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The secant varieties of nilpotent orbits

By

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Abstract

Let $\mathfrak{g}$ be a complex simple Lie algebra. We have the adjoint representation of the adjoint group G on $\mathfrak{g}$. Then G acts on the projective space $\mathbb{P}\mathfrak{g}$. We consider the closure X of the image of a nilpotent orbit in $\mathbb{P}\mathfrak{g}$. The i-secant variety $Sec^{(i)}X$ of a projective variety X is the closure of the union of projective subspaces of dimension i in the ambient space $\mathbb{P}$ spanned by $i+1$ points on X. In particular we call the 1-secant variety the secant variety. In this paper we give explicit descriptions of the secant and the higher secant varieties of nilpotent orbits of complex classical simple Lie algebras.

1. Introduction

Let G be a connected complex simple algebraic group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{h}$ its Cartan subalgebra, W its Weyl group. When we want to exhibit corresponding Lie algebra, we denote $\mathfrak{h}$ by $\mathfrak{h}_{\mathfrak{g}}$ and W by $W_{\mathfrak{g}}$. We have the adjoint action of G on $\mathfrak{g}$. Taking a composition of the categorical quotient $\mathfrak{g} \to \mathfrak{g}//G$ and an isomorphism $\mathfrak{g}//G \simeq \mathfrak{h}/W$ given by Chevalley's theorem for which we refer to [5], we have a map

$$p : \mathfrak{g} \to \mathfrak{h}/W.$$

This map is called the adjoint quotient. For the detail about the adjoint quotient we refer to [10]. The Lie group G acts on the projective space $\mathbb{P}\mathfrak{g}$ naturally under the projection,

$$\pi : \mathfrak{g}\setminus\{0\} \to \mathbb{P}\mathfrak{g}.$$

For any nonzero nilpotent element $x \in \mathfrak{g}$ we set

$$X := \overline{\pi(G \cdot x)} \subset \mathbb{P}\mathfrak{g}.$$

Hence X is an irreducible G-invariant projective variety in $\mathbb{P}\mathfrak{g}$. The i-secant variety $Sec^{(i)}X$ of the projective variety X is the closure of the union of projective subspaces of dimension i in the ambient space $\mathbb{P}\mathfrak{g}$ spanned by $i+1$ points

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on X. If X is irreducible, then $Sec^{(i)}X$ is also irreducible. In particular we call the 1-secant variety the secant variety $SecX$. We refer to the monograph [12] by F. L. Zak for general results about the secant and the higher secant varieties.

In this paper we give explicit descriptions of the secant and the higher secant varieties of nilpotent orbits of complex classical simple Lie algebras except for the nilpotent orbits $[2^r, 1^s]$ in $\mathfrak{so}_n$. The descriptions we give are characterized by the rank of matrices, determinantal varieties. Some results are known about the secant and the higher secant varieties of nilpotent orbits. Kaji-Yasukura [6] proved that if x is a minimal nilpotent element, we have $SecX = \overline{G \cdot \pi(h)}$. Here h is a semisimple element of a $\mathfrak{sl}_2$-triple containing x. Baur-Draisma [2] presented explicit descriptions of the higher secant varieties of the minimal nilpotent orbits of classical simple Lie algebras.

We remark that Theorem 4.7 is the result of K.Nishiyama [9]. The author wishes like to express his hearty thanks to K.Nishiyama for permitting the author to describe his result in this paper.

2. Preparation

In this section we recall well known results about nilpotent elements and their orbits in $\mathfrak{g}$. (See e.g. [3].) Nilpotent elements and corresponding orbits in Lie algebras $\mathfrak{sl}_n, \mathfrak{so}_n, \mathfrak{sp}_n$ are parametrized by partitions $[r_1, r_2, \ldots, r_s]$ of n

$$r_1 \geq r_2 \geq \cdots \geq r_s, \qquad \sum_{i=1}^{s} r_i = n.$$

Then we denote the nilpotent elements and corresponding orbits by $[r_1, r_2, \ldots, r_s]$. This parametrization is given by using the Jordan normal forms as follows,

$$\begin{pmatrix} D_{r_1} & & & \mathbf{0} \\ & D_{r_2} & & \\ & & \ddots & \\ \mathbf{0} & & & D_{r_s} \end{pmatrix}.$$

Here D_i is the $i \times i$ Jordan block whose eigenvalue is zero.

Proposition 2.1. *Nilpotent orbits in $\mathfrak{sl}_n$ are parametrized by the partitions of n.*

Proposition 2.2. *Nilpotent orbits in $\mathfrak{so}_{2n+1}$ are parametrized by the partitions of $2n+1$ in which even parts occur with even multiplicity.*

Proposition 2.3. *Nilpotent orbits in $\mathfrak{so}_{2n}$ correspond to partitions of $2n$ in which all even parts occur with even multiplicity, but each "very even" partition which has only even parts with even multiplicity comes from two orbits.*

Proposition 2.4. *Nilpotent orbits in $\mathfrak{sp}_{2n}$ are parametrized by the partitions of $2n$ in which odd parts occur with even multiplicity.*

We define a partial order on the set of partitions of n. Given two partitions $\mathbf{u} = [u_1, u_2, \ldots, u_s]$ and $\mathbf{v} = [v_1, v_2, \ldots, v_t]$, we say that $\mathbf{u} \leq \mathbf{v}$ if the following condition holds:

$$\sum_{1 \leq j \leq k} u_j \leq \sum_{1 \leq j \leq k} v_j \quad \text{for any } k \geq 1.$$

Moreover if $\mathbf{u} \neq \mathbf{v}$, we say that $\mathbf{u} < \mathbf{v}$. Then we have the following proposition.

Proposition 2.5. *Let $\mathfrak{g}$ be a classical Lie algebra over $\mathbf{C}$. Assume that the nilpotent orbits $\mathcal{O}_{\mathbf{u}}, \mathcal{O}_{\mathbf{v}}$ correspond to the partitions $\mathbf{u}, \mathbf{v}$. Then the closure of $\mathcal{O}_{\mathbf{v}}$ contains $\mathcal{O}_{\mathbf{u}}$ and $\mathcal{O}_{\mathbf{v}} \neq \mathcal{O}_{\mathbf{u}}$ if and only if $\mathbf{u} < \mathbf{v}$.*

Definition 2.6. Let $\mathfrak{g}$ be a complex simple Lie algebra. We define an element x to be regular if and only if its orbit in $\mathfrak{g}$ has maximal dimension.

Next proposition is well known. (See e.g. [10].)

Proposition 2.7. *Let $\mathfrak{g}$ be a complex simple Lie algebra. When we consider the adjoint quotient $p : \mathfrak{g} \to \mathfrak{h}/W$, p is flat and each fiber $p^{-1}(t)$ for $t \in \mathfrak{h}/W$ contains exactly one orbit of regular elements which is dense and open in $p^{-1}(t)$.*

We use the same notation $[r_1, r_2, \ldots, r_s]$ for the projective variety $X = \overline{\pi([r_1, r_2, \ldots, r_s])}$ obtained from a nilpotent orbit $\overline{[r_1, r_2, \ldots, r_s]}$. Moreover we use the same notation $SecX$ for cone varieties $\overline{\pi^{-1}(SecX)}$ in $\mathfrak{g}$ as $SecX$ in $\mathbb{P}\mathfrak{g}$.

We give two propositions and one corollary which are used many times in this paper.

Proposition 2.8. *Let $\mathfrak{g}$ be a simple Lie subalgebra in $\mathfrak{gl}_n$. (Here $\mathfrak{gl}_n$ is a Lie algebra consisting of all $n \times n$ matrices.) If x is a nilpotent element with rank k, Then we have*

$$Sec^{(i)}X \subset \{A \in \mathfrak{g} | \text{ rank } A \leq k(i+1)\}.$$

Proof. The rank of the sum of $i+1$ matrices with rank k is not greater than $k(i+1)$. Then we obtain the proposition by the definition of $Sec^{(i)}X$. $\square$

Next proposition is well known. (See e.g. [1].)

Proposition 2.9. *Let X be a closed G-variety in $\mathfrak{g}$. If X contains $\mathfrak{h}$, we have $X = \mathfrak{g}$.*

Then Proposition 2.7 shows the following corollary.

Corollary 2.10. *Let X be a closed G-variety in $\mathfrak{g}$ and $p : \mathfrak{g} \to \mathfrak{h}/W$ the adjoint quotient. If $p(X) = \mathfrak{h}/W$, we have $X = \mathfrak{g}$.*

3. Distinguished case

We recall the definition and the properties of distinguished nilpotent elements and their orbits in $\mathfrak{g}$. (See e.g. [3].)

Definition 3.1. Let $\mathfrak{g}$ be a complex simple Lie algebra. We define a nilpotent element x or its orbit to be distinguished if and only if the only Levi subalgebra of $\mathfrak{g}$ containing x is $\mathfrak{g}$ itself.

For any nilpotent element x we have a $\mathfrak{sl}_2$-triple $\{x, h, y\}$ such that

$$[h, x] = 2x, [h, y] = -2y, [x, y] = h.$$

Put eigenspaces of h as

$$\mathfrak{g}_i := \{z \in \mathfrak{g} | \operatorname{ad}(h)z = iz\}.$$

Then we have a finite decomposition,

$$\mathfrak{g} = \bigoplus_i \mathfrak{g}_i, \quad x \in \mathfrak{g}_2.$$

Under this decomposition we have following maps

$$\operatorname{ad}(x) : \mathfrak{g}_i \to \mathfrak{g}_{i+2}.$$

Then the next propositions are well known.

Proposition 3.2. *Let $\mathfrak{g}^x$ be the centralizer of a nilpotent element x in $\mathfrak{g}$. Then we have*

$$\mathfrak{g}^x \subset \bigoplus_{i \geq 0} \mathfrak{g}_i.$$

Proposition 3.3. *x is distinguished if and only if $\operatorname{ad}(x) : \mathfrak{g}_0 \to \mathfrak{g}_2$ is bijective.*

Combining these propositions and the equality $\dim \mathfrak{g}_{-2} = \dim \mathfrak{g}_2$, we have the following proposition.

Proposition 3.4. *If x is distinguished, we have $\operatorname{ad}(x) : \mathfrak{g}_{-2} \to \mathfrak{g}_0$ is bijective.*

If a nilpotent element x is distinguished, from the above proposition we obtain

$$SecX \supset TX \supset [\mathfrak{g}, x] \supset [\mathfrak{g}_{-2}, x] = \mathfrak{g}_0 \supset \mathfrak{h}.$$

Since $X = \overline{\pi(G \cdot x)}$ is closed and G-invariant, by Proposition 2.9 we have

$$SecX \supset \overline{G \cdot \mathfrak{h}} = \mathfrak{g}.$$

Then we have the following proposition.

Proposition 3.5. *If a nilpotent element x is distinguished, we have $SecX = \mathfrak{g}$.*

In particular the regular nilpotent orbit is distinguished. Then we have the following corollary.

Corollary 3.6. *If a nilpotent element x is regular, then we have $SecX = \mathfrak{g}$.*

Example 3.7. Let $\mathfrak{g}$ be $\mathfrak{sl}_2$ and let x be $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $SecX = \mathfrak{sl}_2$.

4. The case of $\mathfrak{sl}_n$

In this section we consider the case of $\mathfrak{sl}_n$;

$$\mathfrak{sl}_n = \{A = (a_{ij}) \in M_n(\mathbb{C}) |\ \mathrm{tr} A = 0\}.$$

Here $M_n(\mathbb{C})$ is the set of all $n \times n$ complex matrices. Let $\mathfrak{h}_{\mathfrak{sl}_n}$ be the set of all diagonal matrices in $\mathfrak{sl}_n$. If $n_1 + n_2 + \cdots + n_s \leq n$, for $(X_1, \ldots, X_s) \in \mathfrak{sl}_{n_1} \times \cdots \times \mathfrak{sl}_{n_s}$ the mapping from $\mathfrak{sl}_{n_1} \times \cdots \times \mathfrak{sl}_{n_s}$ to $\mathfrak{sl}_n$ defined by,

$$(X_1, \ldots, X_s) \mapsto \begin{pmatrix} X_1 & & & \mathbf{0} \\ & \ddots & & \\ & & X_s & \\ \mathbf{0} & & & \mathbf{0} \end{pmatrix}$$

gives a natural embedding $\mathfrak{sl}_{n_1} \times \cdots \times \mathfrak{sl}_{n_s} \hookrightarrow \mathfrak{sl}_n$. By this embedding we identify $\mathfrak{sl}_{n_1} \times \cdots \times \mathfrak{sl}_{n_s}$ with the image in $\mathfrak{sl}_n$. We also consider corresponding embeddings of the Cartan subalgebra $\mathfrak{h}_{\mathfrak{sl}_{n_1}} \times \cdots \times \mathfrak{h}_{\mathfrak{sl}_{n_s}}$ into $\mathfrak{h}_{\mathfrak{sl}_n}$ and the Lie group $SL_{n_1} \times \cdots \times SL_{n_s}$ into SL_n.

Lemma 4.1.

$$\overline{SL_N \cdot \mathfrak{h}_{\mathfrak{sl}_n}} = \{x \in \mathfrak{sl}_N |\ \mathrm{rank}(x) \leq n\}. \quad (2 \leq n \leq N)$$

Proof. If $N = n$, the assertion holds by Proposition 2.9. Then we may assume $N \geq n+1$. Let x be an element of $\mathfrak{sl}_N$ whose rank is n. Let $D_{i_k}(a_i)$ be the $i_k \times i_k$ Jordan block whose eigenvalue is a_i. Consider the Jordan normal form of x

$$\begin{pmatrix} D_{i_1}(a_1) & & & \mathbf{0} \\ & D_{i_2}(a_2) & & \\ & & \ddots & \\ \mathbf{0} & & & D_{i_s}(a_s) \end{pmatrix}, \quad |a_1| \geq |a_2| \geq \cdots \geq |a_s|.$$

Let k be the number such that $a_k \neq 0$ and $a_{k+1} = 0$. If $i_{k+1} = \cdots = i_s = 1$, then we have $x \in \overline{SL_N \cdot \mathfrak{sl}_n}$. Otherwise we put

$$A := \sum_{l=k+1}^{s} \mathrm{rank} D_{i_l}(a_l) = \sum_{l=k+1}^{s} (i_l - 1).$$

Let y be an element in $\mathfrak{sl}_{n+1} \subset \mathfrak{sl}_N$ whose Jordan normal form is

$$\begin{pmatrix} D_{i_1}(a_1) & & & & \\ & \ddots & & \mathbf{0} & \\ & & D_{i_k}(a_k) & & \\ & \mathbf{0} & & D_{A+1}(0) & \\ & & & & \mathbf{0} \end{pmatrix}$$

Hence we have

$$x \in \overline{SL_N \cdot y} \subset \overline{SL_N \cdot \mathfrak{sl}_{n+1}}.$$

Then it is enough to prove the case where $n = N - 1$. The inclusion:

$$\overline{SL_N \cdot \mathfrak{h}_{\mathfrak{sl}_{N-1}}} \subset \ \{x \in \mathfrak{sl}_N | \ \mathrm{rank}(x) \leq N - 1\}$$

is obvious. Next we shall prove the inverse inclusion. We consider the adjoint quotient $p : \mathfrak{sl}_N \to \mathfrak{h}_{\mathfrak{sl}_N}/W_{\mathfrak{sl}_N}$. Since generic elements in $\mathfrak{h}_{\mathfrak{sl}_{N-1}}$ are regular in $\mathfrak{sl}_N$, by Proposition 2.7 we have

$$\overline{SL_N \cdot \mathfrak{h}_{\mathfrak{sl}_{N-1}}} \supset p^{-1}(p(\mathfrak{h}_{\mathfrak{sl}_{N-1}})).$$

Then we want to show $p^{-1}(p(\mathfrak{h}_{\mathfrak{sl}_{N-1}})) \supset \{x \in \mathfrak{sl}_N | \ \mathrm{rank}(x) \leq N - 1\}$. We consider any element of $\{x \in \mathfrak{sl}_N | \ \mathrm{rank}(x) \leq N-1\}$. The rank of its semisimple part is not more than $N-1$. So the image by the adjoint quotient p is contained in $p(\mathfrak{h}_{\mathfrak{sl}_{N-1}})$. Hence we have

$$p^{-1}(p(\mathfrak{h}_{\mathfrak{sl}_{N-1}})) \supset \{x \in \mathfrak{sl}_N | \ \mathrm{rank}(x) \leq N - 1\}.$$

Then we have

$$\overline{SL_N \cdot \mathfrak{h}_{\mathfrak{sl}_{N-1}}} \supset \ \{x \in \mathfrak{sl}_N | \ \mathrm{rank}(x) \leq N - 1\}.$$

Then the result has be shown. □

Lemma 4.2. *Let $\mathfrak{g}$ be $\mathfrak{sl}_n$ $(n \geq 3)$.*

$$Sec[3, 1^{n-3}] = \{x \in \mathfrak{sl}_n | \ \mathrm{rank}(x) \leq 4\}.$$

Proof. When $n = 3$ a nilpotent element $[3]$ is regular. Then $Sec[3] = \mathfrak{sl}_3$. Proposition 2.8 shows

$$Sec[3, 1^{n-3}] \subset \{x \in \mathfrak{sl}_n | \ \mathrm{rank}(x) \leq 4\}.$$

If the result holds when $n = 4$, we have $Sec[3, 1] = \mathfrak{sl}_4$. Under the natural embedding $\mathfrak{sl}_4 \subset \mathfrak{sl}_n$ we can regard a nilpotent element $[3, 1^{n-3}] \in \mathfrak{sl}_n$ as a nilpotent element $[3, 1] \in \mathfrak{sl}_4 \subset \mathfrak{sl}_n$. Then we have

$$\mathfrak{sl}_n \supset Sec[3, 1^{n-3}] \supset \mathfrak{sl}_4 \supset \mathfrak{h}_{\mathfrak{sl}_4}.$$

Hence by Lemma 4.1 we obtain the required result. So it is enough to prove the case where $n = 4$. We can regard a nilpotent element $[3, 1]$ as a nilpotent element $[3] \in \mathfrak{sl}_3 \subset \mathfrak{sl}_4$ under the natural embedding $\mathfrak{sl}_3 \subset \mathfrak{sl}_4$. Since a nilpotent element $[3]$ is regular in $\mathfrak{sl}_3$, $Sec[3] = \mathfrak{sl}_3$. Then we have

$$\mathfrak{sl}_4 \supset Sec[3,1] \supset \mathfrak{sl}_3 \supset \mathfrak{h}_{\mathfrak{sl}_3} = \{a_{11} + a_{22} + a_{33} = 0, a_{44} = 0\}.$$

The condition that there is some number i such that $a_{ii} = 0$ is stable under the action of the Weyl group $W_{\mathfrak{sl}_4}$ in $\mathfrak{h}_{\mathfrak{sl}_4}$. Then the image $p(\mathfrak{h}_{\mathfrak{sl}_3})$ of $\mathfrak{h}_{\mathfrak{sl}_3}$ of the adjoint quotient $p : \mathfrak{sl}_4 \to \mathfrak{h}_{\mathfrak{sl}_4}/W_{\mathfrak{sl}_4}$ is an irreducible subvariety of codimension 1 in $\mathfrak{h}_{\mathfrak{sl}_4}/W_{\mathfrak{sl}_4}$ and does not contain the image of any element whose determinant is not equal to zero. If there exists an element in $Sec[3, 1]$ whose determinant is non-zero, we obtain the result by the irreducibility of $Sec[3, 1]$. Actually we can find such element as follows. We consider the nilpotent orbit $[2^2]$. The nilpotent orbit $[2^2]$ is included in the closure of the nilpotent orbit $[3, 1]$. So the sum of any two elements of the nilpotent orbit $[2^2]$ is an element of $Sec[3, 1]$. Now we take

$$y = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in Sec[3,1].$$

The determinant of y is non-zero. So $p(\mathfrak{h}_{\mathfrak{sl}_3})$ never contain $p(y)$. By the irreducibility of $Sec[3, 1]$ we obtain $p(Sec[3, 1]) = \mathfrak{h}_{\mathfrak{sl}_4}/W_{\mathfrak{sl}_4}$. Then by Corollary 2.10 we have $Sec[3, 1] = \mathfrak{sl}_4$. □

Lemma 4.3. *Let* $\mathfrak{g}$ *be* $\mathfrak{sl}_5$.

$$Sec[3, 2] = \mathfrak{sl}_5.$$

Proof. The closure of the nilpotent orbit $[3, 2]$ contains the nilpotent orbit $[3, 1^2]$. We can regard a nilpotent element $[3, 1^2]$ as a nilpotent element $[3, 1] \in \mathfrak{sl}_4$ under the natural embedding $\mathfrak{sl}_4 \subset \mathfrak{sl}_5$. Since $Sec[3, 1] = \mathfrak{sl}_4$ by Lemma 4.2, we have

$$\mathfrak{sl}_5 \supset Sec[3,2] \supset Sec[3,1^2] \supset \mathfrak{sl}_4 \supset \mathfrak{h}_{\mathfrak{sl}_4} = \{a_{11} + a_{22} + a_{33} + a_{44} = 0, a_{55} = 0\}.$$

The condition that there is some number i such that $a_{ii} = 0$ is stable under the action of the Weyl group $W_{\mathfrak{sl}_5}$ in $\mathfrak{h}_{\mathfrak{sl}_5}$. Then the image $p(\mathfrak{h}_{\mathfrak{sl}_4})$ of $\mathfrak{h}_{\mathfrak{sl}_4}$ of the adjoint quotient $p : \mathfrak{sl}_5 \to \mathfrak{h}_{\mathfrak{sl}_5}/W_{\mathfrak{sl}_5}$ is an irreducible subvariety of codimension 1 in $\mathfrak{h}_{\mathfrak{sl}_5}/W_{\mathfrak{sl}_5}$ and does not contain the image of any element whose determinant is not equal to zero. We can regard a nilpotent element $[3, 2] \in \mathfrak{sl}_5$ as a nilpotent element $[3] \times [2] \in \mathfrak{sl}_3 \times \mathfrak{sl}_2 \subset \mathfrak{sl}_5$. Since $Sec[3] = \mathfrak{sl}_3$ and $Sec[2] = \mathfrak{sl}_2$, we have

$$\mathfrak{sl}_5 \supset Sec[3,2] \supset \mathfrak{h}_{\mathfrak{sl}_3} \times \mathfrak{h}_{\mathfrak{sl}_2} = \{a_{11} + a_{22} + a_{33} = 0, a_{44} + a_{55} = 0\}.$$

The image $p(\mathfrak{h}_{\mathfrak{sl}_3} \times \mathfrak{h}_{\mathfrak{sl}_2})$ contains some elements whose determinants are not equal to zero. Then the image $p(\mathfrak{h}_{\mathfrak{sl}_4})$ is not a subset of $p(\mathfrak{h}_{\mathfrak{sl}_3} \times \mathfrak{h}_{\mathfrak{sl}_2})$. Since $Sec[3, 2]$ is irreducible, we obtain $p(Sec[3, 2]) = \mathfrak{h}_{\mathfrak{sl}_5}/W_{\mathfrak{sl}_5}$. Then by Corollary 2.10 we obtain $Sec[3, 2] = \mathfrak{sl}_5$. □

Lemma 4.4. *Let $\mathfrak{g}$ be $\mathfrak{sl}_n$ $(n \geq 5)$.*

$$Sec[3,2,1^{n-5}] = \{x \in \mathfrak{sl}_n | \operatorname{rank}(x) \leq 6\}.$$

Proof. If $n = 5$, the statement is proved in Lemma 4.3. Then by the similar argument as the argument in the proof of Lemma 4.2 it is enough to prove the case where $n = 6$. We can regard a nilpotent element $x = [3,2,1] \in \mathfrak{sl}_6$ as a nilpotent element $[3,2] \in \mathfrak{sl}_5$ under the natural embedding $\mathfrak{sl}_5 \subset \mathfrak{sl}_6$. Since $Sec[3,2] = \mathfrak{sl}_5$ by Lemma 4.3, we have

$$\mathfrak{sl}_6 \supset Sec[3,2,1] \supset \mathfrak{sl}_5 \supset \mathfrak{h}_{\mathfrak{sl}_5} = \{a_{11} + a_{22} + a_{33} + a_{44} + a_{55} = 0, a_{66} = 0\}.$$

The condition that there is some number i such that $a_{ii} = 0$ is stable under the action of the Weyl group $W_{\mathfrak{sl}_6}$ in $\mathfrak{h}_{\mathfrak{sl}_6}$. The image $p(\mathfrak{h}_{\mathfrak{sl}_5})$ of $\mathfrak{h}_{\mathfrak{sl}_5}$ of the adjoint quotient $p : \mathfrak{sl}_6 \to \mathfrak{h}_{\mathfrak{sl}_6}/W_{\mathfrak{sl}_6}$ is an irreducible subvariety of codimension 1 in $\mathfrak{h}_{\mathfrak{sl}_6}/W_{\mathfrak{sl}_6}$ and does not contain the image of any element whose determinant is not equal to zero. We take a sum of two nilpotent elements of the type of $[3,2,1]$,

$$y = \begin{pmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} + \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0 \end{pmatrix} \in Sec[3,2,1].$$

The determinant of y is non-zero. So $p(\mathfrak{h}_{\mathfrak{sl}_5})$ never contain $p(y)$. By the irreducibility of $Sec[3,2,1]$ we obtain $p(Sec[3,2,1]) = \mathfrak{h}_{\mathfrak{sl}_6}/W_{\mathfrak{sl}_6}$. Hence by Corollary 2.10 we obtain

$$Sec[3,2,1] = \mathfrak{sl}_6.$$

Then we proved the assertion. □

Proposition 4.5. *Let $\mathfrak{g}$ be $\mathfrak{sl}_n$. $(n = 2r + s + 3)$*

$$Sec[3,2^r,1^s] = \{x \in \mathfrak{sl}_n | \operatorname{rank}(x) \leq 2r + 4\}.$$

Proof. By the similar argument as the argument in the proof of Lemma 4.2 it is enough to prove the cases where $s = 0, 1$. We prove this proposition by induction on r. By Lemma 4.3 and 4.4 the cases where $r = 1$ and $s = 0, 1$ are proved. We assume that the assertion holds if $r = k$ and $s = 0, 1$. First we study the case where $r = k + 1$ and $s = 0$. We consider a nilpotent element $x = [3, 2^k, 1^2] \in \mathfrak{sl}_{2k+5}$. The closure of the orbit $[3, 2^{k+1}]$ contains the orbit $[3, 2^k, 1^2]$. Since $Sec[3, 2^k, 1] = \mathfrak{sl}_{2k+4}$ by the assumption, we have

$$\mathfrak{sl}_{2k+5} \supset Sec[3, 2^{k+1}] \supset Sec[3, 2^k, 1^2] \supset \mathfrak{sl}_{2k+4} \supset \mathfrak{h}_{\mathfrak{sl}_{2k+4}},$$

and

$$\mathfrak{h}_{\mathfrak{sl}_{2k+4}} = \{a_{11} + a_{22} + \cdots + a_{2k+4,2k+4} = 0, a_{2k+5,2k+5} = 0\}.$$

The condition that there is some number i such that $a_{ii} = 0$ is stable under the action of the Weyl group $W_{\mathfrak{sl}_{2k+5}}$ in $\mathfrak{h}_{\mathfrak{sl}_{2k+5}}$. Then the image $p(\mathfrak{h}_{\mathfrak{sl}_{2k+4}})$ of $\mathfrak{h}_{\mathfrak{sl}_{2k+4}}$ of the adjoint quotient $p : \mathfrak{sl}_{2k+5} \to \mathfrak{h}_{\mathfrak{sl}_{2k+5}}/W_{\mathfrak{sl}_{2k+5}}$ is an irreducible subvariety of codimension 1 in $\mathfrak{h}_{\mathfrak{sl}_{2k+5}}/W_{\mathfrak{sl}_{2k+5}}$ and does not contain the image of any element whose determinant is not equal to zero.

We can regard a nilpotent element $x = [3, 2^{k+1}] \in \mathfrak{sl}_{2k+5}$ as a nilpotent element $[3, 2^k] \times [2] \in \mathfrak{sl}_{2k+3} \times \mathfrak{sl}_2 \subset \mathfrak{sl}_{2k+5}$. Since $Sec[3, 2^k] = \mathfrak{sl}_{2k+3}$ by the assumption and $Sec[2] = \mathfrak{sl}_2$, we have

$$\mathfrak{sl}_{2k+5} \supset Sec[3, 2^{k+1}] \supset \mathfrak{sl}_{2k+3} \times \mathfrak{sl}_2 \supset \mathfrak{h}_{\mathfrak{sl}_{2k+3}} \times \mathfrak{h}_{\mathfrak{sl}_2},$$

and

$$\begin{aligned}\mathfrak{h}_{\mathfrak{sl}_{2k+5}} \supset{}& \mathfrak{h}_{\mathfrak{sl}_{2k+3}} \times \mathfrak{h}_{\mathfrak{sl}_2} \\ &= \{a_{11} + a_{22} + \cdots + a_{2k+3,2k+3} = 0, a_{2k+4,2k+4} + a_{2k+5,2k+5} = 0\}.\end{aligned}$$

Since the set $p(\mathfrak{h}_{\mathfrak{sl}_{2k+3}} \times \mathfrak{h}_{\mathfrak{sl}_2})$ contains the image of elements whose determinant is not equal to zero, the set $p(\mathfrak{h}_{\mathfrak{sl}_{2k+3}} \times \mathfrak{h}_{\mathfrak{sl}_2})$ is not a subset of $p(\mathfrak{h}_{\mathfrak{sl}_{2k+4}})$. Then by the irreducibility of $Sec[3, 2^{k+1}]$ we obtain

$$p(Sec[3, 2^{k+1}]) = \mathfrak{h}_{\mathfrak{sl}_{2k+5}}/W_{2k+5}.$$

Hence by Corollary 2.10 we have

$$Sec[3, 2^{k+1}] = \mathfrak{sl}_{2k+5}.$$

Next we study the case where $r = k + 1, s = 1$. We can regard a nilpotent element $x = [3, 2^{k+1}, 1] \in \mathfrak{sl}_{2k+6}$ as a nilpotent element $[3, 2^{k+1}] \in \mathfrak{sl}_{2k+5} \subset \mathfrak{sl}_{2k+6}$. Since we proved $Sec[3, 2^{k+1}] = \mathfrak{sl}_{2k+5}$, we have

$$\mathfrak{sl}_{2k+6} \supset Sec[3, 2^{k+1}, 1] \supset \mathfrak{h}_{\mathfrak{sl}_{2k+5}} = \{a_{11} + \cdots + a_{2k+5,2k+5} = 0, a_{2k+6,2k+6} = 0\}.$$

The condition that there is some number i such that $a_{ii} = 0$ is stable under the action of the Weyl group $W_{\mathfrak{sl}_{2k+6}}$ in $\mathfrak{h}_{\mathfrak{sl}_{2k+6}}$. Then the image $p(\mathfrak{h}_{\mathfrak{sl}_{2k+5}})$ of $\mathfrak{h}_{\mathfrak{sl}_{2k+5}}$ of the adjoint quotient $p : \mathfrak{sl}_{2k+6} \to \mathfrak{h}_{\mathfrak{sl}_{2k+6}}/W_{\mathfrak{sl}_{2k+6}}$ is an irreducible subvariety of codimension 1 in $\mathfrak{h}_{\mathfrak{sl}_{2k+6}}/W_{\mathfrak{sl}_{2k+6}}$ and does not contain the image of any element whose determinant is not equal to zero. On the other hand we regard a nilpotent element $x = [3, 2^{k+1}, 1]$ as $[3, 2^k, 1] \times [2] \in \mathfrak{sl}_{2k+4} \times \mathfrak{sl}_2 \subset \mathfrak{sl}_{2k+6}$. Similarly we have

$$\mathfrak{sl}_{2k+6} \supset Sec[3, 2^{k+1}, 1] \supset \mathfrak{sl}_{2k+4} \times \mathfrak{sl}_2 \supset \mathfrak{h}_{\mathfrak{sl}_{2k+4}} \times \mathfrak{h}_{\mathfrak{sl}_2},$$

and

$$\mathfrak{h}_{\mathfrak{sl}_{2k+4}} \times \mathfrak{h}_{\mathfrak{sl}_2} = \{a_{11} + a_{22} + \cdots + a_{2k+4,2k+4} = 0, a_{2k+5,2k+5} + a_{2k+6,2k+6} = 0\}.$$

The image $p(\mathfrak{h}_{\mathfrak{sl}_{2k+4}} \times \mathfrak{h}_{\mathfrak{sl}_2})$ contains the images of some elements whose determinants are not equal to zero. Then the image $p(\mathfrak{h}_{\mathfrak{sl}_{2k+5}})$ is not a subset of $p(\mathfrak{h}_{\mathfrak{sl}_{2k+4}} \times \mathfrak{h}_{\mathfrak{sl}_2})$. Then by the irreducibility of $Sec[3, 2^{k+1}, 1]$ we obtain

$$p(Sec[3, 2^{k+1}, 1]) = \mathfrak{h}_{\mathfrak{sl}_{2k+6}}/W_{2k+6}.$$

Then by Corollary 2.10 we obtain

$$Sec[3, 2^{k+1}, 1] = \mathfrak{sl}_{2k+6}.$$

This shows the result. $\square$

Theorem 4.6. *Let x be a nilpotent element ($\neq [2^k, 1^{n-2k}]$) in $\mathfrak{sl}_n$ with* $\mathrm{rank}(x) = k$. *Then we have*

$$SecX = \{x \in \mathfrak{sl}_n | \ \mathrm{rank}(x) \leq 2k\}.$$

Proof. With respect to the closure relation the smallest nilpotent orbit with fixed rank k except $[2^k, 1^{n-2k}]$ is $[3, 2^{k-1}, 1^{n-2k-1}]$. Then Lemma 4.1 and 4.5 show the result. $\square$

K.Nishiyama[9] proved the next theorem.

Theorem 4.7. *Let x be $[2^r, 1^s] \in \mathfrak{sl}_n$. ($n = 2r + s$)*

$$SecX = \{x \in \mathfrak{sl}_n | \ \mathrm{rank}(x) \leq 2r, \ \det(\lambda I_n - x) = \lambda^{n-2r} f(\lambda), f(\lambda) \text{ is an even function.}\}.$$

Here I_n is the $n \times n$ unit matrix.

Proof. We consider any element whose rank is $2r$ and whose eigen polynomial is an even function. By the similar argument as the argument in the proof of Lemma 4.1, it is included in the closure of an orbit of some element with same properties in $\mathfrak{sl}_{2r+1} \subset \mathfrak{sl}_n$. Hence it is enough to prove the case where $s = 0, 1$. First we study the case where $s = 0$. Let $x = [2^r]$ be a matrix which decomposes into blocks of 2×2 matrices such that if i is not equal to j, (i,j)-block is $A_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and otherwise, $A_{ii} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then

$$\mathfrak{sl}_{2r} \supset SecX \supset \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \cdots \times \mathfrak{sl}_2 \supset \mathfrak{h}_{\mathfrak{sl}_2} \times \mathfrak{h}_{\mathfrak{sl}_2} \times \cdots \times \mathfrak{h}_{\mathfrak{sl}_2} \text{ (r times)}.$$

Generic elements of $\mathfrak{h}_{\mathfrak{sl}_2} \times \cdots \times \mathfrak{h}_{\mathfrak{sl}_2}$ (r times) are regular in $\mathfrak{sl}_{2r}$. When we consider the adjoint quotient $p : \mathfrak{sl}_{2r} \to \mathfrak{h}_{\mathfrak{sl}_{2r}}/W_{\mathfrak{sl}_{2r}}$, by Proposition 2.7 we have

$$\mathfrak{sl}_{2r} \supset SecX \supset p^{-1}(p(\mathfrak{h}_{\mathfrak{sl}_2} \times \mathfrak{h}_{\mathfrak{sl}_2} \times \cdots \times \mathfrak{h}_{\mathfrak{sl}_2})).$$

This shows

$$SecX \supset \{x \in \mathfrak{sl}_{2r} | \ \mathrm{rank}(x) \leq 2r, \ \det(\lambda I_{2r} - x) \text{ is an even function}\}.$$

We calculate the eigen polynomial of $x + g^{-1}xg$ for the above nilpotent element $x = [2^r]$ in $\mathfrak{sl}_{2r}$ and $g \in SL_{2r}$,

$$\det(\lambda I_{2r} - x - g^{-1}xg) = \det(\lambda g - gx - xg).$$

We divide $\lambda g - gx - xg$ into blocks of 2×2 matrices. Put $g = (g_{kl})$. Then (i,j)-block is as follows.

$$\begin{pmatrix} \lambda g_{2i-1,2i-1} - g_{2i,2i-1} & \lambda g_{2i-1,2i} - g_{2i-1,2i-1} - g_{2i,2i} \\ \lambda g_{2i,2i-1} & \lambda g_{2i,2i} - g_{2i,2i-1} \end{pmatrix}.$$

We add each even row $\times \frac{1}{\lambda}$ to the upper odd row. Then (i,j)-block is

$$\begin{pmatrix} \lambda g_{2i-1,2i-1} & \lambda g_{2i-1,2i} - g_{2i-1,2i-1} - \frac{1}{\lambda} g_{2i,2i-1} \\ \lambda g_{2i,2i-1} & \lambda g_{2i,2i} - g_{2i,2i-1} \end{pmatrix}.$$

Next we add each odd column $\times \frac{1}{\lambda}$ to the right even column. Then (i,j)-block is

$$\begin{pmatrix} \lambda g_{2i-1,2i-1} & \lambda g_{2i-1,2i} - \frac{1}{\lambda} g_{2i,2i-1} \\ \lambda g_{2i,2i-1} & \lambda g_{2i,2i} \end{pmatrix}.$$

Finally we multiply all odd columns by $\frac{1}{\lambda}$ and all even columns by λ. Then (i,j)-block becomes

$$\begin{pmatrix} g_{2i-1,2i-1} & \lambda^2 g_{2i-1,2i} - g_{2i,2i-1} \\ g_{2i,2i-1} & \lambda^2 g_{2i,2i} \end{pmatrix}.$$

Then the eigen polynomial is an even function. In the case where $s = 0$, we have

$$SecX \subset \{x \in \mathfrak{sl}_{2r} |\ \mathrm{rank}(x) \leq 2r,\ \det(\lambda I_{2r} - x) \text{ is an even function}\}.$$

Next we consider the case where $s = 1$ i.e. $x = [2^r, 1]$. Generic elements of $\mathfrak{h}_{\mathfrak{sl}_2} \times \cdots \times \mathfrak{h}_{\mathfrak{sl}_2}$($r$ times) are regular in $\mathfrak{sl}_{2r+1}$. When we consider the adjoint quotient $p : \mathfrak{sl}_{2r+1} \to \mathfrak{h}_{\mathfrak{sl}_{2r+1}}/W_{\mathfrak{sl}_{2r+1}}$, by Proposition 2.7 we have

$$\overline{SL_{2r+1} \cdot (\mathfrak{h}_{\mathfrak{sl}_2} \times \cdots \times \mathfrak{h}_{\mathfrak{sl}_2})} = p^{-1}(p(\mathfrak{h}_{\mathfrak{sl}_2} \times \cdots \times \mathfrak{h}_{\mathfrak{sl}_2})).$$

This shows

$$SecX \subset \{x \in \mathfrak{sl}_{2r+1} |\ \mathrm{rank}(x) \leq 2r,\ \det(\lambda I_n - x) = \lambda f(\lambda), f(\lambda) \text{ is an even function}\}.$$

By the similar calculations as the case where $s = 0$ we can prove the inverse inclusion. Then the proof is complete. □

For two varieties X, Y in $\mathbb{P}\mathfrak{g}$ we define the join $X + Y$ by

$$X + Y := \overline{\pi(\{x + y \mid x \in \pi^{-1}(X),\ y \in \pi^{-1}(Y)\})},\ \pi : \mathfrak{g} \setminus \{0\} \to \mathbb{P}\mathfrak{g}.$$

We recall that we use the same notation $[r_1, \ldots, r_s]$ for the corresponding projective and the corresponding cone varieties to the nilpotent orbit $[r_1, \ldots, r_s]$.

Lemma 4.8. *Let $\mathfrak{g}$ be $\mathfrak{sl}_n$. For nilpotent orbits $[r, 1^{n-r}], [2, 1^{n-2}] (r \geq 2)$, we have*

$$[r, 1^{n-r}] + [2, 1^{n-2}] = \{x \in \mathfrak{sl}_n | \operatorname{rank}(x) \leq r\} \quad (n \geq 2).$$

Proof. By Lemma 4.1 it is enough to prove the case where $n = r$. Example 3.7 shows the result of the case where $n = 2$. Assume the assertion holds when $n = k$. The closure of the nilpotent orbit $[k+1]$ contains the nilpotent orbit $[k, 1]$. Then we have

$$[k+1] + [2, 1^{k-1}] \supset [k, 1] + [2, 1^{k-1}].$$

Since $[k] + [2, 1^{k-2}] = \{x \in \mathfrak{sl}_k | \operatorname{rank}(x) \leq k\} = \mathfrak{sl}_k$ by the assumption, under the natural embedding $\mathfrak{sl}_k \subset \mathfrak{sl}_{k+1}$, we have

$$\mathfrak{sl}_{k+1} \supset [k+1] + [2, 1^{k-1}] \supset \mathfrak{sl}_k \supset \mathfrak{h}_{\mathfrak{sl}_k},$$

and

$$\mathfrak{sl}_{k+1} \supset \mathfrak{h}_{\mathfrak{sl}_{k+1}} \supset \mathfrak{h}_{\mathfrak{sl}_k} = \{a_{11} + a_{22} + \cdots + a_{k,k} = 0, a_{k+1,k+1} = 0\}.$$

The condition that there is some number i such that $a_{ii} = 0$ is stable under the action of the Weyl group $W_{\mathfrak{sl}_{k+1}}$ in $\mathfrak{h}_{\mathfrak{sl}_{k+1}}$. Then the image $p(\mathfrak{h}_{\mathfrak{sl}_k})$ of $\mathfrak{h}_{\mathfrak{sl}_k}$ of the adjoint quotient $p : \mathfrak{sl}_{k+1} \to \mathfrak{h}_{\mathfrak{sl}_{k+1}}/W_{\mathfrak{sl}_{k+1}}$ is an irreducible subvariety of codimension 1 in $\mathfrak{h}_{\mathfrak{sl}_{k+1}}/W_{\mathfrak{sl}_{k+1}}$ and does not contain the image of any element whose determinant is not equal to zero. Take nilpotent elements $x = [k+1]$ and $y = [2, 1^{k-1}]$,

$$x = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & & & \ddots & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Then the determinant of $x + y$ is not equal to zero. So $p(\mathfrak{h}_{\mathfrak{sl}_k})$ never contain $p(x+y)$. By the irreducibility of $[k+1]+[2, 1^{k-1}]$ we obtain $p([k+1]+[2, 1^{k-2}]) = \mathfrak{h}_{\mathfrak{sl}_{k+1}}/W_{\mathfrak{sl}_{k+1}}$. Then by Corollary 2.10 we obtain

$$[k+1] + [2, 1^{k-2}] = \mathfrak{sl}_{k+1}.$$

Then the result follows. □

Lemma 4.9. *Let $\mathfrak{g}$ be $\mathfrak{sl}_n$. For the nilpotent orbits $[r, 1^{n-r}], [2^s, 1^{n-2s}]$ $(r \geq s+1)$, we have*

$$[r, 1^{n-r}] + [2^s, 1^{n-2s}] = \{x \in \mathfrak{sl}_n | \operatorname{rank}(x) \leq r + s - 1\}.$$

Proof. When $t \le s$, we have $[2^t, 1^{n-2t}] \subset \overline{[2^s, 1^{n-2s}]}$. Then if $r+s > n$, it is enough to prove the statement for nilpotent orbits $[r, 1^{n-r}], [2^{n-r+1}, 1^{2r-n-2}]$. Hence we can assume that $r+s \le n$. Moreover by Lemma 4.1 it is enough to prove the case where $n = r+s-1$. Lemma 4.8 shows the result of the case where $s = 1$. Assume the assertion holds in the case where $s = k$. We consider the case where $s = k+1$. In this case $n = r+(k+1)-1 = r+k$. The closure of the nilpotent orbit $[2^{k+1}, 1^{n-2k-2}]$ contains the nilpotent orbit $[2^k, 1^{n-2k}]$. Then by the assumption we have

$$\begin{aligned}[r, 1^{n-r}] + [2^{k+1}, 1^{n-2k-2}] &\supset [r, 1^{n-r}] + [2^k, 1^{n-2k}] \\ &\supset \{x \in \mathfrak{sl}_n |\ \mathrm{rank}(x) \le r+k-1 = n-1\}.\end{aligned}$$

Since $\{x \in \mathfrak{sl}_n |\ \mathrm{rank}(x) \le r+k-1 = n-1\} \supset \mathfrak{h}_{\mathfrak{sl}_{n-1}}$, we obtain,

$$\mathfrak{sl}_n \supset [r, 1^{n-r}] + [2^{k+1}, 1^{n-2k-2}] \supset \mathfrak{h}_{\mathfrak{sl}_{n-1}} = \{a_{11} + \cdots + a_{n-1,n-1} = 0, a_{n,n} = 0\}.$$

The condition that there is some number i such that $a_{ii} = 0$ is stable under the action of the Weyl group $W_{\mathfrak{sl}_n}$ in $\mathfrak{h}_{\mathfrak{sl}_n}$. The image $p(\mathfrak{h}_{\mathfrak{sl}_{n-1}})$ of $\mathfrak{h}_{\mathfrak{sl}_{n-1}}$ of the adjoint quotient $p : \mathfrak{sl}_n \to \mathfrak{h}_{\mathfrak{sl}_n}/W_{\mathfrak{sl}_n}$ is an irreducible subvariety of codimension 1 in $\mathfrak{h}_{\mathfrak{sl}_n}/W_{\mathfrak{sl}_n}$ and does not contain the image of any element whose determinant is not equal to zero. The closure of the nilpotent orbit $[r, 1^{n-r}]$ contains the orbits $[r-k, 2^k, 1^{n-r-k}] = [r-k, 2^k]$. Take nilpotent elements $x = [r-k, 2^k]$ and $y = [2^{k+1}, 1^{n-2k-2}]$ as follows,

$$x = \begin{pmatrix} D_2 & & & & \\ & D_2 & & \mathbf{0} & \\ & & \ddots & & \\ & & & D_2 & \\ & \mathbf{0} & & & D_{r-k} \end{pmatrix}, \quad y = \begin{pmatrix} 0 & & & & & \\ & D_2 & & & \mathbf{0} & \\ & & \ddots & & & \\ & & & D_2 & & \\ & & & & 0 & \\ & \mathbf{0} & & & & \ddots \\ 1 & & & & & & 0 \end{pmatrix}.$$

Here D_i is the $i \times i$ nilpotent Jordan block. Then the determinant of $x + y \in [r, 1^{n-r}] + [2^{k+1}, 1^{n-2k-2}]$ is not equal to zero. Since $p(\mathfrak{h}_{\mathfrak{sl}_{n-1}})$ never contain $p(x+y)$ and $[r, 1^{n-r}] + [2^{k+1}, 1^{n-2k-2}]$ is irreducible, we obtain $p([r, 1^{n-r}] + [2^{k+1}, 1^{n-2k-2}]) = \mathfrak{h}_{\mathfrak{sl}_n}/W_{\mathfrak{sl}_n}$. Then by Corollary 2.10 we obtain

$$[r, 1^{n-r}] + [2^{k+1}, 1^{n-2k-2}] = \mathfrak{sl}_n.$$

Then the required result follows. □

Hence we have the following proposition.

Proposition 4.10. *Let $\mathfrak{g}$ be $\mathfrak{sl}_n$.*

$$Sec^{(i)}[2^k, 1^{n-2k}] = \{x \in \mathfrak{sl}_n |\ \mathrm{rank}(x) \le k(i+1)\} \quad (i \ge 2).$$

Proof. It is enough to consider the case of $k(i+1) \le n$. We can construct a nilpotent element $[ki+1, 1^{n-ki-1}]$ as a sum of i nilpotent elements of the type of $[2^k, 1^{n-2k}]$. Then we obtain the theorem by Lemma 4.9. □

Then we have the following theorem.

Theorem 4.11. *Let $\mathfrak{g}$ be $\mathfrak{sl}_n$ and x a nilpotent element whose* rank *is k.*

$$Sec^{(i)}X = \{x \in \mathfrak{sl}_n | \operatorname{rank}(x) \leq k(i+1)\} \quad (i \geq 2).$$

Proof. The nilpotent orbit $[2^k, 1^{n-2k}]$ is minimal among orbits which are constructed from nilpotent elements with fixed rank k. Hence $Sec^{(i)}X$ contains $Sec^{(i)}[2^k, 1^{n-2k}]$ for any nilpotent element x whose rank is k. Then we obtain the theorem. □

5. The case of $\mathfrak{so}_n$

We may realize $\mathfrak{so}_n$ as follows,

$$\mathfrak{so}_{2n} = \left\{ \begin{pmatrix} A_1 & A_2 \\ A_3 & -{}^tA_1 \end{pmatrix} \mid A_i \in M_n, A_2 \text{ and } A_3 \text{ are skew symmetric.} \right\},$$

and

$$\mathfrak{so}_{2n+1} = \left\{ \begin{pmatrix} 0 & -{}^tv & -{}^tu \\ u & A_1 & A_2 \\ v & A_3 & -{}^tA_1 \end{pmatrix} \mid u, v \in \mathbb{C}^n,\ A_i \in M_n, A_2 \text{ and } A_3 \text{ are skew symmetric.} \right\}.$$

In this realization the Cartan subalgebra $\mathfrak{h}_{\mathfrak{so}_n}$ is realized as the set of all diagonal matrices in $\mathfrak{so}_n$. In the case of $\mathfrak{so}_n$ we shall use similar embeddings as the case of $\mathfrak{sl}_n$.

Lemma 5.1. *If $N \geq 2n$, in $\mathfrak{so}_N$ we have*

$$\overline{SO_N \cdot \mathfrak{so}_{2n}} = \{x \in \mathfrak{so}_N \mid \operatorname{rank}(x) \leq 2n\}.$$

Proof. The case where $N = 2n$ is obvious. Then we consider the case where $N \geq 2n+1$. The inclusion:

$$\overline{SO_N \cdot \mathfrak{so}_{2n}} \subset \{x \in \mathfrak{so}_N \mid \operatorname{rank}(x) \leq 2n\}$$

is obvious. Then we shall prove the inverse inclusion. Under the natural embedding $\mathfrak{so}_{2n} \subset \mathfrak{so}_{2n+1}$ as the 1-st row and column are zero, we can identify $\mathfrak{h}_{\mathfrak{so}_{2n}}$ with $\mathfrak{h}_{\mathfrak{so}_{2n+1}}$. Then we have

$$\overline{SO_N \cdot \mathfrak{so}_{2n}} = \overline{SO_N \cdot \mathfrak{so}_{2n+1}}.$$

We consider an element $x \in \mathfrak{so}_N$ whose rank is $2n$. Let $D_i(a_i)$ be the $i \times i$ Jordan block whose eigenvalue is a_i. Consider the Jordan normal form of x

$$\begin{pmatrix} D_{i_1}(a_1) & & & \mathbf{0} \\ & D_{i_2}(a_2) & & \\ & & \ddots & \\ \mathbf{0} & & & D_{i_s}(a_s) \end{pmatrix}, \quad |a_1| \geq |a_2| \geq \cdots \geq |a_s|.$$

Let k be the number such that $a_k \neq 0$ and $a_{k+1} = 0$. If $i_{k+1} = \cdots = i_s = 1$, we have $x \in \overline{SO_N \cdot \mathfrak{so}_{2n}}$. Otherwise we put

$$A := \sum_{l=k+1}^{s} \mathrm{rank} D_{i_l}(a_l) = \sum_{l=k+1}^{s} (i_l - 1).$$

Then A is even. Moreover for any nonzero $l_j \in \{a_1, \ldots, a_k\}$ we put

$$b_{l_j} := \sum_{a_t = l_j} i_t.$$

Then we have the set of non-zero complex numbers $\{l_1, \ldots, l_u\}$ and corresponding non negative integers $\{b_{l_1}, \ldots, b_{l_u}\}$. Let y be an element in $\mathfrak{so}_{2n+1} \subset \mathfrak{so}_N$ whose Jordan normal form is

$$\begin{pmatrix} D_{b_{l_1}}(l_1) & & & & \\ & \ddots & & \mathbf{0} & \\ & & D_{b_{l_u}}(l_u) & & \\ & \mathbf{0} & & D_{A+1}(0) & \\ & & & & \mathbf{0} \end{pmatrix}.$$

Then x is contained in $\overline{SO_N \cdot y}$. So we have

$$x \in \overline{SO_N \cdot \mathfrak{so}_{2n+1}} = \overline{SO_N \cdot \mathfrak{so}_{2n}}.$$

Then this shows

$$\overline{SO_N \cdot \mathfrak{so}_{2n}} \supset \{x \in \mathfrak{so}_N \mid \mathrm{rank}(x) \leq 2n\}.$$

Hence the result follows. $\square$

Lemma 5.2. *Let $\mathfrak{g}$ be $\mathfrak{so}_N$ $(N = 3, 4, 5, 6)$.*

$$Sec[3] = \mathfrak{so}_3,\ Sec[3,1] = \mathfrak{so}_4,\ Sec[3,1^2] = \mathfrak{so}_5,\ Sec[3^2] = \mathfrak{so}_6.$$

Proof. If $n = 3$ the nilpotent orbit $[3]$ is regular. Then corollary 3.6 shows

$$Sec[3] = \mathfrak{so}_3.$$

When $n = 4$ there is an isomorphism $\mathfrak{so}_4 \simeq \mathfrak{sl}_2 \times \mathfrak{sl}_2$. Under this isomorphism the nilpotent orbit $[3,1]$ in $\mathfrak{so}_4$ corresponds to the nilpotent orbit $[2] \times [2]$ in $\mathfrak{sl}_2 \times \mathfrak{sl}_2$. Since $Sec([2] \times [2]) = \mathfrak{sl}_2 \times \mathfrak{sl}_2$, we have

$$Sec[3,1] = \mathfrak{so}_4.$$

When $n = 5$ under the natural embedding $\mathfrak{so}_4 \subset \mathfrak{so}_5$ we can identify their Cartan subalgebras. Since $Sec[3,1] = \mathfrak{so}_4$, we have

$$\mathfrak{so}_5 \supset Sec[3,1^2] \supset \mathfrak{so}_4 \supset \mathfrak{h}_{\mathfrak{so}_4} = \mathfrak{h}_{\mathfrak{so}_5}.$$

Then by Proposition 2.9 we have

$$Sec[3,1^2] = \mathfrak{so}_5.$$

When $n = 6$ there is an isomorphism $\mathfrak{so}_6 \simeq \mathfrak{sl}_4$. Under this isomorphism the nilpotent orbit $[3^2]$ in $\mathfrak{so}_6$ corresponds to the nilpotent orbit $[3,1]$ in $\mathfrak{sl}_4$. Since $Sec[3,1] = \mathfrak{sl}_4$, we have

$$Sec[3^2] = \mathfrak{so}_6.$$

□

Lemma 5.3. *Let $\mathfrak{g}$ be $\mathfrak{so}_N$ $(N \geq 3)$.*

$$Sec[3, 2^{2(n-1)}] = \mathfrak{so}_{4n-1},\ \ Sec[3, 2^{2(n-1)}, 1] = \mathfrak{so}_{4n},$$

$$Sec[3, 2^{2(n-1)}, 1^2] = \mathfrak{so}_{4n+1},\ \ Sec[3^2, 2^{2(n-1)}] = \mathfrak{so}_{4n+2}.$$

Proof. Lemma 5.2 shows the result of the case where $n = 1$. We prove this lemma by induction on n. We assume that the assertion holds if $n = k-1$. First we study the case of $\mathfrak{so}_{4k-1}$. We consider $\mathfrak{so}_{4(k-1)}$ as a Lie subalgebra of all matrices in $\mathfrak{so}_{4k-1}$ whose 1-,$2k$- and $(4k-1)$-th rows and columns are zero. Let x be $[3, 2^{2(k-1)}]$ in $\mathfrak{so}_{4k-1}$. The closure of the nilpotent orbit $[3, 2^{2(k-1)}]$ contains the nilpotent orbit $[3, 2^{2(k-2)}, 1^4]$. We can regard a nilpotent element $[3, 2^{2(k-2)}, 1^4] \in \mathfrak{so}_{4k-1}$ as a nilpotent element $[3, 2^{2(k-2)}, 1] \in \mathfrak{so}_{4(k-1)} \subset \mathfrak{so}_{4k-1}$. Since $Sec[3, 2^{2(k-2)}, 1] = \mathfrak{so}_{4(k-1)}$ by the assumption, we have

$$\mathfrak{so}_{4k-1} \supset Sec[3, 2^{2(k-1)}] \supset \mathfrak{so}_{4(k-1)} \supset \mathfrak{h}_{\mathfrak{so}_{4(k-1)}}.$$

The Cartan subalgebra $\mathfrak{h}_{\mathfrak{so}_{4(k-1)}}$ is realized as a subalgebra of all diagonal matrices in $\mathfrak{so}_{4k-1}$ whose $(2k, 2k)$- and $(4k-1, 4k-1)$-entries are zero. The condition that there is some number i except $i = 1$ such that $a_{ii} = 0$ is stable under the action of the Weyl group $W_{\mathfrak{so}_{4k-1}}$ in $\mathfrak{h}_{\mathfrak{so}_{4k-1}}$. Then the image $p(\mathfrak{h}_{\mathfrak{so}_{4(k-1)}})$ of $\mathfrak{h}_{\mathfrak{so}_{4(k-1)}}$ of the adjoint quotient $p : \mathfrak{so}_{4k-1} \to \mathfrak{h}_{\mathfrak{so}_{4k-1}}/W_{\mathfrak{so}_{4k-1}}$ is an irreducible subvariety of codimension 1 in $\mathfrak{h}_{\mathfrak{so}_{4k-1}}/W_{\mathfrak{so}_{4k-1}}$ and does not contain the image of any element which has the eigenvalue 0 whose multiplicity is less than 3. Take a following matrix as a nilpotent element $[3, 2^{2(k-1)}]$,

$$x = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & & & & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & A_{k-1} & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & & & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & & & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & -{}^tA_{k-1} & \\ 0 & 0 & 0 & \cdots & 0 & 0 & & & \end{pmatrix}.$$

Here A_i is the following $2i \times 2i$ matrix

$$A_i = \begin{pmatrix} D_2 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & D_2 \end{pmatrix}.$$

Taking a sum $y = x +^t x \in Sec[3, 2^{2(k-1)}]$, we have

$$\det(\lambda I_{4k-1} - y) = \lambda(\lambda^2 - 2)(\lambda^2 - 1)^{2(k-1)}.$$

Since the multiplicity of eigenvalue 0 of y is 1, the image $p(\mathfrak{h}_{\mathfrak{so}_{4(k-1)}})$ never contain $p(y)$. By the irreducibility of $Sec[3, 2^{2(k-1)}]$ we obtain

$$p(Sec[3, 2^{2(k-1)}]) = \mathfrak{h}_{\mathfrak{so}_{4k-1}}/W_{\mathfrak{sl}_{4k-1}}.$$

Then by Corollary 2.10 we have

$$Sec[3, 2^{2(k-1)}] = \mathfrak{so}_{4k-1}.$$

We consider the case of $\mathfrak{so}_{4k}$. Let x be $[3, 2^{2(k-1)}, 1] \in \mathfrak{so}_{4k}$. Since $Sec[3, 2^{2(k-1)}] = \mathfrak{so}_{4k-1}$, under the natural embedding $\mathfrak{so}_{4(k-1)+2} \subset \mathfrak{so}_{4k}$, we have

$$\mathfrak{so}_{4k} \supset Sec[3, 2^{2(k-1)}, 1] \supset \mathfrak{so}_{4k-1} \supset \mathfrak{so}_{4(k-1)+2} \supset \mathfrak{h}_{\mathfrak{so}_{4(k-1)+2}}.$$

The Cartan subalgebra $\mathfrak{h}_{\mathfrak{so}_{4(k-1)+2}}$ is realized as a subalgebra of all diagonal matrices in $\mathfrak{so}_{4k}$ whose $(2k, 2k)$- and $(4k, 4k)$-entries are zero. The condition that there is some number i such that $a_{ii} = 0$ is stable under the action of the Weyl group $W_{\mathfrak{so}_{4k}}$ in $\mathfrak{h}_{\mathfrak{so}_{4k}}$. Then the image $p(\mathfrak{h}_{\mathfrak{so}_{4(k-1)+2}})$ of $\mathfrak{h}_{\mathfrak{so}_{4(k-1)+2}}$ of the adjoint quotient $p : \mathfrak{so}_{4k} \to \mathfrak{h}_{\mathfrak{so}_{4k}}/W_{\mathfrak{so}_{4k}}$ is an irreducible subvariety of codimension 1 in $\mathfrak{h}_{\mathfrak{so}_{4k}}/W_{\mathfrak{so}_{4k}}$ and does not contain the image of any element whose determinant is not equal to zero. We consider a nilpotent element $[2^{2k}] \in \mathfrak{so}_{4k}$. The closure of the orbit $[3, 2^{2(k-1)}, 1]$ contains the orbit $[2^{2k}]$. Let's take a following y as a nilpotent element $[2^{2k}]$,

$$y = \begin{pmatrix} A_k & \mathbf{0} \\ \mathbf{0} & -{}^tA_k \end{pmatrix}.$$

Take a sum $z = y +^t y \in Sec[3, 2^{2(k-1)}, 1]$. Then the determinant of this element z is non-zero. So $p(\mathfrak{h}_{\mathfrak{so}_{4(k-1)+2}})$ never contain $p(z)$. By the irreducibility of $Sec[3, 2^{2(k-1)}, 1]$ we obtain

$$p(Sec[3, 2^{2(k-1)}, 1]) = \mathfrak{h}_{\mathfrak{so}_{4k}}/W_{\mathfrak{sl}_{4k}}.$$

Then by Corollary 2.10 we have

$$Sec[3, 2^{2(k-1)}, 1] = \mathfrak{so}_{4k}.$$

We consider the case of $\mathfrak{so}_{4k+1}$. Under the natural embedding $\mathfrak{so}_{4k} \subset \mathfrak{so}_{4k+1}$ we can identify $\mathfrak{h}_{\mathfrak{so}_{4k+1}}$ with $\mathfrak{h}_{\mathfrak{so}_{4k}}$. Since $Sec[3, 2^{2(k-1)}, 1] = \mathfrak{so}_{4k}$, we obtain

$$\mathfrak{so}_{4k+1} \supset Sec[3, 2^{2(k-1)}, 1^2] \supset \mathfrak{so}_{4k} \supset \mathfrak{h}_{\mathfrak{so}_{4k}} = \mathfrak{h}_{\mathfrak{so}_{4k+1}}.$$

Then by Proposition 2.9 we have

$$Sec[3, 2^{2(k-1)}, 1^2] = \mathfrak{so}_{4k+1}.$$

Finally we consider the case of $\mathfrak{so}_{4k+2}$. The closure of the nilpotent orbit $[3^2, 2^{2(k-1)}]$ contains the nilpotent orbit $[3^2, 2^{2(k-2)}, 1^4]$. The closure of the nilpotent orbit $[3^2, 2^{2(k-2)}, 1^4]$ contains the nilpotent orbit $[3, 2^{2(k-1)}, 1^3]$. Since $Sec[3, 2^{2(k-1)}, 1] = \mathfrak{so}_{4k}$, we have

$$\mathfrak{so}_{4k+2} \supset Sec[3^2, 2^{2(k-1)}] \supset \mathfrak{so}_{4k} \supset \mathfrak{h}_{\mathfrak{so}_{4k}}.$$

The Cartan subalgebra $\mathfrak{h}_{\mathfrak{so}_{4k}}$ is realized as a subalgebra of all diagonal matrices in $\mathfrak{so}_{4k+2}$ whose $(2k+1, 2k+1)$- and $(4k+2, 4k+2)$-entries are zero. The condition that there is some number i such that $a_{ii} = 0$ is stable under the action of $W_{\mathfrak{so}_{4k+2}}$ in $\mathfrak{h}_{\mathfrak{so}_{4k+2}}$. Then the image $p(\mathfrak{h}_{\mathfrak{so}_{4k}})$ of $\mathfrak{h}_{\mathfrak{so}_{4k}}$ of the adjoint quotient $p : \mathfrak{so}_{4k+2} \to \mathfrak{h}_{\mathfrak{so}_{4k+2}}/W_{\mathfrak{so}_{4k+2}}$ is an irreducible subvariety of codimension 1 in $\mathfrak{h}_{\mathfrak{so}_{4k+2}}/W_{\mathfrak{so}_{4k+2}}$ and does not contain the image of any element whose determinant is not equal to zero. We consider the embedding

$$\mathfrak{sl}_{2k+1} = \left\{ \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & -{}^tA \end{pmatrix} \mid A \in \mathfrak{sl}_{2k+1} \right\} \subset \mathfrak{so}_{4k+2}.$$

Using a nilpotent element $[3, 2^{k-1}] \in \mathfrak{sl}_{2k+1}$ we can realize a nilpotent element $x = [3^2, 2^{2(k-1)}] \in \mathfrak{so}_{4k+2}$ as

$$x = \begin{pmatrix} [3, 2^{k-1}] & \mathbf{0} \\ \mathbf{0} & -{}^t[3, 2^{k-1}] \end{pmatrix}.$$

In $\mathfrak{sl}_{2k+1}$ we have $Sec[3, 2^{k-1}] = \mathfrak{sl}_{2k-1}$. Then in $\mathfrak{so}_{4k+2}$ we have

$$Sec[3^2, 2^{2(k-1)}] \supset \mathfrak{sl}_{2k+1} = \left\{ \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & -{}^tA \end{pmatrix} |A \in \mathfrak{sl}_{2k+1} \right\}.$$

Hence $Sec[3^2, 2^{2(k-1)}]$ contains an element whose determinant is not zero. Then the image of this element is not contained in $p(\mathfrak{h}_{\mathfrak{so}_{4k}})$. By the irreducibility of $Sec[3^2, 2^{2(k-1)}]$ we obtain

$$p(Sec[3^2, 2^{2(k-1)}]) = \mathfrak{h}_{\mathfrak{so}_{4k+2}}/W_{\mathfrak{so}_{4k+2}}.$$

Then by Corollary 2.10 we obtain

$$Sec[3^2, 2^{2(k-1)}] = \mathfrak{so}_{4k+2}.$$

Then we proved the lemma. □

Theorem 5.4. *Let x be a nilpotent element ($\neq [2^{2k}, 1^{n-4k}]$) in $\mathfrak{so}_n$ of* rank$(x) = 2k$*.*

$$SecX = \{x \in \mathfrak{so}_n | \operatorname{rank}(x) \leq 2k\}$$

Proof. When $2k > \frac{n}{2}$, X contain some element y whose rank is $[\frac{n}{2}]$ and which is not $[2^{[\frac{n}{2}]}, 1^{n-2[\frac{n}{2}]}]$. If $Sec(\overline{SO_n \cdot y}) = \mathfrak{so}_n$, we have $SecX = \mathfrak{so}_n$. Then it is enough to prove the case where $2k \leq \frac{n}{2}$. Hence we assume that $2k \leq \frac{n}{2}$. Then the minimal nilpotent orbit with a fixed rank $2k$ except $[2^{2k}, 1^s]$ is $[3, 2^{2(k-1)}, 1^{s+1}]$ or $[3^2, 2^{2(k-2)}]$. Then by Lemma 5.1 and 5.3 we obtain the theorem. □

By Theorem 5.4 we have the following corollary.

Corollary 5.5. *Let x be a nilpotent element x ($\neq [2^{2k}, 1^{n-4k}]$) in $\mathfrak{so}_n$ of* $\mathrm{rank}(x) = 2k$.

$$Sec^{(i)}X = \{x \in \mathfrak{so}_n |\ \mathrm{rank}(x) \leq 2k(i+1)\} \quad (i \geq 1).$$

Proof. We prove the assertion by induction on i. When $i = 1$, the assertion is Theorem 5.4. We assume that the theorem holds when $i = s$. We consider the case where $i = s+1$. By the assumption we have

$$Sec^{(s)}X = \{x \in \mathfrak{so}_n |\ \mathrm{rank}(x) \leq 2k(s+1)\} = \overline{SO_n \cdot \mathfrak{so}_{2k(s+1)}}.$$

Then we have

$$\mathfrak{so}_n \supset Sec^{(s+1)}X = Sec^{(s)}X + X \supset \overline{SO_n \cdot \mathfrak{so}_{2k(s+1)}} + X.$$

Hence if $2k(s+2) \leq n$, we have

$$\overline{SO_n \cdot \mathfrak{so}_{2k(s+1)}} + X \supset \mathfrak{h}_{\mathfrak{so}_{2k(s+2)}}.$$

Otherwise we have

$$\overline{SO_n \cdot \mathfrak{so}_{2k(s+1)}} + X \supset \mathfrak{h}_{\mathfrak{so}_n}.$$

Then by Lemma 5.1 we obtain

$$Sec^{(s+1)}X = \{x \in \mathfrak{so}_n |\ \mathrm{rank}(x) \leq 2k(s+2)\}.$$

This show the assertion of the corollary. □

In the case $[2^{2r}, 1^{n-4r}] \in \mathfrak{so}_n$ we have only partial results. We know the following inequality for any projective variety X,

$$\dim SecX \leq 2\dim X + 1.$$

(See e.g. [12].) Then we have

$$Sec[2^{2r}, 1^{n-4r}] \neq \mathfrak{so}_n,$$

because the data about the dimensions of orbits in [3] shows

$$2\dim[2^{2r}, 1^{n-4r}] < \dim \mathfrak{so}_n$$

for any r.

6. The case of $\mathfrak{sp}_{2n}$

We may realize $\mathfrak{sp}_{2n}$ as the following set of matrices,

$$\mathfrak{sp}_{2n} = \left\{ \begin{pmatrix} A_1 & A_2 \\ A_3 & -{}^tA_1 \end{pmatrix} \mid A_i \in M_n, A_2 \text{ and } A_3 \text{ are symmetric} \right\}.$$

In this realization the Cartan subalgebra $\mathfrak{h}_{\mathfrak{sp}_{2n}}$ is realized as the set of all diagonal matrices. In the case of $\mathfrak{sp}_{2n}$ we shall use similar embeddings as the case of $\mathfrak{sl}_n$ and $\mathfrak{so}_n$.

Proposition 6.1. *Let $\mathfrak{g}$ be $\mathfrak{sp}_{2n}$.*

$$Sec[2^n] = \mathfrak{sp}_{2n}$$

Proof. We prove this proposition by induction on n. When $n = 1$, a nilpotent element $[2]$ is regular. Then by Corollary 3.6 we have $Sec[2] = \mathfrak{sp}_2$. We assume that the assertion holds if $n = k - 1$. We consider $\mathfrak{sp}_{2(k-1)}$ as a Lie subalgebra of all matrices in $\mathfrak{sp}_{2k}$ whose (k,k)- and $(2k,2k)$-th rows and columns are zero. Let x be $[2^k]$ in $\mathfrak{sp}_{2k}$. The closure of nilpotent orbit $[2^k]$ contains the nilpotent orbit $[2^{k-1}, 1^2]$. We can regard a nilpotent element $x = [2^{k-1}, 1^2] \in \mathfrak{sp}_{2k}$ as a nilpotent element $[2^{k-1}] \in \mathfrak{sp}_{2(k-1)} \subset \mathfrak{sp}_{2k}$. Since $Sec[2^{k-1}] = \mathfrak{sp}_{2(k-1)}$ by the assumption, we have

$$\mathfrak{sp}_{2k} \supset Sec[2^k] \supset \mathfrak{sp}_{2(k-1)} \supset \mathfrak{h}_{\mathfrak{sp}_{2(k-1)}}.$$

The Cartan subalgebra $\mathfrak{h}_{\mathfrak{sp}_{2(k-1)}}$ is realized as a subalgebra of all diagonal matrices of $\mathfrak{sp}_{2k}$ whose (k,k)- and $(2k,2k)$-entries are zero. The condition that there is some number i such that $a_{ii} = 0$ is stable under the action of the Weyl group $W_{\mathfrak{sp}_{2k}}$ in $\mathfrak{h}_{\mathfrak{sp}_{2k}}$. The image $p(\mathfrak{h}_{\mathfrak{sp}_{2(k-1)}})$ of $\mathfrak{h}_{\mathfrak{sp}_{2(k-1)}}$ of the adjoint quotient $p : \mathfrak{sp}_{2k} \to \mathfrak{h}_{\mathfrak{sp}_{2k}}/W_{\mathfrak{sp}_{2k}}$ is an irreducible subvariety of codimension 1 in $\mathfrak{h}_{\mathfrak{sp}_{2k}}/W_{\mathfrak{sp}_{2k}}$ and does not contain the image of any element whose determinant is not equal to zero. Let $x = [2^k] \in \mathfrak{sp}_{2k}$ be

$$x = \begin{pmatrix} \mathbf{0} & I_k \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Here I_k is $k \times k$ unit matrix. Take a sum $y = x +^t x \in Sec[2^k]$. Then the determinant of y is non-zero. So $p(\mathfrak{h}_{\mathfrak{sp}_{2(k-1)}})$ never contain $p(y)$. By the irreducibility of $Sec[2^k]$ we obtain $p(Sec[2^k]) = \mathfrak{h}_{\mathfrak{sp}_{2k}}/W_{\mathfrak{sp}_{2k}}$. Then by Corollary 2.10 we obtain

$$Sec[2^k] = \mathfrak{sp}_{2k}.$$

Then the proof is complete. □

Lemma 6.2. *Let $\mathfrak{g}$ be $\mathfrak{sp}_{2n}$.*

$$[2^{n-1},1^2]+[2^{n-1},1^2]\supset[2n-2,2].\ (n\geq 2)$$

Proof. Let's take two nilpotent elements x, y of the type of $[2^{n-1},1^2]$. If n is odd,

$$x=\begin{pmatrix} 0 &&&&&&& \\ & A_{\frac{n-3}{2}} &&& \mathbf{0} &&& \\ && 0 &&&& 1 & 0 \\ &&& 0 &&& 0 & 1 \\ &&&& 0 && 0 & 0 \\ &&&&& -{}^tA_{\frac{n-3}{2}} && \\ & \mathbf{0} &&&&& 0 & \\ &&&&&&& 0 \end{pmatrix}, y=\begin{pmatrix} A_{\frac{n-1}{2}} &&& \mathbf{0} \\ & 0 && \\ && -{}^tA_{\frac{n-1}{2}} & \\ \mathbf{0} &&& 0 \end{pmatrix}.$$

If n is even,

$$x=\begin{pmatrix} 0 &&&&& \\ & A_{\frac{n-2}{2}} &&& \mathbf{0} & \\ && 0 &&& 1 \\ &&& 0 && 0 \\ & \mathbf{0} &&& -{}^tA_{\frac{n-2}{2}} & \\ &&&&& 0 \end{pmatrix}, y=\begin{pmatrix} A_{\frac{n-2}{2}} &&&& \mathbf{0} && \\ & 0 &&&& 1 & 0 \\ && 0 &&& 0 & 0 \\ &&& -{}^tA_{\frac{n-2}{2}} &&& \\ & \mathbf{0} &&&& 0 & \\ &&&&&& 0 \end{pmatrix}.$$

Here we used the same notation as the case of $\mathfrak{so}_N$. Then $x+y$ is a nilpotent element of the type of $[2n-2,2]$. $\square$

Theorem 6.3. *Let x be a nilpotent element in $\mathfrak{sp}_{2n}$ of* $\mathrm{rank}(x)=r$*.*

$$SecX=\{x\in\mathfrak{sp}_{2n}|\ \mathrm{rank}(x)\leq 2r\}.$$

Proof. The minimal nilpotent orbit which is constructed from a nilpotent element with rank r with respect to the closure relation is the nilpotent orbit $[2^r,1^{2n-2r}]$. Then it is enough to prove the assertion of the case of a nilpotent element $[2^r,1^{2n-2r}]$. The inclusion:

$$SecX\subset\{x\in\mathfrak{sp}_{2n}|\ \mathrm{rank}(x)\leq 2r\}$$

is obvious, since $\mathrm{rank}(x)$ is r. Next we shall prove the inverse inclusion. We consider an element y whose rank is $2r$. Take a Jordan normal form of y

$$\begin{pmatrix} D_{i_1}(a_1) &&& \mathbf{0} \\ & D_{i_2}(a_2) && \\ && \ddots & \\ \mathbf{0} &&& D_{i_s}(a_s) \end{pmatrix},\ |a_1|\geq|a_2|\geq\cdots\geq|a_s|.$$

Here we used the same notation as the case of $\mathfrak{so}_N$. Let k be the number such that $a_k\neq 0$ and $a_{k+1}=0$. If $i_{k+1}=\cdots=i_s=1$, we have $x\in\overline{SP_{2n}\cdot\mathfrak{sp}_{2r}}$. Otherwise we put

$$A:=\sum_{l=k+1}^{s}\mathrm{rank}D_{i_l}(a_l)=\sum_{l=k+1}^{s}(i_l-1).$$

Then A is even. Moreover for any nonzero $l_j \in \{a_1, \ldots, a_k\}$ we put

$$b_{l_j} := \sum_{a_t = l_j} i_t.$$

Then we have the set of non-zero complex numbers $\{l_1, \ldots, l_u\}$ and corresponding non negative integers $\{b_{l_1}, \ldots, b_{l_u}\}$. Let z be an element in $\mathfrak{sp}_{2(r+1)} \subset \mathfrak{sp}_{2n}$ whose Jordan normal form is

$$z = \begin{pmatrix} D_{b_{l_1}}(l_1) & & & & & & \\ & \ddots & & & \mathbf{0} & & \\ & & D_{b_{l_u}}(l_u) & & & & \\ & & & D_A(0) & & & \\ & \mathbf{0} & & & D_2(0) & & \\ & & & & & \mathbf{0} & \end{pmatrix}.$$

Then $\overline{SP_{2n} \cdot z}$ contains y. Put $l_1 + \cdots + l_u = 2m$. Under the natural embedding $\mathfrak{sp}_{2m} \times \mathfrak{sp}_{2(n-m)} \subset \mathfrak{sp}_{2n}$ we can regard a nilpotent element $[2^r, 1^{2n-2r}]$ as a nilpotent element $[2^m] \times [2^{r-m}, 1^{2n-2r}] \in \mathfrak{sp}_{2m} \times \mathfrak{sp}_{2(n-m)} \subset \mathfrak{sp}_{2n}$. Then Lemma 6.1 and 6.2 show that $z \in Sec[2^r, 1^{2n-2r}]$. Hence we obtain

$$Sec[2^r, 1^{2n-2r}] \supset \{x \in \mathfrak{sp}_{2n} |\ \mathrm{rank}(x) \leq 2r\}.$$

Then we proved the theorem. $\square$

By Theorem 6.3 and the similar arguments as the argument in the proof of Theorem 5.4 we have the following corollary.

Corollary 6.4. *Let x be a nilpotent element x in $\mathfrak{sp}_{2n}$ with* $\mathrm{rank}(x) = r$.

$$Sec^{(i)} X = \{x \in \mathfrak{sp}_{2n} |\ \mathrm{rank}(x) \leq r(i+1)\}.\ (i \geq 1)$$

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