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On the groups $[X, Sp(n)]$ with $\dim X \leq 4n+2$

By

Tomoaki Nagao

1. Introduction

Let G be a group-like space, that is, G satisfies all the axioms of groups up to homotopy, and let X be a based space. The based homotopy set $[X, G]$ becomes a group by the pointwise multiplication and moreover, when G is connected, G.W. Whitehead [15] shows that $[X, G]$ is a nilpotent group of class $\leq \operatorname{cat} X$, where $\operatorname{cat} X$ stands for the L-S category of X normalized as $\operatorname{cat}(*) = 0$. However, in general it is hard to understand the group $[X, G]$ further. It is of particular interest the case that G is a compact Lie group and it has been studied by many ([16], [2], [11], [12]). In particular, when $G = U(n)$ and X is a CW-complex with $\dim X \leq 2n$, Hamanaka and Kono [8] give an explicit method to calculate $U_n(X) = [X, U(n)]$. Note that $U_n(X)$ is naturally isomorphic to $\widetilde{K}^{-1}(X)$ when $\dim X < 2n$. Then, when $\dim X = 2n$, $U_n(X)$ may contain the first unstable property and, in fact, Hamanaka and Kono [8] show that $U_n(X)$ is given by a central extension of $\widetilde{K}^{-1}(X)$. Moreover, the commutator in $U_n(X)$ is explicitly calculated. Later, Hamanaka and Kono developed this method further and give applications ([5], [9], [6], [7]).

The aim of this paper is to study the group $Sp_n(X) = [X, Sp(n)]$ when $\dim X \leq 4n+2$ following Hamanaka and Kono [8]. In this paper, all cohomology groups have integral coefficients. We will prove:

Theorem 1.1. *Let X be a CW-complex with $\dim X \leq 4n+2$. There is an exact sequence*

$$\widetilde{KSp}^{-2}(X) \xrightarrow{\Theta_{\mathbb{H}}} H^{4n+2}(X) \to Sp_n(X) \to \widetilde{KSp}^{-1}(X) \to 0 \tag{1.1}$$

which is natural with respect to X. Moreover, the induced sequence

$$0 \to \mathbf{N}_n(X) \xrightarrow{\iota} Sp_n(X) \to \widetilde{KSp}^{-1}(X) \to 0.$$

is a central extension, where $\mathbf{N}_n(X) = \operatorname{Coker} \Theta_{\mathbb{H}}$.

As in the case of $U_n(X)$ noted above, we can give the commutator in $Sp_n(X)$ explicitly as follows. The cohomology of $Sp(n)$ is:

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(1.2) $$H^*(Sp(n)) = \Lambda(y_3, y_7, \ldots, y_{4n-1}),\ y_{4i-1} = \sigma(q_i),$$

where σ and q_i denote the cohomology suspension and the universal i-th symplectic Pontrjagin class respectively.

Theorem 1.2. *Let X be a CW-complex with $\dim X \leq 4n+2$ and let $\iota\colon \mathbf{N}_n(X) \to Sp_n(X)$ be as in Theorem 1.1. For $\alpha, \beta \in Sp_n(X)$, the commutator $[\alpha, \beta]$ in $Sp_n(X)$ is given as*

$$[\alpha, \beta] = \iota\left(\left[\sum_{i+j=n+1} \alpha^*(y_{4i-1})\beta^*(y_{4j-1})\right]\right).$$

Denote by $\mathbf{c}'$ both the canonical inclusion $Sp(n) \hookrightarrow U(2n)$ and the induced map $\widetilde{KSp}^*(-) \to \widetilde{K}^*(-)$. We also denote by $\mathbf{c}'$ the composition of the inclusions

$$Sp(n) \overset{\mathbf{c}'}{\hookrightarrow} U(2n) \hookrightarrow U(2n+1).$$

By using the above maps $\mathbf{c}'$, we compare $Sp_n(X)$ with $U_{2n+1}(X)$ as:

Theorem 1.3. *Let X be a CW-complex with $\dim X \leq 4n+2$. Then there is a commutative diagram*

(1.3)
$$\begin{array}{ccccccccc}
\widetilde{KSp}^{-2}(X) & \xrightarrow{\Theta_{\mathbb{H}}} & H^{4n+2}(X) & \longrightarrow & Sp_n(X) & \longrightarrow & \widetilde{KSp}^{-1}(X) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \mathbf{c}'} & & \downarrow{\scriptstyle (-1)^{n+1}} & & \downarrow{\scriptstyle \mathbf{c}'} & & \downarrow{\scriptstyle \mathbf{c}'} & & \\
\widetilde{K}^{-2}(X) & \xrightarrow{\Theta_{\mathbb{C}}} & H^{4n+2}(X) & \longrightarrow & U_{2n+1}(X) & \longrightarrow & \widetilde{K}^{-1}(X) & \longrightarrow & 0
\end{array}$$

which is natural with respect to X, where the top and the bottom rows are the exact sequences in Theorem 1.1 *and in* [8, Theorem 1.1] *respectively.*

As an application of the above results, we will give some special calculation (For a further application, see [10].).

Proposition 1.4. $Sp_n(\Sigma^2\mathbb{H}P^n) \cong \mathbb{Z}/4(2n+1)$.

Proposition 1.5. *Let Q_2 be the quasi-projective space of $Sp(2)$. Denote by ϵ and ϵ_3 respectively the inclusions $Q_2 \to Sp(2)$ and $S^3 \to Sp(2)$. Then the order of the Samelson product $\langle \epsilon_3, \epsilon \rangle$ is* 40.

Theorem 1.6. *Let $S^{4n-1} \xrightarrow{i} X \xrightarrow{p} S^{4m-1}$ be a sphere bundle over a sphere such that $m+n$ is odd. Then $Sp_{m+n-1}(X)$ is generated by three elements α, β, ϵ subject to the relations*

$$[\alpha, \epsilon] = [\beta, \epsilon] = (2(m+n)-1)!\epsilon = 0,\ [\alpha, \beta] = 2(2m-1)!(2n-1)!\epsilon.$$

By applying Theorem 1.6 to the fiber bundle $Sp(1) \to Sp(2) \to S^7$, we obtain the following.

Corollary 1.7 (Mimura and Ōshima [14]). *The group* $[Sp(2), Sp(2)]$ *is generated by three elements* α, β, ϵ *subject to the relations*

$$[\alpha, \epsilon] = [\beta, \epsilon] = 5!\epsilon = 0, \ [\alpha, \beta] = 12\epsilon.$$

The organization of this paper is as follows. In Section 2, we first recall some results of Hamanaka and Kono [8]. We follow their methods to prove Theorem 1.1 and Theorem 1.3. We also estimate the order of elements in $\mathbf{N}_n(X)$. In Section 3, we prove Theorem 1.2 quite similarly to the proof of [8, Theorem 1.4]. In Section 4, by exploiting the results obtained so far, we give the above special calculation as an application.

2. Exact sequences

Let us first recall some results of Hamanaka and Kono [8]. Let X be a CW-complex with $\dim X \leq 2n$ and let W_n denote the infinite Stiefel manifold $U(\infty)/U(n)$. By applying $[X, -]$ to the fibration sequence

$$\Omega U(\infty) \to \Omega W_n \to U(n) \xrightarrow{i} U(\infty) \xrightarrow{p} W_n,$$

we obtain the exact sequence

$$\widetilde{K}^{-2}(X) \to [X, \Omega X_n] \to U_n(X) \xrightarrow{i_*} \widetilde{K}^{-1}(X) \to [X, W_n], \tag{2.1}$$

here we use the isomorphism

$$\widetilde{K}^{-i}(X) \cong [\Sigma^i X, BU(\infty)].$$

Since W_n is $2n$-connected and $\dim X \leq 2n$, $[X, W_n]$ is trivial. Then i_* is epic.

It is well known that the cohomology of $U(n)$ is given by

$$H^*(U(n)) = \Lambda(x_1, \dots, x_{2n-1}), \ x_{2i-1} = \sigma(c_i)$$

where σ and c_i are the cohomology suspension and the universal i-th Chern class respectively. The cohomology of W_n is given as

$$H^*(W_n) = \Lambda(\bar{x}_{2n+1}, \bar{x}_{2n+3}, \dots), \ p^*(\bar{x}_{2i-1}) = x_{2i-1} \in H^*(U(\infty)).$$

Since W_n is $2n$-connected, one can see that $H^{2n}(\Omega W_n) \cong \mathbb{Z}$ which is generated by $a_{2n} = \sigma(\bar{x}_{2n+1})$. We ambiguously write the representing map of a_{2n}, that is, $\Omega W_n \to K(\mathbb{Z}, 2n)$, by the same symbol a_{2n}. Then, by definition, $a_{2n} \colon \Omega W_n \to K(\mathbb{Z}, 2n)$ is a loop map. On the other hand, $a_{2n} \colon \Omega W_n \to K(\mathbb{Z}, 2n)$ is a $(2n+1)$-equivalence. Then, by the J.H.C. Whitehead theorem, we have a group isomorphism

$$(a_{2n})_* \colon [X, \Omega W_n] \xrightarrow{\cong} H^{2n}(X)$$

and hence the exact sequence (2.1) can be reformulated as

$$\widetilde{K}^{-2}(X) \xrightarrow{\Theta_{\mathbb{C}}} H^{2n}(X) \to U_n(X) \to \widetilde{K}^{-1}(X) \to 0. \tag{2.2}$$

This exact sequence is, of course, the bottom row sequence of (1.3).

Let ω_1 be the canonical line bundle over $S^2 = \mathbb{C}P^1$ and let $\eta \in \widetilde{K}^0(S^2)$ denote $\omega_1 - 1_{\mathbb{C}}$, where $1_{\mathbb{C}}$ is the trivial complex line bundle. Then it is well known that

$$\bar{\eta}\wedge : \widetilde{K}^0(X) \to \widetilde{K}^{-2}(X)$$

is an isomorphism for any X, which is Bott periodicity.

We write the representing map of $\alpha \in \widetilde{K}^0(X)$, namely $X \to BU(\infty)$, by the same symbol α. Hamanaka and Kono [8] explicitly give the formula of $\Theta_{\mathbb{C}}$ in the above exact sequence (2.2) as:

Proposition 2.1 (Hamanaka and Kono [8, Proposition 3.1]). *Let X be a CW-complex with $\dim X \leq 2n$ and let $s_n \in H^{2n}(BU(\infty))$ be the n-th power sum. Then, for $\alpha \in \widetilde{K}^0(X)$, $\Theta_{\mathbb{C}}$ in (2.2) is given by*

$$\Theta_{\mathbb{C}}(\bar{\eta} \wedge \alpha) = (-1)^n s_n(\alpha),$$

where $s_n(\alpha) = \alpha^*(s_n)$.

In order to make Proposition 2.1 more applicable, we give a formula of the power sum s_n.

Proposition 2.2 (Hamanaka and Kono [8, Lemma 3.2]). *For $\theta_1 \in \widetilde{K}^0(X_1)$, $\theta_2 \in \widetilde{K}^0(X_2)$, we have*

$$s_j(\theta_1 \wedge \theta_2) = \sum_{k=1}^{j-1} \binom{j}{k} s_k(\theta_1) \times s_{j-k}(\theta_2).$$

Following the above method of constructing the exact sequence (2.2), we prove Theorem 1.1 and Theorem 1.3. Let X be a CW-complex with $\dim X \leq 4n+2$. Consider the fibration sequence

$$\Omega Sp(\infty) \to \Omega X_n \xrightarrow{\Omega\delta} Sp(n) \xrightarrow{i} Sp(\infty) \xrightarrow{p} X_n,$$

where $X_n = Sp(\infty)/Sp(n)$. By applying $[X, -]$ to the above fibration sequence, we obtain the exact sequence

$$(2.3) \qquad \widetilde{KSp}^{-2}(X) \to [X, \Omega X_n] \xrightarrow{\Omega\delta_*} Sp_n(X) \xrightarrow{i_*} \widetilde{KSp}^{-1}(X) \to [X, X_n]$$

as well as the above case of $U(n)$, where we use the isomorphism $\widetilde{KSp}^{-i}(X) \cong [\Sigma^i X, BSp(\infty)]$. Since X_n is $(4n+2)$-connected and $\dim X \leq 4n+2$, $[X, X_n]$ is trivial and hence i_* in (2.3) is epic.

The cohomology of $Sp(n)$ is given as (1.2). It is easily seen that

$$H^*(X_n) = \Lambda(\bar{y}_{4n+3}, \bar{y}_{4n+7}, \dots),\ p^*(\bar{y}_{4i+3}) = y_{4i+3} \in H^*(Sp(\infty)).$$

Since X_n is $(4n+2)$-connected, one has that $H^{4n+2}(\Omega X_n) \cong \mathbb{Z}$ which is generated by $b_{4n+2} = \sigma(\bar{y}_{4n+3})$. As above, we write the representing map of

b_{4n+2}, that is, $\Omega X_n \to K(\mathbb{Z}, 4n+2)$, by the same symbol b_{4n+2} and then, by definition, $b_{4n+2} \colon \Omega X_n \to K(\mathbb{Z}, 4n+2)$ is a loop map. On the other hand, $b_{4n+2} \colon \Omega X_n \to K(\mathbb{Z}, 4n+2)$ is a $(4n+3)$-equivalence. Then, by the J.H.C. Whitehead theorem, we have a group isomorphism

$$(b_{4n+2})_* \colon [X, \Omega X_n] \xrightarrow{\cong} H^{4n+2}(X)$$

and hence, from (2.3), we obtain the exact sequence

$$\widetilde{KSp}^{-2}(X) \xrightarrow{\Theta_{\mathbb{H}}} H^{4n+2}(X) \to Sp_n(X) \xrightarrow{i_*} \widetilde{KSp}^{-1}(X) \to 0. \tag{2.4}$$

Thus we have established the first part of Theorem 1.1.

Note that we have the homotopy commutative diagram

$$\begin{array}{ccccccc}
\Omega Sp(\infty) & \xrightarrow{\Omega p} & \Omega X_n & \longrightarrow & Sp(n) & \longrightarrow & Sp(\infty) \\
\downarrow{\scriptstyle \Omega \mathbf{c}'} & & \downarrow{\scriptstyle \Omega \bar{\mathbf{c}'}} & & \downarrow{\scriptstyle \mathbf{c}'} & & \downarrow{\scriptstyle \mathbf{c}'} \\
\Omega U(\infty) & \xrightarrow{\Omega p'} & \Omega W_{2n+1} & \longrightarrow & U(2n+1) & \longrightarrow & U(\infty),
\end{array}$$

where $\bar{\mathbf{c}'} \colon X_n \to W_{2n+1}$ is the map induced by $\mathbf{c}'$. Since $(B\mathbf{c}')^*(c_{2n+2}) = (-1)^{n+1} q_{n+1}$, one has $(\bar{\mathbf{c}'})^*(\bar{x}_{4n+3}) = (-1)^{n+1} \bar{y}_{4n+3}$. Then it follows that

$$\begin{aligned}
(\Omega \bar{\mathbf{c}'})^*(a_{4n+2}) &= (\Omega \bar{\mathbf{c}'})^*(\sigma(\bar{x}_{4n+3})) = \sigma((\bar{\mathbf{c}'})^*(\bar{x}_{4n+3})) = (-1)^{n+1}\sigma(\bar{y}_{4n+3}) \\
&= (-1)^{n+1} b_{4n+2}.
\end{aligned}$$

Hence, by the construction of the exact sequences (2.2) and (2.4), the proof of Theorem 1.3 is accomplished.

We continue to denote by X a CW-complex with $\dim X \leq 4n+2$. Next, we prove the rest part of Theorem 1.1, that is,

$$0 \to \mathbf{N}_n(X) \xrightarrow{\iota} Sp_n(X) \xrightarrow{i_*} \widetilde{KSp}^{-1}(X) \to 0$$

is a central extension, where $\mathbf{N}_n(X) = \operatorname{Coker} \Theta_{\mathbb{H}}$. For $\alpha \colon X \to Sp(n)$ and $\beta \colon X \to \Omega X_n$, the commutator $[\alpha, \Omega\delta \circ \beta]$ in $Sp_n(X)$ is the composition

$$X \xrightarrow{\Delta} X \wedge X \xrightarrow{\alpha \wedge \beta} Sp(n) \wedge \Omega X_n \xrightarrow{1 \wedge \Omega\delta} Sp(n) \wedge Sp(n) \xrightarrow{\gamma} Sp(n), \tag{2.5}$$

where Δ and γ denote the diagonal map and the commutator map of $Sp(n)$ respectively. Since $Sp(n) \wedge \Omega X_n$ is $(4n+4)$-connected and $\dim X \leq 4n+2$, the map $(\alpha \wedge \beta) \circ \Delta \colon X \to Sp(n) \wedge \Omega X_n$ is null-homotopic. Then the commutator $[\alpha, \Omega\delta \circ \beta]$ is trivial and hence the proof of Theorem 1.1 is completed.

Remark 2.1. Let X be a CW-complex X with $\dim X \leq 4n+4$. Then it follows from the above proof that

$$0 \to N_n(X) \to Sp_n(X) \to \operatorname{Im}\{i_* \colon Sp_n(X) \to \widetilde{KSp}^{-1}(X)\} \to 0$$

is a central extension and hence $Sp_n(X)$ is a nilpotent group of class less than or equal to 2.

For the last of this section, we estimate the order of elements in $\mathbf{N}_n(X)$.

Proposition 2.3. *Let X and $\mathbf{N}_n(X)$ be as in Theorem 1.1. Then each element in the group $\mathbf{N}_n(X)$ is of order dividing $2(2n+1)!$ when n is odd and $(2n+1)!$ when n is even.*

Proof. Consider the cofibration sequence

$$X^{(4n+1)} \to X \xrightarrow{p} \bigvee_\alpha S_\alpha^{4n+2},$$

where $X^{(4n+1)}$ denotes the $(4n+1)$-skeleton of X and p is the pinching map. Then it follows from Theorem 1.1 that, in the diagram

$$\begin{array}{ccccc}
0 & \longleftarrow & H^{4n+2}(X) & \xleftarrow{p^*} & \prod_\alpha H^{4n+2}(S_\alpha^{4n+2}) \\
\downarrow & & \downarrow{\scriptstyle \tilde{\imath}} & & \downarrow \\
Sp_n(X^{(4n+1)}) & \longleftarrow & Sp_n(X) & \xleftarrow{p^*} & \prod_\alpha \pi_{4n+2}(Sp(n)) \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{KSp}^{-1}(X^{(4n+1)}) & \longleftarrow & \widetilde{KSp}^{-1}(X) & \longleftarrow & 0,
\end{array}$$

each row and column sequence is exact. Hence we have

$$\begin{aligned}
\mathbf{N}_n(X) &\cong \mathrm{Im}\left\{\tilde{\imath}\colon H^{4n+2}(X) \to Sp_n(X)\right\} \\
&= \mathrm{Im}\left\{\tilde{\imath} \circ p^*\colon \prod_\alpha H^{4n+2}(S_\alpha^{4n+2}) \to Sp_n(X)\right\} \\
&= \mathrm{Im}\left\{p^*\colon \prod_\alpha \pi_{4n+2}(Sp(n)) \to Sp_n(X)\right\}.
\end{aligned}$$

One can easily deduce from the result of Borel and Hirzebruch [4] that

$$\pi_{4n+2}(Sp(n)) \cong \begin{cases} \mathbb{Z}/(2n+1)! & n \text{ is even} \\ \mathbb{Z}/2(2n+1)! & n \text{ is odd} \end{cases}$$

and then we have established Proposition 2.3. □

3. The commutator in $Sp_n(X)$

Hamanaka and Kono [8] investigated the commutator in $U_n(X)$ by constructing a lift of the commutator map $U(n)\wedge U(n) \to U(n)$ to ΩW_n. We follow this procedure to study the commutator in $Sp_n(X)$. Let $\gamma\colon Sp(n) \wedge Sp(n) \to Sp(n)$ be the commutator of $Sp(n)$ as in the previous section. Consider the fibration

$$\Omega X_n \xrightarrow{\Omega\delta} Sp(n) \xrightarrow{i} Sp(\infty).$$

Since $Sp(\infty)$ is homotopy abelian, $i \circ \gamma$ is null-homotopic. Then, by the homotopy lifting property of $i\colon Sp(n) \to Sp(\infty)$, we have a map $\tilde{\gamma}\colon Sp(n) \wedge Sp(n) \to \Omega X_n$ satisfying the following homotopy commutative diagram.

$$\begin{array}{ccc} & & \Omega X_n \\ & \overset{\tilde{\gamma}}{\nearrow} & \downarrow{\scriptstyle \Omega\delta} \\ Sp(n) \wedge Sp(n) & \xrightarrow{\gamma} & Sp(n) \end{array}$$

We shall construct a special lift $\tilde{\gamma}$ to prove Theorem 1.2.

Define a map $\bar{\omega}\colon Sp(n) * Sp(n) \to \Sigma Sp(n) \vee \Sigma Sp(n)$ by

$$\bar{\omega}(t, x, y) = \begin{cases} ((1-2t, x), e) & 0 \leq t \leq \frac{1}{2} \\ (e, (2t-1, y)) & \frac{1}{2} \leq t \leq 1, \end{cases}$$

where $X * Y$ denotes the join of X and Y, and e is the basepoint of $\Sigma Sp(n)$. Let $\omega\colon \Sigma Sp(n) \wedge Sp(n) \to \Sigma Sp(n) \vee \Sigma Sp(n)$ be a homotopy inverse of the canonical map $Sp(n) * Sp(n) \to \Sigma Sp(n) \wedge Sp(n)$ followed by $\bar{\omega}$. Then the induced map

$$\omega^*\colon [\Sigma Sp(n), X] \times [\Sigma Sp(n), X] \to [\Sigma Sp(n) \wedge Sp(n), X]$$

gives the generalized Whitehead product in the sense of Arkowitz [1]. Hence it follows that, for $\alpha, \beta \in [\Sigma Sp(n), X]$, one has

$$\text{(3.1)} \qquad \mathrm{ad}(\omega^*(\alpha, \beta)) = \gamma \circ (\mathrm{ad}(\alpha) \wedge \mathrm{ad}(\beta)),$$

where $\mathrm{ad}\colon [\Sigma X, Y] \xrightarrow{\cong} [X, \Omega Y]$ takes the adjoint (See [1] for details).

Let I_ω and C_ω denote the mapping cylinder and the mapping cone of ω respectively. Arkowitz [1] showed that there is a homotopy equivalence $\phi\colon C_\omega \xrightarrow{\simeq} \Sigma Sp(n) \times \Sigma Sp(n)$ which satisfies the following homotopy commutative diagram.

$$\begin{array}{ccc} I_\omega & \xrightarrow{p_1} & C_\omega \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \phi} \\ \Sigma Sp(n) \vee \Sigma Sp(n) & \subset & \Sigma Sp(n) \times \Sigma Sp(n), \end{array}$$

where p_1 and p_2 are the pinching map and the projection respectively. Let j and k be the compositions

$$\Sigma Sp(n) \vee \Sigma Sp(n) \xrightarrow{\mathrm{ad}^{-1}(1) \vee \mathrm{ad}^{-1}(1)} BSp(n) \vee BSp(n) \xrightarrow{\nabla} BSp(n)$$

and

$$\Sigma Sp(n) \times \Sigma Sp(n) \xrightarrow{\mathrm{ad}^{-1}(1) \times \mathrm{ad}^{-1}(1)} BSp(n) \times BSp(n) \xrightarrow{D} BSp(2n) \xrightarrow{Bi} BSp(\infty)$$

respectively, where ∇ denotes the folding map and D is the induced map from the diagonal inclusion $Sp(n) \times Sp(n) \to Sp(2n)$. Let us consider the homotopy commutative diagram:

$$\begin{array}{ccc} I_\omega & \xrightarrow{p_1} & C_\omega \\ {\scriptstyle j\circ p_2}\downarrow & & \downarrow{\scriptstyle k\circ\phi} \\ BSp(n) & \xrightarrow{Bi} & BSp(\infty) \end{array}$$

Here we choose $k \circ \phi$ to be basepoint preserving. By applying the homotopy lifting property of the fibration $Bi\colon BSp(n) \to BSp(\infty)$ to the homotopy $Bi \circ j \circ p_2 \sim k \circ \phi \circ p_1$, we can get a map $j'\colon I_\omega \to BSp(n)$ satisfying $j' \sim j \circ p_2$ and the *strictly* commutative diagram:

$$\begin{array}{ccc} I_\omega & \xrightarrow{p_1} & C_\omega \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle k\circ\phi} \\ BSp(n) & \xrightarrow{Bi} & BSp(\infty) \end{array}$$

Then, since $X_n = Bi^{-1}(*)$ for the basepoint $*$ of $BSp(\infty)$, one has the *strictly* commutative diagram

$$\begin{array}{ccccc} \Sigma Sp(n) \wedge Sp(n) & \subset & I_\omega & \xrightarrow{p_1} & C_\omega \\ {\scriptstyle j''}\downarrow & & {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle k\circ\phi} \\ X_n & \xrightarrow{\delta} & BSp(n) & \xrightarrow{Bi} & BSp(\infty). \end{array}$$

By definition, $j \circ \omega$ represents the generalized Whitehead product $\omega^*(\mathrm{ad}^{-1}(1), \mathrm{ad}^{-1}(1))$ and then it follows from (3.1) that $\mathrm{ad}(j \circ \omega)$ represents the commutator γ. Thus, since $\delta \circ j'' \sim j \circ \omega$, we can put

$$\tilde{\gamma} = \mathrm{ad}(j'').$$

Now let us show the cohomological property of the above $\tilde{\gamma}$. Consider the commutative diagram

$$\begin{array}{ccccc} \widetilde{H}^{4n+3}(\Sigma Sp(n) \wedge Sp(n)) & \xrightarrow{\partial} & H^{4n+4}(I_\omega, \Sigma Sp(n) \wedge Sp(n)) & \xleftarrow[\cong]{p_1^*} & \widetilde{H}^{4n+4}(C_\omega) \\ {\scriptstyle (j'')^*}\uparrow & & {\scriptstyle (j')^*}\uparrow & & \cong\uparrow{\scriptstyle (k\circ\phi)^*} \\ \widetilde{H}^{4n+3}(X_n) & \xrightarrow{\partial'} & H^{4n+4}(BSp(n), X_n) & \xleftarrow{Bi^*} & \widetilde{H}^{4n+4}(BSp(\infty)), \end{array}$$

where ∂ and ∂' are the connecting homomorphisms. By definition, one has

$$\partial'(\bar{y}_{4n+3}) = Bi^*(q_{n+1})$$

and then

$$\begin{aligned}\partial \circ (j'')^*(\bar{y}_{4n+3}) &= (j')^* \circ \partial'(\bar{y}_{4n+3}) = (j')^* \circ Bi^*(q_{n+1}) = p_1^* \circ (k \circ \phi)^*(q_{n+1}) \\ &= p_1^* \circ \phi^* \left(\sum_{i+j=n+1} \Sigma(y_{4i-1}) \times \Sigma(y_{4j-1}) \right),\end{aligned}$$

where q_i and Σ denote the universal i-th symplectic Pontrjagin class and the suspension isomorphism respectively. Let $T\colon \Sigma^2 Sp(n) \wedge Sp(n) \to \Sigma Sp(n) \wedge \Sigma Sp(n)$ be the alternating map $T(s,t,x,y) = (t,x,s,y)$ for $s,t \in S^1$ and $x, y \in Sp(n)$. Then, for the construction of the homotopy equivalence ϕ, one has the following commutative diagram (See [1]).

$$\begin{array}{ccccc} \widetilde{H}^{4n+3}(\Sigma Sp(n)\wedge Sp(n)) & \xrightarrow{\partial} & H^{4n+4}(I_\omega, \Sigma Sp(n)\wedge Sp(n)) & \xleftarrow[\cong]{p_1^*} & \widetilde{H}^{4n+4}(C_\omega) \\ \Sigma \big\downarrow \cong & & & & \cong \big\uparrow \phi^* \\ \widetilde{H}^{4n+4}(\Sigma^2 Sp(n)\wedge Sp(n)) & \xleftarrow[\cong]{T^*} & \widetilde{H}^{4n+4}(\Sigma Sp(n)\wedge\Sigma Sp(n)) & \xrightarrow{\pi^*} & \widetilde{H}^{4n+4}(\Sigma Sp(n) \times \Sigma Sp(n)) \end{array}$$

where $\pi\colon \Sigma Sp(n) \times \Sigma Sp(n) \to \Sigma Sp(n) \wedge \Sigma Sp(n)$ is the projection. Then it follows that

$$\partial\left(\Sigma\left(\sum_{i+j=n+1} y_{4i-1} \times y_{4j-1}\right)\right) = \partial \circ (j'')^*(\bar{y}_{4n+3}).$$

Since π^* is monic, so is ∂. Then one can see that

$$(j'')^*(\bar{y}_{4n+3}) = \Sigma\left(\sum_{i+j=n+1} y_{4i-1} \times y_{4j-1}\right)$$

and hence

$$(\mathrm{ad}(j''))^*(b_{4n+2}) = \sum_{i+j=n+1} y_{4i-1} \times y_{4j-1}.$$

Therefore we have obtained:

Lemma 3.1. *There exists a map* $\tilde{\gamma}\colon Sp(n) \wedge Sp(n) \to \Omega X_n$ *such that* $\Omega\delta \circ \tilde{\gamma} \sim \gamma$ *and that*

$$\tilde{\gamma}^*(b_{4n+2}) = \sum_{i+j=n+1} y_{4i-1} \times y_{4j-1}.$$

Proof of Theorem 1.2. Note that, for $\alpha, \beta \in Sp_n(X)$, the commutator $[\alpha, \beta]$ in $Sp_n(X)$ is represented by the composition $\gamma \circ (\alpha \wedge \beta) \circ \Delta \sim \Omega\delta \circ \tilde{\gamma} \circ (\alpha \wedge \beta) \circ \Delta$ as above, where $\tilde{\gamma}$ is as in Lemma 3.1. For the construction of the exact sequence (1.1), one can see that

$$\iota([(\tilde{\gamma} \circ (\alpha \wedge \beta) \circ \Delta)^*(b_{4n+2})]) = [\alpha, \beta],$$

where ι is as in Theorem 1.1. Then Theorem 1.2 follows from Lemma 3.1. $\square$

4. Applications

As an application of the above results, we give three example calculations using Theorem 1.1, Theorem 1.2 and Theorem 1.3.

4.1. $Sp_n(\Sigma^2\mathbb{H}P^n)$

Proof of Proposition 1.4. We calculate $Sp_n(\Sigma^2\mathbb{H}P^n)$. Consider the exact sequence

$$\cdots \to \widetilde{KSp}^*(S^{4n+2}) \to \widetilde{KSp}^*(\Sigma^2\mathbb{H}P^n) \to \widetilde{KSp}^*(\Sigma^2\mathbb{H}P^{n-1}) \to \widetilde{KSp}^{*+1}(S^{4n+2}) \to \cdots$$

induced from the cofibration sequence $\Sigma^2\mathbb{H}P^{n-1} \to \Sigma^2\mathbb{H}P^n \to S^{4n+2}$. Then it follows from $\widetilde{KSp}^{-1}(S^{4n+2}) = 0$ that $\widetilde{KSp}^{-1}(\Sigma^2\mathbb{H}P^n) = 0$ inductively. Hence, for Theorem 1.1, one has

$$Sp_n(\Sigma^2\mathbb{H}P^n) \cong \mathbf{N}_n(\Sigma^2\mathbb{H}P^n).$$

Thus we shall calculate $\mathbf{N}_n(\Sigma^2\mathbb{H}P^n)$.

For Theorem 1.3, we have the following commutative diagram.

$$\begin{array}{ccc}
\widetilde{KSp}^{-2}(\Sigma^2\mathbb{H}P^n) & \xrightarrow{\Theta_{\mathbb{H}}} & H^{4n+2}(\Sigma^2\mathbb{H}P^n) \\
\downarrow{\scriptstyle \mathbf{c}'} & & \downarrow{\scriptstyle (-1)^{n+1}} \\
\widetilde{K}^{-2}(\Sigma^2\mathbb{H}P^n) & \xrightarrow{\Theta_{\mathbb{C}}} & H^{4n+2}(\Sigma^2\mathbb{H}P^n)
\end{array}$$

Then one can deduce $\mathbf{N}_n(\Sigma^2\mathbb{H}P^n) = \operatorname{Coker}\Theta_{\mathbb{H}}$ from $\Theta_{\mathbb{C}}$ and $\mathbf{c}'$ in the above diagram.

By using Proposition 2.1, we calculate $\Theta_{\mathbb{C}}\colon \widetilde{K}^{-2}(\Sigma^2\mathbb{H}P^n) \to H^{4n+2}(\Sigma^2\mathbb{H}P^n)$. Let ξ_n be the canonical quaternionic line bundle over $\mathbb{H}P^n$ and let $\gamma_n \in \widetilde{K}^0(\mathbb{H}P^n)$ be $\mathbf{c}'(\xi_n - 1_{\mathbb{H}})$, where $1_{\mathbb{H}}$ denotes the trivial quaternionic line bundle. It is straightforward to see that

$$K^0(\mathbb{H}P^n) = \mathbb{Z}[\gamma_n]/(\gamma_n^{n+1}). \tag{4.1}$$

Let $\pi\colon \mathbb{C}P^{2n+1} \to \mathbb{H}P^n$ be the standard surjection and let ω_n be the canonical line bundle over $\mathbb{C}P^n$. Since π is the restriction of $BU(1) \to BSp(1)$, $\pi^*(\mathbf{c}'(\xi_n)) = \omega_{2n+1} \oplus \bar{\omega}_{2n+1}$. In the commutative diagram

$$\begin{array}{ccc}
\widetilde{K}^0(\mathbb{H}P^n) & \xrightarrow{\pi'^*} & \widetilde{K}^0(\mathbb{C}P^{2n+1}) \\
\downarrow{\scriptstyle s_{2n}} & & \downarrow{\scriptstyle s_{2n}} \\
H^{4n}(\mathbb{H}P^n) & \xrightarrow{\pi^*} & H^{4n}(\mathbb{C}P^{2n+1}),
\end{array}$$

we have

$$\begin{aligned}\pi^*(s_{2n}(\gamma_n)) &= s_{2n}(\pi'^*(\gamma_n)) \\ &= s_{2n}(\omega_{2n+1} \oplus \bar{\omega}_{2n+1} - 2_{\mathbb{C}}) \\ &= s_{2n}(\omega_{2n+1}) + s_{2n}(\bar{\omega}_{2n+1}) \\ &= c_1(\omega_{2n+1})^{2n} + (-c_1(\omega_{2n+1}))^{2n} \\ &= 2c_1(\omega_{2n+1})^{2n}\end{aligned}$$

for $n \geq 1$.

Let q denote the first symplectic Pontrjagin class of ξ_n. Since $\pi^*(q) = c_1(\omega_{2n+1})^2$, π^* is monic and $s_{2l}(\gamma_n) = 2q^l$. For a dimensional reason, $s_{2l+1}(\gamma_n) = 0$. Then it follows that

$$ch(\gamma_n^k) = (ch(\gamma_n))^k = \left(\sum_{l=1}^{\infty} \frac{s_{2l}(\gamma_n)}{2l!}\right)^k = \sum_{l=1}^{\infty} \sum_{\substack{i_1+\cdots+i_k=l \\ i_1,\ldots,i_k>0}} \frac{2^k q^l}{(2i_1)!\cdots(2i_k)!}.$$

Hence we obtain

$$s_{2n}(\gamma_n^k) = 2^k \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k>0}} \frac{(2n)!}{(2i_1)!\cdots(2i_k)!} q^n.$$

Thus, for Proposition 2.1 and Proposition 2.2, we have

$$\Theta_{\mathbb{C}}(\bar{\eta} \wedge \bar{\eta} \wedge \gamma_n^k) = -2^k \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k>0}} \frac{(2n+1)!}{(2i_1)!\cdots(2i_k)!} s_1(\bar{\eta}) \times q^n. \tag{4.2}$$

Here, for the result of Atiyah and Hirzebruch [3], $s_1(\bar{\eta})$ is a generator of $H^2(S^2)$.

Note that $\mathrm{Im}\{\mathbf{c}' \colon \widetilde{KSp}^{-2}(\Sigma^2\mathbb{H}P^1) \to \widetilde{K}^{-2}(\Sigma^2\mathbb{H}P^1)\} = 2\widetilde{K}^{-2}(\Sigma^2\mathbb{H}P^1)$ and that, for (4.1), $\mathrm{Ker}\{i^* \colon \widetilde{K}^{-2}(\Sigma^2\mathbb{H}P^n) \to \widetilde{K}^{-2}(\Sigma^2\mathbb{H}P^1)\}$ is a free abelian group generated by $\bar{\eta} \wedge \bar{\eta} \wedge \gamma_n^2, \ldots, \bar{\eta} \wedge \bar{\eta} \wedge \gamma_n^n$, where $\bar{\eta}$ is as in Section 2. Then it follows from the commutative diagram

$$\begin{array}{ccc} \widetilde{KSp}^{-2}(\Sigma^2\mathbb{H}P^n) & \xrightarrow{i^*} & \widetilde{KSp}^{-2}(\Sigma^2\mathbb{H}P^1) \\ \downarrow{\mathbf{c}'} & & \downarrow{\mathbf{c}'} \\ \widetilde{K}^{-2}(\Sigma^2\mathbb{H}P^n) & \xrightarrow{i^*} & \widetilde{K}^{-2}(\Sigma^2\mathbb{H}P^1) \end{array}$$

that

$$\bar{\eta} \wedge \bar{\eta} \wedge \gamma_n \notin \mathrm{Im}\left\{\mathbf{c}' \colon \widetilde{KSp}^{-2}(\Sigma^2\mathbb{H}P^n) \to \widetilde{K}^{-2}(\Sigma^2\mathbb{H}P^n)\right\}.$$

On the other hand, there is $\alpha \in \widetilde{KO}^0(S^4)$ such that $\mathbf{c}(\alpha) = 2\bar{\eta} \wedge \bar{\eta}$, where $\mathbf{c} \colon \widetilde{KO}^0(S^4) \to \widetilde{K}^0(S^4)$ is the complexification. Then one has

$$\mathbf{c}'(\alpha \wedge (\xi_n - 1_{\mathbb{H}})) = 2\bar{\eta} \wedge \bar{\eta} \wedge \gamma_n \in \mathrm{Im}\left\{\mathbf{c}' \colon \widetilde{KSp}^{-2}(\Sigma^2\mathbb{H}P^n) \to \widetilde{K}^{-2}(\Sigma^2\mathbb{H}P^n)\right\}$$

and hence, for (4.2),

$$\mathbf{N}_n(\Sigma^2\mathbb{H}P^n) = \operatorname{Coker}\Theta_{\mathbb{C}} \cong \mathbb{Z}/4(2n+1).$$

Therefore we have established Proposition 1.4. □

4.2. Samelson product $\langle \epsilon_3, \epsilon \rangle$

Proof of Proposition 1.5. Let Q_2 be the quasi-projective space of $Sp(2)$, that is, Q_2 is the 9-skeleton of $Sp(2) = S^3 \cup e^7 \cup e^{10}$. Denote the inclusions $S^3 \hookrightarrow Sp(2)$ and $Q_2 \hookrightarrow Sp(2)$ by ϵ_3 and ϵ respectively. We calculate the order of the Samelson product $\langle \epsilon_3, \epsilon \rangle$. For Theorem 1.3, we have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{KSp}^{-2}(S^3 \wedge Q_2) & \xrightarrow{\Theta_{\mathbb{H}}} & H^{10}(S^3 \wedge Q_2) \\ \downarrow{\scriptstyle \mathbf{c}'} & & \downarrow{\scriptstyle -1} \\ \widetilde{K}^{-2}(S^3 \wedge Q_2) & \xrightarrow{\Theta_{\mathbb{C}}} & H^{10}(S^3 \wedge Q_2). \end{array}$$

Then, in order to calculate the Coker $\Theta_{\mathbb{H}}$, we first consider the map $\mathbf{c}' \colon \widetilde{KSp}^{-2}(S^3 \wedge Q_2) \to \widetilde{K}^{-2}(S^3 \wedge Q_2)$. Consider the following commutative diagram of exact sequences induced from the cofibration sequence $S^6 \to S^3 \wedge Q_2 \to S^{10}$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{KSp}^{-2}(S^{10}) & \longrightarrow & \widetilde{KSp}^{-2}(S^3 \wedge Q_2) & \longrightarrow & \widetilde{KSp}^{-2}(S^6) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathbf{c}'} & & \downarrow{\scriptstyle \mathbf{c}'} & & \downarrow{\scriptstyle \mathbf{c}'} & & \\ 0 & \longrightarrow & \widetilde{K}^{-2}(S^{10}) & \longrightarrow & \widetilde{K}^{-2}(S^3 \wedge Q_2) & \longrightarrow & \widetilde{K}^{-2}(S^6) & \longrightarrow & 0 \end{array}$$

Since $\widetilde{KSp}^{-2}(S^{4n+2}) \cong \mathbb{Z}$ and $\widetilde{K}^{-2}(S^{2n}) \cong \mathbb{Z}$, $\widetilde{KSp}^{-2}(S^3 \wedge Q_2) = \mathbb{Z}\langle \alpha, \beta \rangle$ and $\widetilde{K}^{-2}(S^3 \wedge Q_2) = \mathbb{Z}\langle \alpha', \beta' \rangle$, where $\mathbb{Z}\langle a, b, \dots \rangle$ denote the free abelian group with a basis $a, b, \dots$. Moreover, since $\mathbf{c}' = 1 \colon \widetilde{KSp}^{-2}(S^{10}) \to \widetilde{K}^{-2}(S^{10})$ and $\mathbf{c}' = 2 \colon \widetilde{KSp}^{-2}(S^6) \to \widetilde{K}^{-2}(S^6)$, we can choose $\alpha, \beta, \alpha', \beta'$ such that $\mathbf{c}'(\alpha) = 2\alpha'$ and $\mathbf{c}'(\beta) = \beta'$.

We next calculate $\Theta_{\mathbb{C}} \colon \widetilde{K}^{-2}(S^3 \wedge Q_2) \to H^{10}(S^3 \wedge Q_2)$. Let $\hat{\mathbf{c}}' \colon Q_2 \to \Sigma\mathbb{C}P^3$ be the restriction of $\mathbf{c}' \colon Sp(2) \to SU(4)$ to their quasi-projective spaces. Then

$$H^*(Q_2) = \mathbb{Z}\langle \hat{y}_3, \hat{y}_7 \rangle,\ H^*(\Sigma\mathbb{C}P^3) = \mathbb{Z}\langle \hat{x}_3, \hat{x}_5, \hat{x}_7 \rangle$$

such that

$$\hat{\mathbf{c}}'(\hat{x}_3) = \hat{y}_3,\ \hat{\mathbf{c}}'(\hat{x}_5) = 0,\ \hat{\mathbf{c}}'(\hat{x}_7) = \hat{y}_7.$$

Let $\mu \in \widetilde{K}^0(\mathbb{C}P^3)$ denote $\omega_3 - 1_{\mathbb{C}}$, where ω_3 is as in the previous subsection. $\widetilde{K}^0(\Sigma^6\mathbb{C}P^3) = \widetilde{K}^{-2}(\Sigma^4\mathbb{C}P^3)$ has three generators $\bar{\eta} \wedge \bar{\eta} \wedge \bar{\eta} \wedge \mu^i$ $(i = 1, 2, 3)$, where $\bar{\eta}$ is as in Section 2. We can put α', β' as

$$\alpha' = \hat{\mathbf{c}}'(\bar{\eta} \wedge \bar{\eta} \wedge \bar{\eta} \wedge \mu),\ \beta' = \hat{\mathbf{c}}'(\bar{\eta} \wedge \bar{\eta} \wedge \bar{\eta} \wedge \mu^3).$$

Consider the commutative diagram

$$\begin{array}{ccc} \widetilde{K}^{-2}(\Sigma^4\mathbb{C}P^3) & \xrightarrow{\Theta'_{\mathbb{C}}} & H^{10}(\Sigma^4\mathbb{C}P^3) \\ \downarrow{\hat{\mathbf{c}}'} & & \downarrow{\cong} \\ \widetilde{K}^{-2}(S^3 \wedge Q_2) & \xrightarrow{\Theta_{\mathbb{C}}} & H^{10}(S^3 \wedge Q_2). \end{array}$$

By Proposition 2.1, $\Theta'_{\mathbb{C}}(\bar\eta \wedge \bar\eta \wedge \bar\eta \wedge \mu^i) = -s_5(\bar\eta \wedge \bar\eta \wedge \mu^i)$ $(i = 1, 2, 3)$. Since

$$ch(\bar\eta \wedge \bar\eta \wedge \mu) = s_1(\bar\eta) \otimes s_1(\bar\eta) \otimes \left(c_1 + \frac{c_1^2}{2} + \frac{c_1^3}{6}\right)$$
$$ch(\bar\eta \wedge \bar\eta \wedge \mu^3) = s_1(\bar\eta) \otimes s_1(\bar\eta) \otimes c_1^3,$$

it follows that $\Theta'_{\mathbb{C}}(\bar\eta \wedge \bar\eta \wedge \bar\eta \wedge \mu) = -20 s_1(\bar\eta) \otimes s_1(\bar\eta) \otimes c_1^3$ and $\Theta'_{\mathbb{C}}(\bar\eta \wedge \bar\eta \wedge \bar\eta \wedge \mu^3) = -120 s_1(\bar\eta) \otimes s_1(\bar\eta) \otimes c_1^3$, where c_1 is the first Chern class of ω_3. Since $s_1(\bar\eta) \otimes s_1(\bar\eta) \otimes c_1^3 \in H^{10}(\Sigma^4\mathbb{C}P^3)$ is a generator, we have $\Theta_{\mathbb{H}}(\alpha) = \pm 40 u_3 \otimes \hat{y}_7$ and $\Theta_{\mathbb{H}}(\beta) = \pm 120 u_3 \otimes \hat{y}_7$.

Since $(pr_1 \wedge pr_2) \circ \bar\Delta = 1\colon S^3 \wedge Q_2 \to S^3 \wedge Q_2 \wedge S^3 \wedge Q_2 \to S^3 \wedge Q_2$, the Samelson product $\langle \epsilon_3, \epsilon \rangle$ is equal to the commutator $[\epsilon_3 \circ pr_1, \epsilon \circ pr_2]$ in the group $[S^3 \wedge Q_2, Sp(2)]$, where $\bar\Delta$ is the reduced diagonal and pr_1 and pr_2 are the first and the second projections respectively. By Theorem 1.2, the latter is given as $[\epsilon_3 \circ pr_1, \epsilon \circ pr_2] = \iota([\epsilon_3^*(y_3) \otimes \epsilon^*(y_7)]) = \iota([u_3 \otimes \hat{y}_7])$. Hence the order of $\langle \epsilon_3, \epsilon \rangle$ is 40 and we have accomplished the proof of Proposition 1.5. ☐

4.3. $Sp_n(X)$ when X is a sphere bundle over a sphere

We calculate $Sp_n(X)$ when X is a specific sphere bundle over a sphere. Recall the cell decomposition of a sphere bundle over a sphere due to James and Whitehead [13].

Proposition 4.1 (James and Whitehead [13]). *Let X be a sphere bundle over a sphere $S^k \xrightarrow{i} X \xrightarrow{p} S^l$. Then X has a cell decomposition*

$$X = S^k \cup e^l \cup e^{k+l} \tag{4.3}$$

such that p restricts to the map $S^k \cup e^l \to S^l$ which pinches $S^k \subset S^k \cup e^l$ to the basepoint.

Proof. Let $p_i\colon D^i \to S^i$ be the map which pinches the boundary of D^i to the basepoint of S^i. Since D^l is contractible, the induced bundle $p_l^{-1}(X)$ is the product bundle $D^l \times S^k$. Let $\psi\colon D^l \times S^k = p_l^{-1}(X) \to X$ denote the bundle map. Then the composition $h\colon D^l \times D^k \xrightarrow{1 \times p_k} D^l \times S^k \xrightarrow{\psi} X$ is a surjection. One can see that $h|_{S^{l-1} \times D^k}$ is a surjection onto the fiber $p^{-1}(*) = S^k$, where $*$ is the basepoint of S^l. One can also see that $h|_{S^{l-1} \times S^{k-1}}$ is the composition $S^{l-1} \times S^{k-1} \to S^{l-1} \to p^{-1}(*) = S^k$. Since $\partial(D^l \times D^k) = S^{l-1} \times D^k \cup D^l \times S^{k-1}$, we have obtained the cell decomposition (4.1). For the construction of this cell decomposition, p restricts to the pinching map $S^k \cup e^l \to S^l$. ☐

In order to calculate $Sp_n(X)$ when X is a sphere bundle over a sphere, we calculate $\widetilde{KSp}^{-1}(X)$ by using Proposition 4.1.

Lemma 4.2. *Let X be a sphere bundle over a sphere $S^{4n-1} \xrightarrow{i} X \xrightarrow{p} S^{4m-1}$ such that $m+n$ is odd. Then we have*

$$\widetilde{KSp}^{-1}(X) = \mathbb{Z}\langle\tilde{\alpha}, \tilde{\beta}\rangle$$

such that

$$i^*(\tilde{\alpha}) = t_n,\ p^*(t_m) = \tilde{\beta},$$

where $\mathbb{Z}\langle\alpha, \beta, \dots\rangle$ denotes the free abelian group with a basis $\alpha, \beta, \dots$ and t_j is a generator of $\widetilde{KSp}^{-1}(S^{4j-1}) \cong \mathbb{Z}$.

Proof. We fix $N = m+n-1$. For Proposition 4.1, X has a cell decomposition

$$X = S^{4n-1} \cup e^{4m-1} \cup e^{4N+2}$$

and p restricts to the pinching map $S^{4n-1} \cup e^{4m-1} \to S^{4m-1}$. Let $X^{(4N+1)}$ denote the $(4N+1)$-skeleton of X. Then, for Proposition 4.1, the restriction of p,

$$S^{4n-1} \xrightarrow{i} X^{(4N+1)} \xrightarrow{p|_{X^{(4N+1)}}} S^{4m-1}, \tag{4.4}$$

is a cofibration sequence and hence it induces the exact sequence

$$\cdots \to \widetilde{KSp}^*(S^{4m-1}) \xrightarrow{(p|_{X^{(4N+1)}})^*} \widetilde{KSp}^*(X^{(4N+1)}) \to \\ \xrightarrow{i^*} \widetilde{KSp}^*(S^{4n-1}) \to \widetilde{KSp}^{*+1}(S^{4m-1}) \to \cdots .$$

Since $\widetilde{KSp}^0(S^{4m-1}) = 0$, $\widetilde{KSp}^{-1}(S^{4n-1}) \cong \widetilde{KSp}^{-1}(S^{4m-1}) \cong \mathbb{Z}$ and $\widetilde{KSp}^{-2}(S^{4n-1}) \cong 0$ or $\mathbb{Z}/2$, one has

$$\widetilde{KSp}^{-1}(X^{(4N+1)}) = \langle\alpha, \beta\rangle \tag{4.5}$$

such that $i^*(\alpha) = t_n$ and $(p|_{X^{(4N+1)}})^*(t_m) = \beta$. Similarly the cofibration sequence

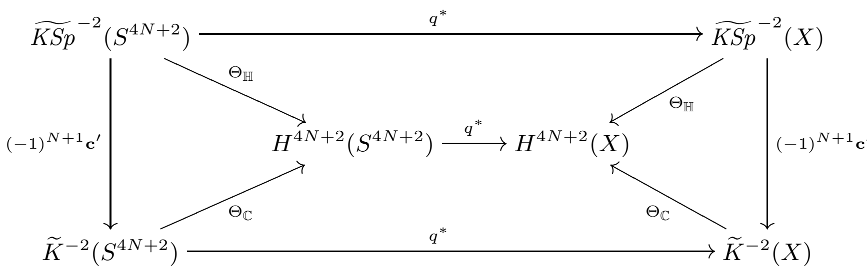
$$X^{(4N+1)} \xrightarrow{j} X \to S^{4N+2}, \tag{4.6}$$

induces the exact sequence

$$\cdots \to \widetilde{KSp}^*(S^{4N+2}) \to \widetilde{KSp}^*(X) \xrightarrow{j^*} \widetilde{KSp}^*(X^{(4N+1)}) \\ \to \widetilde{KSp}^{*+1}(S^{4N+2}) \to \cdots .$$

Since N is even, $\widetilde{KSp}^{-1}(S^{4N+2}) = 0$ and $\widetilde{KSp}^{0}(S^{4N+2}) = 0$. Then we have $j^*\colon \widetilde{KSp}^{-1}(X) \cong \widetilde{KSp}^{-1}(X^{(4N+1)})$ and hence Lemma 4.2 follows from (4.5). ☐

Proof of Theorem 1.6. Fix $N = m+n-1$. Since the diagram (1.3) is natural for the pinching map $q\colon X \to S^{4N+2}$, we have the following commutative diagram.



The left vertical arrow $\mathbf{c}'$ is an isomorphism since N is even. The cofibration sequence (4.4) induces the exact sequence

$$\cdots \to \widetilde{K}^{-2}(S^{4m-1}) \to \widetilde{K}^{-2}(X^{(4N+1)}) \to \widetilde{K}^{-2}(S^{4n-1}) \to \cdots .$$

Then it follows from $\widetilde{K}^{-2}(S^{4m-1}) = \widetilde{K}^{-2}(S^{4n-1}) = 0$ that $\widetilde{K}^{-2}(X^{(4N+1)}) = 0$. Hence the bottom horizontal arrow q^* is epic since we have the exact sequence

$$\cdots \to \widetilde{K}^{-2}(S^{4N+2}) \xrightarrow{q^*} \widetilde{K}^{-2}(X) \to \widetilde{K}^{-2}(X^{(4N+1)}) \to \cdots$$

induced from the cofibration sequence (4.6). Thus the right vertical arrow $\mathbf{c}'$ is epic and one has

$$\begin{aligned}
&\mathrm{Coker}\{\Theta_{\mathbb{H}}\colon \widetilde{KSp}^{-2}(X) \to H^{4N+2}(X)\} \\
&\quad = \mathrm{Coker}\{\Theta_{\mathbb{C}}\colon \widetilde{K}^{-2}(X) \to H^{4N+2}(X)\} \\
&\quad = \mathrm{Coker}\{\Theta_{\mathbb{C}} \circ q^*\colon \widetilde{K}^{-2}(S^{4N+2}) \to H^{4N+2}(X)\} \\
&\quad = \mathrm{Coker}\{q^* \circ \Theta_{\mathbb{C}}\colon \widetilde{K}^{-2}(S^{4N+2}) \to H^{4N+2}(X)\} \\
&\quad \cong \mathrm{Coker}\{\Theta_{\mathbb{C}}\colon \widetilde{K}^{-2}(S^{4N+2}) \to H^{4N+2}(S^{4N+2})\},
\end{aligned}$$

here we use the fact that $q^*\colon H^{4N+2}(S^{4N+2}) \to H^{4N+2}(X)$ is an isomorphism. For the result of Atiyah and Hirzebruch [3], we have $\mathrm{Coker}\{\Theta_{\mathbb{C}}\colon \widetilde{K}^{-2}(S^{4N+2}) \to H^{4N+2}(S^{4N+2})\} \cong \mathbb{Z}/(2N+1)!$. Therefore we have obtained

$$\mathbf{N}_N(X) = \mathrm{Coker}\{\Theta_{\mathbb{H}}\colon \widetilde{KSp}^{-2}(X) \to H^{4N+2}(X)\} \cong \mathbb{Z}/(2N+1)!.$$

For Theorem 1.1, we have the central extension

$$0 \to \mathbb{Z}/(2N+1)! \xrightarrow{\iota} Sp_N(X) \xrightarrow{\pi} \widetilde{KSp}^{-1}(X) \to 0.$$

Then we have only to calculate the order of $[\alpha,\beta]$ in $\mathbb{Z}/(2N+1)! \subset Sp_N(X)$, where $\alpha,\beta \in Sp(X)$ satisfy $\pi(\alpha)=\tilde{\alpha}, \pi(\beta)=\tilde{\beta}$ and $\tilde{\alpha},\tilde{\beta} \in \widetilde{KSp}^{-1}(X)$ are as in Lemma 4.2.

It is obvious that

$$H^*(X) \cong \Lambda(u'_{4n-1}, u'_{4m-1})$$

such that $i^*(u'_{4n-1}) = u_{4n-1}$ and $u'_{4m-1} = p^*(u_{4m-1})$, where $u_i \in H^i(S^i)$ is a generator. Let $\epsilon \in Sp_N(X)$ be a generator of $\mathrm{Coker}\{\Theta_{\mathbb{C}}\colon \widetilde{K}^{-2}(X) \to H^{4N+2}(X)\} \cong \mathbb{Z}/(2N+1)!$ represented by $u'_{4m-1}u'_{4n-1}$.

From Theorem 1.2, it follows that $[\alpha,\beta] = \iota([u])$ such that

$$u = \sum_{i+j=m+n} \alpha^*(y_{4i-1})\beta^*(y_{4j-1}) \in H^{4N+2}(X).$$

Let t'_i be a generator of $\pi_{4i-1}(Sp(N)) \cong \mathbb{Z}$ for $i \le N$. Then we have

$$i^* \circ \alpha^*(y_{4i-1}) = (t'_n)^*(y_{4i-1}),\ \beta^*(y_{4i-1}) = p^* \circ (t'_m)^*(y_{4i-1}).$$

Let v_i be a generator of $\pi_{2i-1}(U(2N)) \cong \mathbb{Z}$ for $i \le 2N$. Atiyah and Hirzebruch [3] showed that

$$v_i^*(x_{2i-1}) = \pm(i-1)!u_{2i-1},$$

where x_{2i-1} is as in Section 2. Since

$$\mathbf{c}'(t'_i) = \begin{cases} \pm v_{2i} & i \text{ is odd} \\ \pm 2v_{2i} & i \text{ is even} \end{cases}$$

and $(\mathbf{c}')^*(x_{4i-1}) = (-1)^i y_{4i-1}$, we have

$$u = \pm 2(2n-1)!(2m-1)!u'_{4n-1}u'_{4m-1}$$

and then

$$[\alpha,\beta] = \pm 2(2n-1)!(2m-1)!\epsilon.$$

Therefore the proof of Theorem 1.6 is completed. □

Department of Mathematics
Kyoto University
Kyoto 606-8502, Japan
e-mail: tnagao@math.kyoto-u.ac.jp

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