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Subsheaves of a hermitian torsion free coherent sheaf on an arithmetic variety

By

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Introduction

Let K be a number field and O_K the ring of integers of K. Let (E, h) be a hermitian finitely generated flat O_K-module. For an O_K-submodule F of E, let us denote by $h_{F \hookrightarrow E}$ the submetric of F induced by h. It is well known that the set of all saturated O_K-submodules F with $\widehat{\deg}(F, h_{F\hookrightarrow E}) \geq c$ is finite for any real numbers c (for details, see [4, the proof of Proposition 3.5]).

In this note, we would like to give its generalization on a projective arithmetic variety. Let X be a normal and projective arithmetic variety. Here we assume that X is an arithmetic surface to avoid several complicated technical definitions on a higher dimensional arithmetic variety. Let us fix a nef and big C^∞-hermitian invertible sheaf $\overline{H}$ on X as a polarization of X. Then we have the following finiteness of saturated subsheaves with bounded arithmetic degree, which is also a generalization of a partial result [5, Corollary 2.2].

Theorem A (cf. Theorem 3.1)**.** *Let E be a torsion free coherent sheaf on X and h a C^∞-hermitian metric of E on $X(\mathbb{C})$. For any real number c, the set of all saturated $\mathcal{O}_X$-subsheaves F of E with $\widehat{\deg}(\widehat{c}_1(\overline{H}) \cdot \widehat{c}_1(F, h_{F\hookrightarrow E})) \geq c$ is finite.*

For a non-zero C^∞-hermitian torsion free coherent sheaf $\overline{G}$ on X, the *arithmetic slope* $\hat{\mu}_{\overline{H}}(\overline{G})$ of $\overline{G}$ with respect to $\overline{H}$ is defined by

$$\hat{\mu}_{\overline{H}}(\overline{G}) = \frac{\widehat{\deg}(\widehat{c}_1(\overline{H}) \cdot \widehat{c}_1(\overline{G}))}{\operatorname{rk} G}.$$

As defined in the paper [5], (E, h) is said to be *arithmetically μ-semistable* with respect to $\overline{H}$ if, for any non-zero saturated $\mathcal{O}_X$-subsheaf F of E,

$$\hat{\mu}_{\overline{H}}(F, h_{F\hookrightarrow E}) \leq \hat{\mu}_{\overline{H}}(E, h).$$

The above semistability yields an arithmetic analogue of the Harder-Narasimham filtration of a torsion free sheaf on an algebraic variety as follows: A filtration

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_l = E$$

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of E is called an *arithmetic Harder-Narasimham filtration of* (E, h) *with respect to* $\overline{H}$ if the following properties are satisfied:

(1) E_i/E_{i-1} is torsion free for every $1 \le i \le l$.

(2) Let $h_{E_i/E_{i-1}}$ be a C^∞-hermitian metric of E_i/E_{i-1} induced by h, that is,

$$h_{E_i/E_{i-1}} = (h_{E_i \hookrightarrow E})_{E_i \twoheadrightarrow E_i/E_{i-1}} = (h_{E \twoheadrightarrow E/E_{i-1}})_{E_i/E_{i-1} \hookrightarrow E/E_{i-1}}$$

(for details, see Proposition 1.1.1). Then $(E_i/E_{i-1}, h_{E_i/E_{i-1}})$ is arithmetically μ-semistable with respect to $\overline{H}$.

(3) $\hat{\mu}_{\overline{H}}(E_1/E_0, h_{E_1/E_0}) > \hat{\mu}_{\overline{H}}(E_2/E_1, h_{E_2/E_1}) > \cdots > \hat{\mu}_{\overline{H}}(E_l/E_{l-1}, h_{E_l/E_{l-1}})$.

As a consequence of the above theorem, we can show the unique existence of an arithmetic Harder-Narasimham filtration:

Theorem B (cf. Theorem 5.1)**.** *There is a unique arithmetic Harder-Narasimham filtration of* (E, h).

1. Preliminaries

1.1. Hermitian vector space

In this subsection, let us recall several basic facts of hermitian complex vector spaces.

Let (V, h) be a finite dimensional hermitian complex vector space, i.e., V is a finite dimensional vector space over $\mathbb{C}$ and h is a hermitian metric of V. Let $\phi : V' \to V$ be an injective homomorphism of complex vector spaces. If we set $h'(x, y) = h(\phi(x), \phi(y))$, then h' is a hermitian metric of V'. This metric h' is called the *submetric of* V' *induced by* h *and* $V' \to V$, and it is denoted by $h_{V' \hookrightarrow V}$.

Let $\psi : V \to V''$ be a surjective homomorphism of complex vector spaces. Let W be the orthogonal complement of $\mathrm{Ker}(\psi)$ with respect to h. Let $h_{W \hookrightarrow V}$ be the submetric of W induced by h and $W \to V$. Then there is a unique hermitian metric h'' of V'' such that the isomorphism $\psi|_W : W \to V''$ gives rise to an isometry $(W, h_{W \hookrightarrow V}) \xrightarrow{\sim} (V'', h'')$. The metric h'' is called the *quotient metric of* V'' *induced by* h *and* $V \to V''$, and it is denoted by $h_{V \twoheadrightarrow V''}$.

For simplicity, the submetric $h_{V' \hookrightarrow V}$ and the quotient metric $h_{V \twoheadrightarrow V''}$ are often denoted by $h_{V'}$ and $h_{V''}$ respectively. It is easy to see the following proposition:

Proposition 1.1.1. *Let* V, V', V'' *be finite dimensional complex vector spaces with* $V'' \subseteq V' \subseteq V$. *Let* h *be a hermitian metric of* V. *Then*

$$(h_{V' \hookrightarrow V})_{V' \twoheadrightarrow V'/V''} = (h_{V \twoheadrightarrow V/V''})_{V'/V'' \hookrightarrow V/V''}$$

as hermitian metrics of V'/V''.

More generally, we have the following lemma:

Lemma 1.1.2. *Let (V, h) be a finite dimensional hermitian complex vector space. Let W and U be subspaces of V. Let us consider a natural homomorphism*

$$\phi : W \hookrightarrow V \to V/U$$

of complex vector spaces. Let Q be the image of ϕ. Let us consider two natural hermitian metrics h_1 and h_2 of Q given by

$$h_1 = (h_{W \hookrightarrow V})_{W \twoheadrightarrow Q} \quad \text{and} \quad h_2 = (h_{V \twoheadrightarrow V/U})_{Q \hookrightarrow V/U}.$$

Then $h_1(x, x) \geq h_2(x, x)$ for all $x \in Q$. In particular, if $\{x_1, \ldots, x_s\}$ is a basis of Q, then $\det(h_1(x_i, x_j)) \geq \det(h_2(x_i, x_j))$.

Proof. Let T be the orthogonal complement of $\operatorname{Ker}(\phi : W \to Q)$ with respect to $h_{W \hookrightarrow V}$. Then $h(v, v) = h_1(\phi(v), \phi(v))$ for all $v \in T$. Let $U^{\perp}$ be the orthogonal complement of U with respect to h. Then, for $v \in T$, we can set $v = u + u'$ with $u \in U$ and $u' \in U^{\perp}$. Then $h_2(\phi(v), \phi(v)) = h(u', u')$. Thus

$$h_2(\phi(v), \phi(v)) = h(u', u') \leq h(v, v) = h_1(\phi(v), \phi(v)).$$

For the last assertion, see [4, Lemma 3.4]. □

Let $e_1, \ldots, e_n$ be an orthonormal basis of V with respect to h. Let $V^{\vee}$ be the dual space of V and $e_1^{\vee}, \ldots, e_n^{\vee}$ the dual basis of $e_1, \ldots, e_n$. For $\phi, \psi \in V^{\vee}$, we set

$$h^{\vee}(\phi, \psi) = \sum_{i=1}^{n} a_i \bar{b}_i,$$

where $\phi = a_1 e_1^{\vee} + \cdots + a_n e_n^{\vee}$ and $\psi = b_1 e_1^{\vee} + \cdots + b_n e_n^{\vee}$. It is easy to see that $h^{\vee}$ does not depend on the choice of the orthonormal basis of V, so that the hermitian metric $h^{\vee}$ of $V^{\vee}$ is called the *dual hermitian metric of h*. Moreover we can easily check the following facts:

Proposition 1.1.3.

(1) $h^{\vee}(\phi, \phi) = \sup_{x \in V \setminus \{0\}} \dfrac{|\phi(x)|^2}{h(x, x)}$.

(2) *Let $x_1, \ldots, x_n$ be a basis of V and $x_1^{\vee}, \ldots, x_n^{\vee}$ be the dual basis of $V^{\vee}$. If we set $H = (h(x_i, x_j))$ and $H^{\vee} = (h^{\vee}(x_i^{\vee}, x_j^{\vee}))$, then $H^{\vee} = \overline{H}^{-1}$.*

(3) *Let $0 \to V_1 \to V_2 \to V_3 \to 0$ be an exact sequence of finite dimensional complex vector spaces and h_1, h_2, h_3 hermitian metrics of V_1, V_2, V_3 respectively. We assume that $h_1 = (h_2)_{V_1 \hookrightarrow V_2}$ and $h_3 = (h_2)_{V_2 \twoheadrightarrow V_3}$. Let us consider the dual exact sequence $0 \to V_3^{\vee} \to V_2^{\vee} \to V_1^{\vee} \to 0$ of $0 \to V_1 \to V_2 \to V_3 \to 0$ and the dual hermitian metrics $h_1^{\vee}, h_2^{\vee}, h_3^{\vee}$ of h_1, h_2, h_3 respectively. Then $h_3^{\vee} = (h_2^{\vee})_{V_3^{\vee} \hookrightarrow V_2^{\vee}}$ and $h_1^{\vee} = (h_2^{\vee})_{V_2^{\vee} \twoheadrightarrow V_1^{\vee}}$.*

Let (U, h_U) and (W, h_W) be finite dimensional hermitian vector spaces over $\mathbb{C}$. Then $U \otimes_{\mathbb{C}} W$ has the standard hermitian metric $h_U \otimes h_W$ defined by

$$(h_U \otimes h_W)(u \otimes w, u' \otimes w') = h_U(u, u') h_W(w, w').$$

Thus the standard hermitian metric of $\bigotimes^r V$ is given by

$$\left(\bigotimes^r h\right)(v_1 \otimes \cdots v_r, v_1' \otimes \cdots \otimes v_r') = h(v_1, v_1') \cdots h(v_r, v_r').$$

Let $\pi : \bigotimes^r V \to \bigwedge^r V$ be the natural surjective homomorphism and $\bigwedge^r h$ a hermitian metric of $\bigwedge^r V$ given by

$$\bigwedge^r h = r! \left(\bigotimes^r h\right)_{\bigotimes^r V \twoheadrightarrow \bigwedge^r V}.$$

Then we have the following:

Proposition 1.1.4. $(\bigwedge^r h)(x_1 \wedge \cdots \wedge x_r, x_1 \wedge \cdots \wedge x_r) = \det(h(x_i, x_j))$.

Proof. For $a_1, \ldots, a_r \in V$, we set

$$\phi(a_1, \ldots, a_r) = \frac{1}{r!} \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(r)}.$$

Then, by an easy calculation, for $\sigma \in S_r$ and $a_1, \ldots, a_r, b_1, \ldots, b_r \in V$, we can see

$$(1.1.4.1) \quad \left(\bigotimes^r h\right)(a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(r)}, \phi(b_1, \ldots, b_r)) = \operatorname{sgn}(\sigma) \left(\bigotimes^r h\right)(a_1 \otimes \cdots \otimes a_r, \phi(b_1, \ldots, b_r))$$

Note that $\operatorname{Ker}(\pi)$ is generated by elements of type

$$a_1 \otimes \cdots \otimes a_r,$$

where $a_i = a_j$ for some $i \neq j$. Therefore, by (1.1.4.1), $\phi(x_1, \ldots, x_r) \in \operatorname{Ker}(\pi)^\perp$ for all $x_1, \ldots, x_r \in V$. Thus, since

$$\pi(\phi(x_1, \ldots, x_r)) = x_1 \wedge \cdots \wedge x_r,$$

we have

$$\left(\bigotimes^r h\right)_{\bigotimes^r V \twoheadrightarrow \bigwedge^r V}(x_1 \wedge \cdots \wedge x_r, x_1 \wedge \cdots \wedge x_r) = \left(\bigotimes^r h\right)(\phi(x_1, \ldots, x_r), \phi(x_1, \ldots, x_r)).$$

On the other hand, by using (1.1.4.1) again, we can check

$$\left(\bigotimes^r h\right)(\phi(x_1,\ldots,x_r),\phi(x_1,\ldots,x_r)) = \frac{1}{r!}\det(h(x_i,x_j)).$$

Therefore we get our assertion. □

1.2. Finitely generated modules over a 1-dimensional noetherian integral domain

Let R be a noetherian integral domain with $\dim R = 1$, and K the quotient field of R. For $a \in R \setminus \{0\}$, we set $\operatorname{ord}_R(a) = \operatorname{length}_R(R/aR)$, which yields a homomorphism $\operatorname{ord}_R : R \setminus \{0\} \to \mathbb{Z}$, that is, $\operatorname{ord}_R(ab) = \operatorname{ord}_R(a) + \operatorname{ord}_R(b)$ for $a, b \in R \setminus \{0\}$. Thus it extends to a homomorphism on $K^\times$ given by $\operatorname{ord}_R(a/b) = \operatorname{ord}_R(a) - \operatorname{ord}_R(b)$.

Proposition 1.2.1. *Let E be a finitely generated R-module. Let $s_1,\ldots,s_r$ and $s'_1,\ldots,s'_r$ be sequences of elements of E such that $s_1,\ldots,s_r$ and $s'_1,\ldots,s'_r$ form bases of $E \otimes_R K$ respectively. Let $A = (a_{ij})$ be an $r \times r$-matrix such that $a_{ij} \in K$ for all i, j and $s'_i = \sum_{j=1}^r a_{ij}s_j$ in $E \otimes_R K$ for all i. Then*

$$\operatorname{length}_R(E/Rs'_1 + \cdots + Rs'_r) = \operatorname{length}_R(E/Rs_1 + \cdots + Rs_r) + \operatorname{ord}_R(\det(A)).$$

Proof. We set $M = Rs_1 + \cdots + Rs_r$ and $M' = Rs'_1 + \cdots + Rs'_r$. First we assume that $M' \subseteq M$. Then $a_{ij} \in R$. An exact sequence

$$0 \to M/M' \to E/M' \to E/M \to 0.$$

yields

$$\operatorname{length}_R(E/M') = \operatorname{length}_R(E/M) + \operatorname{length}_R(M/M').$$

Note that M is a free R-module. Let $\phi : M \to M$ be an endomorphism given by $\phi(s_i) = s'_i$. Then, by [EGA IV, Lemme 21.10.17.3], $\operatorname{length}_R(M/\phi(M)) = \operatorname{length}_R(R/\det(\phi)R)$. Thus we get

$$\operatorname{length}_R(E/M') = \operatorname{length}_R(E/M) + \operatorname{length}_R(R/\det(A)R).$$

Next we consider a general case. Since E/M is a torsion module, there is $b \in R \setminus \{0\}$ with $bM' \subseteq M$. Thus, by the previous observation,

$$\operatorname{length}_R(E/bM') = \operatorname{length}_R(E/M) + \operatorname{length}_R(R/\det(bA)R)$$

because $bs_i = \sum_{j=1}^r ba_{ij}s_j$ in $E \otimes_R K$ for all i. Moreover

$$\operatorname{length}_R(E/bM') = \operatorname{length}_R(E/M') + \operatorname{length}_R(R/b^rR).$$

Hence the proposition follows. □

Corollary 1.2.2.

(1) *Let $\{x_1,\ldots,x_r\}$ be a basis of $E \otimes_R K$. Let $s_1,\ldots,s_r \in E$ and $a \in R \setminus \{0\}$ such that $ax_i = s_i$ in $E \otimes_R K$ for all i. Then the number*

$$\operatorname{length}_R(E/Rs_1 + \cdots + Rs_r) - r\operatorname{ord}_R(a)$$

does not depend on the choice of $s_1,\ldots,s_r$ and a, so that it is denoted by $\ell_R(E; x_1,\ldots,x_r)$.

(2) *Let* $\{x_1, \ldots, x_r\}$ *and* $\{x'_1, \ldots, x'_r\}$ *be bases of* $E \otimes_R K$. *Let* $B = (b_{ij})$ *be an* $r \times r$ *matrix such that* $x'_i = \sum_{j=1}^r b_{ij} x_j$ *for all* i. *Then*

$$\ell_R(E; x'_1, \ldots, x'_r) = \ell_R(E; x_1, \ldots, x_r) + \mathrm{ord}_R(\det(B)).$$

Proof. (1) Let $s'_1, \ldots, s'_r \in E$ and $a' \in R \setminus \{0\}$ be another choice with $a' x_i = s'_i$ in $E \otimes_R K$ for all i. Then $s'_i = (a'/a) s_i$ in $E \otimes_R K$. Thus, by the previous proposition,

$$\mathrm{length}_R(E/Rs'_1 + \cdots + Rs'_r) = \mathrm{length}_R(E/Rs_1 + \cdots + Rs_r) + \mathrm{ord}_R((a'/a)^r),$$

which yields the assertion.

(2) Let us choose $a, b \in R \setminus \{0\}$ and $s_1, \ldots, s_r \in E$ such that $ax_i = s_i$ in $E \otimes_R K$ for all i and $bb_{ij} \in R$ for all i, j. If we set $s'_i = \sum_j (bb_{ij}) s_i$, then $abx'_i = s'_i$ in $E \otimes_R K$ for all i. Thus

$$\begin{aligned} \ell_R(E; x_1, \ldots, x_r) &= \mathrm{length}_R(E/Rs_1 + \cdots + Rs_r) - r\,\mathrm{ord}_R(a) \\ \ell_R(E; x'_1, \ldots, x'_r) &= \mathrm{length}_R(E/Rs'_1 + \cdots + Rs'_r) - r\,\mathrm{ord}_R(ab). \end{aligned}$$

On the other hand, by the previous proposition,

$$\mathrm{length}_R(E/Rs'_1 + \cdots + Rs'_r) = \mathrm{length}_R(E/Rs_1 + \cdots + Rs_r) + \mathrm{ord}_R(\det(bB)).$$

Hence we obtain (2). □

1.3. Subsheaves of a torsion free coherent sheaf

In this subsection, we consider how we can get a saturated subsheaf.

Proposition 1.3.1. *Let* X *be an irreducible noetherian integral scheme,* η *the generic point of* X, *and* $K = \mathcal{O}_{X,\eta}$ *the function field of* X. *Let* E *be a torsion free coherent sheaf on* X. *Let* $\Sigma(X, E)$ *be the set of all saturated* $\mathcal{O}_X$*-subsheaves of* E *and* $\Sigma(K, E_\eta)$ *the set of all vector subspaces of* E_η *over* K. *Then the map* $\gamma : \Sigma(X, E) \to \Sigma(K, E_\eta)$ *given by* $\gamma(F) = F_\eta$ *is bijective. For a vector subspace* W *of* E_η *over* K, *the subsheaf given by* $\gamma^{-1}(W)$ *is called the* saturated $\mathcal{O}_X$-subsheaf of E induced by W *and is denoted by* $\mathcal{O}_X(W; E)$.

Proof. Let us begin with the following lemma:

Lemma 1.3.2. *Let* F, G *be* $\mathcal{O}_X$*-subsheaves of* E *such that* F *is saturated in* E *and* $F_\eta = G_\eta$. *Then* $F \supseteq G$.

Proof. Let us consider a homomorphism $\phi : G \to E \to E/F$. Then $\phi_\eta = 0$. Since E/F is torsion free, we have $\phi = 0$, which means that $G \subseteq F$. □

The injectivity of γ is a consequence of the above lemma. Let W be a vector subspace of E_η over K. We set $F(U) = W \cap E(U)$ for any Zariski open set U of X. Then $F_\eta = W$. We need to see that F is saturated in E. Since F is the kernel of the natural homomorphism $E \to E_\eta \to E_\eta/W$, we have an injection $E/F \hookrightarrow E_\eta/W$, so that E/F is torsion free. □

Proposition 1.3.3. *Let X be a noetherian scheme and E a locally free coherent sheaf on X. Let $\pi : P = \operatorname{Proj}(\bigoplus_{d \geq 0} \operatorname{Sym}^d(E^\vee)) \to X$ be the projective bundle and $\mathcal{O}_P(1)$ the tautological line bundle of $P \to X$. Let $\Gamma(X, P)$ be the set of all sections of $\pi : P \to X$. Moreover let $\Sigma'_1(X, E)$ be the set of all $\mathcal{O}_X$-subsheaves L such that L is invertible and E/L is locally free. For $s \in \Gamma(X, P)$, let*

$$\phi_s : s^*(\mathcal{O}_P(-1)) \to s^*\pi^*(E) = E$$

be a homomorphism obtained from the dual homomorphism $\mathcal{O}_P(-1) \to \pi^*(E)$ of the natural homomorphism $\pi^*(E^\vee) \to \mathcal{O}_P(1)$ by applying s^*. We denote the image of $\phi_s : s^*(\mathcal{O}_P(-1)) \to E$ by $L(s)$. Then $L(s) \in \Sigma'_1(X, E)$ for all $s \in \Gamma(X, P)$ and a map

$$\Gamma(X, P) \to \Sigma'_1(X, E)$$

given by $s \mapsto L(s)$ is bijective.

Proof. See [1, Theorem 7.1 and Proposition 7.12]. □

1.4. Hermitian locally free coherent sheaf on a smooth variety

Let X be a smooth variety over $\mathbb{C}$, η be the generic point of X, and $K = \mathcal{O}_{X,\eta}$ the function field of X.

Proposition 1.4.1. *Let (E, h) and (E', h') be C^∞-hermitian locally free coherent sheaves on X. If there is a dense Zariski open set U of X such that $(E, h)|_U$ is isometric to $(E', h')|_U$, then this isometry extends to an isometry over X.*

Proof. Since $V = E_\eta$ is isomorphic to E'_η, we may assume that E' is a subsheaf of V. Then $(E, h)|_U$ coincides with $(E', h')|_U$.

First let us see that $E = E'$. For this purpose, it is sufficient to see that $E_\gamma = E'_\gamma$ for all codimension one points γ. Let $\{\omega_1, \ldots, \omega_r\}$ and $\{\omega'_1, \ldots, \omega'_r\}$ be local bases of E_γ and E'_γ respectively. Then there are $r \times r$-matrices (a_{ij}) and (b_{ij}) such that $a_{ij}, b_{ij} \in K$ for all i, j and

$$\omega'_i = \sum_{j=1}^{r} a_{ij}\omega_j, \quad \omega_i = \sum_{j=1}^{r} b_{ij}\omega'_j$$

for all i. Clearly $(a_{ij})(b_{ij}) = (b_{ij})(a_{ij}) = (\delta_{ij})$.

Claim 1.4.1.1. *$a_{ij}, b_{ij} \in \mathcal{O}_{X,\gamma}$ for all i, j.*

For each i, we set $e_i = \min_{1 \leq j \leq r}\{\operatorname{ord}_\gamma(a_{ij})\}$. We assume that $e_i < 0$. Let t be a local parameter of $\mathcal{O}_{X,\gamma}$. Then $t^{-e_i}a_{ij} \in \mathcal{O}_{X,\gamma}$ for all j. Thus $t^{-e_i}\omega'_i \in E_\gamma$ and $t^{-e_i}\omega'_i \neq 0$ in $E_\gamma \otimes \kappa(\gamma)$. Let Γ be the Zariski closure of $\{\gamma\}$. If we choose a general closed point x_0 of Γ, then $\omega'_i \neq 0$ in $E'_{x_0} \otimes \kappa(x_0)$ and $t^{-e_i}\omega'_i \neq 0$ in $E_{x_0} \otimes \kappa(x_0)$. On the other hand, there is an open neighborhood U_{x_0} of x_0 such that

$$h(t^{-e_i}\omega'_i, t^{-e_i}\omega'_i)(x) = h'(t^{-e_i}\omega'_i, t^{-e_i}\omega'_i)(x)$$

for $x \in U_{x_0} \cap U$. Thus if we set

$$f(x) = h(t^{-e_i}\omega'_i, t^{-e_i}\omega'_i)(x) = |t|^{-2e_i} h'(\omega'_i, \omega'_i)(x)$$

on $U_{x_0} \cap U$, then $\lim_{x \to x_0} f(x) = h(t^{-e_i}\omega'_i, t^{-e_i}\omega'_i)(x_0) = 0$ because $t = 0$ at x_0. This is a contradiction because $t^{-e_i}\omega'_i \neq 0$ in $E_{x_0} \otimes \kappa(y)$. Therefore we can see that $a_{ij} \in \mathcal{O}_{X,\gamma}$ for all i, j. In the same way, $b_{ij} \in \mathcal{O}_{X,\gamma}$ for all i, j.

By the above claim, $\{\omega_1, \ldots, \omega_r\}$ and $\{\omega'_1, \ldots, \omega'_r\}$ generate the same $\mathcal{O}_{X,\gamma}$-module in V. Thus $E_\gamma = E'_\gamma$. Hence we get $E = E'$.

Let x be an arbitrary closed point of X. Let $v, v' \in E_x \otimes \kappa(x)$. Choose $\omega, \omega' \in E_x$ such that ω and ω' give rise to v and v' in $E_x \otimes \kappa(x)$. Then there is a neighborhood U_x of x such that $h(\omega, \omega')(y) = h'(\omega, \omega')(y)$ for all $y \in U_x \cap U$. Thus

$$h(\omega, \omega')(x) = \lim_{y \to x} h(\omega, \omega')(y) = \lim_{y \to x} h'(\omega, \omega')(y) = h'(\omega, \omega')(x),$$

which means that $h_x(v, v') = h'_x(v, v')$. □

Proposition 1.4.2. *Let (E, h) be a C^∞-hermitian locally free coherent sheaf on X. Let $x_1, \ldots, x_r$ be a K-linearly independent elements of E_η. Then $\log(\det(h(x_i, x_j)))$ is a locally integrable function.*

Proof. Let W be a vector subspace of E_η generated by $x_1, \ldots, x_r$. By Proposition 1.3.1, there is a saturated $\mathcal{O}_X$-subsheaf F of E with $F_\eta = W$. First we assume that F and E/F are locally free. For a closed point $x \in X$, let $\{\omega_1, \ldots, \omega_r\}$ be a local basis of F_x. Then we can find a matrix $A = (a_{ij})$ such that $a_{ij} \in K$ for all i, j and $x_i = \sum_{j=1}^r a_{ij}\omega_j$ for all i. Then

$$\det(h(x_i, x_j)) = |\det(A)|^2 \det(h(\omega_i, \omega_j)).$$

Since F and E/F are locally free, $\det(h(\omega_i, \omega_j))$ is a non-zero C^∞-function around x and $\det(A)$ is a non-zero rational function on X. Thus $\log(\det(h(x_i, x_j)))$ is locally integrable around x.

In general, if we set $Q = E/F$, then there is a proper birational morphism $\mu : Y \to X$ of smooth algebraic varieties over $\mathbb{C}$ such that

$$\mu^*(Q)/(\text{the torsion part of } \mu^*(Q))$$

is locally free. We set $F' = \operatorname{Ker}(\mu^*(E) \to \mu^*(Q)/(\text{the torsion part of } \mu^*(Q)))$. Then F' and $\mu^*(E)/F'$ are locally free. Thus, since $F'_\eta = W$,

$$\log(\det(\mu^*(h)(x_i, x_j))) = \mu^*(\log(\det(h(x_i, x_j))))$$

is a locally integrable function on Y. Therefore so is $\log\det(h(x_i, x_j))$ on X by virtue of [3, Proposition 1.2.5] □

1.5. Arakelov geometry

For basic definitions concerning Arakelov geometry, we refer to [6, Section 1]. Let X be a projective arithmetic variety. We use several kinds of positivity of a C^∞-hermitian invertible sheaf on X (like ampleness, nefness and bigness) as defined in [6, Section 2]. Let $\overline{H} = (\overline{H}_1, \ldots, \overline{H}_d)$ be a sequence of nef C^∞-hermitian invertible sheaves on X, where $d = \dim X_{\mathbb{Q}}$. Note that the sequence is empty in the case of $d = 0$. We say $\overline{H}$ is *fine* if $(X; \overline{H}_1, \ldots, \overline{H}_d)$ gives rise to a fine polarization of the function field of X (for details, see [7, Section 6.1]). For example, if $\overline{H}_i$'s are nef and big, then $\overline{H}$ is fine. Finally we consider the following lemma.

Lemma 1.5.1. *Let X be a generically smooth arithmetic variety and U a Zariski open set of X with* $\operatorname{codim}(X \setminus U) \geq 2$*. Then the natural homomorphism*

$$\widehat{\operatorname{CH}}^1_D(X) \to \widehat{\operatorname{CH}}^1_D(U)$$

is injective.

Proof. Let (D, T) be an arithmetic cycle of codimension one on X. We assume that $(\left.D\right|_U, \left.T\right|_U) = \widehat{(\left.\phi\right|_U)}$ for some non-zero rational function ϕ on X. Then, since $\operatorname{codim}(X \setminus U) \geq 2$, we have $(D, T) = \widehat{(\phi)}$. □

2. Birationally C^∞-hermitian torsion free coherent sheaves on a normal arithmetic variety

Let X be a normal arithmetic variety. Let E be a torsion free coherent sheaf on X. We say a pair (E, h) is called a *birationally C^∞-hermitian torsion free coherent sheaf* on X if there are a proper birational morphism $\mu : X' \to X$ of normal arithmetic varieties, a C^∞-hermitian locally free coherent sheaf (E', h') on X', and a Zariski open set U of X with the following properties:

(1) X' and U are generically smooth.

(2) $\operatorname{codim}(X \setminus U) \geq 2$.

(3) $\mu : X' \to X$ is an isomorphism over U, that is, if we set $U' = \mu^{-1}(U)$, then $\left.\mu\right|_{U'} : U' \xrightarrow{\sim} U$.

(4) E is locally free on U and h is a C^∞-hermitian metric of $\left.E\right|_U$ over $U(\mathbb{C})$.

(5) $(\left.\mu\right|_{U'})^*(\left.(E, h)\right|_U)$ is isometric to $\left.(E', h')\right|_{U'}$.

This C^∞-hermitian locally free coherent sheaf (E', h') is called a *model of* (E, h) *in terms of* $\mu : X' \to X$. Note that if $\mu' : X'' \to X'$ is a proper birational morphism of normal and generically smooth arithmetic varieties, then ${\mu'}^*(E', h')$ is also a model of (E, h) in terms of $\mu \circ \mu' : X'' \to X$. For, let X'_0 be the maximal Zariski open set over which μ' is an isomorphism. Then $\operatorname{codim}(X' \setminus X_0) \geq 2$. Thus if we set $V = \mu(U' \cap X'_0)$, then we can see the above properties for V.

Proposition 2.1. *Let X be a normal arithmetic variety and (E, h) a birationally C^∞-hermitian torsion free coherent sheaf on X. Let F be a saturated $\mathcal{O}_X$-subsheaf of E. Let $h_{F \hookrightarrow E}$ (resp. $h_{E \twoheadrightarrow E/F}$) be the submetric of*

F induced by $F \hookrightarrow E$ and h (resp. the quotient metric of E/F induced by $E \twoheadrightarrow E/F$ and h) on a big Zariski open set of X, i.e., a Zariski open set whose complement has the codimension greater than or equal to 2. Then $(F, h_{F\hookrightarrow E})$ and $(E/F, h_{E\twoheadrightarrow E/F})$ are also birationally C^∞-hermitian torsion free coherent sheaves on X.

Proof. Let η be the generic point of X. Let (E', h') be a model of (E, h) in terms of $\mu : X' \to X$. Let F' be a saturated $\mathcal{O}_{X'}$-subsheaf F' of E' with $F'_\eta = F_\eta$ (cf. Proposition 1.3.1). We set $Q = E'/F'$. By [8, Theorem 1 in Chapter 4], there is a proper birational morphism $\mu' : X'' \to X'$ of normal and generically smooth arithmetic varieties such that ${\mu'}^*(Q)/(\text{torsion})$ is locally free. Let

$$F'' = \operatorname{Ker}({\mu'}^*(E') \to {\mu'}^*(Q)/(\text{torsion})).$$

Then F'' and ${\mu'}^*(E')/F''$ are locally free. Thus

$$(F'', {\mu'}^*(h')_{F''\hookrightarrow{\mu'}^*(E')}) \quad \text{and} \quad ({\mu'}^*(E')/F'', {\mu'}^*(h')_{{\mu'}^*(E')\twoheadrightarrow{\mu'}^*(E')/F''})$$

yield models of $(F, h_{F\hookrightarrow E})$ and $(E/F, h_{E\twoheadrightarrow E/F})$ respectively because ${\mu'}^*(E', h')$ gives rise to a model of (E, h). □

Proposition 2.2. *We assume that X is projective. Let $\overline{H} = (\overline{H}_1, \ldots, \overline{H}_d)$ be a sequence of nef C^∞-hermitian invertible sheaves on X, where $d = \dim X_{\mathbb{Q}}$. Then the quantity*

$$\widehat{\deg}(\widehat{c}_1(\mu^*(\overline{H}_1)) \cdots \widehat{c}_1(\mu^*(\overline{H}_d)) \cdot \widehat{c}_1(E', h'))$$

does not depend on the choice of a model (E', h') in terms of $\mu : X' \to X$. It is denoted by $\widehat{\deg}_{\overline{H}}(E, h)$ and is called the arithmetic degree of (E, h) with respect to $\overline{H}$.

Proof. Let us begin with the following lemma.

Lemma 2.3. *Let $\nu : Y \to X$ be a birational morphism of normal and projective arithmetic varieties such that Y is generically smooth. Let (E, h) and (E', h') be C^∞-hermitian locally free coherent sheaves on Y. We assume that there is a Zariski open set U of X such that $\operatorname{codim}(X \setminus U) \geq 2$ and ν is an isomorphism over U, that is, if we set $V = \nu^{-1}(U)$, then $\nu|_V : V \xrightarrow{\sim} U$. Let $\overline{L}_1, \ldots, \overline{L}_d$ be C^∞-hermitian invertible sheaves on X, where $d = \dim X_{\mathbb{Q}}$. If $(E, h)|_V$ is isometric to $(E', h')|_V$, then*

$$\widehat{\deg}(\widehat{c}_1(\nu^*(\overline{L}_1)) \cdots \widehat{c}_1(\nu^*(\overline{L}_d)) \cdot \widehat{c}_1(E, h)) = \widehat{\deg}(\widehat{c}_1(\nu^*(\overline{L}_1)) \cdots \widehat{c}_1(\nu^*(\overline{L}_d)) \cdot \widehat{c}_1(E', h')).$$

Proof. Let η be the generic point of Y and $x_1, \ldots, x_r$ a basis of E_η. Let $x'_1, \ldots, x'_r$ be the corresponding basis of E'_η with $x_1, \ldots, x_r$. Let $Y^{(1)}$ be the set of all codimension one points of Y. Then $\widehat{c}_1(E,h)$ and $\widehat{c}_1(E',h')$ are represented by

$$\left(\sum_{\gamma \in Y^{(1)}} \ell_{\mathcal{O}_{Y,\gamma}}(E; x_1, \ldots, x_r)\overline{\{\gamma\}},\ -\log(\det(h(x_i, x_j)))\right)$$

and

$$\left(\sum_{\gamma \in Y^{(1)}} \ell_{\mathcal{O}_{Y,\gamma}}(E'; x'_1, \ldots, x'_r)\overline{\{\gamma\}},\ -\log(\det(h'(x'_i, x'_j)))\right)$$

respectively. By Proposition 1.4.1, we can see that

$$\det(h(x_i, x_j)) = \det(h'(x'_i, x'_j))$$

on $Y(\mathbb{C})$. Here

$$\ell_{\mathcal{O}_{Y,\gamma}}(E; x_1, \ldots, x_r) = \ell_{\mathcal{O}_{Y,\gamma}}(E'; x'_1, \ldots, x'_r)$$

for all $\gamma \in V^{(1)}$. Moreover, for $\gamma \in Y^{(1)} \setminus V^{(1)}$, since $\operatorname{codim}(\nu(\overline{\{\gamma\}})) \geq 2$,

$$\widehat{\deg}(\widehat{c}_1(\nu^*(\overline{L}_1)) \cdots \widehat{c}_1(\nu^*(\overline{L}_d)) \cdot (\overline{\{\gamma\}}, 0)) = 0$$

by the projection formula (cf. [6, Proposition 1.2 and Proposition 1.3]). Thus we have our lemma. □

Let us go back to the proof of Proposition 2.2. Let (E_1, h_1) and (E_2, h_2) be two models of (E, h) in terms of $\mu_1 : X_1 \to X$ and $\mu_2 : X_2 \to X$ respectively. We can choose a normal, projective and generically smooth arithmetic variety Y and birational morphisms $\pi_1 : Y \to X_1$ and $\pi_2 : Y \to X_2$ with $\mu_1 \circ \pi_1 = \mu_2 \circ \pi_2$. We set $\nu = \mu_1 \circ \pi_1 = \mu_2 \circ \pi_2$. First of all, by the projection formula, we have

$$\begin{aligned}&\widehat{\deg}(\widehat{c}_1(\mu_1^*(\overline{H}_1)) \cdots \widehat{c}_1(\mu_1^*(\overline{H}_d)) \cdot \widehat{c}_1(E_1, h_1)) \\ &\qquad = \widehat{\deg}(\widehat{c}_1(\nu^*(\overline{H}_1)) \cdots \widehat{c}_1(\nu^*(\overline{H}_d)) \cdot \widehat{c}_1(\pi_1^*(E_1, h_1)))\end{aligned}$$

and

$$\begin{aligned}&\widehat{\deg}(\widehat{c}_1({\mu_2}^*(\overline{H}_1)) \cdots \widehat{c}_1({\mu_2}^*(\overline{H}_d)) \cdot \widehat{c}_1(E_2, h_2)) \\ &\qquad = \widehat{\deg}(\widehat{c}_1(\nu^*(\overline{H}_1)) \cdots \widehat{c}_1(\nu^*(\overline{H}_d)) \cdot \widehat{c}_1(\pi_2^*(E_2, h_2))).\end{aligned}$$

Moreover, by Lemma 2.3,

$$\begin{aligned}&\widehat{\deg}(\widehat{c}_1(\nu^*(\overline{H}_1)) \cdots \widehat{c}_1(\nu^*(\overline{H}_d)) \cdot \widehat{c}_1(\pi_1^*(E_1, h_1))) \\ &\qquad = \widehat{\deg}(\widehat{c}_1(\nu^*(\overline{H}_1)) \cdots \widehat{c}_1(\nu^*(\overline{H}_d)) \cdot \widehat{c}_1(\pi_2^*(E_2, h_2))).\end{aligned}$$

Thus we get the assertion. □

Let X be a normal arithmetic variety and (E, h) a birationally C^∞-hermitian torsion free sheaf on X. Let $\pi : X' \to X$ be a proper birational morphism of normal arithmetic varieties and (E', h') a birationally C^∞-hermitian torsion free sheaf on X'. We say (E, h) is *birationally dominated by* (E', h') *by means of* $\pi : X' \to X$ if there is a Zariski open set U of X with the following properties:

(1) $\operatorname{codim}(X \setminus U) \geq 2$ and U is generically smooth.

(2) (E, h) is a C^∞-hermitian locally free sheaf over U.

(3) If we set $U' = \pi^{-1}(U)$, then $\pi|_{U'} : U' \overset{\sim}{\longrightarrow} U$.

(4) $(\pi|_{U'})^*((E, h)|_U)$ is isometric to $(E', h')|_{U'}$.

Then we have the following:

Proposition 2.4. *The notation is the same as above. We assume that* (E, h) *is birationally dominated by* (E', h') *by means of* $\pi : X' \to X$.

(1) *Let F be a saturated $\mathcal{O}_X$-subsheaf of E and F' the corresponding saturated $\mathcal{O}_{X'}$-subsheaf of E' with F. Then $(F, h_{F \hookrightarrow E})$ and $(E/F, h_{E \twoheadrightarrow E/F})$ are birationally dominated by $(F', h'_{F' \hookrightarrow E'})$ and $(E'/F', h'_{E' \twoheadrightarrow E'/F'})$ respectively.*

(2) *We assume that X and X' are projective. Let $\overline{H} = (\overline{H}_1, \ldots, \overline{H}_d)$ be a sequence of nef C^∞-hermitian invertible sheaves on X, where $d = \dim X_{\mathbb{Q}}$. Then $\widehat{\deg}_{\overline{H}}(E, h) = \widehat{\deg}_{\pi^*(\overline{H})}(E', h')$.*

Proof. (1) There is a Zariski open set U_1 such that $U_1 \subseteq U$, $\operatorname{codim}(X \setminus U_1) \geq 2$ and that $E|_{U_1}$ and $E/F|_{U_1}$ are locally free. We set $U_1' = \pi^{-1}(U_1)$. Then $(\pi|_{U'})^*((F, h_{F \hookrightarrow E})|_{U_1})$ is isometric to $(F', h'_{F' \hookrightarrow E'})|_{U_1'}$. Thus our assertions follow.

(2) Let (E'', h'') be a model of (E', h') in terms of a birational morphism $\mu : Y \to X'$. Then it is easy to see that (E'', h'') is a model of (E, h) in terms of $\pi \circ \mu : Y \to X$. Thus we have (2) by Proposition 2.2. □

3. Finiteness of subsheaves with bounded arithmetic degree

In this section, we would like to give the proof of the main theorem of this note.

Theorem 3.1. *Let X be a normal projective arithmetic variety and (E, h) a birationally C^∞-hermitian torsion free coherent sheaf on X. Let $\overline{H} = (\overline{H}_1, \ldots, \overline{H}_d)$ be a fine sequence of nef C^∞-hermitian invertible sheaves on X, where $d = \dim X_{\mathbb{Q}}$. For any real number c, the set of all non-zero saturated $\mathcal{O}_X$-subsheaf F of E with $\widehat{\deg}_{\overline{H}}(\widehat{c}_1(F, h_{F \hookrightarrow E})) \geq c$ is finite, where $h_{F \hookrightarrow E}$ is the submetric of F induced by h over a big open set.*

Proof. Let (E', h') be a model of (E, h) in terms of $\mu : X' \to X$. Let η be the generic point of X. For each vector subspace W of E_η, let F (resp. F') be a saturated $\mathcal{O}_X$-subsheaf of E (resp. $\mathcal{O}_{X'}$-subsheaf of E') induced by W. Then, by Proposition 2.4,

$$\widehat{\deg}_{\overline{H}}(F, h_{F \hookrightarrow E}) = \widehat{\deg}_{\mu^*(\overline{H})}(F', h_{F' \hookrightarrow E'}).$$

Therefore we may assume that X is generically smooth, E is locally free and h is a C^{∞}-hermitian metric of E.

For each $0 < s < \operatorname{rk} E$, let $\Sigma_s(X, E)$ be the set of all saturated rank s $\mathcal{O}_X$-subsheaves of E. First let us see that, for any real number c, the set

$$\{L \in \Sigma_1(X, E) \mid \widehat{\deg}_{\overline{H}}(F, h_{F \hookrightarrow E}) \geq c\}$$

is finite. Let $\pi : P = \operatorname{Proj}(\bigoplus_{d \geq 0} \operatorname{Sym}^d(E^{\vee})) \to X$ be the projective bundle and $\mathcal{O}_P(1)$ the tautological line bundle of P. Let h_P be the quotient hermitian metric of $\mathcal{O}_P(1)$ by using the surjective homomorphism $\pi^*(E^{\vee}) \to \mathcal{O}_P(1)$ and the hermitian metric $\pi^*(h^{\vee})$. In other words, the metric h_P^{-1} of $\mathcal{O}_P(-1)$ is the submetric induced by the injective homomorphism $\mathcal{O}_P(-1) \to \pi^*(E)$ and $\pi^*(h)$ (cf. (3) of Proposition 1.1.3). Let P_η be the generic fiber of $\pi : P \to X$, and K the function field of X.

For a K-rational point x of P_η, let us introduce Δ_x, U_x, V_x and s_x as follows: Δ_x is the Zariski closure of x in P and U_x is the maximal open set of X over which $\pi|_{\Delta_x} : \Delta_x \to X$ is an isomorphism. Further $V_x = (\pi|_{\Delta_x})^{-1}(U_x)$ and $s_x : U_x \to P$ is the section induced by the isomorphism $\pi|_{V_x} : V_x \to U_x$

Let $\Sigma_1(K, E_\eta)$ be the set of all 1-dimensional vector subspaces of E_η over K. Then, by Proposition 1.3.3, there is a natural bijection

$$P_\eta(K) \to \Sigma_1(K, E_\eta).$$

Moreover let $\Sigma_1(X, E)$ be the set of all saturated rank one $\mathcal{O}_X$-subsheaves of E. By Proposition 1.3.1, we have a bijective map

$$\Sigma_1(X, E) \to \Sigma_1(K, E_\eta).$$

Therefore there is a natural bijection between $P_\eta(K)$ and $\Sigma_1(X, E)$. For a K-rational point x of P_η, the corresponding saturated rank one $\mathcal{O}_X$-subsheaf of E is denoted by $L(x)$. Then, by using Proposition 1.3.3, we can see that $L(x)$ has the following property: Let $s_x^*(\mathcal{O}_P(-1)) \to s_x^*\pi^*(E) = E|_{U_x}$ be the homomorphism from the natural homomorphism $\mathcal{O}_P(-1) \to \pi^*(E)$ by applying s_x^*. Then the image of $s_x^*(\mathcal{O}_P(-1)) \to E|_{U_x}$ is $L(x)|_{U_x}$. Let h_x be the submetric of $L(x)$ induced by h.

Claim 3.1.1. $\widehat{c}_1(L(x), h_x) = (\pi|_{\Delta_x})_* \left(\widehat{c}_1 \left((\mathcal{O}_P(-1), h_P^{-1})\big|_{\Delta_x} \right) \right).$

Since the metric h_P^{-1} is the submetric of $\mathcal{O}_P(-1)$ induced by $\pi^*(h)$, we can see that $s_x^*(\mathcal{O}_P(-1), h_P^{-1})$ is isometric to $(L(x), h_x)|_{U_x}$. Thus $(\mathcal{O}_P(-1), h_P^{-1})\big|_{V_x}$ is isometric to $(\pi|_{V_x})^*((L(x), h_x)|_{U_x})$, which implies that

$$\begin{aligned} (\pi|_{V_x})_* \left(\widehat{c}_1 \left((\mathcal{O}_P(-1), h_P^{-1})\big|_{V_x} \right) \right) &= (\pi|_{V_x})_* \left(\widehat{c}_1 \left((\pi|_{V_x})^*((L(x), h_x)|_{U_x}) \right) \right) \\ &= \widehat{c}_1((L(x), h_x)|_{U_x}). \end{aligned}$$

This means that the assertion of the claim holds over U_x. Thus so does over X by Lemma 1.5.1.

For a K-rational point x of P_η, the height $h_{\mathcal{O}(1)}(x)$ with respect to $\mathcal{O}_P(1)$ and $(X, \overline{H})$ is given by

$$h_{\mathcal{O}(1)}(x) = \widehat{\deg}\left(\widehat{c}_1((\pi|_{\Delta_x})^*(\overline{H}_1)) \cdots \widehat{c}_1((\pi|_{\Delta_x})^*(\overline{H}_d)) \cdot \widehat{c}_1\left((\mathcal{O}_P(1), h_P)|_{\Delta_x}\right)\right).$$

By using the above claim and the projection formula,

$$\begin{aligned} & - h_{\mathcal{O}_P(1)}(x) \\ & \quad = \widehat{\deg}\left(\widehat{c}_1((\pi|_{\Delta_x})^*(\overline{H}_1)) \cdots \widehat{c}_1((\pi|_{\Delta_x})^*(\overline{H}_d)) \cdot \widehat{c}_1\left((\mathcal{O}_P(-1), h_P^{-1})|_{\Delta_x}\right)\right) \\ & \quad = \widehat{\deg}\left(\widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_d) \cdot \widehat{c}_1(L(x), h_x)\right) = \widehat{\deg}_{\overline{H}}(L(x), h_x). \end{aligned}$$

Thus we have a bijective correspondence between

$$\{L \in \Sigma_1(X, E) \mid \widehat{\deg}_{\overline{H}}(F, h_{F \hookrightarrow E}) \geq c\}$$

and

$$\{x \in P_\eta(K) \mid h(x) \leq -c\}.$$

On the other hand, by virtue of Northcott's theorem over finitely generated field (cf. [6, Theorem 4.3]), $\{x \in P_\eta(K) \mid h(x) \leq -c\}$ is a finite set. Therefore we get the case where $s = 1$.

For $F \in \Sigma_s(X, E)$, let $\lambda(F)$ be the saturation of

$$\bigwedge^s F/(\text{the torsion part of } \bigwedge^s F)$$

in $\bigwedge^s E$.

Claim 3.1.2. *If $\lambda(F) = \lambda(F')$, then $F = F'$.*

We assume that $\lambda(F) = \lambda(F')$. Let K be the function field of X. Then, using Plücker coordinates over K, we can see that $F \otimes K = F' \otimes K$. Thus, by Lemma 1.3.2, $F' = F$.

Let $h_{\lambda(F)} = (\bigwedge^s h)_{\lambda(F) \hookrightarrow \bigwedge^s E}$. Then, by Proposition 1.1.4,

$$\widehat{c}_1(F, h_F) = \widehat{c}_1(\lambda(F), h_{\lambda(F)}).$$

Therefore, by using the above claim and the case where $s = 1$, our theorem follows. □

Let X be a normal and projective arithmetic variety and (E, h) a birationally C^∞-hermitian torsion free coherent sheaf on X. Let $\overline{H} = (\overline{H}_1, \ldots, \overline{H}_d)$ be a fine sequence of nef C^∞-hermitian invertible sheaves on X. For a non-zero saturated $\mathcal{O}_X$-subsheaf G of E, we set

$$\hat{\mu}_{\overline{H}}(G, h_{G \hookrightarrow E}) = \frac{\widehat{\deg}_{\overline{H}}(G, h_{G \hookrightarrow E})}{\operatorname{rk} G}.$$

A saturated $\mathcal{O}_X$-subsheaf F of E is called a *maximal slope sheaf of* (E,h) *with respect to* $\overline{H}$ if $\hat{\mu}_{\overline{H}}(F, h_{F\hookrightarrow E})$ gives rise to the maximal value of the set

$$\{\hat{\mu}_{\overline{H}}(G, h_{G\hookrightarrow E}) \mid G \text{ is a non-zero saturated } \mathcal{O}_X\text{-subsheaf of } E\}.$$

Moreover a maximal slope sheaf F of (E,h) is called a *maximal destabilizing sheaf of* (E,h) *with respect to* $\overline{H}$ if $\operatorname{rk} F$ is maximal among all maximal slope sheaves of (E,h). As a corollary of Theorem 3.1, we have the following:

Corollary 3.2. *There is a maximal destabilizing sheaf of* (E,h) *with respect to* $\overline{H}$.

4. Arithmetic first Chern class of a subsheaf

Let X be a normal and generically smooth arithmetic variety and η the generic point of X. Let (E,h) be a C^∞-hermitian locally free sheaf on X. Let F be an $\mathcal{O}_X$-subsheaf of E. Let $x_1,\ldots,x_r$ be a basis of F_η. Let us consider an arithmetic codimension one cycle $z(F;x_1,\ldots,x_r)$ (i.e., an element of $\in \widehat{Z}^1_D(X)$) given by

$$z(F;x_1,\ldots,x_r) = \left(\sum_{\Gamma} \ell_{\mathcal{O}_{X,\Gamma}}(F_\Gamma;x_1,\ldots,x_r)\Gamma, -\log\det(h(x_i,x_j))\right).$$

Note that $\log\det(h(x_i,x_j))$ is locally integrable on $X(\mathbb{C})$ by Proposition 1.4.2. Let $x'_1,\ldots,x'_r$ be another basis of F_η. There is an $r\times r$-matrix $A=(a_{ij})$ with $x'_i=\sum_{j=1}^r a_{ij}x_j$. Using (2) of Corollary 1.2.2, we can see that

$$z(F;x'_1,\ldots,x'_r) = z(F;x_1,\ldots,x_r) + (\widehat{\det(A)}).$$

Therefore the class of $z(F;x_1,\ldots,x_r)$ in $\widehat{\mathrm{CH}}^1_D(X)$ does not depend on the choice of $x_1,\ldots,x_r$. We denote the class of $z(F;x_1,\ldots,x_r)$ in $\widehat{\mathrm{CH}}^1_D(X)$ by $\widehat{c}_1(F\hookrightarrow E,h)$. If $F=E$, then $\widehat{c}_1(E\hookrightarrow E,h)$ is equal to the usual $\widehat{c}_1(E,h)$. Note that

$$\widehat{c}_1(F\hookrightarrow E,h) = \widehat{c}_1(F,h_{F\hookrightarrow E})$$

if F is saturated in E. More generally, we have the following:

Proposition 4.1. *Let F be an $\mathcal{O}_X$-subsheaf of E and $\widetilde{F}$ the saturation of F in E. Then $\widehat{c}_1(\widetilde{F},h_{\widetilde{F}\hookrightarrow E}) - \widehat{c}_1(F\hookrightarrow E,h)$ is represented by an arithmetic divisor*

$$\left(\sum_{\Gamma\,:\,\text{prime divisor}} \mathrm{length}_{\mathcal{O}_{X,\Gamma}}(\widetilde{F}_\Gamma/F_\Gamma)\Gamma, 0\right).$$

In particular, if $\overline{H}=(\overline{H}_1,\ldots,\overline{H}_d)$ is a sequence of nef C^∞-hermitian invertible sheaves on X, then

$$\widehat{\deg}(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\cdot\widehat{c}_1(F\hookrightarrow E,h)) \le \widehat{\deg}(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\cdot\widehat{c}_1(\widetilde{F},h_{\widetilde{F}\hookrightarrow E})).$$

Proof. Let η be the generic point of X. Let $\{x_1, \ldots, x_r\}$ be a basis of F_η. Then $\{x_1, \ldots, x_r\}$ also gives rise to a basis of $\widetilde{F}_\eta$. Thus $\widehat{c}_1(\widetilde{F}, h_{\widetilde{F}\hookrightarrow E}) - \widehat{c}_1(F \hookrightarrow E, h)$ is represented by

$$\left(\sum_\Gamma (\ell_{\mathcal{O}_{X,\Gamma}}(\widetilde{F}_\Gamma; x_1, \ldots, x_r) - \ell_{\mathcal{O}_{X,\Gamma}}(F_\Gamma; x_1, \ldots, x_r))\Gamma, 0\right).$$

Hence it is sufficient to see that

$$\ell_{\mathcal{O}_{X,\Gamma}}(\widetilde{F}_\Gamma; x_1, \ldots, x_r) - \ell_{\mathcal{O}_{X,\Gamma}}(F_\Gamma; x_1, \ldots, x_r) = \operatorname{length}_{\mathcal{O}_{X,\Gamma}}(\widetilde{F}_\Gamma / F_\Gamma)$$

for all Γ. Let a be an element of $\mathcal{O}_{X,\Gamma} \setminus \{0\}$ such that $ax_i \in \mathcal{O}_{X,\Gamma}$ for all i. Then

$$\ell_{\mathcal{O}_{X,\Gamma}}(\widetilde{F}_\Gamma; x_1, \ldots, x_r) = \operatorname{length}_{\mathcal{O}_{X,\Gamma}}(\widetilde{F}_\Gamma / \mathcal{O}_{X,\Gamma}ax_1 + \cdots + \mathcal{O}_{X,\Gamma}ax_r) - r \operatorname{ord}_\Gamma(a),$$

$$\ell_{\mathcal{O}_{X,\Gamma}}(F_\Gamma; x_1, \ldots, x_r) = \operatorname{length}_{\mathcal{O}_{X,\Gamma}}(F_\Gamma / \mathcal{O}_{X,\Gamma}ax_1 + \cdots + \mathcal{O}_{X,\Gamma}ax_r) - r \operatorname{ord}_\Gamma(a).$$

Therefore we get our proposition. □

5. Arithmetic Harder-Narasimham filtration

Let X be a normal and projective arithmetic variety and $\overline{H} = (\overline{H}_1, \ldots, \overline{H}_d)$ a fine sequence of nef C^∞-hermitian invertible sheaves. Let (E, h) be a birationally C^∞-hermitian torsion free coherent sheaf on X. (E, h) is said to be *arithmetically μ-semistable* with respect to $\overline{H}$ if, for any non-zero saturated $\mathcal{O}_X$-subsheaf F of E,

$$\hat{\mu}_{\overline{H}}(F, h_{F\hookrightarrow E}) \le \hat{\mu}_{\overline{H}}(E, h).$$

A filtration

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_l = E$$

of $\mathcal{O}_X$-subsheaves of E is called a *saturated filtration of* E if E_i/E_{i-1} is torsion free for every $1 \le i \le l$. Moreover we say a saturated filtration $0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_l = E$ of E is an *arithmetic Harder-Narasimham filtration of* (E, h) *with respect to* $\overline{H}$ if the following properties are satisfied:

(1) Let $h_{E_i/E_{i-1}}$ be a C^∞-hermitian metric of E_i/E_{i-1} induced by h, that is,

$$h_{E_i/E_{i-1}} = (h_{E_i\hookrightarrow E})_{E_i \twoheadrightarrow E_i/E_{i-1}} = (h_{E\twoheadrightarrow E/E_{i-1}})_{E_i/E_{i-1}\hookrightarrow E/E_{i-1}}.$$

Then $(E_i/E_{i-1}, h_{E_i/E_{i-1}})$ is arithmetically μ-semistable with respect to $\overline{H}$.

(2) $\hat{\mu}_{\overline{H}}(E_1/E_0, h_{E_1/E_0}) > \hat{\mu}_{\overline{H}}(E_2/E_1, h_{E_2/E_1}) > \cdots > \hat{\mu}_{\overline{H}}(E_l/E_{l-1}, h_{E_l/E_{l-1}})$.

In the case where X is generically smooth and (E, h) is a C^∞-hermitian locally free coherent sheaf on X, for a non-zero $\mathcal{O}_X$-subsheaf G of E, we set

$$\hat{\mu}_{\overline{H}}(G \hookrightarrow E, h) = \frac{\widehat{\deg}(\widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_d) \cdot \widehat{c}_1(G \hookrightarrow E, h))}{\operatorname{rk} G}.$$

The purpose of this section is to prove the following unique existence of an arithmetic Harder-Narasimham filtration:

Theorem 5.1. *Let X be a normal and projective arithmetic variety. Let (E, h) be a birationally C^∞-hermitian torsion free coherent sheaf on X. Let $\overline{H} = (\overline{H}_1, \dots, \overline{H}_d)$ be a fine sequence of nef C^∞-hermitian invertible sheaves. Then there exists uniquely an arithmetic Harder-Narasimham filtration of (E, h) with respect to $\overline{H}$. Moreover, if (E, h) is not arithmetically μ-semistable with respect to $\overline{H}$, then a maximal destabilizing sheaf of (E, h) is unique.*

We need several lemmas to prove the above theorem.

Lemma 5.2. *Let (E, h) and (E', h') be birationally C^∞-hermitian torsion free coherent sheaves on normal projective arithmetic varieties X and X' respectively. Let $\overline{H} = (\overline{H}_1, \dots, \overline{H}_d)$ be a fine sequence of nef C^∞-hermitian invertible sheaves on X. We assume that there is a birational morphism $\pi : X' \to X$ and (E, h) is dominated by (E', h') by means of $\pi : X' \to X$. Then we have the followings*:

(1) *(E, h) is arithmetically μ-semistable with respect to $\overline{H}$ if and only if so is (E', h') with respect to $\pi^*(\overline{H})$.*

(2) *Let F be a saturated $\mathcal{O}_X$-subsheaf of E and F' the corresponding saturated $\mathcal{O}_{X'}$-subsheaf of E'. Then F is a maximal destabilizing sheaf of (E, h) with respect to $\overline{H}$ if and only if so is F' with respect to $\pi^*(\overline{H})$.*

(3) *Let $0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_l = E$ be a saturated filtration of E and $0 = E'_0 \subsetneq E'_1 \subsetneq \cdots \subsetneq E'_l = E'$ the corresponding saturated filtration of E'. Then $0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_l = E$ is a Harder-Narasimham filtration with respect to $\overline{H}$ if and only if so is $0 = E'_0 \subsetneq E'_1 \subsetneq \cdots \subsetneq E'_l = E'$ with respect to $\pi^*(\overline{H})$.*

Proof. This is a consequence of Proposition 2.4. □

Lemma 5.3. *Let (E, h) be a birationally C^∞-hermitian torsion free coherent sheaf on a normal projective arithmetic variety X. If (E, h) is not arithmetically μ-semistable with respect to $\overline{H}$ and F is a maximal slope sheaf of (E, h), then*

$$\hat{\mu}_{\overline{H}}(F, h_{F \hookrightarrow E}) > \hat{\mu}_{\overline{H}}(E/F, h_{E \twoheadrightarrow E/F}).$$

Proof. We set $a = \mathrm{rk}(F)$ and $b = \mathrm{rk}(E/F)$. Then

$$\hat{\mu}_{\overline{H}}(E, h) = \frac{a}{a+b}\hat{\mu}_{\overline{H}}(F, h_{F \hookrightarrow E}) + \frac{b}{a+b}\hat{\mu}_{\overline{H}}(E/F, h_{E \twoheadrightarrow E/F}).$$

Thus, since $\hat{\mu}_{\overline{H}}(F, h_{F \hookrightarrow E}) > \hat{\mu}_{\overline{H}}(E, h)$, we get our lemma. □

Lemma 5.4. *Let (E, h) be a birationally C^∞-hermitian torsion free coherent sheaf on a normal projective arithmetic variety X. Let $\overline{H} = (\overline{H}_1, \dots, \overline{H}_d)$ be a fine sequence of nef C^∞-hermitian invertible sheaves. Then there are a* *model (E', h') of (E, h) in terms of a birational morphism $\mu : Y \to X$ of normal projective arithmetic varieties and a Harder-Narasimham filtration*

$$0 = E'_0 \subsetneq E'_1 \subsetneq \cdots \subsetneq E'_l = E'$$

of (E', h') with respect to $\mu^*(\overline{H})$ such that E'_i/E'_{i-1} is locally free for every $i = 1, \ldots, l$.

Proof. Let (E', h') be a model of (E, h) in terms of $\mu : Y \to X$. By Proposition 2.4, (E, h) is arithmetically μ-semistable with respect to $\overline{H}$ if and only if so is (E', h') with respect to $\mu^*(\overline{H})$. Thus we may assume that (E, h) is not arithmetically μ-semistable with respect to $\overline{H}$. Let E'_1 be a maximal destabilizing sheaf of (E', h'). Considering Proposition 2.4 and a suitable birational morphism $\mu' : Y' \to Y$ of normal, projective and generically smooth arithmetic varieties to remove the pinching points of E'/E'_1, we may assume that E'_1 and E'/E'_1 are locally free. If $(E'/E'_1, h'_{E' \twoheadrightarrow E'/E'_1})$ is arithmetically μ-semistable, then we are done. Otherwise, let E'_2 be a saturated $\mathcal{O}_Y$-subsheaf of E' such that $E'_1 \subsetneq E'_2$ and E'_2/E'_1 is a maximal destabilizing sheaf of $(E'/E'_1, h'_{E' \twoheadrightarrow E'/E'_1})$. Changing Y as before, we may assume that E'_2 and E'/E'_2 are locally free. Moreover, by Lemma 5.3,

$$\begin{aligned}\hat{\mu}_{\mu^*(\overline{H})}(E'_1, h_{E'_1 \hookrightarrow E'}) &= \hat{\mu}_{\mu^*(\overline{H})}(E'_1, (h_{E'_2 \hookrightarrow E})_{E'_1 \hookrightarrow E'_2}) \\ &> \hat{\mu}_{\mu^*(\overline{H})}(E'_2/E'_1, (h_{E'_2 \hookrightarrow E})_{E'_2 \twoheadrightarrow E'_2/E'_1}).\end{aligned}$$

Thus, continuing this construction, we have our lemma. □

Lemma 5.5. *Let (E, h) be a C^∞-hermitian locally free coherent sheaf on a normal projective and generically smooth arithmetic variety X. Let $\overline{H} = (\overline{H}_1, \ldots, \overline{H}_d)$ be a fine sequence of nef C^∞-hermitian invertible sheaves. Let $0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_l = E$ be an arithmetic Harder-Narasimham filtration of (E, h) such that E_i/E_{i-1} is locally free for every $i = 1, \ldots, l$. If F is a maximal slope sheaf of (E, h), then $F \subseteq E_1$ and $\hat{\mu}_{\overline{H}}(F \hookrightarrow E, h) = \hat{\mu}_{\overline{H}}(E_1 \hookrightarrow E, h)$.*

Proof. We choose i such that $F \subseteq E_i$ and $F \not\subseteq E_{i-1}$. We assume that $i \geq 2$. Let Q be the image of $F \to E_i/E_{i-1}$. Let h_Q be the quotient metric of Q induced by $h_{F \hookrightarrow E}$ and $F \to Q$, that is, $h_Q = (h_{F \hookrightarrow E})_{F \twoheadrightarrow Q}$. Then, by virtue of Lemma 1.1.2,

$$\hat{\mu}_{\overline{H}}(Q, h_Q) \leq \hat{\mu}_{\overline{H}}(Q \hookrightarrow E_i/E_{i-1}, h_{E_i/E_{i-1}}).$$

On the other hand, since $(F, h_{F \hookrightarrow E})$ and $(E_i/E_{i-1}, h_{E_i/E_{i-1}})$ are arithmetically μ-semistable,

$$\hat{\mu}_{\overline{H}}(F, h_{F \hookrightarrow E}) \leq \hat{\mu}_{\overline{H}}(Q, h_Q)$$

and

$$\hat{\mu}_{\overline{H}}(Q \hookrightarrow E_i/E_{i-1}, h_{E_i/E_{i-1}}) \leq \hat{\mu}_{\overline{H}}(E_i/E_{i-1}, h_{E_i/E_{i-1}}).$$

Therefore,

$$\hat{\mu}_{\overline{H}}(F, h_{F\hookrightarrow E}) \leq \hat{\mu}_{\overline{H}}(E_i/E_{i-1}, h_{E_i/E_{i-1}}) < \hat{\mu}_{\overline{H}}(E_1, h_{E_1\hookrightarrow E}),$$

which contradicts to the maximality of $\hat{\mu}_{\overline{H}}(F, h_{F\hookrightarrow E})$. Thus $F \subseteq E_1$. Moreover, since $(E_1, h_{E_1\hookrightarrow E})$ is arithmetically μ-semistable, $\hat{\mu}_{\overline{H}}(F, h_{F\hookrightarrow E}) \leq \hat{\mu}_{\overline{H}}(E_1, h_{E_1\hookrightarrow E})$. Therefore $\hat{\mu}_{\overline{H}}(F, h_{F\hookrightarrow E}) = \hat{\mu}_{\overline{H}}(E_1, h_{E_1\hookrightarrow E})$ by the maximality of $\hat{\mu}_{\overline{H}}(F, h_{F\hookrightarrow E})$. □

Let us start the proof of Theorem 5.1. The existence of a Harder-Narasimham filtration is a consequence of Lemma 5.4 and Proposition 2.4. Let us see the uniqueness of a Harder-Narasimham filtration. Clearly we may assume that (E, h) is not arithmetically μ-semistable. Let $0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_l = E$ and $0 = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_{l'} = E$ be Harder-Narasimham filtration of (E, h). Let (E', h') be a model of (E, h) in terms of $\mu : Y \to X$. Let $0 = E'_0 \subsetneq E'_1 \subsetneq \cdots \subsetneq E'_l = E'$ and $0 = G'_0 \subsetneq G'_1 \subsetneq \cdots \subsetneq G'_{l'} = E'$ be corresponding Harder-Narasimham filtration of (E', h') with $0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_l = E$ and $0 = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_{l'} = E$ respectively. By taking a birational morphism $\mu' : Y' \to Y$, we may assume that E'_i/E'_{i-1} and G'_j/G'_{j-1} are locally free for all $i = 1, \ldots, l$ and $j = 1, \ldots, l'$. Let F' be a maximal destabilizing sheaf of (E', h'). Then, by Lemma 5.5, $F' \subseteq E'_1$ and $\hat{\mu}_{\mu^*(\overline{H})}(F', h_{F'\hookrightarrow E'}) = \hat{\mu}_{\mu^*(\overline{H})}(E'_1, h_{E'_1\hookrightarrow E'})$. Thus $F' = E'_1$. In the same way, $F' = G'_1$. Hence, by considering a Harder-Narasimham filtration of $(E'/F', h_{E'\twoheadrightarrow E'/F'})$ and induction on the rank, we have $l = l'$ and $E'_i = G'_i$ for all i.

The above observation also show the uniqueness of a maximal destabilizing sheaf. □

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