambda)Q^{(m)}_\lambda(\xi)\mathcal{U}f(x) \\ &= \mathcal{U}^{-1}\psi(\xi)\frac{m!}{2\pi i}\oint\frac{(L(\xi)-z'+z'-\lambda)R(\xi,z')}{(z'-\lambda)^{m+1}}dz'\mathcal{U}f(x) \\ &= m\mathcal{U}^{-1}\psi(\xi)Q^{(m-1)}_\lambda(\xi)\mathcal{U}f(x) = mI''^{(m-1)}_{4,\lambda+i0}f(x). \end{aligned}$$

We shall show that for any positive integer N there exists $C_{N,m}$ independent of $\rho>0$ such that

$$|I^{(m)}_{4,\lambda+i0}\chi_{\rho,y}(x)| \leq C_{N,m}|x-y|^{-N}, \tag{3.7}$$

where $\chi_{\rho,y}$ is defined in the proof of Lemma 2.10. Once this is proved, for any positive integer N the integral kernel $I^{(m)}_{4,\lambda+i0}(x,y)$ of $I^{(m)}_{4,\lambda+i0}$ satisfies the estimate

$$I^{(m)}_{4,\lambda+i0}(x,y) = O(|x-y|^{-N}), \tag{3.8}$$

which can be proved in the same way as in the proof of Lemma 2.10. We show (3.7) by induction on $m\geq 0$. We have already shown (3.7) in the case $m=0$ in Section 2, since $I^{(0)}_{4,\lambda+i0}\chi_{\rho,y}(x) = I_{4,\lambda+i0}\chi_{\rho,y}(x)$. Let $m\geq 1$. In the same way as in the proof of Lemma 2.10, using (2.18) and Lemma 3.3, we obtain that for any positive integers N, N', there exists $C_{N+N'}$ such that for $l \in \mathbf{Z}^d$, $|l| \gg 1$, $z \in (-1/2, 1/2]^d$,

$$
\begin{aligned}
&|I^{(m)}_{4,\lambda+i0}\chi_{\rho,y}(y+l+z)| \\
&\le \begin{cases} C_{N+N'}|l|^{-N-N'}C_m, & d = 2, \\ C_{N+N'}|l|^{-N-N'}|z|^{-(d-2-2m)_+}, & d \ge 3,\ d \ne 2+2m, \\ C_{N+N'}|l|^{-N-N'}(1+|\log|z||), & d \ge 3,\ d = 2+2m. \end{cases}
\end{aligned}
$$

Hence, indeed, we have (3.7) in the case $d = 2$ or the case $d < 2+2m$. In the following we assume that $d > 2+2m$. The case $d = 2+2m$ is similarly shown. In the case $|z| \ge r := |l|^{-N'/(d-2-2m)}$, we have $|I^{(m)}_{4,\lambda+i0}\chi_{\rho,y}(y+l+z)| \le C_N|l|^{-N}$. Consider the case $|z| < r$. By the induction hypothesis, we can choose a constant M_m which is independent of l, $|l| \gg 1$, and satisfies

$$
M_m \ge m\|I^{(m-1)}_{4,\lambda+i0}\chi_{\rho,y}(y+l+\cdot)\|_{L^p((-1/2,1/2]^d)}, \quad p > d/2.
$$

Since $I^{(m)}_{4,\lambda+i0}\chi_{\rho,y}$ belongs to $H^1_{loc}(\mathbf{R}^d)$, and satisfies (3.6) with f replaced by $\chi_{\rho,y}$, the Hölder continuity of solutions implies that there exist α, $K > 0$ independent of $y \in \mathbf{R}^d$, $l \in \mathbf{Z}^d$, $|l| \gg 1$, $\rho > 0$, such that for z, $z' \in (-1/2, 1/2]^d$

$$
\begin{aligned}
&|I^{(m)}_{4,\lambda+i0}\chi_{\rho,y}(y+l+z) - I^{(m)}_{4,\lambda+i0}\chi_{\rho,y}(y+l+z')| \\
&\quad \le K|z-z'|^\alpha(\sup_{z\in(-1/2,1/2]^d}|I^{(m)}_{4,\lambda+i0}\chi_{\rho,y}(y+l+z)| + M_m).
\end{aligned}
\tag{3.9}
$$

Take z' such that $|z'| = r$ in (3.9). Then

$$
\begin{aligned}
&\sup_{|z|<r}|I^{(m)}_{4,\lambda+i0}\chi_{\rho,y}(y+l+z)| \\
&\le C|l|^{-N} + K(2r)^\alpha(\sup_{z\in(-1/2,1/2]^d}|I^{(m)}_{4,\lambda+i0}\chi_{\rho,y}(y+l+z)| + M_m) \\
&\le C|l|^{-N} + K(2r)^\alpha \max(\sup_{|z|<r}|I^{(m)}_{4,\lambda+i0}\chi_{\rho,y}(y+l+z)|, C|l|^{-N}) + KM_m(2r)^\alpha.
\end{aligned}
$$

Choose N' so large that $-\alpha N'/(d-2-2m) \le -N$, and take $|l|$ so large that $K(2r)^\alpha < 1/2$. Then we have

$$
\sup_{|z|<r}|I^{(m)}_{4,\lambda+i0}\chi_{\rho,y}(y+l+z)| \le C_N|l|^{-N}.
$$

We thus obtain (3.7). We have proved Theorem 3.2. □

4. The limiting absorption principle

We denote by B_s, $s \in \mathbf{R}$, the space

$$
B_s := \left\{ v \in L^2_{loc}(\mathbf{R}^d); \|v\|_{B_s} := \sum_{j=1}^{\infty} R_j^s \left(\int_{R_{j-1}<|x|<R_j} |v(x)|^2 dx \right)^{1/2} < \infty \right\},
$$

where $R_0 = 0$, $R_j = 2^{j-1}$ for $j > 0$. The dual space B_s^* is the set

$$B_s^* = \left\{ v \in L^2_{loc}(\mathbf{R}^d); \|v\|_{B_s^*} := \sup_{j\geq 1} R_j^{-s} \left(\int_{R_{j-1}<|x|<R_j} |v(x)|^2 dx \right)^{1/2} < \infty \right\}.$$

The main theorem of this section is the following

Theorem 4.1. *Let $\Lambda(0) < \lambda < \lambda_{conv}$. The operator $\left(\frac{d}{d\lambda}\right)^m R(\lambda \pm i0)$, $m \geq 0$, is bounded from $B_{\frac{1}{2}+m}$ to $B^*_{\frac{1}{2}+m}$.*

Proof. Since for $|x-y| < 1$

$$|G^{(m)}_{\lambda+i0}(x,y)| \leq \begin{cases} C_m(1+|\log|x-y||)^{(1-m)_+}, & d=2, \\ C_m|x-y|^{-(d-2-2m)_+}, & d\geq 3, \quad d\neq 2+2m, \\ C_m(1+|\log|x-y||), & d\geq 3, \quad d=2+2m, \end{cases}$$

the operator with the integral kernel $G^{(m)}_{\lambda+i0}(x,y)\chi_1(|x-y|)$ is bounded on $L^2(\mathbf{R}^d)$, where $\chi_1(r)$ is a $C^\infty([0,\infty))$-function such that $\chi_1(r) = 1$ for $0 \leq r \leq 1/2$, and $\chi_1(r) = 0$ for $1 \leq r$.

For $|x-y|$ large, we use the decomposition $G^{(m)}_{\lambda+i0}(x,y) = \sum_{j=1}^{4} I^{(m)}_{j,\lambda+i0}(x,y)$, $m \geq 1$, as in Section 3. (The case $m = 0$ is proved in the same way by using the decomposition as in Section 2.) Let $\omega \in \mathbf{S}^{d-1}$, and V_ω be such a conic neighborhood of ω as in Theorem 3.2. Let χ_2 be a C^∞-function on $\mathbf{S}^{d-1}$ such that $\chi_2 = 1$ near ω and $\chi_2 = 0$ outside of $V_\omega \cap \mathbf{S}^{d-1}$. By (3.4), (3.5), and (3.8), we have for any positive N and $x - y \in V_\omega$,

$$\sum_{j=2}^{4} I^{(m)}_{j,\lambda+i0}(x,y) = O(|x-y|^{-N}). \tag{4.1}$$

Thus the operator with the kernel $(1-\chi_1(|x-y|))\chi_2(\frac{x-y}{|x-y|})\sum_{j=2}^4 I^{(m)}_{j,\lambda+i0}(x,y)$ is bounded on $L^2(\mathbf{R}^d)$.

By using the coordinates $((x-y)_1, (x-y)')$ as in (2.8), we have by (3.3) that for any positive N and $x-y \in V_\omega$, $I^{(m)}_{1,\lambda+i0}(x,y)$ has the expression

$$\begin{aligned} &I^{(m)}_{1,\lambda+i0}(x,y) \\ &\quad = \int_{\mathbf{R}^{d-1}} e^{i[(x-y)_1\xi_1(0,\zeta')+(x-y)'\cdot\zeta']} \sum_{j=0}^{m} (x-y)_1^j \tilde{a}^{(m)}_j(\zeta';x,y)\, d\zeta' \\ &\qquad + O(|x-y|^{-N}). \end{aligned} \tag{4.2}$$

Here, since $p(\xi;x,y) = \varphi_\xi(x)\overline{\varphi_\xi(y)}/\|\varphi_\xi\|^2$ for some $C(\mathbf{T}^d)$-valued smooth function φ_ξ, $\tilde{a}^{(m)}_j(\zeta';x,y)$ is written as

$$\tilde{a}^{(m)}_j(\zeta';x,y) = \sum_{l=1}^{L} v^{(m)}_{j,l}(\zeta',x)w^{(m)}_{j,l}(\zeta',y),$$

where $v_{j,l}^{(m)}(\zeta', x)$, $w_{j,l}^{(m)}(\zeta', x) \in C_0^\infty(\mathbf{R}^{d-1}; C(\mathbf{T}^d))$ and for any multi-index α,

$$\sup_{\zeta'} \|\partial_{\zeta'}^\alpha v_{j,l}^{(m)}(\zeta', \cdot)\|_{C(\mathbf{T}^d)} + \sup_{\zeta'} \|\partial_{\zeta'}^\alpha w_{j,l}^{(m)}(\zeta', \cdot)\|_{C(\mathbf{T}^d)} \le C_{m,\alpha}. \tag{4.3}$$

Let

$$\begin{aligned} Kf(x) := &\int_{\mathbf{R}^d} (1 - \chi_1(|x-y|))\chi_2(\frac{x-y}{|x-y|})(x-y)_1^j \\ &\times \left(\int_{\mathbf{R}^{d-1}} e^{i[(x-y)_1\xi_1(0,\zeta') + (x-y)'\cdot\zeta']} v_{j,l}^{(m)}(\zeta', x) w_{j,l}^{(m)}(\zeta', y) d\zeta' \right) f(y) dy. \end{aligned}$$

By (4.1) and (4.2), the theorem follows from the following estimate: for any $f, g \in C_0^\infty(\mathbf{R}^d)$,

$$|\langle Kf, g\rangle| = \left| \int_{\mathbf{R}^d} Kf(x) g(x) dx \right| \le C_{m,j} \|f\|_{B_{\frac{1}{2}+j}} \|g\|_{B_{\frac{1}{2}+j}}.$$

Let us show this estimate. For simplicity, we write $\chi_3(x-y) := (1 - \chi_1(|x-y|))\chi_2(\frac{x-y}{|x-y|})$, $v(\zeta', x) := v_{j,l}^{(m)}(\zeta', x)$, and $w(\zeta', y) := w_{j,l}^{(m)}(\zeta', y)$. Put

$$\begin{aligned} \hat{\chi}_3(x_1, \xi') &:= (2\pi)^{-(d-1)/2} \int_{\mathbf{R}^{d-1}} e^{-ix'\cdot\xi'} \chi_3(x_1, x') dx', \\ \hat{F}(\zeta', y_1, \xi') &:= (2\pi)^{-(d-1)/2} \int_{\mathbf{R}^{d-1}} e^{-iy'\cdot\xi'} w(\zeta', y_1, y') f(y_1, y') dy', \\ \hat{G}(\zeta', x_1, \xi') &:= (2\pi)^{-(d-1)/2} \int_{\mathbf{R}^{d-1}} e^{-ix'\cdot\xi'} \overline{v(\zeta', x_1, x') g(x_1, x')} dx'. \end{aligned}$$

Then by Planchrel's formula with respect to x'-variable,

$$\begin{aligned} \langle Kf, g\rangle = &(2\pi)^{(d-1)/2} \int dx_1 dy_1 d\zeta' d\xi' (x-y)_1^j e^{i(x-y)_1\xi_1(0,\zeta')} \hat{\chi}_3(x_1 - y_1, \xi' - \zeta') \\ &\times \hat{F}(\zeta', y_1, \xi') \overline{\hat{G}(\zeta', x_1, \xi')} \\ = &(2\pi)^{(d-1)/2} \int dx_1 dy_1 d\eta' d\xi' (x-y)_1^{j-d+1} e^{i(x-y)_1\xi_1(0,\xi' - \eta'/(x-y)_1)} \\ &\times \hat{\chi}_3(x_1 - y_1, \eta'/(x-y)_1) \hat{F}(\xi' - \eta'/(x-y)_1, y_1, \xi') \\ &\times \overline{\hat{G}(\xi' - \eta'/(x-y)_1, x_1, \xi')}. \end{aligned}$$

Hence

$$\begin{aligned} |\langle Kf, g\rangle| \le C &\int dx_1 dy_1 |x_1 - y_1|^{j-d+1} \|\hat{\chi}_3((x-y)_1, \cdot/(x-y)_1)\|_{L^1(\mathbf{R}^{d-1})} \\ \times &\sup_{\eta'} \|\hat{F}(\cdot - \eta'/(x-y)_1, y_1, \cdot)\|_{L^2(\mathbf{R}^{d-1})} \sup_{\eta'} \|\hat{G}(\cdot - \eta'/(x-y)_1, x_1, \cdot)\|_{L^2(\mathbf{R}^{d-1})}. \end{aligned}$$

We claim that for some constant C independent of y_1 and x_1,

$$\sup_{\eta'} \|\hat{F}(\cdot - \eta'/(x-y)_1, y_1, \cdot)\|_{L^2(\mathbf{R}^{d-1})} \le C\|f(y_1, \cdot)\|_{L^2(\mathbf{R}^{d-1})},$$
$$\sup_{\eta'} \|\hat{G}(\cdot - \eta'/(x-y)_1, x_1, \cdot)\|_{L^2(\mathbf{R}^{d-1})} \le C\|g(x_1, \cdot)\|_{L^2(\mathbf{R}^{d-1})}.$$

In fact, for any $h \in C_0^\infty(\mathbf{R}^{d-1})$,

$$\begin{aligned}
&\int_{\mathbf{R}^{d-1}} \overline{h(\xi')}\hat{F}(\xi' - \eta'/(x-y)_1, y_1, \xi')d\xi' \\
&\quad = (2\pi)^{-(d-1)/2} \int dy' f(y_1, y') \int e^{-iy'\cdot\xi'} w(\xi' - \eta'/(x-y)_1, y_1, y')\overline{h(\xi')}d\xi' \\
&\quad = (2\pi)^{-(d-1)} \int dy' f(y_1, y') \\
&\qquad \times \iint e^{-i(y'-x')\cdot\xi'} w(\xi' - \eta'/(x-y)_1, y_1, y')\overline{\check{h}(x')}dx'd\xi',
\end{aligned}$$

where $\check{h}$ is the inverse Fourier transform of h. By (4.3) and the integration by parts, we have for any positive N

$$\left|\int e^{-i(y'-x')\cdot\xi'} w(\xi' - \eta'/(x-y)_1, y_1, y')d\xi'\right| \le C(1 + |x' - y'|)^{-N}.$$

Thus we have

$$\left|\int_{\mathbf{R}^{d-1}} \overline{h(\xi')}\hat{F}(\xi' - \eta'/(x-y)_1, y_1, \xi')d\xi'\right| \le C\|h\|_{L^2(\mathbf{R}^{d-1})}\|f(y_1, \cdot)\|_{L^2(\mathbf{R}^{d-1})}.$$

The inequality for $\hat{G}$ follows in the same way as above. Furthermore, we claim that for some constant C independent of $(x-y)_1$,

$$(x-y)_1^{1-d}\|\hat{\chi}_3((x-y)_1, \cdot/(x-y)_1)\|_{L^1(\mathbf{R}^{d-1})} \le C.$$

In fact, we have for $t := (x-y)_1$ large

$$\begin{aligned}
t^{1-d}\hat{\chi}_3(t, \eta'/t) &= t^{1-d} \int e^{-i\eta'\cdot z'/t} \chi_2\Big(\frac{t}{(t^2 + z'^2)^{1/2}}, \frac{z'}{(t^2 + z'^2)^{1/2}}\Big)dz' \\
&= \int e^{-i\eta'\cdot z'} \chi_2\Big(\frac{1}{(1 + z'^2)^{1/2}}, \frac{z'}{(1 + z'^2)^{1/2}}\Big)dz',
\end{aligned}$$

which is a rapidly decreasing function of η' independent of t. Hence we have

$$\begin{aligned}
&|\langle Kf, g\rangle| \\
&\quad \le C \int dx_1 dy_1 (x-y)_1^j \|f(y_1, \cdot)\|_{L^2(\mathbf{R}^{d-1})}\|g(x_1, \cdot)\|_{L^2(\mathbf{R}^{d-1})} \\
&\quad \le C \int (1 + |x_1|)^j \|g(x_1, \cdot)\|_{L^2(\mathbf{R}^{d-1})}dx_1 \int (1 + |y_1|)^j \|f(y_1, \cdot)\|_{L^2(\mathbf{R}^{d-1})}dy_1 \\
&\quad \le C\|f\|_{B_{\frac{1}{2}+j}}\|g\|_{B_{\frac{1}{2}+j}},
\end{aligned}$$

where the last inequality is shown in the same way as Theorem 2.4 of [AH]. We have thus proved the theorem. □

5. The one dimensional case

In the one dimensional case we shall show that the Green functions $G_{\lambda\pm i0}$ are written as a product of an exponential function and a periodic function, and that the limiting absorption principle holds for all λ in the interior of the spectrum. We shall also calculate the resolvent kernel for all $\lambda \in \mathbf{R}$ in the resolvent set.

In this section, let

$$L = -\frac{d}{dx}\left(a(x)\frac{d}{dx}\right) + c(x),$$

where $a(x)$ and $c(x)$ are real-valued periodic functions with period 1. Assume that $a \in L^\infty(\mathbf{R})$ and $0 < \mu \le a(x) \le \mu^{-1}$ for some constant μ, and that $c \in L^1_{loc}(\mathbf{R})$. Corresponding to this operator, we consider the equation

$$\frac{d}{dx}\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} 0 & a(x)^{-1} \\ c(x) - z & 0 \end{pmatrix}\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} \tag{5.1}$$

for $z \in \mathbf{C}$. By the standard iteration method of ordinary differential equations, we can find unique solutions to (5.1), $(c_1(x,z), c_2(x,z))$ and $(s_1(x,z), s_2(x,z))$ with the initial conditions

$$\begin{pmatrix} c_1(0,z) \\ c_2(0,z) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} s_1(0,z) \\ s_2(0,z) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

respectively, in the space of $\mathbf{C}^2$-valued absolutely continuous functions $AC(\mathbf{R})^2$. We can also see that $c_j(x,z)$ and $s_j(x,z)$ are $C([-R,R])$-valued entire functions of z for any R.

For each $\zeta \in \mathbf{C}$, the eigenvalue problem

$$\begin{cases} y \in H^1_{loc}(\mathbf{R}) \\ Ly = zy \\ y(x+1) = e^{i\zeta}y(x) \quad (\zeta\text{-periodicity}) \end{cases} \tag{5.2}$$

is equivalent to

$$\begin{cases} (y_1, y_2) \in AC(\mathbf{R})^2 \\ (y_1, y_2) \text{ satisfies (5.1) and } y_1 \text{ satisfies the } \zeta\text{-periodicity} \end{cases}$$

under the relation $y_1 = y$, $y_2 = ay'$. Writing a solution to (5.2) as $y(x) = \alpha_1 c_1(x,z) + \alpha_2 s_1(x,z)$, $|\alpha_1|^2 + |\alpha_2|^2 \neq 0$, by the ζ-periodicity we have $(M(z) - e^{i\zeta}I)\alpha = 0$, where

$$M(z) := \begin{pmatrix} c_1(1,z) & s_1(1,z) \\ c_2(1,z) & s_2(1,z) \end{pmatrix}, \quad \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

We see that $\det(M(z) - e^{i\zeta} I) = 0$ if and only if

$$D(z) = e^{i\zeta} + e^{-i\zeta}, \tag{5.3}$$

where $D(z) := c_1(1, z) + s_2(1, z)$ is the discriminant, which is an entire function. Hence the existence of non-trivial solution of (5.2) is equivalent to (5.3).

A function y is an eigenfunction of (5.2) if and only if $u(x) = e^{-ix\zeta} y(x)$ is an eigenfunction of $L(\zeta)$ with the same eigenvalue. Here $L(\zeta) = e^{-ix\zeta} L e^{ix\zeta}$ is an operator on $L^2(\mathbf{T})$ with compact resolvent with the domain $D(L(\zeta)) = \{u \in H^1(\mathbf{T}); L(\zeta)u \in L^2(\mathbf{T})\}$. Regarding L as the selfadjoint operator on $L^2(\mathbf{R})$ with the domain $D(L) = \{u \in H^1(\mathbf{R}); Lu \in L^2(\mathbf{R})\}$, we have the direct integral decomposition $\mathcal{U} L \mathcal{U}^{-1} = \int_{(-\pi,\pi]}^{\oplus} L(\xi) d\xi$, where $\mathcal{U}$ is the unitary operator defined in Section 2 with $d = 1$ (cf. [RS]).

We denote the eigenvalues of $L(\xi)$ by $\lambda_1(\xi) \le \lambda_2(\xi) \le \cdots$ for $\xi \in \mathbf{R}$ counted with multiplicities. Each $\lambda_n(\xi)$ is known to be continuous on $\mathbf{R}$. We summarize several facts, which can be proved in ways similar to those in [E], [Ku], [Ma], and [RS]. Each $\lambda_n(\xi)$ is real analytic on $(0, \pi)$, and for $\xi \in (0, \pi)$, $\lambda_n(\xi)$ is a nondegenerate eigenvalue of $L(\xi)$. There exists a sequence of real numbers

$$-\infty < \mu_1 < \nu_1 \le \nu_2 < \mu_2 \le \mu_3 < \nu_3 \le \cdots$$

such that it tends to infinity and has the following properties:

(i) The spectrum $\sigma(L)$ of L is $\cup_{n=1}^{\infty} ([\mu_{2n-1}, \nu_{2n-1}] \cup [\nu_{2n}, \mu_{2n}])$; and $|D(\lambda)| \le 2$, $\lambda \in \mathbf{R}$, if and only if $\lambda \in \sigma(L)$.

(ii) $D(\lambda) = 2$ only at $\lambda = \mu_j$, and $D(\lambda) = -2$ only at $\lambda = \nu_j$.

(iii) $D'(\lambda) < 0$ on $(-\infty, \nu_1)$ and (μ_{2n-1}, ν_{2n-1}), and $D'(\lambda) > 0$ on (ν_{2n}, μ_{2n}).

(iv) $\lambda'_{2n-1}(\xi) > 0$ and $\lambda'_{2n}(\xi) < 0$ on $(0, \pi)$; in the interval $[0, \pi]$, $\lambda_{2n-1}(\xi)$ increases from μ_{2n-1} to ν_{2n-1}, and $\lambda_{2n}(\xi)$ decreases from μ_{2n} to ν_{2n}; $\lambda_n(k\pi + \xi) = \lambda_n(k\pi - \xi)$ for any integer k and real ξ.

(v) If $\lambda_{2n-1}(\pi) = \lambda_{2n}(\pi)$, then $\lambda_{2n-1}(\pi - 0) \neq 0$; if $\lambda_{2n}(0) = \lambda_{2n+1}(0)$, then $\lambda_{2n+1}(0 + 0) \neq 0$

(vi) If $\nu_{2n-1} \neq \nu_{2n}$, then $D'(\nu_{2n-1}) \neq 0$ and $D'(\nu_{2n}) \neq 0$, and ν_{2n-1} and ν_{2n} are nondegenerate eigenvalues of $L(\pi)$; if $\mu_{2n} \neq \mu_{2n+1}$, then $D'(\mu_{2n}) \neq 0$ and $D'(\mu_{2n+1}) \neq 0$ and μ_{2n} and μ_{2n+1} are nondegenerate eigenvalues of $L(0)$; if $\nu_{2n-1} = \nu_{2n}$ or $\mu_{2n} = \mu_{2n+1}$, then $D' = 0$ at these points, and these are doubly degenerate eigenvalues of $L(\pi)$ or $L(0)$, respectively; if $D(\lambda) \ge 2$ (≤ -2) and $D'(\lambda) = 0$, then $D''(\lambda) < 0$ (> 0).

We denote by $G_z(x, y)$ the integral kernel of the resolvent $R(z) := (L - z)^{-1}$ for z in the resolvent set. We use the notations $(u, v) = \int_0^1 u(x)\overline{v(x)}\, dx$ and $\|u\|^2 = (u, u)$.

First, let λ be in the interior of $\sigma(L)$. Then the only one of the following four cases holds:

(I) $\lambda = \lambda_{2n-1}(\xi) \in (\mu_{2n-1}, \nu_{2n-1})$ for some $\xi \in (0, \pi)$,

(II) $\lambda = \lambda_{2n}(\xi) \in (\nu_{2n}, \mu_{2n})$ for some $\xi \in (-\pi, 0)$,

(III) $\lambda = \lambda_{2n-1}(\pi) = \lambda_{2n}(\pi) = \nu_{2n-1} = \nu_{2n}$,

(IV) $\lambda = \lambda_{2n}(0) = \lambda_{2n+1}(0) = \mu_{2n} = \mu_{2n+1}$.

Theorem 5.1. *Assume that λ is in the interior of $\sigma(L)$. There exists the limit $\lim_{\varepsilon\downarrow 0}\big(\frac{d}{d\lambda}\big)^m R(\lambda \pm i\varepsilon)f(x)$ in $L^2_{loc}(\mathbf{R})$ for $m \geq 0$ and $f \in L^2(\mathbf{R})$ with compact support, and the convergence is locally uniform in the interior of $\sigma(L)$. The integral kernels $G_{\lambda+i0}(x,y)$ and $G^{(m)}_{\lambda+i0}(x,y)$ of $\lim_{\varepsilon\downarrow 0} R(\lambda+i\varepsilon)$ and $\lim_{\varepsilon\downarrow 0}\big(\frac{d}{d\lambda}\big)^m R(\lambda+i\varepsilon)$, $m \geq 1$, admit the following expressions:*

Case (I).

$$G_{\lambda+i0}(x,y) = G_{\lambda+i0}(y,x) = \frac{ie^{i(x-y)\xi}}{\lambda'_{2n-1}(\xi)}\frac{u_\xi(x)\overline{u_\xi(y)}}{\|u_\xi\|^2}, \qquad y \leq x,$$

$$G^{(m)}_{\lambda+i0}(x,y) = G^{(m)}_{\lambda+i0}(y,x)$$
$$= \left(\frac{i}{\lambda'_{2n-1}(\xi)}\right)^{m+1}(x-y)^m e^{i(x-y)\xi}\frac{u_\xi(x)\overline{u_\xi(y)}}{\|u_\xi\|^2}(1+O(|x-y|^{-1})), \quad y \leq x.$$

Here u_ξ is an eigenfunction corresponding to the eigenvalue $\lambda_{2n-1}(\xi)$.

Case (II). *$G_{\lambda+i0}(x,y)$ and $G^{(m)}_{\lambda+i0}(x,y)$ admit the same expressions as in* (I) *with $\lambda'_{2n-1}(\xi)$ replaced by $\lambda'_{2n}(\xi)$, and with u_ξ being an eigenfunction corresponding to the eigenvalue $\lambda_{2n}(\xi)$.*

Case (III). *With u_ξ being a $C(\mathbf{T})$-valued holomorphic function in a neighborhood of π such that $\|u_\xi\| \neq 0$, $(L(\xi)-\lambda_{2n-1}(\xi))u_\xi = 0$ for $\xi \leq \pi$, and $(L(\xi)-\lambda_{2n}(\xi))u_\xi = 0$ for $\pi < \xi$,*

$$G_{\lambda+i0}(x,y) = G_{\lambda+i0}(y,x) = \frac{ie^{i(x-y)\pi}}{\lambda'_{2n-1}(\pi-0)}\frac{u_\pi(x)\overline{u_\pi(y)}}{\|u_\pi\|^2}, \qquad y \leq x,$$

$$\begin{aligned} G^{(m)}_{\lambda+i0}(x,y) &= G^{(m)}_{\lambda+i0}(y,x) \\ &= \left(\frac{i}{\lambda'_{2n-1}(\pi-0)}\right)^{m+1}(x-y)^m e^{i(x-y)\pi}\frac{u_\pi(x)\overline{u_\pi(y)}}{\|u_\pi\|^2} \\ &\quad \times (1+O(|x-y|^{-1})), \quad y \leq x. \end{aligned}$$

Case (IV). *With u_ξ being a $C(\mathbf{T})$-valued holomorphic function in a neighborhood of 0 such that $\|u_\xi\| \neq 0$, $(L(\xi)-\lambda_{2n+1}(\xi))u_\xi = 0$ for $0 \leq \xi$, and $(L(\xi)-\lambda_{2n}(\xi))u_\xi = 0$ for $\xi < 0$,*

$$G_{\lambda+i0}(x,y) = G_{\lambda+i0}(y,x) = \frac{i}{\lambda'_{2n+1}(0+0)}\frac{u_0(x)\overline{u_0(y)}}{\|u_0\|^2}, \qquad y \leq x,$$

$$G^{(m)}_{\lambda+i0}(x,y) = G^{(m)}_{\lambda+i0}(y,x)$$
$$= \left(\frac{i}{\lambda'_{2n+1}(0+0)}\right)^{m+1}(x-y)^m\frac{u_0(x)\overline{u_0(y)}}{\|u_0\|^2}(1+O(|x-y|^{-1})), \qquad y \leq x.$$

Proof. (I) Since $D'(\lambda) < 0$ on (μ_{2n-1}, ν_{2n-1}), there exists a holomorphic inverse function D^{-1} of D on an open set containing $(-2, 2)$. Put $\lambda(\zeta) := D^{-1}(e^{i\zeta} + e^{-i\zeta})$ for ζ in an open set containing $(0, \pi)$. We have $\lambda(\xi) = \lambda_{2n-1}(\xi)$ for $\xi \in (0, \pi)$. Let

$$\alpha(\zeta) = (\alpha_1(\zeta), \alpha_2(\zeta)) := (-s_1(1, \lambda(\zeta)), c_1(1, \lambda(\zeta)) - e^{i\zeta}).$$

Since $\alpha(\xi) \neq 0$ for $\xi \in (0, \pi)$, $\alpha(\zeta)$ is an eigenvector of $M(\lambda(\zeta))$ corresponding to the eigenvalue $e^{i\zeta}$ for ζ in an open set containing $(0, \pi)$. Thus $y_\zeta(x) := \alpha_1(\zeta) c_1(x, \lambda(\zeta)) + \alpha_2(\zeta) s_1(x, \lambda(\zeta))$ satisfies (5.2) with z replaced by $\lambda(\zeta)$. So $u_\zeta(x) := e^{-i\zeta x} y_\zeta(x)$ is a $C(\mathbf{T})$-valued holomorphic eigenfunction of $L(\zeta)$ corresponding to the eigenvalue $\lambda(\zeta)$. Since $\lambda'_{2n-1}(\xi) > 0$ on $(0, \pi)$, the inverse function theorem implies that there exists a holomorphic function $\zeta(z)$ on an open set containing (μ_{2n-1}, ν_{2n-1}) such that $\lambda(\zeta(z)) = z$. For each $\lambda \in (\mu_{2n-1}, \nu_{2n-1})$, if $\varepsilon > 0$ is small enough, $y_{\zeta(\lambda+i\varepsilon)}(x)$ is a solution to the equation $Ly = (\lambda + i\varepsilon)y$. Taking the complex conjugate of this equation and replacing ε by $-\varepsilon$, we obtain that $\overline{y_{\zeta(\lambda-i\varepsilon)}(x)}$ is also a solution. Since $\zeta'(\lambda) > 0$, we obtain the linearly independent solutions to $Ly = (\lambda + i\varepsilon)y$:

$$\begin{aligned} y_{\zeta(\lambda+i\varepsilon)}(x) &= e^{i\zeta(\lambda+i\varepsilon)x} u_{\zeta(\lambda+i\varepsilon)}(x) \\ &= \exp[(i\zeta(\lambda) - \varepsilon\zeta'(\lambda) + O(\varepsilon^2))x] u_{\zeta(\lambda+i\varepsilon)}(x), \\ \overline{y_{\zeta(\lambda-i\varepsilon)}(x)} &= e^{-i\overline{\zeta(\lambda-i\varepsilon)}x} \overline{u_{\zeta(\lambda-i\varepsilon)}(x)} \\ &= \exp[(-i\zeta(\lambda) + \varepsilon\zeta'(\lambda) + O(\varepsilon^2))x] \overline{u_{\zeta(\lambda-i\varepsilon)}(x)}. \end{aligned}$$

Let $[y, \tilde{y}](x) := a(x)(y(x)\tilde{y}'(x) - y'(x)\tilde{y}(x))$ be the Wronskian of two solutions y and $\tilde{y}$. Then

$$G_{\lambda+i\varepsilon}(x, y) = \begin{cases} y_{\zeta(\lambda+i\varepsilon)}(x)\overline{y_{\zeta(\lambda-i\varepsilon)}(y)} / [y_{\zeta(\lambda+i\varepsilon)}, \overline{y_{\zeta(\lambda-i\varepsilon)}}](0), & y \le x, \\ y_{\zeta(\lambda+i\varepsilon)}(y)\overline{y_{\zeta(\lambda-i\varepsilon)}(x)} / [y_{\zeta(\lambda+i\varepsilon)}, \overline{y_{\zeta(\lambda-i\varepsilon)}}](0), & x \le y, \end{cases}$$

(cf. §5.3 in [E]). Since $[y_{\zeta(\lambda+i\varepsilon)}, \overline{y_{\zeta(\lambda-i\varepsilon)}}](x)$ is a constant independent of x and $\zeta(\lambda + i\varepsilon) = \overline{\zeta(\lambda - i\varepsilon)}$, it follows that

$$\begin{aligned} &[y_{\zeta(\lambda+i\varepsilon)}, \overline{y_{\zeta(\lambda-i\varepsilon)}}](0) \\ &\quad = \int_0^1 \big([u_{\zeta(\lambda+i\varepsilon)}, \overline{u_{\zeta(\lambda-i\varepsilon)}}](x) - 2i\zeta(\lambda+i\varepsilon)a(x)u_{\zeta(\lambda+i\varepsilon)}(x)\overline{u_{\zeta(\lambda-i\varepsilon)}(x)}\big)dx. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_0^1 \bigg[a(x) \left(\frac{d}{dx} + i\zeta(\lambda+i\varepsilon)\right) u_{\zeta(\lambda+i\varepsilon)}(x) \left(\frac{d}{dx} - i\zeta(\lambda+i\varepsilon)\right) \overline{u_{\zeta(\lambda-i\varepsilon)}(x)} \\ + c(x) u_{\zeta(\lambda+i\varepsilon)}(x)\overline{u_{\zeta(\lambda-i\varepsilon)}(x)} \bigg] dx = (\lambda+i\varepsilon)(u_{\zeta(\lambda+i\varepsilon)}, u_{\zeta(\lambda-i\varepsilon)}). \end{aligned}$$

Differentiating both sides of this equation with respect to λ, we have

$$\begin{aligned} i\zeta'(\lambda+i\varepsilon)\int_0^1 \Big(& [u_{\zeta(\lambda+i\varepsilon)}, \overline{u_{\zeta(\lambda-i\varepsilon)}}](x) \\ & - 2i\zeta(\lambda+i\varepsilon)a(x)u_{\zeta(\lambda+i\varepsilon)}(x)\overline{u_{\zeta(\lambda-i\varepsilon)}(x)}\Big)dx \\ &= (u_{\zeta(\lambda+i\varepsilon)}, u_{\zeta(\lambda-i\varepsilon)}). \end{aligned}$$

Thus

$$i\zeta'(\lambda+i\varepsilon)[y_{\zeta(\lambda+i\varepsilon)}, \overline{y_{\zeta(\lambda-i\varepsilon)}}](0) = (u_{\zeta(\lambda+i\varepsilon)}, u_{\zeta(\lambda-i\varepsilon)}).$$

Therefore we have for $y \le x$

$$G_{\lambda+i\varepsilon}(x,y) = G_{\lambda+i\varepsilon}(y,x) = i\zeta'(\lambda+i\varepsilon)e^{i\zeta(\lambda+i\varepsilon)(x-y)}\frac{u_{\zeta(\lambda+i\varepsilon)}(x)\overline{u_{\zeta(\lambda-i\varepsilon)}(y)}}{(u_{\zeta(\lambda+i\varepsilon)}, u_{\zeta(\lambda-i\varepsilon)})}.$$

Taking the limit $\varepsilon \downarrow 0$, we have the existence of the limit $\lim_{\varepsilon\downarrow 0} R(\lambda \pm i\varepsilon)f(x)$ and

$$G_{\lambda+i0}(x,y) = \lim_{\varepsilon\downarrow 0} G_{\lambda+i\varepsilon}(x,y) = \frac{ie^{i(x-y)\xi}}{\lambda'_{2n-1}(\xi)}\frac{u_\xi(x)\overline{u_\xi(y)}}{\|u_\xi\|^2}, \qquad y \le x,$$

where $\xi = \zeta(\lambda)$, i.e., $\lambda_{2n-1}(\xi) = \lambda$. Furthermore, we can see that for any integer $m \ge 1$, the limit $\lim_{\varepsilon\downarrow 0} \left(\frac{d}{d\lambda}\right)^m R(\lambda \pm i\varepsilon)f(x)$ exists and

$$\begin{aligned} &G^{(m)}_{\lambda+i0}(x,y) = \lim_{\varepsilon\downarrow 0}\left(\frac{d}{d\lambda}\right)^m G_{\lambda+i\varepsilon}(x,y) \\ &= \left(\frac{i}{\lambda'_{2n-1}(\xi)}\right)^{m+1}(x-y)^m e^{i(x-y)\xi}\frac{u_\xi(x)\overline{u_\xi(y)}}{\|u_\xi\|^2}(1+O(|x-y|^{-1})), \qquad y \le x. \end{aligned}$$

We have thus proved the case (I). The case (II) is proved in the same way as (I).

(III) Assume that $\lambda_{2n-1}(\pi) = \lambda_{2n}(\pi) = \nu_{2n-1} = \nu_{2n}$. Since ν_{2n} is a doubly degenerate eigenvalue and $L(\xi)$ is selfadjoint for ξ real, Theorem XII.13 in [RS] implies that there exist holomorphic eigenvalues $E_1(\zeta)$ and $E_2(\zeta)$ of $L(\zeta)$ near $\zeta = \pi$ such that $E_1(\pi) = E_2(\pi) = \nu_{2n}$. If $\xi \in \mathbf{R}$, each of $\lambda_{2n-1}(\xi)$ and $\lambda_{2n}(\xi)$ must be equal to one of $E_j(\xi)$, $j = 1, 2$. Since $D(E_j(\xi)) = 2\cos\xi$ near $\xi = \pi$, we have

$$D''(E_j(\xi))E'_j(\xi)^2 + D'(E_j(\xi))E''_j(\xi) = -2\cos\xi.$$

So, since $D'(\nu_{2n}) = 0$ and $D''(\nu_{2n}) > 0$, we obtain that $E'_j(\pi) \ne 0$ (which implies the fact (v) stated before Theorem 5.1). Since

$$\begin{cases} \lambda'_{2n-1}(\xi) > 0, & \xi < \pi, \\ \lambda'_{2n}(\xi) > 0, & \pi < \xi, \end{cases} \quad \text{and} \quad \begin{cases} \lambda'_{2n-1}(\xi) < 0, & \pi < \xi, \\ \lambda'_{2n}(\xi) < 0, & \xi < \pi, \end{cases}$$

we conclude that there exist holomorphic functions $E_1(\zeta)$ and $E_2(\zeta)$ on an open set containing $(0, 2\pi)$ such that

$$E_1(\xi) = \begin{cases} \lambda_{2n-1}(\xi), & 0 \le \xi \le \pi, \\ \lambda_{2n}(\xi), & \pi \le \xi \le 2\pi, \end{cases} \qquad E_2(\xi) = \begin{cases} \lambda_{2n}(\xi), & 0 \le \xi \le \pi, \\ \lambda_{2n-1}(\xi), & \pi \le \xi \le 2\pi. \end{cases}$$

Since $E_1'(\xi) > 0$ on $(0, 2\pi)$, the inverse function theorem implies that there exists a holomorphic function $\zeta(z)$ on an open set containing (μ_{2n-1}, μ_{2n}) such that $E_1(\zeta(z)) = z$.

Let $p(\xi)$ be the eigenprojection for the eigenvalue $e^{i\xi}$ of $M(E_1(\xi))$ for $\xi \in (0, \pi) \cup (\pi, 2\pi)$:

$$\begin{aligned} p(\xi) &:= (-2\pi i)^{-1} \oint_{|z-e^{i\xi}|=\delta} (M(E_1(\xi)) - z)^{-1} dz \\ &= \frac{-1}{e^{i\xi} - e^{-i\xi}} \begin{pmatrix} s_2(1, E_1(\xi)) - e^{i\xi} & -s_1(1, E_1(\xi)) \\ -c_2(1, E_1(\xi)) & c_1(1, E_1(\xi)) - e^{i\xi} \end{pmatrix}, \end{aligned}$$

where $\delta > 0$ is taken so that $e^{i\xi}$ is the only eigenvalue of $M(E_1(\xi))$ inside the circle $|z - e^{i\xi}| = \delta$. Since $s_2(1, \nu_{2n}) + 1 = c_1(1, \nu_{2n}) + 1 = s_1(1, \nu_{2n}) = c_2(1, \nu_{2n}) = 0$ (cf. [E, p.7 and p.29]), $\xi = \pi$ is a removable singularity of $p(\xi)$. We have $(p(\xi))_{11} \neq 0$ on $(0, 2\pi)$, since

$$(p(\pi))_{11} = (2i)^{-1} \partial_\xi (s_2(1, E_1(\xi)) - e^{i\xi})|_{\xi=\pi} = (2i)^{-1} (\partial_z s_2(1, \nu_{2n}) E_1'(\pi) + i) \neq 0.$$

Thus $p(\xi)$ is a real analytic rank one matrix on $(0, 2\pi)$. Note that the holomorphically extended $p(\zeta)$ to an open set containing $(0, 2\pi)$ is the eigenprojection for the eigenvalue $e^{i\zeta}$ of $M(E_1(\zeta))$. Thus the function

$$y_\zeta(x) := (p(\zeta))_{11} c_1(x, E_1(\zeta)) + (p(\zeta))_{21} s_1(x, E_1(\zeta))$$

is a solution to (5.2) with z replaced by $E_1(\zeta)$; and so $u_\zeta(x) = e^{-i\zeta x} y_\zeta(x)$ is a $C(\mathbf{T})$-valued holomorphic eigenfunction of $L(\zeta)$ corresponding to $E_1(\zeta)$ on an open set containing $(0, 2\pi)$. Thus as in the case (I), since $\zeta'(\lambda) > 0$ for $\lambda \in (\mu_{2n-1}, \mu_{2n})$, $y_{\zeta(\lambda+i\varepsilon)}(x)$ and $\overline{y_{\zeta(\lambda-i\varepsilon)}(x)}$ are linearly independent solutions to $Ly = (\lambda + i\varepsilon)y$. Hence, as in the proof of (I) we have

$$G_{\nu_{2n}+i0}(x, y) = \lim_{\varepsilon \downarrow 0} G_{\nu_{2n}+i\varepsilon}(x, y) = \frac{ie^{i(x-y)\pi}}{E_1'(\pi)} \frac{u_\pi(x)\overline{u_\pi(y)}}{\|u_\pi\|^2}, \qquad y \le x,$$

and for any integer $m \ge 1$,

$$\begin{aligned} &G^{(m)}_{\nu_{2n}+i0}(x, y) = \lim_{\varepsilon \downarrow 0} \left(\frac{d}{d\lambda} \right)^m G_{\nu_{2n}+i\varepsilon}(x, y) \\ &= \left(\frac{i}{E_1'(\pi)} \right)^{m+1} (x - y)^m e^{i(x-y)\pi} \frac{u_\pi(x)\overline{u_\pi(y)}}{\|u_\pi\|^2} (1 + O(|x - y|^{-1})), \qquad y \le x. \end{aligned}$$

Note that $E_1'(\pi) = \lambda_{2n-1}'(\pi - 0)$. We have thus proved (III). (IV) is proved similarly. From the proof above it follows that the convergence $\lim_{\varepsilon \downarrow 0} (\frac{d}{d\lambda})^m R(\lambda \pm i\varepsilon) f(x)$ is locally uniform with respect to λ. $\square$

The following is a direct consequence of Theorem 5.1.

Corollary 5.2. *Let λ be in the interior of $\sigma(L)$. Then $\left(\frac{d}{d\lambda}\right)^m R(\lambda \pm i0)$, $m \geq 0$, is bounded from $B_{\frac{1}{2}+m}$ to $B^*_{\frac{1}{2}+m}$.*

Proof. Let $f \in C_0^\infty(\mathbf{R})$. Since Theorem 5.1 yields that

$$\left|\left(\frac{d}{d\lambda}\right)^m R(\lambda+i0)f(x)\right| \leq C_m(1+|x|)^m \int_{\mathbf{R}} (1+|y|)^m |f(y)|dy \leq C_m(1+|x|)^m \|f\|_{B_{\frac{1}{2}+m}},$$

it follows that

$$\left\|\left(\frac{d}{d\lambda}\right)^m R(\lambda+i0)f(x)\right\|_{B^*_{\frac{1}{2}+m}} \leq C_m\|(1+|x|)^m\|_{B^*_{\frac{1}{2}+m}}\|f\|_{B_{\frac{1}{2}+m}} \leq C_m\|f\|_{B_{\frac{1}{2}+m}}.$$

□

Next we study the case that the parameter $\lambda \in \mathbf{R}$ is in the resolvent set of L. This case is equivalent to $|D(\lambda)| > 2$. $D(\lambda) > 2$ if and only if $\lambda \in A_+ := (-\infty, \mu_1) \cup [\cup_{n=1}^\infty(\mu_{2n}, \mu_{2n+1})]$; and $D(\lambda) < -2$ if and only if $\lambda \in A_- := \cup_{n=1}^\infty(\nu_{2n-1}, \nu_{2n})$. Consider a function $e^\eta + e^{-\eta}$ on $(0, \infty)$, and solve the equation

$$e^\eta + e^{-\eta} = D(\lambda)$$

with respect to η, where $\lambda \in A_+$. By the implicit function theorem, we have a unique solution $\eta(\lambda)$ which is real analytic on A_+. Similarly, define $\eta(\lambda)$ on A_- by $e^\eta + e^{-\eta} = -D(\lambda)$. Note that $\dim \mathrm{Ker}\,(L(\pm i\eta(\lambda)) - \lambda) = 1$ for $\lambda \in A_+$ and $\dim \mathrm{Ker}\,(L(\pi \pm i\eta(\lambda)) - \lambda) = 1$ for $\lambda \in A_-$ (cf. [E, p.6]).

Theorem 5.3. (i) *Let $\lambda \in A_+$. Let u_λ and v_λ be real-valued eigenfunctions of $L(i\eta(\lambda))$ and $L(-i\eta(\lambda))$ corresponding to the eigenvalue λ, respectively.*

Suppose $D'(\lambda) \neq 0$. Then $(u_\lambda, v_\lambda) \neq 0$ and

$$G_\lambda(x,y) = G_\lambda(y,x) = -\eta'(\lambda)e^{-\eta(\lambda)(x-y)}\frac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, v_\lambda)}, \quad y \leq x. \tag{5.4}$$

Suppose $D'(\lambda) = 0$. Then there exists a solution $\psi_{v_\lambda} \in H^1(\mathbf{T})$ of the equation $(L(-i\eta(\lambda)) - \lambda)\psi = v_\lambda$ such that $(u_\lambda, \psi_{v_\lambda}) \neq 0$, and

$$G_\lambda(x,y) = G_\lambda(y,x) = -\frac{\eta''(\lambda)}{2}e^{-\eta(\lambda)(x-y)}\frac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, \psi_{v_\lambda})}, \quad y \leq x. \tag{5.5}$$

(ii) *Let $\lambda \in A_-$. Let u_λ and v_λ be eigenfunctions of $L(\pi + i\eta(\lambda))$ and $L(\pi - i\eta(\lambda))$ corresponding to the eigenvalue λ, respectively.*

Suppose $D'(\lambda) \neq 0$. Then $(u_\lambda, v_\lambda) \neq 0$ and

$$G_\lambda(x,y) = G_\lambda(y,x) = -\eta'(\lambda)e^{(i\pi-\eta(\lambda))(x-y)}\frac{u_\lambda(x)\overline{v_\lambda(y)}}{(u_\lambda, v_\lambda)}, \quad y \leq x.$$

Suppose $D'(\lambda) = 0$. Then there exists a solution $\psi_{v_\lambda} \in H^1(\mathbf{T})$ of the equation $(L(\pi - i\eta(\lambda)) - \lambda)\psi = v_\lambda$ such that $(u_\lambda, \psi_{v_\lambda}) \neq 0$, and

$$G_\lambda(x,y) = G_\lambda(y,x) = -\frac{\eta''(\lambda)}{2} e^{(i\pi - \eta(\lambda))(x-y)} \frac{u_\lambda(x)\overline{v_\lambda(y)}}{(u_\lambda, \psi_{v_\lambda})}, \quad y \leq x.$$

Proof. Let $\lambda \in A_+$. Since $c_1(1,\lambda) - e^{\pm\eta(\lambda)}$ and $s_2(1,\lambda) - e^{\pm\eta(\lambda)} = e^{\mp\eta(\lambda)} - c_1(1,\lambda)$ do not vanish simultaneously on a neighborhood of each $\lambda \in A_+$, there exist nonzero real analytic eigenvectors $\alpha_\pm(\lambda) = (\alpha_{\pm,1}(\lambda), \alpha_{\pm,2}(\lambda))$ of $M(\lambda)$ corresponding to the eigenvalues $e^{\eta(\lambda)}$ and $e^{-\eta(\lambda)}$, respectively. Then $y_\lambda(x) := \alpha_{-,1}(\lambda)c_1(x,\lambda) + \alpha_{-,2}(\lambda)s_1(x,\lambda)$ and $z_\lambda(x) := \alpha_{+,1}(\lambda)c_1(x,\lambda) + \alpha_{+,2}(\lambda)s_1(x,\lambda)$ are solutions to (5.2) with ζ replaced by $i\eta(\lambda)$ and $-i\eta(\lambda)$. Thus $u_\lambda(x) := e^{\eta(\lambda)x}y_\lambda(x)$ and $v_\lambda(x) := e^{-\eta(\lambda)x}z_\lambda(x)$ are $C(\mathbf{T})$-valued real analytic eigenfunctions on A_+ of $L(i\eta(\lambda))$ and $L(i\eta(\lambda))^* = L(-i\eta(\lambda))$ corresponding to the eigenvalue λ, respectively. Hence $y_\lambda(x) = e^{-\eta(\lambda)x}u_\lambda(x)$ and $z_\lambda(x) = e^{\eta(\lambda)x}v_\lambda(x)$ are linearly independent solutions, and so

$$G_\lambda(x,y) = \begin{cases} y_\lambda(x)z_\lambda(y)/[y_\lambda, z_\lambda](0), & y \leq x, \\ y_\lambda(y)z_\lambda(x)/[y_\lambda, z_\lambda](0), & x \leq y. \end{cases}$$

Since $[y_\lambda, z_\lambda](x)$ is a constant independent of x, it follows that

$$[y_\lambda, z_\lambda](0) = \int_0^1 ([u_\lambda, v_\lambda](x) + 2\eta(\lambda)a(x)u_\lambda(x)v_\lambda(x))dx.$$

On the other hand, we have

$$\int_0^1 \left[a(x)\left(\frac{d}{dx} - \eta(\lambda)\right)u_\lambda(x)\left(\frac{d}{dx} + \eta(\lambda)\right)v_\lambda(x) + c(x)u_\lambda(x)v_\lambda(x)\right]dx$$
$$= \lambda(u_\lambda, v_\lambda).$$

Differentiating both sides of this equation with respect to λ, we have

$$-\eta'(\lambda)\int_0^1 ([u_\lambda, v_\lambda](x) + 2\eta(\lambda)a(x)u_\lambda(x)v_\lambda(x))dx = (u_\lambda, v_\lambda).$$

Hence

$$-\eta'(\lambda)[y_\lambda, z_\lambda](0) = (u_\lambda, v_\lambda). \tag{5.6}$$

Suppose $D'(\lambda) \neq 0$. Then $\eta'(\lambda) = D'(\lambda)/(e^{\eta(\lambda)} - e^{-\eta(\lambda)}) \neq 0$ and

$$G_\lambda(x,y) = -\eta'(\lambda)e^{-\eta(\lambda)(x-y)}u_\lambda(x)v_\lambda(y)/(u_\lambda, v_\lambda), \quad y \leq x.$$

Suppose $D'(\lambda) = 0$. Then $\eta'(\lambda) = 0$ and $\eta''(\lambda) = D''(\lambda)/(e^{\eta(\lambda)} - e^{-\eta(\lambda)}) < 0$. Differentiating (5.6), we have

$$\eta''(\lambda)[y_\lambda, z_\lambda](0) = -(u_\lambda, v_\lambda)'. \tag{5.7}$$

Therefore

$$G_\lambda(x,y) = -\eta''(\lambda)e^{-\eta(\lambda)(x-y)}u_\lambda(x)v_\lambda(y)/(u_\lambda, v_\lambda)', \quad y \le x.$$

By (5.6), $(u_\lambda, v_\lambda) = 0$. Moreover, since $\eta'(\lambda) = 0$,

$$(L(i\eta(\lambda)) - \lambda)\partial_\lambda u_\lambda = u_\lambda \text{ and } (L(-i\eta(\lambda)) - \lambda)\partial_\lambda v_\lambda = v_\lambda. \tag{5.8}$$

Put $\psi_{v_\lambda} = \partial_\lambda v_\lambda$. Then ψ_{v_λ} is a solution of $(L(-i\eta(\lambda)) - \lambda)\psi = v_\lambda$. By (5.8), we have

$$\begin{aligned}(\partial_\lambda u_\lambda, v_\lambda) &= (\partial_\lambda u_\lambda, (L(-i\eta(\lambda)) - \lambda)\partial_\lambda v_\lambda) \\ &= ((L(i\eta(\lambda)) - \lambda)\partial_\lambda u_\lambda, \partial_\lambda v_\lambda) = (u_\lambda, \partial_\lambda v_\lambda).\end{aligned}$$

Thus $(u_\lambda, v_\lambda)' = 2(u_\lambda, \psi_{v_\lambda})$, which together with (5.7) implies that $(u_\lambda, \psi_{v_\lambda}) \neq 0$. Therefore we have (5.5). The assertion (ii) is proved similarly. □

We have seen that in the formula (5.4) and (5.5) the different factor $\dfrac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, v_\lambda)}$ or $\dfrac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, \psi_{v_\lambda})}$ appears according to whether $D'(\lambda)$ does not vanish or not. This is related to the Laurent expansion of $(L(i\eta(\lambda)) - z)^{-1}$ with respect to z around λ.

Proposition 5.4. *Let $\lambda \in A_+$. If $D'(\lambda) \neq 0$, the eigenvalue λ of $L(i\eta(\lambda))$ is nondegenerate and its eigenprojection has the integral kernel $\dfrac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, v_\lambda)}$; and if $D'(\lambda) = 0$, the eigenvalue λ of $L(i\eta(\lambda))$ is degenerate and its eigennilpotent has the integral kernel $\dfrac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, \psi_{v_\lambda})}$. Similar statement holds for $\lambda \in A_-$.*

Proof. We shall represent the integral kernel $R(\zeta, z; x, y)$ of the resolvent $R(\zeta, z) := (L(\zeta) - z)^{-1}$, by using $c_j(x,z)$ and $s_j(x,z)$. Let $(\zeta, z) \in \Gamma := \{(\zeta, z) \in \mathbf{C}^2; z \notin \sigma(L(\zeta))\}$. Put

$$k(z; x, y) := \begin{cases} c_1(x,z)s_1(y,z), & y \le x, \\ s_1(x,z)c_1(y,z), & x \le y. \end{cases}$$

For $f \in C_0^\infty(0,1)$, put

$$K_z f(x) := \int k(z; x, y) f(y) dy.$$

Since $(L - z)K_z f(x) = f(x)$ and $(L - z)e^{ix\zeta}R(\zeta, z)e^{-ix\zeta}f(x) = f(x)$ on $(0,1)$, $e^{ix\zeta}R(\zeta, z)e^{-ix\zeta}f(x) - K_z f(x)$ is a solution to $Ly = zy$. Thus

$$e^{ix\zeta}R(\zeta, z)e^{-ix\zeta}f(x) - K_z f(x) = \alpha c_1(x,z) + \beta s_1(x,z) \tag{5.9}$$

for some α and β. Since $R(\zeta, z)e^{-ix\zeta}f(x) \in D(L(\zeta))$ has the periodicity, we get

(5.10)
$$K_zf(x)+\alpha c_1(x,z)+\beta s_1(x,z) = e^{-i\zeta}(K_zf(x+1)+\alpha c_1(x+1,z)+\beta s_1(x+1,z)),$$

so putting $x = 0$, we have

$$\alpha = e^{-i\zeta}\left[c_1(1,z)\int_0^1 s_1(y,z)f(y)dy + \alpha c_1(1,z) + \beta s_1(1,z)\right]. \tag{5.11}$$

Differentiating both sides of (5.10) with respect to x and putting $x = 0$, we have

$$\begin{aligned}&\int_0^1 c_1(y,z)f(y)dy + \beta \\ &\quad = e^{-i\zeta}\left[c_2(1,z)\int_0^1 s_1(y,z)f(y)dy + \alpha c_2(1,z) + \beta s_2(1,z)\right].\end{aligned} \tag{5.12}$$

Note that $(\zeta, z) \in \Gamma$ if and only if $\delta(\zeta, z) := D(z) - e^{i\zeta} - e^{-i\zeta} \neq 0$. Solving (5.11) and (5.12) with respect to (α, β), we have

$$\begin{aligned}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \delta(\zeta,z)^{-1}\int_0^1 \Bigg[&\begin{pmatrix} s_1(1,z) \\ e^{i\zeta} - c_1(1,z)\end{pmatrix} c_1(y,z) \\ &+ \begin{pmatrix} e^{-i\zeta} - c_1(1,z) \\ -c_2(1,z)\end{pmatrix} s_1(y,z)\Bigg] f(y)dy.\end{aligned}$$

Combining this with (5.9), we obtain that

$$R(\zeta, z; x, y) = e^{i\zeta(y-x)}k(z;x,y) + \frac{e^{i\zeta(y-x)}s(\zeta,z;x,y)}{D(z) - e^{i\zeta} - e^{-i\zeta}},$$

where

$$\begin{aligned}s(\zeta,z;x,y) :=&[s_1(1,z)c_1(x,z) + (e^{i\zeta} - c_1(1,z))s_1(x,z)]c_1(y,z) \\ &+ [(e^{-i\zeta} - c_1(1,z))c_1(x,z) - c_2(1,z)s_1(x,z)]s_1(y,z).\end{aligned}$$

Suppose $D'(\lambda) \neq 0$. For z near λ, we have $D(z) - e^{\eta(\lambda)} - e^{-\eta(\lambda)} = (z - \lambda)F_\lambda(z)$ for some $F_\lambda(z)$ such that $F_\lambda(\lambda) = D'(\lambda) \neq 0$. Thus $R(i\eta(\lambda), z; x, y)$ has a pole λ of order one with the residue

$$r_1(\lambda; x, y) := D'(\lambda)^{-1}e^{(x-y)\eta(\lambda)}s(i\eta(\lambda), \lambda; x, y).$$

This implies that the eigenvalue λ of $L(i\eta(\lambda))$ is nondegenerate and its eigenprojection has the integral kernel $-r_1(\lambda; x, y)$. On the other hand, the eigenprojection and its adjoint are projections onto the spaces $\mathrm{Ker}\,(L(i\eta(\lambda)) - \lambda)$ and $\mathrm{Ker}\,(L(-i\eta(\lambda)) - \lambda)$, respectively, so the eigenprojection has the integral kernel $\dfrac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, v_\lambda)}$. Therefore $\dfrac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, v_\lambda)} = -r_1(\lambda; x, y)$.

Let $\lambda_0 \in \mathbf{R}$ satisfy $D'(\lambda_0) = 0$. For z near λ_0, we have $D(z) - e^{\eta(\lambda_0)} - e^{-\eta(\lambda_0)} = (z-\lambda_0)^2 H(z)$ for some $H(z)$ such that $H(\lambda_0) = D''(\lambda_0)/2 \neq 0$. Thus $R(i\eta(\lambda_0), z; x, y)$ has a pole λ_0 of order two:

$$R(i\eta(\lambda_0), z; x, y) = r_2(x, y)(z - \lambda_0)^{-2} + O((z - \lambda_0)^{-1}),$$

where

$$r_2(x, y) := 2D''(\lambda_0)^{-1} e^{(x-y)\eta(\lambda_0)} s(i\eta(\lambda_0), \lambda_0; x, y).$$

Hence the eigenvalue λ_0 of $L(i\eta(\lambda_0))$ is degenerate and its eigennilpotent has the integral kernel $-r_2(x, y)$. We shall show that $\dfrac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, \psi_{v_\lambda})} = -r_2(x, y)$ at $\lambda = \lambda_0$. Since

$$\partial_z c_1(x, z) = \int_0^x (c_1(x, z)s_1(t, z) - s_1(x, z)c_1(t, z))c_1(t, z)\,dt,$$
$$\partial_z s_2(x, z) = \int_0^x (c_2(x, z)s_1(t, z) - s_2(x, z)c_1(t, z))s_1(t, z)\,dt$$

(cf. [E]), we have for $\lambda \in A_+$

$$\begin{aligned} D'(\lambda) &= \partial_\lambda c_1(1, \lambda) + \partial_\lambda s_2(1, \lambda) \\ &= \int_0^1 [c_2(1, \lambda)s_1(x, \lambda)^2 + (c_1(1, \lambda) - s_2(1, \lambda))c_1(x, \lambda)s_1(x, \lambda) \\ &\qquad - s_1(1, \lambda)c_1(x, \lambda)^2]dx \\ &= -\int_0^1 s(i\eta(\lambda), \lambda; x, x)dx. \end{aligned}$$

As eigenfunctions of $L(i\eta(\lambda))$ and $L(-i\eta(\lambda))$ for $\lambda \in A_+$ near λ_0, we can choose u_λ and v_λ as follows: (i) when $c_1(1, \lambda_0) - e^{-\eta(\lambda_0)} \neq 0$,

$$\begin{aligned} u_\lambda(x) &:= e^{\eta(\lambda)x}[-s_1(1, \lambda)c_1(x, \lambda) + (c_1(1, \lambda) - e^{-\eta(\lambda)})s_1(x, \lambda)], \\ v_\lambda(x) &:= e^{-\eta(\lambda)x}[(c_1(1, \lambda) - e^{-\eta(\lambda)})c_1(x, \lambda) + c_2(1, \lambda)s_1(x, \lambda)]; \end{aligned}$$

(ii) when $c_1(1, \lambda_0) - e^{\eta(\lambda_0)} \neq 0$,

$$\begin{aligned} u_\lambda(x) &:= e^{\eta(\lambda)x}[(c_1(1, \lambda) - e^{\eta(\lambda)})c_1(x, \lambda) + c_2(1, \lambda)s_1(x, \lambda)], \\ v_\lambda(x) &:= e^{-\eta(\lambda)x}[-s_1(1, \lambda)c_1(x, \lambda) + (c_1(1, \lambda) - e^{\eta(\lambda)})s_1(x, \lambda)]. \end{aligned}$$

Let us treat the former case. (The latter is done similarly.) We have

$$\begin{aligned} s_1(1, \lambda)c_2(1, \lambda) &= c_1(1, \lambda)s_2(1, \lambda) - 1 \\ &= c_1(1, \lambda)(e^{\eta(\lambda)x} + e^{-\eta(\lambda)x} - c_1(1, \lambda)) - 1 \\ &= (e^{\eta(\lambda)x} - c_1(1, \lambda))(c_1(1, \lambda) - e^{-\eta(\lambda)x}). \end{aligned}$$

Thus

$$\begin{aligned} u_\lambda(x)v_\lambda(y) &= -e^{\eta(\lambda)(x-y)}(c_1(1, \lambda) - e^{-\eta(\lambda)})s(i\eta(\lambda), \lambda; x, y), \\ (u_\lambda, v_\lambda) &= (c_1(1, \lambda) - e^{-\eta(\lambda)})D'(\lambda). \end{aligned}$$

So $(u_\lambda, v_\lambda)' = (c_1(1,\lambda) - e^{-\eta(\lambda)})D''(\lambda)$ at $\lambda = \lambda_0$. Therefore

$$\frac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, \psi_{v_\lambda})} = 2\frac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, v_\lambda)'} = -2\frac{e^{\eta(\lambda)(x-y)}s(i\eta(\lambda), \lambda; x, y)}{D''(\lambda)} = -r_2(x,y)$$

at $\lambda = \lambda_0$. We have thus shown the proposition. □

Finally, we give an asymptotic expansion of the Green function $G_z(x,y)$ as the spectral parameter z approaches one of edges of the spectrum of L. We show it in a direct and elementary way, although the expansion of resolvents for Schrödinger operators with periodic potentials is given by [G, Corollary 4.2]. Let $\Delta_+ := \mathbf{C} \setminus [0,\infty)$. We denote by $z^{\frac{1}{2}}$ a branch of the square root of $z \in \Delta_+$ such that $z^{\frac{1}{2}} = \sqrt{r}e^{i\theta/2}$ for $z = re^{i\theta}$, $0 < \theta < 2\pi$, $r > 0$. Note that λ is an edge of the spectrum of L if and only if $|D(\lambda)| = 2$ and $D'(\lambda) \neq 0$. If $D(\lambda) = 2$ and $D'(\lambda) \neq 0$, there exist real-valued linearly independent solutions u and ψ of $Ly = \lambda y$ such that u is a real-valued periodic function with period 1 and $\psi(x) = xu(x) + v(x)$ for some real-valued periodic function v with period 1; if $D(\lambda) = -2$ and $D'(\lambda) \neq 0$, there exist real-valued linearly independent solutions u and ψ of $Ly = \lambda y$ such that u is a real-valued semi-periodic function with semi-period 1, i.e., $u(x+1) = -u(x)$, and $\psi(x) = xu(x) + v(x)$ for some real-valued semi-periodic function v with semi-period 1 (cf. [E, p.7 and p.29]).

Theorem 5.5. *Assume that μ_{2n-1} is an edge of the spectrum of L. Then for any integer $m \geq -1$ one has the expansion for small $z - \mu_{2n-1} \in \Delta_+$*

$$G_z(x,y) = \sum_{j=-1}^{m} (z - \mu_{2n-1})^{\frac{j}{2}} q_j(x,y) + r_m(z; x, y),$$

where $r_m(z;x,y)$ satisfies the estimate: for any $0 \leq \theta \leq 1$

$$|r_m(z;x,y)| \leq C_m |z - \mu_{2n-1}|^{(m+\theta)/2}(|x-y|+1)^{m+1+\theta}.$$

Furthermore, $q_j(x,y)$ is of the form

$$q_j(x,y) = q_j(y,x) = \sum_{k=0}^{j+1} (x-y)^k q_{j,k}(x,y), \quad y \leq x,$$

for some $q_{j,k}(x,y) \in C(\mathbf{T} \times \mathbf{T})$. In particular,

$$q_{-1}(x,y) = \frac{i}{\sqrt{2\lambda''_{2n-1}(0)}} \frac{u(x)u(y)}{\|u\|^2},$$
$$q_0(x,y) = q_0(y,x) = \lambda''_{2n-1}(0)^{-1}(u(x)\psi(y) - \psi(x)u(y))/\|u\|^2, \quad y \leq x,$$

where $\lambda''_{2n-1}(0) > 0$, and u and ψ are real-valued linearly independent solutions of $Ly = \mu_{2n-1}y$ such that u is a periodic function with period 1 and $\psi(x) = xu(x) + v(x)$ for some periodic function v with period 1.

Remark 5.6. If ν_{2n-1}, ν_{2n}, or μ_{2n} is an edge of the spectrum, a similar expansion holds around it.

Proof. Since $D(\mu_{2n-1}) = 2$ and $D'(\mu_{2n-1}) < 0$, there exists a holomorphic inverse function D^{-1} of D near $D = 2$. Put $\lambda(\zeta) = D^{-1}(e^{i\zeta} + e^{-i\zeta})$ near $\zeta = 0$. Then $\lambda(\xi) = \lambda_{2n-1}(\xi) \geq \mu_{2n-1}$ for small $\xi \in \mathbf{R}$ and $\lambda'(0) = 0$. Furthermore, since $D(\lambda(\xi)) = 2\cos\xi$, we have

$$D''(\lambda(\xi))\lambda'(\xi)^2 + D'(\lambda(\xi))\lambda''(\xi) = -2\cos\xi.$$

This implies that $\lambda''(0) = -2/D'(\mu_{2n-1}) > 0$. Therefore we can choose a sufficiently small positive number R such that the set $\{\lambda(\zeta); \operatorname{Im}\zeta > 0,\ |\zeta| < R\}$ is a subdomain of $\mathbf{C} \setminus [\mu_{2n-1}, \infty)$. We have also that $s_1(1, \mu_{2n-1})$ and $c_2(1, \mu_{2n-1})$ are not both zero (cf. [E, p.29]). So we can choose a holomorphic eigenvector $(\alpha_1(\zeta), \alpha_2(\zeta))$ of $M(\lambda(\zeta))$ corresponding to the eigenvalue $e^{i\zeta}$ near $\zeta = 0$. Put $y_\zeta(x) := \alpha_1(\zeta)c_1(x, \lambda(\zeta)) + \alpha_2(\zeta)s_1(x, \lambda(\zeta))$. Then $u_\zeta(x) := e^{-i\zeta x}y_\zeta(x)$ is a holomorphic eigenfunction of $L(\zeta)$ corresponding to the eigenvalue $\lambda(\zeta)$ near $\zeta = 0$. Let $\mathbf{C}_+ := \{\zeta \in \mathbf{C}; \operatorname{Im}\zeta > 0\}$. For small $\zeta \in \mathbf{C}_+$, since $\overline{\lambda(\zeta)} = \lambda(\bar{\zeta})$, it follows that $y_\zeta = e^{i\zeta x}u_\zeta$ and $\overline{y_{\bar{\zeta}}} = e^{-i\zeta x}\overline{u_{\bar{\zeta}}}$ are linearly independent solutions to $Ly = \lambda(\zeta)y$. Hence as in the proof of Theorem 5.1, since $i[y_\zeta, \overline{y_{\bar{\zeta}}}](0) = \lambda'(\zeta)(u_\zeta, u_{\bar{\zeta}})$, we have for $y \leq x$ and small $\zeta \in \mathbf{C}_+$

(5.13)
$$G_{\lambda(\zeta)}(x,y) = G_{\lambda(\zeta)}(y,x) = y_\zeta(x)\overline{y_{\bar{\zeta}}(y)}/[y_\zeta, \overline{y_{\bar{\zeta}}}](0) = i\lambda'(\zeta)^{-1}e^{i(x-y)\zeta}p_\zeta(x,y),$$

where $p_\zeta(x,y) := u_\zeta(x)\overline{u_{\bar{\zeta}}(y)}/(u_\zeta, u_{\bar{\zeta}})$ is a $C(\mathbf{T} \times \mathbf{T})$-valued holomorphic function near $\zeta = 0$. Let $y \leq x$. We write the Taylor expansion of $e^{i(x-y)\zeta}p_\zeta(x,y)$ with respect to ζ as follows:

$$e^{i(x-y)\zeta}p_\zeta(x,y) = \sum_{j=0}^{m} \tilde{q}_j(x,y)\zeta^j + \tilde{r}_m(\zeta; x, y), \tag{5.14}$$

where

$$\tilde{q}_j(x,y) = \sum_{k=0}^{j} (x-y)^k \tilde{q}_{j,k}(x,y) \tag{5.15}$$

for some $\tilde{q}_{j,k}(x,y) \in C(\mathbf{T} \times \mathbf{T})$, and $\tilde{r}_m(\zeta; x, y)$ satisfies the estimate: for any $0 \leq \theta \leq 1$

$$|\tilde{r}_m(\zeta; x, y)| \leq C_m|\zeta|^{m+\theta}(|x-y|+1)^{m+\theta}. \tag{5.16}$$

Let us show this remainder estimate. We have

$$e^{i(x-y)\zeta} = \sum_{j=0}^{m} \frac{(i(x-y)\zeta)^j}{j!} + \frac{(i(x-y)\zeta)^{m+1}}{m!}\int_0^1 (1-t)^m e^{it(x-y)\zeta}\,dt.$$

Thus

$$\left| e^{i(x-y)\zeta} - \sum_{j=0}^{m} \frac{(i(x-y)\zeta)^j}{j!} \right| \le \frac{(|x-y||\zeta|)^{m+1}}{(m+1)!},$$

since $\mathrm{Re}\,[it(x-y)\zeta] \le 0$. This implies that

$$|\tilde{r}_m(\zeta; x, y)| \le C_m |\zeta|^{m+1} (|x-y|+1)^{m+1}.$$

On the other hand, since

$$\tilde{r}_m(\zeta; x, y) = \tilde{r}_{m-1}(\zeta; x, y) - \tilde{q}_m(x, y)\zeta^m,$$

we have

$$|\tilde{r}_m(\zeta; x, y)| \le C_m |\zeta|^m (|x-y|+1)^m.$$

Hence we get the desired estimate (5.16). We see that $\tilde{q}_0(x, y) = p_0(x, y)$ and $\tilde{q}_1(x, y) = i(x-y)p_0(x, y) + \partial_\zeta p_\zeta(x, y)|_{\zeta=0}$. We shall show that $\tilde{q}_1(x, y) = i(\psi(x)u(y) - u(x)\psi(y))/\|u\|^2$, where $u(x)$ and $\psi(x) = xu(x) + v(x)$ are linearly independent solutions stated in the theorem. We have

$$\begin{aligned} \partial_\zeta y_\zeta|_{\zeta=0} &= \alpha_1'(0)c_1(x, \mu_{2n-1}) + \alpha_2'(0)s_1(x, \mu_{2n-1}) = ixu_0 + \partial_\zeta u_\zeta|_{\zeta=0}, \\ \partial_\zeta \overline{y_{\bar\zeta}}|_{\zeta=0} &= \overline{\alpha_1'(0)}c_1(x, \mu_{2n-1}) + \overline{\alpha_2'(0)}s_1(x, \mu_{2n-1}) = -ix\overline{u_0} + \partial_\zeta \overline{u_{\bar\zeta}}|_{\zeta=0}. \end{aligned}$$

So $\partial_\zeta y_\zeta|_{\zeta=0}$ and $\partial_\zeta \overline{y_{\bar\zeta}}|_{\zeta=0} = \overline{\partial_\zeta y_\zeta}|_{\zeta=0}$ are solutions of $Ly = \mu_{2n-1}y$, and we have $u_0 = cu$ and $\partial_\zeta y_\zeta|_{\zeta=0} = ic\psi + c'u$ for some $c, c' \in \mathbf{C}$. Hence

$$\partial_\zeta u_\zeta|_{\zeta=0} = icv(x) + c'u(x), \quad \partial_\zeta \overline{u_{\bar\zeta}}|_{\zeta=0} = -i\bar{c}v(x) + \overline{c'}u(x).$$

Using this we have

$$\begin{aligned} &\tilde{q}_1(x, y) = i(x-y)p_0(x, y) + \partial_\zeta p_\zeta(x, y)|_{\zeta=0} \\ &= i(x-y)p_0(x, y) + \frac{\partial_\zeta(u_\zeta(x)\overline{u_{\bar\zeta}(y)})|_{\zeta=0}}{\|u_0\|^2} - p_0(x, y)\frac{(u_\zeta, u_{\bar\zeta})'|_{\zeta=0}}{\|u_0\|^2} \\ &= i(x-y)\frac{u(x)u(y)}{\|u\|^2} + \frac{(icv(x) + c'u(x))\bar{c}u(y) + cu(x)(-i\bar{c}v(y) + \overline{c'}u(y))}{|c|^2\|u\|^2} \\ &\qquad - \frac{u(x)u(y)}{\|u\|^2}\frac{2\mathrm{Re}(icv + c'u, cu)}{|c|^2\|u\|^2} \\ &= i(x-y)u(x)u(y)/\|u\|^2 + i(v(x)u(y) - u(x)v(y))/\|u\|^2 \\ &= i(\psi(x)u(y) - u(x)\psi(y))/\|u\|^2. \end{aligned}$$

There exists an entire function $F(z)$ such that $F(\zeta^2) = e^{i\zeta} + e^{-i\zeta} - 2$; $F(z)$ is real for real z, $F(0) = 0$, and $F'(0) = -1$. So there exists an inverse function F^{-1} of F near the origin. Thus for $\delta > 0$ small, the map $z \in \{z \in \Delta_+ + \mu_{2n-1};\ |z - \mu_{2n-1}| < \delta\} \mapsto \zeta(z) := (F^{-1}(D(z) - 2))^{\frac{1}{2}} \in \mathbf{C}_+$ is conformal from the disc with the cut to the intersection of a neighborhood of the origin and $\mathbf{C}_+$.

Note that $\lambda(\zeta(z)) = z$. Noting that $D(z) - 2 = D'(\mu_{2n-1})(z - \mu_{2n-1}) + O((z - \mu_{2n-1})^2)$ and $F^{-1}(w) = -w + O(w^2)$, we have the Puiseux series

$$\zeta(z) = \sum_{j=0}^{\infty} a_j (z - \mu_{2n-1})^{j+\frac{1}{2}}, \tag{5.17}$$

where $a_0 = \sqrt{|D'(\mu_{2n-1})|} = \sqrt{2/\lambda''_{2n-1}(0)}$. Note that $\lambda'(\zeta(z))^{-1} = \zeta'(z)$. By (5.13), (5.14) and (5.17),

$$\begin{aligned} G_z(x,y) &= i\zeta'(z) e^{i(x-y)\zeta(z)} p_{\zeta(z)}(x,y) \\ &= i \left[\sum_{j=0}^{\infty} a_j \left(j + \frac{1}{2} \right) (z - \mu_{2n-1})^{j-1/2} \right] \left[\sum_{j=0}^{m} \tilde{q}_j(x,y) \zeta(z)^j + \tilde{r}_m(\zeta(z); x, y) \right] \\ &= \sum_{j=-1}^{m} (z - \mu_{2n-1})^{j/2} q_j(x,y) + r_m(z; x, y). \end{aligned}$$

This together with (5.15) and (5.16) yields the desired expansion. □

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