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On category $\mathcal{O}$ for the rational Cherednik algebra of $G(m, 1, n)$: the almost semisimple case

By

Richard Vale

Abstract

We determine the structure of category $\mathcal{O}$ for the rational Cherednik algebra of the wreath product complex reflection group $G(m, 1, n)$ in the case where the $\mathtt{KZ}$ functor satisfies a condition called *separating simples*. As a consequence, we show that the property of having exactly $N-1$ simple modules, where N is the number of simple modules of $G(m, 1, n)$, determines the Ariki-Koike algebra up to isomorphism.

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1. The rational Cherednik algebra

In this section, we recall the basic facts about the rational Cherednik algebra, before stating our main theorem. Throughout this paper, we fix a complex square root of -1, which we denote $\sqrt{-1}$. The group $W = \mathbb{Z}/m\mathbb{Z} \wr S_n$ may be realised as a complex reflection group as follows. Let $\mathfrak{h}$ be an n–dimensional complex vector space. Let $\{y_1, \ldots, y_n\}$ be the standard basis of $\mathfrak{h}$. With respect to this basis, W may be regarded as the group $G(m, 1, n)$ of $n \times n$ matrices with exactly one nonzero entry in each row and column, the nonzero entries being powers of $\varepsilon := e^{\frac{2\pi\sqrt{-1}}{m}}$. We also let $\{x_1, \ldots, x_n\}$ denote the basis of $\mathfrak{h}^*$ which is dual to $\{y_1, \ldots, y_n\}$.

The complex reflections in W are then the elements s_i^t, $1 \leq i \leq n$, $1 \leq t \leq m-1$ and $\sigma_{ij}^{(k)}$, $1 \leq i < j \leq n$, $0 \leq k \leq m-1$ defined as follows: for $1 \leq i \leq n$

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and $1 \le t \le m-1$, we define

$$\begin{aligned} s_i^t : \mathfrak{h} &\to \mathfrak{h} \\ y_i &\mapsto \varepsilon^t y_i \\ y_j &\mapsto y_j, \qquad j \neq i \end{aligned}$$

and for $1 \le i < j \le n$ and $0 \le k \le m-1$, define

$$\begin{aligned} \sigma_{ij}^{(k)} : \mathfrak{h} &\to \mathfrak{h} \\ y_i &\mapsto \varepsilon^{-k} y_j \\ y_j &\mapsto \varepsilon^k y_i \\ y_r &\mapsto y_r, \qquad r \neq i, j. \end{aligned}$$

Each of these elements has a reflecting hyperplane H. The reflecting hyperplane of s_i^t is $\{v : x_i(v) = 0\}$ while the reflecting hyperplane of $\sigma_{ij}^{(k)}$ is $\{v : x_i(v) = \varepsilon^{-k} x_j(v)\}$. Let $\mathcal{A}$ be the set of these reflecting hyperplanes. For each $H \in \mathcal{A}$, let α_H be a linear functional on $\mathfrak{h}$ with kernel H.

Let $\kappa = (\kappa_{00}, \kappa_0, \kappa_1, \ldots, \kappa_{m-1}) \in \mathbb{C}^m$ be a vector of complex numbers. Let $T(\mathfrak{h} \oplus \mathfrak{h}^*)$ denote the tensor algebra on $\mathfrak{h} \oplus \mathfrak{h}^*$. Then the rational Cherednik algebra $H_\kappa = H_\kappa(W)$ of W is the quotient of the $\mathbb{C}$–algebra $T(\mathfrak{h} \oplus \mathfrak{h}^*) * W$ by the relations $[x_1, x_2] = 0$ for $x_1, x_2 \in \mathfrak{h}^*$, $[y_1, y_2] = 0$ for $y_1, y_2 \in \mathfrak{h}$, together with the commutation relations

$$\begin{aligned} [y, x] = y(x) + \sum_{i=1}^{n} y(x_i) x(y_i) \sum_{j=0}^{m-1} (\kappa_{j+1} - \kappa_j) \sum_{r=0}^{m-1} \varepsilon^{rj} s_i^r \\ + \kappa_{00} \sum_{1 \le i < j \le n} \sum_{k=0}^{m-1} y(x_i - \varepsilon^k x_j) x(y_i - \varepsilon^{-k} y_j) \sigma_{ij}^{(k)} \end{aligned}$$

for all $x \in \mathfrak{h}^*$ and all $y \in \mathfrak{h}$. In this paper, we assume $\kappa_0 = 0$ throughout.

1.1. The Dunkl representation

For a $\mathbb{C}$–algebra A equipped with a W–action, we denote by $A * W$ the skew group algebra of W with coefficients in A. Let $\mathfrak{h}^{\text{reg}} = \mathfrak{h} \setminus (\cup_{H \in \mathcal{A}} H)$ and let $\mathcal{D}(\mathfrak{h}^{\text{reg}})$ denote the ring of differential operators on $\mathfrak{h}^{\text{reg}}$. It is well–known (see for instance, [DO03], [EG02, Proposition 4.5]) that there is an injective homomorphism

$$H_\kappa \hookrightarrow \mathcal{D}(\mathfrak{h}^{\text{reg}}) * W$$

called the Dunkl representation. If $\delta = \prod_{H \in \mathcal{A}} \alpha_H \in \mathbb{C}[\mathfrak{h}]$, then $\mathbb{C}[\mathfrak{h}^{\text{reg}}]$ is the localization $\mathbb{C}[\mathfrak{h}]_\delta$ and the induced map

$$H_\kappa|_{\mathfrak{h}^{\text{reg}}} := H_\kappa \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}[\mathfrak{h}^{\text{reg}}] \to \mathcal{D}(\mathfrak{h}^{\text{reg}}) * W$$

is an isomorphism ([GGOR03, Theorem 5.6]).

1.2. Category $\mathcal{O}$

Following [BEG03a], let $\mathcal{O}$ be the abelian category of finitely-generated H_κ–modules M such that for $P \in \mathbb{C}[\mathfrak{h}^*]^W$, the action of $P - P(0)$ is locally nilpotent. Category $\mathcal{O}$ is by definition a full subcategory of the category of all H_κ–modules, so for objects $X, Y \in \mathcal{O}$ we usually write $\mathrm{Hom}(X, Y)$ for $\mathrm{Hom}_{\mathcal{O}}(X, Y) = \mathrm{Hom}_{H_\kappa}(X, Y)$. Let $\mathsf{Irrep}(W)$ denote the set of isoclasses of simple W–modules. Given $\tau \in \mathsf{Irrep}(W)$, define the standard module $M(\tau)$ by:

$$M(\tau) = H_\kappa \otimes_{(\mathbb{C}[\mathfrak{h}^*]*W)} \tau$$

where τ is made into a $\mathbb{C}[\mathfrak{h}^*] * W$–module by setting for $p \in \mathbb{C}[\mathfrak{h}^*], w \in W$ and $v \in \tau$, $pw \cdot v := p(0)wv$.

In [DO03, Corollary 2.28], it is proved that $M(\tau)$ has a unique simple quotient $L(\tau)$, and [GGOR03, Proposition 2.11, Corollary 2.16] prove that $\{L(\tau) | \tau \in \mathsf{Irrep}(W)\}$ is a complete set of nonisomorphic simple objects of $\mathcal{O}$, and that every object of $\mathcal{O}$ has finite length. Furthermore, it is proved in [GGOR03, Theorem 2.19] that category $\mathcal{O}$ is a *highest weight category* in the sense of [CPS88]. In particular, every simple object $L(\tau)$ of $\mathcal{O}$ has a projective cover $P(\tau)$ and an injective envelope $I(\tau)$, and *BGG reciprocity* holds, that is, $[P(\tau) : L(\sigma)] = [M(\sigma) : L(\tau)]$ for all σ, τ.

1.3. The KZ functor

The group $B_W := \pi_1(\mathfrak{h}^{\mathrm{reg}}/W)$ is called the braid group of W. In [GGOR03, Section 5.2.5], a functor

$$\mathtt{KZ} : \mathcal{O} \to \mathbb{C}B_W - \mathrm{mod}$$

is constructed as follows. If $M \in \mathcal{O}$ then $M|_{\mathfrak{h}^{\mathrm{reg}}} := \mathbb{C}[\mathfrak{h}^{\mathrm{reg}}] \otimes_{\mathbb{C}[\mathfrak{h}]} M$ is a finitely-generated module over $\mathbb{C}[\mathfrak{h}^{\mathrm{reg}}] \otimes_{\mathbb{C}[\mathfrak{h}]} H_\kappa \cong \mathcal{D}(\mathfrak{h}^{\mathrm{reg}}) * W$. In particular, M is a W–equivariant $\mathcal{D}$–module on $\mathfrak{h}^{\mathrm{reg}}$ and hence corresponds to a W–equivariant vector bundle on $\mathfrak{h}^{\mathrm{reg}}$ with a flat connection ∇. Passing to the monodromy of this vector bundle with flat connection gives a representation of the fundamental group $\pi_1(\mathfrak{h}^{\mathrm{reg}}/W, *)$ where $*$ is any choice of basepoint. By definition, $\mathtt{KZ}(M)$ is the monodromy representation of $\pi_1(\mathfrak{h}^{\mathrm{reg}}/W)$ associated to M. For more details, see [GGOR03].

1.4. The Ariki-Koike algebra

By [GGOR03, Section 5.25], the monodromy representation factors through the Hecke algebra $\mathcal{H}$ of W. This is the quotient of $\mathbb{C}B_W$ by relations given in [GGOR03, Section 5.2,5]. From the braid diagram in [BMR98, Table 1], we see that $\mathcal{H}$ is generated by $T_s, T_{t_2}, \ldots, T_{t_m}$ subject to the relations:

$$\begin{aligned} T_sT_{t_2}T_sT_{t_2} - T_{t_2}T_sT_{t_2}T_s &= 0 & & \\ [T_s, T_{t_i}] &= 0 & & i \geq 3 \\ T_{t_i}T_{t_{i+1}}T_{t_i} - T_{t_{i+1}}T_{t_i}T_{t_{i+1}} &= 0 & & 2 \leq i \leq r \\ [T_{t_i}, T_{t_j}] &= 0 & & |i - j| > 1 \end{aligned}$$

$$(T_{t_i}-1)(T_{t_i}+e^{2\pi\sqrt{-1}\kappa_{00}})=0 \qquad\qquad 2\le i\le r$$

$$(T_s-1)\prod_{j=1}^{m-1}(T_s-\varepsilon^{-j}e^{-2\pi\sqrt{-1}\kappa_j})=0$$

We see that $\mathcal{H}$ is the *Ariki-Koike algebra* of [AK94], specialised at the parameters $q=e^{2\pi\sqrt{-1}\kappa_{00}}$, and $u_i=\varepsilon^{-i}e^{-2\pi\sqrt{-1}\kappa_i}$ for $1\le i\le m$, where as before, $\varepsilon=e^{2\pi\sqrt{-1}/m}$. Note in particular that $u_i\neq 0$ for all i. When we refer to the Ariki-Koike algebra in this paper, we always mean the specialised Ariki-Koike algebra in this sense.

Therefore, $\mathtt{KZ}$ gives a functor $\mathtt{KZ}:\mathcal{O}\to\mathcal{H}-\mathrm{mod}$. By [GGOR03, Section 5.3], $\mathtt{KZ}$ is exact, and if $\mathcal{O}_{\mathrm{tor}}$ is the full subcategory of those M in $\mathcal{O}$ such that $M|_{\mathfrak{h}^{\mathrm{reg}}}=0$ then $\mathtt{KZ}$ gives an equivalence $\mathcal{O}/\mathcal{O}_{\mathrm{tor}}\tilde{\to}\mathcal{H}-\mathrm{mod}$ [GGOR03, Theorem 5.14].

1.5. A useful lemma

We will frequently make use of the following result of Ginzburg, Guay, Opdam and Rouquier.

Lemma 1.1 ([GGOR03, Proposition 5.21])**.** *Suppose $L(\tau)|_{\mathfrak{h}^{\mathrm{reg}}}\neq 0$ (equivalently, $\mathtt{KZ}(L(\tau))\neq 0$). Then $L(\tau)$ is a submodule of $M(\mu)$ for some μ.*

2. A condition on KZ

Our aim is to study category $\mathcal{O}$ in the situation where it is, in some sense, as close as possible to being semisimple. We make the following definition:

Definition 2.1. Say $\mathtt{KZ}:\mathcal{O}\to\mathcal{H}-\mathrm{mod}$ *separates simples* if whenever $S\ncong T$ are simple objects of $\mathcal{O}$, then $\mathtt{KZ}(S)\ncong\mathtt{KZ}(T)$.

Now we state the main theorem.

Theorem 2.1. *Suppose $m>1$ and $n>1$ and $\mathtt{KZ}$ separates simples. Then either $\mathcal{O}$ is semisimple, or the following hold*:

1. *There exists a linear character χ of W (ie. a homomorphism $W\to\mathbb{C}^*$) such that $L(\chi)$ is finite-dimensional and all the other simple objects in $\mathcal{O}$ are infinite-dimensional.*
2. *There exists a positive integer r not divisible by m, such that* $\dim L(\chi)=r^n$.
3. *Let $v\in\mathbb{N}$ be the residue of r modulo m, $1\le v\le m-1$. Then there is a representation $\mathfrak{h}_v$ of W with* $\dim\mathfrak{h}_v=\dim\mathfrak{h}$ *such that if $\tau\notin\{\wedge^i\mathfrak{h}_v\otimes\chi:0\le i\le n\}$, then $M(\tau)=L(\tau)$.*
4. *$\mathcal{O}=\mathcal{O}^\wedge\oplus\mathcal{O}^{ss}$ where $\mathcal{O}^\wedge$ is generated by $\{L(\wedge^i\mathfrak{h}_v\otimes\chi):0\le i\le n\}$ and $\mathcal{O}^{ss}$ is a semisimple category generated by the other simple objects.*

5. *The composition multiplicities in* $\mathcal{O}^\wedge$ *are*

$$[M(\wedge^i \mathfrak{h}_v \otimes \chi) : L(\wedge^j \mathfrak{h}_v \otimes \chi)] = \begin{cases} 1 & \text{if } j = i,\ i+1 \\ 0 & \text{otherwise} \end{cases}$$

Before proving Theorem 2.1, we make some remarks. Theorem 2.1 may be viewed as an analogue in the $G(m, 1, n)$ case of [BEG03b, Theorem 1.2, Theorem 1.3]. In fact, in the case $m = 1$, we show in the proof of Corollary 6.1 below that when the KZ functor separates simples, there is a finite-dimensional simple module in category $\mathcal{O}$ and so [BEG03b, Theorem 1.2, Theorem 1.3] apply. In this case, Theorem 2.1 is true with n replaced by $\dim\mathfrak{h} = n-1$. Also, it can be shown by direct calculations that Theorem 2.1 is true in the $n = 1$ case (that is, when W is a cyclic group). Thus, Theorem 2.1 is true for all values of n. In the proof of Theorem 2.1, it is convenient for us to assume that $m, n > 1$.

Although the methods we use for proving Theorem 2.1 are based on those of [BEG03b], we have to use different arguments to get round the problem that in the $G(m, 1, n)$ case, the functor KZ is not known to take standard modules $M(\lambda)$ in $\mathcal{O}$ to the corresponding Specht modules S^λ for $\mathcal{H}$, even on the level of Grothendieck groups. We also have to do some work to calculate the blocks of the Hecke algebra at the parameters that we are interested in.

One reason why Theorem 2.1 is of interest is that it gives a source of examples of choices of κ such that there is a finite-dimensional object in category $\mathcal{O}$, and yet category $\mathcal{O}$ is completely understood.

The proof of Theorem 2.1 proceeds as follows. In Section 3, we recall some facts about the representations of the Ariki-Koike algebra. We use these facts in Section 4.1 to Section 4.4 to prove parts (1) and (2) of Theorem 2.1. Next, between Section 4.5 and Section 4.8, we compute the blocks of the Ariki-Koike algebra in our situation by a combinatorial argument. This enables us to prove parts (3) and (4) of Theorem 2.1. Finally, in Sections 4.9 and 4.10, we prove part (5) of Theorem 2.1.

3. The Ariki-Koike algebra

Let us recall some facts about the Ariki-Koike algebra. This is the algebra $\mathcal{H}$ introduced in Section 1.4, also called the Hecke algebra of W. It depends on parameters $q, u_1, \ldots, u_m \in \mathbb{C}$ and we are only interested in the case where these parameters are all nonzero.

We use the following conventions. For us, a partition of a positive integer n is a sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ with $\sum \lambda_k = n$. A partition λ will be identified with its Young diagram, and we use the non-Francophone convention for Young diagrams. That is, the Young diagram of λ has λ_i boxes in row i, row 1 being the top row. A multipartition of n is an m–tuple $(\lambda^{(1)}, \ldots, \lambda^{(m)})$ where the $\lambda^{(i)}$ are partitions with $\sum |\lambda^{(i)}| = n$. Following the paper [AM00], we may regard a multipartition as a subset of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ by thinking of it as an m–tuple of Young diagrams. A *node* is any box of λ. More generally, a node will be any element of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$.

It has been shown (see [Mat99]) that for each multipartition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(m)})$ of n, there is a *Specht module* S^λ for $\mathcal{H}$. Each S^λ has a quotient D^λ which is either 0 or simple. Let Π_n^m be the set of multipartitions of n with m parts. The set $\{D^\lambda | D^\lambda \neq 0,\ \lambda \in \Pi_n^m\}$ is a complete set of nonisomorphic simple $\mathcal{H}$–modules. We will need a parametrisation of this set. There are two different parametrisations, depending on whether $q = 1$ or $q \neq 1$.

3.1. Parametrisation of the simple modules

If $q = 1$ then [Mat98, Theorem 3.7] states that $D^\lambda \neq 0$ if and only if $\lambda^{(s)} = \varnothing$ whenever $s < t$ and $u_s = u_t$. If $q \neq 1$ then the description, due to Ariki and stated in [Mat04, Theorem 3.24] is more complicated. The nonzero D^λ are in bijection with the set of *Kleshchev multipartitions*, which we now describe.

Given a multipartition λ, the *residue* of a node x in row i and column j of $\lambda^{(k)}$ is defined to be $u_k q^{j-i}$. A node x in λ with residue a is called a *removable a–node* if $\lambda \setminus \{x\}$ is a multipartition. A node x not in λ with residue a is called an *addable a–node* if $\lambda \cup \{x\}$ is a multipartition.

Say a node $y \in \lambda^{(\ell)}$ is *below* a node $x \in \lambda^{(k)}$ if either $\ell > k$, or $\ell = k$ and y is in a lower row than x.

A removable a–node x is called *normal* if whenever x' is an addable a–node below x then there are more removable a–nodes between x and x' than there are addable a–nodes. The highest normal a–node in λ is called the *good a–node*.

The set of Kleshchev multipartitions is defined inductively as follows: $\varnothing$ is Kleshchev, and otherwise λ is Kleshchev if and only if there is some $a \in \mathbb{C}$ and a good a–node $x \in \lambda$ such that $\lambda \setminus \{x\}$ is Kleshchev. More details plus examples may be found in the introduction to the paper [AM00].

3.2. Blocks of $\mathcal{H}$

Finally we need a description of the blocks of $\mathcal{H}$. This is given in [LM06, Corollary 2.16]. Recall that the Specht modules are partitioned into blocks as follows: two Specht modules S^λ and S^μ are in the same block if and only if there is a sequence $S^{\lambda_1}, S^{\lambda_2}, \dots, S^{\lambda_t}$ with $S^{\lambda_1} = S^\lambda$, $S^{\lambda_t} = S^\mu$ and such that S^{λ_i} and $S^{\lambda_{i+1}}$ have a common composition factor for all i. Define the *content* $\mathrm{cont}(\lambda)$ of a multipartition λ to be the multiset of residues of λ, ie. the set of residues counted according to multiplicity. Then for $q \neq 1$, two Specht modules S^λ and S^μ are in the same block if and only if $\mathrm{cont}(\lambda) = \mathrm{cont}(\mu)$.

4. Proof of Theorem 2.1

4.1.

To begin the proof, suppose KZ separates simples. If $\mathcal{O}$ is not semisimple then we claim there exists $S \in \mathcal{O}$ with $\mathsf{KZ}(S) = 0$. Indeed, if $\mathsf{KZ}(S) \neq 0$ for all simple objects $S \in \mathcal{O}$ then $\mathcal{H}$ has $|\mathsf{Irrep}(W)|$ simple modules, but it is well-known that this implies that $\mathcal{H}$ is semisimple. We give here a proof using the

Cherednik algebra.

Lemma 4.1. *Suppose $\mathcal{H}$ has $|\mathsf{Irrep}(W)|$ simple modules. Then $\mathcal{H}$ is semisimple.*

Proof. By [GGOR03, Theorem 5.14], every object of $\mathcal{H}-\mathrm{mod}$ is the image of an object of $\mathcal{O}$ under $\mathtt{KZ}$. If S is a simple $\mathcal{H}$–module and $\mathtt{KZ}(N)=S$, then using induction on a composition series of N and exactness of $\mathtt{KZ}$, we may assume that N is simple. Thus, each simple $\mathcal{H}$–module is the image of a simple object of $\mathcal{O}$ under $\mathtt{KZ}$. Therefore, the category $\mathcal{O}_{\mathrm{tor}}\subset\mathcal{O}$ is 0. Therefore, $\mathtt{KZ}$ induces an equivalence $\mathcal{O}\to\mathcal{H}-\mathrm{mod}$. We show that $\mathcal{O}$ is a semisimple category. By [DO03, (32)], there is an ordering $\leqslant$ on $\mathsf{Irrep}(W)$ such that $[M(\tau):L(\sigma)]\neq 0$ implies $\tau\leqslant\sigma$. By Lemma 1.5, if $L(\sigma)|_{\mathfrak{h}^{\mathrm{reg}}}\neq 0$ then $L(\sigma)\subset M(\tau)$ for some τ. Combining this fact with induction on the ordering $\leqslant$ yields $M(\tau)=L(\tau)$ for all τ. But it is observed in [BEG03a, Remark following Lemma 2.12] that $M(\tau)=L(\tau)$ for all τ if and only if $\mathcal{O}$ is semisimple. Since there is an equivalence of categories $\mathcal{O}\cong\mathcal{H}-\mathrm{mod}$, $\mathcal{H}-\mathrm{mod}$ is a semisimple category and so $\mathcal{H}$ is a semisimple algebra. $\square$

Remark 1. Note that the above proof works for any complex reflection group W, where $\mathcal{H}$ is the Hecke algebra of W as defined in [GGOR03, Section 5.2.5].

It follows that if $\mathtt{KZ}(S)\neq 0$ for all simples S, then $\mathcal{H}-\mathrm{mod}$ is a semisimple category. But also $S|_{\mathfrak{h}^{\mathrm{reg}}}\neq 0$ for all simples $S\in\mathcal{O}$, and therefore $\mathcal{O}_{\mathrm{tor}}=0$. Therefore, $\mathtt{KZ}$ induces an equivalence $\mathcal{O}\to\mathcal{H}-\mathrm{mod}$ and it follows that $\mathcal{O}$ is semisimple.

Therefore we have shown that if $\mathcal{O}$ is not semisimple then there is some simple $S\in\mathcal{O}$ with $\mathtt{KZ}(S)=0$, and $\mathtt{KZ}(T)\neq 0$ for all simples $T\not\cong S$ by our assumption on $\mathtt{KZ}$. Since $\mathtt{KZ}$ separates simples, we also have that $\#\{\mathtt{KZ}(T) : T\,\mathrm{simple}, T\not\cong S\}=|\mathsf{Irrep}(W)|-1$. Furthermore, if T is simple then so is $\mathtt{KZ}(T)$, because $\mathtt{KZ}$ induces an equivalence $\mathcal{O}/\mathcal{O}_{\mathrm{tor}}\to\mathcal{H}-\mathrm{mod}$, and the localisation to $\mathfrak{h}^{\mathrm{reg}}$ preserves simple objects. Therefore, $\mathcal{H}$ has exactly $|\mathsf{Irrep}(W)|-1$ simple modules.

Next, we show that $q\neq 1$. Suppose $q=1$. Then by Section 3.1, since $\mathcal{H}$ is not semisimple, there must be some $s<t$ with $u_s=u_t$. Under the assumption that $n>1$, there are at least three multipartitions λ with $\lambda^{(s)}\neq\varnothing$. Hence, there are at least three D^λ which are zero and so $\mathcal{H}$ cannot have $|\mathsf{Irrep}(W)|-1$ simple modules and therefore $q\neq 1$. Therefore, the simple $\mathcal{H}$–modules are in bijection with Kleshchev multipartitions.

4.2.

Ariki's semisimplicity criterion [Ari94, Main Theorem] states that

$$[n]_q!\prod_{\substack{i<j\\ -n<c<n}}(u_i-q^c u_j)=0.$$

Therefore, either there are i, j, c with $u_i = q^c u_j$, or else $[n]_q! = 0$. We show $[n]_q! \neq 0$. Suppose that $[n]_q! = 0$. Then there is a k, $1 \leq k \leq n$ with $q^k = 1$ and $q^\ell \neq 1$, $0 < \ell < k$. Since $q \neq 1$, the simple $\mathcal{H}$–modules are in bijection with Kleshchev multipartitions. Let ρ_k be the partition of k with one part, ie. the Young diagram of ρ_k is a row of k boxes. Then ρ_k is not Kleshchev, because the only removable node of ρ_k, call it μ, cannot be good, because it is not normal. Indeed, the node labelled λ in the diagram below is an addable node below μ with the same residue as μ, and there are no removable nodes between them.

$$\overbrace{\begin{array}{|c|c|c|c|c|}\hline & & & & \mu \\ \hline \lambda \\ \cline{1-1} \end{array}}^{k \text{ boxes in row}} .$$

Hence, ρ_k is not Kleshchev and therefore ρ_n, a row of n boxes, is not Kleshchev. We may therefore define multipartitions $\lambda_1 = (\rho_n, \varnothing, \ldots, \varnothing)$ and $\lambda_2 = (\varnothing, \rho_n, \varnothing, \ldots, \varnothing)$, neither of which is Kleshchev (here we use the hypothesis that $m > 1$). This contradicts the fact that there is only one non-Kleshchev multipartition, and so $[n]_q! \neq 0$.

Therefore, there exist integers $1 \leq i, j \leq n$ and $-n < c < n$ such that $u_i = q^c u_j$. Writing what this means in terms of the κ_i, we get

$$m(\kappa_j - \kappa_i) - mc\kappa_{00} - (i - j) \in m\mathbb{Z}. \tag{4.1}$$

The next step is to show that $|c| = n - 1$.

Redefining c if necessary, we have that there are $i < j$ with $q^c u_i = u_j$. Either $c \geq 0$ or $c \leq 0$. Consider the case $c \geq 0$. In this case, let ρ_{c+1} be a row of $c + 1$ boxes, and take a multipartition τ with ρ_{c+1} as its i^{th} part and $\varnothing$ everywhere else. If $c < n - 1$ then consider two multipartitions defined as follows: λ is the multipartition of n whose i^{th} part is ρ_n and μ is the multipartition of n whose i^{th} part is

$$\overbrace{\begin{array}{|c|c|c|c|c|}\hline & & & & \\ \hline \\ \cline{1-1} \end{array}}^{n-1 \text{ boxes in row}} .$$

Then τ is not Kleshchev, and so λ is clearly not Kleshchev. Also, μ is not Kleshchev, essentially because $\mu \supset \tau$ (note that, even after some nodes have been removed from μ, the node at the right hand end of τ can never be a good node, since we have established that $q^{c+1} \neq 1$). Hence there are two non-Kleshchev multipartitions, which contradicts our hypothesis that $\mathcal{H}$ has $|\mathsf{Irrep}(W)| - 1$ simple modules. Therefore $c = n - 1$.

In the $c \leq 0$ case, we take γ_{c+1} to be a column of $-c + 1$ boxes, and do a similar argument to show that $c = -(n - 1)$.

4.3.

The above argument shows that the multiplicative order of q must be at least $2n - 1$. Indeed, suppose $q^{n+a} = 1$ where a is a nonnegative integer. Then if $q^{n-1}u_i = u_j$ for some i, j, we get $q^{-a-1}u_i = u_j$. But the above argument in the $c \le 0$ case shows that $-a-1 \le -n$ or else we would have more than one non-Kleshchev multipartition.

4.4.
Now we may rewrite the condition (4.1) on the parameters as

$$m(\kappa_j - \kappa_i) + (-1)^a m(n-1)\kappa_{00} = (i-j) + mt$$

for some $a \in \{0,1\}$ and some $t \in \mathbb{Z}$. Note that $(i-j)+mt$ cannot be zero because $1 \le i, j \le m$. If it is positive, multiply through by -1 (possibly interchanging the roles of i and j, and changing a), in order to assume that $(i-j)+mt < 0$. Now we do a so-called twist. Consider the multiplicative character of W which sends $\sigma_{rs}^{(\ell)}$ to $(-1)^a$ for all r, s, ℓ, and which sends s_k to ε^{-i}. Explicitly checking with a set of generators and relations of W shows that this is a well-defined character of W. Now by [GGOR03, Section 5.4.1], we have an isomorphism of Cherednik algebras $\psi : H_\kappa \to H_{\kappa'}$ where $\kappa'_{00} = (-1)^a \kappa_{00}$ and $\kappa'_u = \kappa_{u+i} - \kappa_i$ for each u. (These equations for κ'_u follow from writing down the generators and relations for $H_{\kappa'}$.) The twist ψ induces an auotequivalence of category $\mathcal{O}$ which preserves the dimension of the objects [GGOR03, Section 5.4.1]. Our new parameters satisfy

$$m\kappa'_{j-i} + m(n-1)\kappa'_{00} = (i-j) + mt < 0.$$

Now we are in a position where we can use [CE03, Section 4.1]. Translating our parameters into the language of [CE03], we get

$$m(n-1)k + 2\sum_{a=1}^{m-1} c_a \frac{1-\varepsilon^{-av}}{1-\varepsilon^{-a}} = r$$

where $r = (j-i) - mt$ is a positive integer of the form $(p-1)m + v$ for some nonnegative integer p and some $1 \le v \le m-1$. Then from [CE03] we have the module $\tilde{Y}_c$ which is a quotient of $M(\mathsf{triv})$. Furthermore, since $[n]_q! \ne 0$, we may apply [CE03, Theorem 4.3] to conclude that $\tilde{Y}_c$ is finite-dimensional. Therefore, $L(\mathsf{triv})$ is finite-dimensional. By [GGOR03, Section 5.4.1], twisting by ψ sends $L(\chi)$ to $L(\mathsf{triv})$ for some linear character χ of W. Furthermore, $\dim L(\chi) = \dim L(\mathsf{triv}) = r^n$ by [CE03, Theorem 2.3 (iii)]. Since $L(\chi)$ is finite-dimensional, $\mathtt{KZ}(L(\chi)) = 0$, and therefore $\mathtt{KZ}(L(\tau)) \ne 0$ for $\tau \ne \chi$, by our assumption that $\mathtt{KZ}$ separates simples. Therefore $L(\tau)$ is infinite-dimensional if $\tau \ne \chi$. We have proved parts (1) and (2) of Theorem 2.1.

4.5. Blocks
To proceed further, it is necessary to calculate the blocks of the Hecke algebra.

4.6. *Standing assumption*

We have parameters q and $u_1, \dots, u_m$ for the Hecke algebra. We are assuming that there is exactly 1 non-Kleshchev multipartition, and we have already shown that $q^{n-1}u_i = u_j$ for some $i \neq j$.

First, we prove the following lemma.

Lemma 4.2. *If $k \neq i, j$ then for each $\ell \neq k$, we have $u_k/u_\ell \neq q^c$ for any $-n < c < n$.*

Proof. Suppose $u_k = q^c u_\ell$. If $\ell \neq i, j$ then it would follow from the earlier calculations that there is another non-Kleshchev multipartition, so we need only consider the case where $\ell = i$ or $\ell = j$. Suppose $i < j$. If $\ell = i$ then suppose there is $-n < c < n$ with $u_k = q^c u_i$, and $u_j = q^{n-1}u_i$. If $c < 0$ then considering a multipartition whose only nontrivial part is a column γ_n in the i^{th} position, and a multipartition whose only nontrivial part is a row ρ_n in the i^{th} position, we have that there is more than one non-Kleshchev multipartition. On the other hand, if $c \geq 0$ then $u_k = q^c u_i = q^{c-(n-1)}u_j$ and hence there exists a non-Kleshchev multipartition which is $\varnothing$ except in the j^{th} position, and one which is $\varnothing$ except in the i^{th} position. Similarly, if $\ell = j$, we reach the same conclusion, and so such a c cannot exist. Similar arguments deal with the $i > j$ case. $\square$

Recall from Section 3.2 that if α and β are multipartitions then the Specht modules S^α and S^β belong to the same block if and only if $\operatorname{cont}(\alpha) = \operatorname{cont}(\beta)$. The next lemma is needed to study the content of a multipartition.

Lemma 4.3. *Under the assumptions of Section 4.6, let $\alpha = (\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(m)})$ be a multipartition of n. Then $\operatorname{cont}(\alpha^{(r)}) \cap \operatorname{cont}(\alpha^{(s)}) = \varnothing$ for all $r \neq s$.*

Proof. By Lemma 4.2 and our assumption that $q^{n-1}u_i = u_j$, we get that for all r, s, $u_r/u_s \neq q^c$ for any $-(n-1) < c < n-1$. Now, if the residue of some node x in $\alpha^{(r)}$ is equal to the residue of some other node y in $\alpha^{(s)}$, then

$$u_r q^{\operatorname{col}(x)-\operatorname{row}(x)} = u_s q^{\operatorname{col}(y)-\operatorname{row}(y)}.$$

But if $t := \operatorname{col}(x) + \operatorname{row}(y) - \operatorname{row}(x) - \operatorname{col}(y)$ then $u_s/u_r = q^t$ but $t \leqslant n-2$ and $t \geq -(n-2)$, a contradiction. $\square$

The next lemma is useful in determining a multipartition from its content.

Lemma 4.4. *Under the assumptions of Section 4.6, if α and β are multipartitions of n and $1 \leq k \leq m$, then $\operatorname{cont}(\alpha^{(k)}) = \operatorname{cont}(\beta^{(k)})$ implies $\alpha^{(k)} = \beta^{(k)}$.*

Proof. We show that if two nodes of $\alpha^{(k)}$ have the same residue, then they lie on the same diagonal. It will follow that the multiplicity of a residue in $\operatorname{cont}(\alpha)$ is equal to the length of the corresponding diagonal of α. The same is true of β. Thus under the hypothesis, the Young diagrams α and β have diagonals of the same lengths, thus they are equal.

Suppose then that nodes (i,j) and $(i^{'},j^{'})$ in $\alpha^{(k)}$ have the same residue. Then $u_k q^{j-i} = u_k q^{j^{'}-i^{'}}$. Thus $q^{j-i-j^{'}+i^{'}} = 1$ and therefore if $j-i \neq j^{'}-i^{'}$ then either $z := j-i-j^{'}+i^{'} \geq n$ or $z \leq -n$. But $j+i^{'}, j^{'}+i \leq n+1$ and so z cannot be either greater than n or less than $-n$. Therefore, $z=0$ and $j-i = j^{'}-i^{'}$. In other words, (i,j) and $(i^{'},j^{'})$ lie on the same diagonal. □

4.7.

We are finally in a position to calculate the blocks of the Hecke algebra. In order to determine the blocks of $\mathcal{H}$, we first note that if ρ_a denotes a row of length a and γ_b a column of length b, then we may define a multipartition λ_a to have ρ_a in the i^{th} place and γ_{n-a} in the j^{th} place. For example, if $m=3$, $n=3$, $i=3$, $j=2$ then

$$\lambda_0 = \left(\varnothing \quad \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} \quad \varnothing \right), \lambda_1 = \left(\varnothing \quad \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \quad \begin{array}{|c|}\hline \\ \hline \end{array} \right),$$

$$\lambda_2 = \left(\varnothing \quad \begin{array}{|c|}\hline \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline & \\ \hline \end{array} \right), \lambda_3 = \left(\varnothing \quad \varnothing \quad \begin{array}{|c|c|c|}\hline & & \\ \hline \end{array} \right).$$

Then if $q^{n-1}u_i = u_j$, then $\text{cont}(\lambda_a) = \{u_i q^x | 0 \leq x \leq n-1\}$ and hence all the λ_a belong to the same block. It remains to show that if α, β are multipartitions and one of them is not of the form λ_a, then they belong to distinct blocks.

Now we suppose that we have two multipartitions $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(m)})$ and $\beta = (\beta^{(1)}, \ldots, \beta^{(m)})$ and $\text{cont}(\alpha) = \text{cont}(\beta)$. We will show that if $k \neq i,j$ then $\alpha^{(k)} = \beta^{(k)}$.

Lemma 4.5. *Let $k \neq i,j$. If $x \in \text{cont}(\alpha^{(k)})$ then $x \notin \cup_{\ell \neq k}\text{cont}(\beta^{(\ell)})$.*

Proof. There is an integer b with $-n+1 \leq b \leq n-1$ such that $x = u_k q^b$. We consider the cases $b \geq 0$ and $b \leq 0$ separately. In the case $b \geq 0$, we now prove by induction that $x \notin \text{cont}(\beta^{(\ell)})$ for any $\ell \neq k$. The proof for $b \leq 0$ is very similar, so we omit it.

For the base step, suppose $b=0$. Then $u_k \in \text{cont}(\alpha^{(k)})$. Hence u_k is a residue of β. If $u_k \in \text{cont}(\beta^{(\ell)})$ where $\ell \neq k$ then $u_k = u_\ell q^{c-r}$ for some column c and row r of $\beta^{(\ell)}$. But clearly $-n < c-r < n$ which contradicts Lemma 4.2. Therefore $u_k \notin \cup_{\ell \neq k}\beta^{(\ell)}$ and so $u_k \in \text{cont}(\beta^{(k)})$.

Now we do the inductive step. Suppose $b > 0$. Suppose $u_k q^b$ is a residue of $\beta^{(\ell)}$ with $\ell \neq k$. Then $u_k q^b = u_\ell q^{c-r}$ for some c,r. So $u_k/u_\ell = q^{c-r-b}$. Since $c-r < n$ and $b > 0$, we have $c-r-b < n$. By Lemma 4.2, $c-r-b \leq -n$. Therefore, $r \geq n+c-b \geq n+1-b$. But $\beta^{(\ell)}$ contains at least r boxes, by definition of r. Therefore, $|\beta^{(\ell)}| \geq n+1-b$.

Next, we note that since $u_k q^b$ is the residue of a node in $\alpha^{(k)}$, this node must lie on the diagonal containing $(b+1,1)$. Therefore, there are at least $b+1$ boxes in the first row of $\alpha^{(k)}$ and hence there is a node in the first row of $\alpha^{(k)}$ with residue $u_k q^{b-1}$. By induction on b, this is also a residue of $\beta^{(k)}$. It follows that there is a box in column b and row 1 of $\beta^{(k)}$. Therefore, $|\beta^{(k)}| \geq b$. So $|\beta| \geq |\beta^{(k)}| + |\beta^{(\ell)}| \geq n+1$, a contradiction. $\square$

It follows from Lemma 4.5 that if $\mathrm{cont}(\alpha) = \mathrm{cont}(\beta)$ then $\mathrm{cont}(\alpha^{(k)}) = \mathrm{cont}(\beta^{(k)})$ for all $k \neq i, j$. Then applying Lemma 4.4, we get $\alpha^{(k)} = \beta^{(k)}$. It remains to deal with $\alpha^{(i)}$ and $\alpha^{(j)}$. The proof of this case will be very similar to Lemma 4.5, but slightly more complicated.

Given multipartitions $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(m)})$ and $\beta = (\beta^{(1)}, \ldots, \beta^{(m)})$, with $\mathrm{cont}(\alpha) = \mathrm{cont}(\beta)$, let a_1 be the length of the first row of $\alpha^{(i)}$ and a_2 be the length of the first column of $\alpha^{(j)}$ and define b_1, b_2 similarly for β. First we prove a technical lemma.

Lemma 4.6. *Under our assumptions of Section* 4.6*, suppose $a_1 + a_2 < n$. Then $u_i q^{a_1} \notin \mathrm{cont}(\alpha)$.*

Proof. First, we show that $u_i q^{a_1} \notin \mathrm{cont}(\alpha^{(k)})$ when $k \neq i, j$. So let $k \neq i, j$ and suppose there is a node of $\alpha^{(k)}$ with residue $u_i q^{a_1}$. Say this node lies in column c and row r of $\alpha^{(k)}$. Then $u_i q^{a_1} = u_k q^{c-r}$. So $u_i/u_k = q^{a_1-(c-r)}$. We show that $a_1 - (c-r)$ lies between $-n$ and n. If $a_1 - (c-r) \geq n$ then $c + n \leq r + a_1 \leq n$, a contradiction. While if $a_1 - (c-r) \leq -n$ then $c \geq n + a_1 + r \geq n+1$, a contradiction. Therefore, $-n < a_1 - (c-r) < n$, which violates Lemma 4.2. Hence, $u_i q^{a_1}$ is not a residue of $\alpha^{(k)}$.

Next, we show that $u_i q^{a_1}$ is not a residue of $\alpha^{(i)}$. If it is, then there is a node in column c and row r of $\alpha^{(i)}$ whose residue is $u_i q^{a_1} = u_i q^{c-r}$. Therefore $q^{a_1-(c-r)} = 1$. Then by Section 4.3, if $a_1 - (c-r) \neq 0$ then either $a_1 - (c-r) \geq 2n-1$ or $a_1 - (c-r) \leq -(2n-1)$. If $a_1 - (c-r) \leq -(2n-1)$ then $2n \leq a_1 + r - 1 + 2n \leq c$, which is impossible. If $a_1 - (c-r) \geq 2n-1$ then $c + 2n \leq a_1 + r + 1 \leq n+2$, which is impossible if $n > 1$. Therefore, $a_1 = c - r$. But $c \leq a_1$ and $r \geq 1$, so this is also impossible. Therefore, $u_i q^{a_1}$ cannot be a residue of $\alpha^{(i)}$.

The argument that $u_i q^{a_1}$ is not a residue of $\alpha^{(j)}$ is very similar. We use the fact that $a_1 < n - a_2$. $\square$

The claim of Section 4.7 follows from the next lemma. We use the same notation as Section 4.6.

Lemma 4.7. *Under the assumptions of Section* 4.6*, if $a_1 + a_2 < n$ then if $x \in \mathrm{cont}(\alpha^{(i)})$ then $x \notin \mathrm{cont}(\beta^{(j)})$.*

Proof. By Lemma 4.5, $\mathrm{cont}(\alpha^{(k)}) = \mathrm{cont}(\beta^{(k)})$ for $k \neq i, j$. Therefore, by Lemma 4.3, we get $\mathrm{cont}(\alpha^{(i)}) \cup \mathrm{cont}(\alpha^{(j)}) = \mathrm{cont}(\beta^{(i)}) \cup \mathrm{cont}(\beta^{(j)})$. This is a disjoint union.

If $x \in \mathrm{cont}(\alpha^{(i)})$ then $x = u_i q^b$ for some b with $-n+1 \leq b \leq n-1$. As in the proof of Lemma 4.5, we consider the cases $b \geq 0$ and $b \leq 0$ separately. We give the proof only for the $b \geq 0$ case. The proof is by induction on b.

For the base step, if $b = 0$ then u_i is a residue of $\alpha^{(i)}$. If this is a residue of $\beta^{(j)}$, then it has the form $u_i = u_i q^{n-1} q^{c-r}$ for some c, r, and so $q^{n-1+c-r} = 1$.

Now, $n-1+c-r \geq 0$. If $n-1+c-r \geq 2n-1$ then $c-r \geq n$ which is impossible. Therefore $n-1+c-r=0$. Hence, $c=1, r=n$, and $\beta^{(j)}$ must be a column of n boxes. But then $\mathrm{cont}(\beta^{(j)}) = \{u_i q^{n-1}, u_i q^{n-1}, \dots, u_i q, u_i\}$. Since $0 \leq a_1 < n$, we have $u_i q^{a_1} \in \mathrm{cont}(\beta^{(j)}) = \mathrm{cont}(\beta) = \mathrm{cont}(\alpha)$, which contradicts Lemma 4.6. Therefore u_i must be a residue of $\beta^{(i)}$, which proves the base step.

For the inductive step, suppose $b > 0$ and $u_i q^b$ is a residue of $\alpha^{(i)}$. If $u_i q^b$ is a residue of a node in column c and row r of $\beta^{(j)}$, then $u_i q^b = u_i q^{n-1} q^{c-r}$, and so $q^{c-r+n-1-b} = 1$. Since $c-r<n$ and $b>0$, we have $c-r-b<n$. Therefore $c-r-b+n-1 < 2n-1$. Therefore, either $c-r-b+n-1=0$ or $c-r-b+n-1 \leq -(2n-1)$. If the latter holds then $c+3n \leq r+b+2 \leq 2n+1$ since we may take $b \leq n-1$. Hence $1+n \leq c+n \leq 1$, a contradiction. We therefore get $c-r-b+n-1=0$, and so $r \geq n-b$. But $\beta^{(j)}$ has at least r nodes. Therefore, $|\beta^{(j)}| \geq n-b$ and has at least $n-b$ rows. But since $u_i q^b \in \mathrm{cont}(\alpha^{(i)})$, we get $u_i q^{b-1} \in \mathrm{cont}(\alpha^{(i)})$, as in the proof of Lemma 4.5. By induction on b, $u_i q^{b-1} \in \mathrm{cont}(\beta^{(i)})$. So, as in the proof of Lemma 4.5, there is a box in row 1 and column b of $\beta^{(i)}$. Therefore, $|\beta^{(i)}| \geq b$ and $\beta^{(i)}$ has at least b columns. Therefore $\beta = \lambda_b$ in the notation of Section 4.7. Therefore $\mathrm{cont}(\beta) = \{u_i, qu_i, \dots, q^{n-1}u_i\}$, and hence $u_i q^{a_1} \in \mathrm{cont}(\beta) = \mathrm{cont}(\alpha)$. This contradicts Lemma 4.6. Therefore, $u_i q^b$ must be a residue of $\beta^{(i)}$ and this proves the inductive step. $\square$

Now suppose we have a multipartition α not of the form λ_a. Suppose $\beta \neq \alpha$. We show that $\mathrm{cont}(\alpha) \neq \mathrm{cont}(\beta)$. Indeed, if $\beta \neq \lambda_b$ for any b, then by Lemmas 4.5 and 4.7, $\mathrm{cont}(\alpha^{(k)}) = \mathrm{cont}(\beta^{(k)})$ for all k. Therefore, by Lemma 4.4, $\alpha^{(k)} = \beta^{(k)}$ for all k, so $\alpha = \beta$, a contradiction. On the other hand, if $\beta = \lambda_b$ for some b, then $u_i q^{a_1} \in \mathrm{cont}(\beta) \setminus \mathrm{cont}(\alpha)$ by Lemma 4.6, and hence $\mathrm{cont}(\alpha) \neq \mathrm{cont}(\beta)$.

Therefore, S^α is the unique Specht module in its block. Furthermore, $\{S^{\lambda_a} | 0 \leq a \leq n\}$ form a block, by the same reasoning.

4.8.

We get that there is one block of the Hecke algebra containing $n+1$ of the Specht modules, and all the other blocks are singletons. Hence, there are $|\mathsf{Irrep}(W)| - n$ blocks. By [GGOR03, Corollary 5.18], the blocks of $\mathcal{O}$ are in bijection with blocks of $\mathcal{H}$ and hence $\mathcal{O}$ also has $|\mathsf{Irrep}(W)| - n$ blocks. We work in the category $\mathcal{O}(H_{\kappa'})$. Now by [CE03, Theorem 2.3], there is a representation $\mathfrak{h}_v$ of W with $\dim \mathfrak{h}_v = \dim \mathfrak{h}$ such that there is a BGG-resolution of $\tilde{Y}_c$, ie. an exact sequence

$$0 \leftarrow \tilde{Y}_c \leftarrow M(\mathsf{triv}) \leftarrow M(\mathfrak{h}_v) \leftarrow \cdots \leftarrow M(\wedge^n \mathfrak{h}_v) \leftarrow 0. \tag{4.2}$$

As the classes $[M(\tau)]$ form a basis of the Grothendieck group $K_0(\mathcal{O})$, none of the maps in this sequence can be zero, and hence all the $L(\wedge^i \mathfrak{h}_v)$ belong to the same block. There are $n+1$ simples in this block and hence by counting we see that all the other blocks must be singletons. Using the fact that simple objects in $\mathcal{O}$ have no self-extensions ([BEG03b, Proposition 1.12]), we get that these blocks are semisimple. Translating back to category $\mathcal{O}(H_\kappa)$, we get parts (3) and (4) of Theorem 2.1.

In order to prove part (5) of Theorem 2.1, we require the following lemma.

Lemma 4.8. *The module $\tilde{Y}_c$ is isomorphic to $L(\mathsf{triv})$.*

Proof. Since $\tilde{Y}_c$ is finite-dimensional, its only composition factor can be $L(\mathsf{triv})$, by part (1) of Theorem 2.1. But $M(\mathsf{triv}) \twoheadrightarrow \tilde{Y}_c$ and $[M(\mathsf{triv}) : L(\mathsf{triv})] = 1$. □

4.9.

It remains to compute the composition multiplicities in the one nontrivial block $\mathcal{O}^\wedge$. Again we work in the category $\mathcal{O}(H_{\kappa'})$. Lemma 1.5 tells us that each $L(\wedge^i \mathfrak{h}_v)$, $i > 0$ is a submodule of a standard module. Write $L_i = L(\wedge^i \mathfrak{h}_v)$ and $M_i = M(\wedge^i \mathfrak{h}_v)$. Let R_i be the radical of M_i. We cannot have a nonzero map $L_i \to M_j$ if $j > i$ by [DO03, Section 2.5 (32)][*1] and therefore L_1 is a submodule either of M_0 or M_1. It cannot be a submodule of M_1 because $[M_1 : L_1] = 1$, and we therefore have $L_1 \hookrightarrow M_0$. Therefore, $L_1 \hookrightarrow R_0$. But by Lemma 4.8, we have $\tilde{Y}_c \cong L(\mathsf{triv})$. Hence $\tilde{Y}_c$ is simple and it follows that $R_0 = \ker(M_0 \to \tilde{Y}_c) = \mathrm{Im}(M_1 \to M_0)$ is a quotient of M_1. Hence $[R_0 : L_1] = 1$. But R_0 is a quotient of M_1, so R_0 has L_1 both as a submodule and as a quotient. Therefore, $R_0 = L_1$.

4.10.

We have shown that the composition factors of M_0 are L_0 and L_1. To conclude the argument, we show by induction that the composition factors of M_i are L_i and L_{i+1}. Consider first L_{i+1}. Then L_{i+1} is a submodule of some M_j. We cannot have $j \geq i+1$, and by induction, we cannot have $j < i$. Hence, L_{i+1} is a submodule of M_i and so $L_{i+1} \hookrightarrow R_i$. Now $R_i = \ker(M_i \to M_{i-1})$ by induction and so R_i is a quotient of M_{i+1}. Therefore, $[R_i : L_{i+1}] = 1$. If there was a $j > i+1$ with $[R_i : L_j] \neq 0$ then we would have that for some $j > i+1$, L_j would be a quotient of R_i and hence a quotient of M_{i+1}, contradicting the fact that M_{i+1} has a unique simple quotient. Therefore, $R_i = L_{i+1}$ and we are done. This proves part (5) of Theorem 2.1. □

5. Characterisations of separating simples

Now that we have completed the proof of Theorem 2.1, let us turn our attention to the question of when `KZ` separates simples.

Theorem 5.1. *The following are equivalent*

1. `KZ` *separates simples.*
2. *If q, $u_1, \ldots, u_m$ are the parameters of the Ariki-Koike algebra $\mathcal{H}$, then*

$$(q+1)\prod_{i<j}(u_i - u_j) \neq 0,$$

[*1] This is because a calculation very similar to [Gor03, Lemma 4.2] shows that the number denoted $c_{\wedge^t \mathfrak{h}_v}(k)$ in [DO03] equals $-tN$ for some $N \in \mathbb{N}$ which is independent of t. Thus if $[M_j : L_i] \neq 0$ then $-jN + iN \in \mathbb{N}$ and so $i > j$.

and furthermore,

$$\#\{\tau \in \mathsf{Irrep}(W) : \; L(\tau)|_{\mathfrak{h}^{\mathrm{reg}}} \neq 0\} \geq n-1.$$

3. *The algebra $\mathcal{H}$ has at least $|\mathsf{Irrep}(W)|-1$ nonisomorphic simple modules.*

Proof. First, we show that (2) implies (1). We must show that if $L(\sigma)|_{\mathfrak{h}^{\mathrm{reg}}} \cong L(\tau)|_{\mathfrak{h}^{\mathrm{reg}}} \neq 0$ then $\sigma = \tau$. Suppose then that $L(\sigma)|_{\mathfrak{h}^{\mathrm{reg}}} \cong L(\tau)|_{\mathfrak{h}^{\mathrm{reg}}} \neq 0$. By Lemma 1.5, there exists a standard module $M(\lambda)$ such that $L(\sigma) \hookrightarrow M(\lambda)$. Let $t = \dim(\mathrm{Hom}_{\mathcal{O}}(L(\sigma), M(\lambda)))$. Then $M(\lambda)$ must have t submodules isomorphic to $L(\sigma)$, because the only automorphisms of $L(\sigma)$ are the scalars. Therefore, $L(\sigma)^{\oplus t} \subset M(\lambda)$ and $M(\lambda)$ has no submodule isomorphic to $L(\sigma)^{\oplus(t+1)}$. Now since $L(\sigma)|_{\mathfrak{h}^{\mathrm{reg}}} \cong L(\tau)|_{\mathfrak{h}^{\mathrm{reg}}}$, we have $\mathrm{Hom}(L(\tau)|_{\mathfrak{h}^{\mathrm{reg}}}, M(\lambda)|_{\mathfrak{h}^{\mathrm{reg}}}) = \mathrm{Hom}(L(\sigma)|_{\mathfrak{h}^{\mathrm{reg}}}, M(\lambda)|_{\mathfrak{h}^{\mathrm{reg}}}) \neq 0$ and hence by [GGOR03, Proposition 5.9], $\mathrm{Hom}(L(\tau), M(\lambda)) \neq 0$ (using the condition on the parameters). Therefore, $M(\lambda)$ has a submodule isomorphic to $L(\tau)$ and hence a submodule isomorphic to $L(\tau) + L(\sigma)^{\oplus t}$. This sum must be direct if $L(\sigma) \ncong L(\tau)$, hence $M(\lambda)$ has a submodule $L(\tau) \oplus L(\sigma)^{\oplus t}$ and $M(\lambda)|_{\mathfrak{h}^{\mathrm{reg}}}$ has a submodule $L(\tau)|_{\mathfrak{h}^{\mathrm{reg}}} \oplus L(\sigma)|_{\mathfrak{h}^{\mathrm{reg}}}^{\oplus t} = L(\sigma)|_{\mathfrak{h}^{\mathrm{reg}}}^{\oplus(t+1)}$. Therefore,

$$\dim(\mathrm{Hom}(L(\sigma)|_{\mathfrak{h}^{\mathrm{reg}}}, M(\lambda)|_{\mathfrak{h}^{\mathrm{reg}}})) \geq t+1$$

and therefore by [GGOR03, Proposition 5.9], $\dim(\mathrm{Hom}(L(\sigma), M(\lambda))) \geq t+1$, a contradiction. It follows that $L(\sigma) \cong L(\tau)$ and hence $\sigma = \tau$.

Next, (1) implies (3) by Section 4.1.

Finally, to show (3) implies (2), note that under the hypothesis that $\mathcal{H}$ has $|\mathsf{Irrep}(W)|-1$ simple modules, it has already been shown in Section 4.2 that $[n]_q! \neq 0$, hence $q \neq -1$ since we assume $n \geq 2$, and that $u_i \neq u_j$ for all $i \neq j$, so the condition on the parameters holds. Furthermore, since every object of $\mathcal{H}-\mathrm{mod}$ is the image of some object of $\mathcal{O}$ under $\mathtt{KZ}$, and $\mathtt{KZ}$ is exact, if $\mathcal{H}$ has $|\mathsf{Irrep}(W)|-1$ simple modules then there are at least $|\mathsf{Irrep}(W)|-1$ of the $L(\tau)$ with $\mathtt{KZ}(L(\tau)) \neq 0$ and hence with $L(\tau)|_{\mathfrak{h}^{\mathrm{reg}}} \neq 0$. □

6. The Ariki-Koike algebra in the almost-semisimple case

In this section we use the facts proved about category $\mathcal{O}$ in Theorem 2.1 to prove a theorem about the Hecke algebra which does not mention the Cherednik algebra in its hypothesis or conclusion. This theorem is an example of a general philosophy suggested by Rouquier in [Rou05] of using the Cherednik algebra and the $\mathtt{KZ}$ functor as a tool to prove theorems about Hecke algebras.

It is well-known that $\mathcal{H}_\kappa$ is semisimple if and only if the number of irreducible modules $|\mathsf{Irrep}(\mathcal{H}_\kappa)|$ of $\mathcal{H}_\kappa$ equals the number of irreducible modules of $\mathbb{C}W$, and that in this case $\mathcal{H}_\kappa \cong \mathbb{C}W$. So the property of having $|\mathsf{Irrep}(W)|$ simple modules determines the algebra $\mathcal{H}_\kappa$ up to isomorphism. We show that the property of having $|\mathsf{Irrep}(W)|-1$ simple modules also determines $\mathcal{H}_\kappa$ up to isomorphism.

Theorem 6.1. *Suppose $\mathcal{H}_\kappa$ and $\mathcal{H}_\mu$ are Ariki-Koike algebras corresponding to some parameters $\kappa, \mu \in \mathbb{C}^m$ and that $|\mathsf{Irrep}(\mathcal{H}_\kappa)| = |\mathsf{Irrep}(\mathcal{H}_\mu)| = |\mathsf{Irrep}(W)| - 1$. Then there is an isomorphism of algebras $\mathcal{H}_\kappa \cong \mathcal{H}_\mu$.*

Proof. We work in the category $\mathcal{O} = \mathcal{O}_\kappa$ and write $\mathtt{KZ} = \mathtt{KZ}_\kappa$, $M(\tau) = M_\kappa(\tau)$, and so forth. By [GGOR03, Theorem 5.15], there is an algebra isomorphism $\mathcal{H}_\kappa \cong \mathrm{End}_{\mathcal{O}}(P_{\mathtt{KZ}})^{opp}$ where

$$P_{\mathtt{KZ}} = \bigoplus_{\tau \in \mathsf{Irrep}(W)} \dim \mathtt{KZ}(L(\tau)) P(\tau).$$

Here, $P(\tau)$ is the projective cover of $L(\tau)$. The strategy of the proof is to calculate $P_{\mathtt{KZ}}$ in the case where $\mathtt{KZ}_\kappa$ separates simples, and show that its endomorphism ring can be written in a way that does not depend on κ. By Theorem 2.1, there is a one-dimensional representation χ of W with $\mathcal{O} = \mathcal{O}^\wedge \oplus \mathcal{O}^{ss}$, where $\mathcal{O}^\wedge$ is the subcategory of $\mathcal{O}$ generated by $\{L(\wedge^i \mathfrak{h}_v \otimes \chi) : 0 \leq i \leq n\}$. Let $\lambda^i = \wedge^i \mathfrak{h}_v \otimes \chi$ and let $S = \{\lambda^i : 0 \leq i \leq n\}$. Write $M_i = M(\lambda^i)$, $L_i = L(\lambda^i)$ and $P_i = P(\lambda^i)$.

For $\sigma, \tau \in \mathsf{Irrep}(W)$, since $\mathcal{O}$ is a highest weight category, we have

$$\begin{aligned}
\dim \mathrm{Hom}(P(\sigma), P(\tau)) &= [P(\tau) : L(\sigma)] \\
&= \sum_\gamma [P(\tau) : M(\gamma)][M(\gamma) : L(\sigma)] \\
&= \sum_\gamma [M(\gamma) : L(\tau)][M(\gamma) : L(\sigma)] \\
&= \sum_{\gamma \in S} [M(\gamma) : L(\tau)][M(\gamma) : L(\sigma)] \\
&\quad + \sum_{\gamma \notin S} [M(\gamma) : L(\tau)][M(\gamma) : L(\sigma)]
\end{aligned}$$

If $\gamma \notin S$ then $M(\gamma) = L(\gamma)$, and we obtain

$$\dim \mathrm{Hom}(P(\sigma), P(\tau)) = \sum_{i=0}^n [M_i : L(\tau)][M_i : L(\sigma)] + \sum_{\gamma \notin S} \delta_{\gamma\tau}\delta_{\gamma\sigma}.$$

Now, if $\sigma \notin S$ or $\tau \notin S$, this sum must be $\delta_{\sigma\tau}$. Otherwise, $\sigma, \tau \in S$ and hence $\sigma = \lambda^a$, $\tau = \lambda^b$ for some a, b. We get

$$\dim \mathrm{Hom}(P(\lambda^a), P(\lambda^b)) = \sum_{i=0}^n [M_i : L_a][M_i : L_b]$$

which equals 2 if $a = b$ and 1 if $|a - b| = 1$ and 0 otherwise. Therefore, we have

$$\dim \mathrm{Hom}(P(\sigma), P(\tau)) = \begin{cases} 2 & \text{if } \sigma = \tau \in S \\ 1 & \text{if } \sigma = \tau \notin S \\ 1 & \text{if } \{\sigma, \tau\} = \{\lambda^a, \lambda^{a+1}\}, 0 \leq a \leq n-1 \\ 0 & \text{otherwise} \end{cases}$$

The ring $\mathrm{End}_{\mathcal{O}}(P_{\mathtt{KZ}})$ is a matrix algebra with entries in the spaces $\mathrm{Hom}(P(\sigma), P(\tau))$. We calculate the multiplication relations between basis elements of the $\mathrm{Hom}(P(\sigma), P(\tau))$ and show that these relations do not depend on κ. It will follow that the structure constants of $\mathrm{End}_{\mathcal{O}}(P_{\mathtt{KZ}})$ do not depend on κ, which will prove the theorem provided that the multiplicity of each $P(\tau)$ in $P_{\mathtt{KZ}}$ is also independent of κ. But in our situation $P_{\mathtt{KZ}} = \oplus_{\tau \notin S}(\dim \tau) \cdot P(\tau) \oplus \left(\oplus_{1 \le i \le n} \binom{n-1}{i-1} P_i\right)$ since $\dim \mathtt{KZ}(L_i) = \binom{n-1}{i-1}$, as can be readily shown using induction on the BGG-resolution (4.2) of L_0 and the fact that $\dim \mathtt{KZ}(M(\tau)) = \dim(\tau)$ for all τ.

By BGG reciprocity, we have $[P_i : M_i] = [M_i : L_i] = 1 = [M_{i-1} : L_i] = [P_i : M_{i-1}]$, and $[P_i : M(\sigma)] = [M(\sigma) : L_i] = 0$ if $\sigma \neq \lambda^i, \lambda^{i-1}$. Therefore, the factors in any filtration of P_i by standard modules are M_i and M_{i-1}. But by [GGOR03, Corollary 2.10], P_i has a filtration by standard modules with M_i as the top factor, so P_i may be described as $P_i = {}^{M_i}_{M_{i-1}}$, meaning that there is a series $0 = P_i^0 \subset P_i^1 \subset P_i^2 = P_i$ with $P_i^1 \cong M_{i-1}$ and $P_i^2/P_i^1 \cong M_i$. We may write the resulting composition series of P_i as

$$P_i = \begin{matrix} L_i \\ L_{i+1} \\ L_{i-1} \\ L_i \end{matrix}$$

This description of P_i makes it easy to write down the nontrivial maps $P_i \to P_i$.

First, there are two obvious maps $P_i \to P_i$, namely the identity map id_i and the map ξ_i which is projection onto the top composition factor L_i followed by inclusion. Note that $\xi_i^2 = 0$ and therefore $\mathrm{End}_{\mathcal{O}}(P_i) = \mathbb{C}[\xi_i]/(\xi_i^2)$, since we have already shown that $\dim \mathrm{Hom}(P_i, P_i) = 2$.

Next, we describe the map $P_i \to P_{i+1}$. This is a map ${}^{M_i}_{M_{i-1}} \to {}^{M_{i+1}}_{M_i}$. We may construct a map $f_{i,i+1} : P_i \to P_{i+1}$ by factoring out the copy of M_{i-1} and then embedding M_i in P_{i+1}. This map is nonzero, so $\mathrm{Hom}(P_i, P_{i+1}) = \mathbb{C} f_{i,i+1}$, $1 \le i \le n-1$.

Now we describe the map $P_i \to P_{i-1}$, $n \ge i \ge 2$. There are two different descriptions of this map. One way of defining a map $P_i \to P_{i-1}$ is to take the quotient $P_i \to L_i$ and then embed $L_i \hookrightarrow M_{i-1}$. Since M_{i-1} is a quotient of P_{i-1}, this induces a nonzero map $f_{i,i-1} : P_i \to P_{i-1}$. The second way of getting a map $P_i \to P_{i-1}$ is to observe that, by [GGOR03, Proposition 5.2.1 (ii)], $P_i \supset L_i$ is injective and therefore P_i contains the injective envelope $I_i = I(\lambda^i)$ of L_i. Therefore, since P_i is indecomposable, $P_i = I_i$. Now, category $\mathcal{O}$ contains a *costandard module* $\nabla(\tau) \supset L(\tau)$ for every $\tau \in \mathsf{Irrep}(W)$, with $[\nabla(\tau)] = [M(\tau)]$ in $K_0(\mathcal{O})$. Write $\nabla_i = \nabla(\lambda^i)$. Then $L_i \subset \nabla_i$, so ∇_i has a composition series of the form $\nabla_i = {}^{L_{i+1}}_{L_i}$. Furthermore, $\nabla_i \subset I_i$ and so I_i has a filtration by costandard modules of the form $I_i = {}^{\nabla_{i-1}}_{\nabla_i}$ (the existence of such a filtration follows from [CPS88, Definition 3.1, Axiom (c)]). Since $I_i = P_i$, to get a map ${}^{\nabla_{i-1}}_{\nabla_i} = P_i \to P_{i-1} = {}^{\nabla_{i-2}}_{\nabla_{i-1}}$, we may factor out the copy of ∇_i and then embed ∇_{i-1} in P_{i-1}. This gives a nonzero map $f_{i,i-1}$, and therefore $\mathrm{Hom}(P_i, P_{i-1}) = \mathbb{C} f_{i,i-1}$. In particular, this shows that the image of $f_{i,i-1}$ has length 2.

Now we calculate multiplication relations between the various $f_{i,i+1}$, $f_{i,i-1}$ and ξ_i. First, it is immediate from the definitions that $\xi_{i+1}f_{i,i+1} = f_{i,i+1}\xi_i = 0$. We need to do a little more work to show that the same holds for $f_{i,i-1}$. Take the description of I_i as $I_i = {}^{\nabla_{i-1}}_{\nabla_i}$. Then I_i has a composition series

$$I_i = \begin{matrix} L_i \\ L_{i-1} \\ L_{i+1} \\ L_i \end{matrix}$$

and therefore there is a map $\zeta_i : I_i \to I_i$ defined by projection onto the top composition factor L_i followed by the embedding $L_i \hookrightarrow I_i$. Clearly, $\zeta_i f_{i-1,i} = f_{i-1,i}\zeta_{i-1} = 0$. But since $P_i = I_i$, we may regard ζ_i as a map $P_i \to P_i$. Therefore, there are $a, b \in \mathbb{C}$ with $\zeta_i = a\mathrm{id}_i + b\xi_i$. Since $\zeta_i^2 = 0$, we get $a^2 = 0$ and hence ζ_i is a nonzero multiple of ξ_i. This shows that $\xi_i f_{i-1,i} = f_{i-1,i}\xi_{i-1} = 0$.

Finally, we need to calculate $f_{i+1,i}f_{i,i+1}$ and $f_{i-1,i}f_{i,i-1}$. Consider first $f_{i-1,i}f_{i,i-1}$. By the definition of $f_{i,i-1}$ above, we have $[\mathrm{im}(f_{i,i-1}) : L_i] \neq 0$. Hence, $\mathrm{im}(f_{i,i-1})$ cannot be contained in the submodule of P_{i-1} isomorphic to M_{i-2}, and therefore $f_{i-1,i}f_{i,i-1}$ must be nonzero. Since $f_{i-1,i}f_{i,i-1}\xi_i = 0$, $f_{i-1,i}f_{i,i-1}$ must be a nonzero multiple of ξ_i. Let us replace ξ_i by $f_{i-1,i}f_{i,i-1}$. So we may assume that $f_{i-1,i}f_{i,i-1} = \xi_i$, and this does not change any of the relations which have already been calculated. Now consider $f_{i+1,i}f_{i,i+1}$. We show that this composition is nonzero. Indeed, the image $\mathrm{im}(f_{i,i+1})$ has composition factors L_i and L_{i+1}. If $f_{i+1,i}f_{i,i+1}$ were zero, then we would get that $\mathrm{im}(f_{i+1,i})$ could only have composition factors L_{i+1} and L_{i+2}. But we have shown that $\mathrm{im}(f_{i+1,i})$ has length 2, and $[P_i : L_{i+2}] = 0$, a contradiction. Therefore, $f_{i+1,i}f_{i,i+1} \neq 0$ and so there is a nonzero $b_{i,i+1} \in \mathbb{C}$, $n-1 \geq i \geq 1$, such that

$$f_{i+1,i}f_{i,i+1} = b_{i,i+1}\xi_i = b_{i,i+1}f_{i-1,i}f_{i,i-1}.$$

It remains to do some rescaling. Let

$$\begin{aligned} \xi'_i &= \frac{1}{b_{12}b_{23}\cdots b_{i-1,i}}\xi_i, && 1 \leq i \leq n \\ f'_{i,i-1} &= f_{i,i-1} && 2 \leq i \leq n \\ f'_{i,i+1} &= \frac{1}{b_{12}b_{23}\cdots b_{i,i+1}}f_{i,i+1} && 1 \leq i \leq n-1. \end{aligned}$$

Then we have the following relations:

$$\begin{aligned} \xi'_i f'_{i-1,i} &= f'_{i-1,i}\xi'_{i-1} = 0 \\ \xi'_{i+1}f'_{i,i+1} &= f'_{i,i+1}\xi'_i = 0 \\ f'_{i-1,i}f'_{i,i-1} &= f'_{i+1,i}f'_{i,i+1} = \xi'_i. \end{aligned} \tag{6.1}$$

These are the only nontrivial relations between the various $\mathrm{Hom}(P(\sigma), P(\tau))$. This shows that we may choose a basis of $\mathrm{Hom}(P(\sigma), P(\tau))$ for each σ, τ such that the composition relations between the basis elements are independent of κ. Hence, we may choose a basis of the algebra $\mathrm{End}_{\mathcal{O}}(P_{\mathsf{KZ}})$ such that the structure constants are independent of κ. This proves the theorem. $\square$

Remark 2. By variations on the arguments given in the above proof, it is possible to show that

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathcal{O}}(L_i, L_j) = \begin{cases} 1 & j = i+1, i-1 \\ 0 & \text{otherwise} \end{cases}$$

and so the composition series of P_i may be written more symmetrically as $P_i = I_i = \begin{smallmatrix} & L_i & \\ L_{i-1} & \oplus & L_{i+1} \\ & L_i & \end{smallmatrix}$.

Note that since we have shown earlier that the Ariki-Koike algebra has $|\mathsf{Irrep}(W)| - n$ blocks, by counting we get that the algebra $B_n := \mathrm{End}_{\mathcal{O}}(\oplus_{i=1}^n \binom{n-1}{i-1} P_i)$ is a block of the Ariki-Koike algebra. From the relations (6.1), it is clear that B_n is independent both of κ and m. To extend this description of the unique non-semisimple block to $m = 1$, we consider the Cherednik algebra of the group S_{n+1} acting on its reflection representation $\mathfrak{h} = \mathbb{C}^n$. In this case, the Cherednik algebra only depends on one parameter κ_{00} (denoted c in [BEG03b]). We write the Hecke algebra as $\mathcal{H}_c(S_{n+1})$, with parameter $q = e^{2\pi\sqrt{-1}c}$. The simple modules of $\mathcal{H}_c(S_{n+1})$ are in bijection with e–restricted partitions λ of $n+1$, where e is the multiplicative order of q in $\mathbb{C}^*$, and a partition λ is said to be e–restricted if $\lambda_i - \lambda_{i+1} < e$ for all $i \geq 1$. It is clear from this description that $\mathcal{H}_c$ has $|\mathsf{Irrep}(S_{n+1})| - 1$ simple modules if and only if $e = n+1$ if and only if $c = \frac{r}{n+1}$ with $(r, n+1) = 1$. In this case, Theorem 2.1 holds without change by various results of [BEG03b, Section 3], and the proof of Theorem 6.1 also goes through without change in this case. We therefore have the following corollary.

Corollary 6.1. *Let $\ell_1, \ell_2 > 1$ and for $i = 1, 2$ let $\kappa_i \in \mathbb{C}^{\ell_i}$ and suppose $\mathcal{H}_{\kappa_i}(G(\ell_i, 1, n))$ has $|\mathsf{Irrep}(G(\ell_i, 1, n))| - 1$ simple modules. Then the unique nonsemisimple blocks of $\mathcal{H}_{\kappa_1}(G(\ell_1, 1, n))$ and $\mathcal{H}_{\kappa_2}(G(\ell_2, 1, n))$ are isomorphic algebras. Furthermore, they are isomorphic to the principal block B_n of $\mathcal{H}_{\frac{1}{n+1}}(S_{n+1})$.*

Remark 3. The representation theory of the algebra B_n is described in [BEG03b, 5.3] and [EN02, 3.2].

CORNELL UNIVERSITY
e-mail: rvale@math.cornell.edu

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