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The Nielsen-Thurston classification of mapping classes is determined by TQFT

By

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Abstract

For each fixed $n \geq 2$ we show how the Nielsen-Thurston classification of mapping classes for a closed surface of genus $g \geq 2$ is determined by the sequence of quantum $SU(n)$-representations $(\rho_k)_{k\in\mathbb{N}}$. That this is the case is a consequence of the asymptotic faithfulness property proved in [A3]. We here provide explicit conditions on $(\rho_k(\phi))_{k\in\mathbb{N}}$, which determines the Nielsen-Thurston type of any mapping class ϕ.

1. Introduction

The Nielsen-Thurston classification of mapping classes of compact oriented surfaces splits mapping classes into three disjoint types [Th] (see also [FLP] and [BC]). We shall only be interested in the closed surface case here, so we state the Nielsen-Thurston theorem in this case.

Suppose Σ is a closed oriented surface of genus $g \geq 2$ and let Γ be the mapping class group of Σ.

Theorem 1 (Nielsen-Thurston). *A mapping class $\phi \in \Gamma$ has exactly one of the following three properties.*

1. *The mapping class ϕ is* finite order*, that is ϕ is a finite order element in Γ. This is equivalent to ϕ having an automorphism of a Riemann surface as representative.*
2. *The mapping class ϕ is not finite order, but it is* reducible*, meaning there exists a simple closed curve on the surface, whose non-trivial homotopy class is preserved by some power of ϕ.*
3. *The mapping class ϕ is* Pseudo-Anosov*, meaning that there exists $\lambda > 1$, two transverse measured foliations F^s and F^u on Σ and a diffeomorphism f of Σ, which represents ϕ, such that*

$$f_*(F^s) = \lambda^{-1}F^s \text{ and } f_*(F^u) = \lambda F^u.$$

In the Pseudo-Anosov case, λ is uniquely determined by ϕ and it is called the streching factor for ϕ.

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In the reducible case one continues the analysis of ϕ, by cutting Σ along the preserved simple closed curve, to get a mapping class of a surface with boundary. This mapping class is then classified in terms of the Nielsen-Thurston classification of mapping classes of surfaces with boundary. The upshot of this is that there is a diffeomorphism f of Σ, which represents ϕ and which preserve a system of simple closed curves, and when one cuts the surface along these curves, f induces a diffeomeorphism of the resulting cut surface. For each component of the cut surface, there is a smallest power of f, which preserves the component, and this power of f is a diffeomorphism of the component, which is either finite order or Pseudo-Anosov. See [FLP] for further details regarding this.

Fix an $n \geq 2$ and $d \in \mathbb{Z}/n\mathbb{Z}$. Let ρ_k be the quantum $SU(n)$ projective representations of the mapping class group Γ at level k (with twist d as described in [A3]), which arises from the Reshetikhin-Turaev $SU(n)$ Topological Quantum Field Theory. These were first discussed by Witten in [W1]. Then they were rigorously constructed by Reshetikhin and Turaev in [RT1] and [RT2]. Subsequently they were constructed using skein theory in [BHMV1], [BHMV2] and [B1].

The objective of this paper is to provide explicit conditions on the endomorphisms $\rho_k(\phi)$ which determines which Nielsen-Thurston type $\phi \in \Gamma$ belongs to.

Recall that we proved the asymptotic faithfulness property of the sequence ρ_k in [A3]:

Theorem 2. *Assume that n and d are coprime or that $(n, d) = (2, 0)$ when $g = 2$. Then we have that*

$$\bigcap_{k=1}^{\infty} \ker(\rho_k) = \begin{cases} \{1, H\} & g = 2,\ n = 2 \text{ and } d = 0 \\ \{1\} & \text{otherwise.} \end{cases}$$

where H is the hyperelliptic involution.

The non coprime cases is covered in [A4][*1].

Logically it follows from this Theorem, that the Nielsen-Thurston classification of a mapping class ϕ is determined by $(\rho_k(\phi))_{k\in\mathbb{N}}$, since this sequence determines ϕ itself. - However, we would here like to provide explicit conditions on $(\rho_k(\phi))_{k\in\mathbb{N}}$, which separates the three Nielsen-Thurston types.

The asymptotic faithfulness property gives us immediately the following theorem.

Theorem 3. *For any mapping class $\phi \in \Gamma$ we have that there exist an integer M such that*

$$(\rho_k(\phi))^M \in \mathbb{C}\,\mathrm{Id}$$

for all k if and only if $\phi^M = 1$ (or $\phi^{2M} = 1$, in case $(n, d) = (0, 2)$).

[*1] In [FWW] our argument from [A3] was translated to the BHMV-skein model. See also [M2].

This separates the finite order ones from the rest. In order to separate the reducibles from the Pseudo-Anosov, we recall the construction of the quantum representations and the Toeplitz operator construction from [A2] and [A3].

Let p a point on Σ. Let $M^{(d)}$ be the moduli space of flat $SU(n)$-connections on $\Sigma - p$ with holonomy $d \in \mathbb{Z}/n\mathbb{Z} \cong Z_{SU(n)}$ around p. Assume that n and d are coprime or that $(n,d) = (2,0)$ when $g = 2$. Then $M^{(d)}$ is a smooth manifold.

By applying geometric quantization at level k to the moduli space $M^{(d)}$ one gets a vector bundle $\mathcal{H}^{(k)}$ over Teichmüller space $\mathcal{T}$. The fiber of this bundle over a point $\sigma \in \mathcal{T}$ is $\mathcal{H}^{(k)}_\sigma = H^0(M^{(d)}_\sigma, \mathcal{L}^k_\sigma)$, where $M^{(d)}_\sigma$ is $M^{(d)}$ equipped with the complex structure induced from σ and $\mathcal{L}_\sigma$ is an ample generator of the Picard group of $M^{(d)}_\sigma$.

The main result pertaining to this bundle $\mathcal{H}^{(k)}$ is that its projectivization $\mathbb{P}(\mathcal{H}^{(k)})$ supports a natural flat connection. This is a result proved independently by Axelrod, Della Pietra and Witten [ADW] and by Hitchin [H2] (see also [A5], where it is proved that these two constructions gives the same connection). Now, since there is an action of the mapping class group Γ of Σ on $\mathcal{H}^{(k)}$ covering its action on $\mathcal{T}$, which preserves the flat connection in $\mathbb{P}(\mathcal{H}^{(k)})$, we get for each k a finite dimensional projective representation, say ρ_k, of Γ, namely on the covariant constant sections of $\mathbb{P}(\mathcal{H}^{(k)})$ over $\mathcal{T}$. This sequence of projective representations ρ_k, $k \in \mathbb{N}$ is the *quantum $SU(n)$ representations* of the mapping class group Γ.

For each $f \in C^\infty(M^{(d)})$ and each point $\sigma \in \mathcal{T}$, we have the *Toeplitz operator*

$$T^{(k)}_{f,\sigma} : H^0(M^{(d)}_\sigma, \mathcal{L}^k_\sigma) \to H^0(M^{(d)}_\sigma, \mathcal{L}^k_\sigma)$$

which is given by

$$T^{(k)}_{f,\sigma} = \pi^{(k)}_\sigma (fs)$$

for all $s \in H^0(M^{(d)}_\sigma, \mathcal{L}^k_\sigma)$, where $\pi^{(k)}_\sigma$ is the orthogonal projection onto $H^0(M^{(d)}_\sigma, \mathcal{L}^k_\sigma)$ induced from the L_2-inner product on $C^\infty(M^{(d)}, \mathcal{L}^k)$. We get smooth section of $\mathrm{End}(\mathcal{H}^{(k)})$ over $\mathcal{T}$

$$T^{(k)}_f \in C^\infty(\mathcal{T}, \mathrm{End}(\mathcal{H}^{(k)}))$$

by letting $T^{(k)}_f(\sigma) = T^{(k)}_{f,\sigma}$ (see [A3]).

The L_2-inner product on $C^\infty(M^{(d)}, \mathcal{L}^k)$ induces an inner product on $H^0(M^{(d)}_\sigma, \mathcal{L}^k_\sigma)$, which in turn induces the operator norm $\|\cdot\|$ on $\mathrm{End}(H^0(M^{(d)}_\sigma, \mathcal{L}^k_\sigma))$. Hence for any $A \in C^\infty(\mathcal{T}, \mathrm{End}(\mathcal{H}^{(k)}))$ we get the smooth function $\|A\|$ on $\mathcal{T}$.

Suppose now γ is a closed curve on the surface Σ. When we choose an orientation on γ, we have the holonomy function h_γ defined on $M^{(d)}$ associated to γ, given by taking the trace of the holonomy around γ. Note that h_γ only depends on the free homotopy class of γ. Further h_γ is constant if γ is nulhomotopic.

Theorem 4. *For any mapping class $\phi \in \Gamma$ and any homotopy class γ of a simple closed curve on Σ we have that ϕ is reducible along γ, i.e.*

$$\phi(\gamma) = \gamma$$

if and only if $T^{(k)}_{h_\gamma}$ (for any orientation of γ) asymptotically commutes with $\rho_k(\phi)$:

$$\lim_{k \to \infty} \|[\rho_k(\phi), T^{(k)}_{h_\gamma}]\| = 0.$$

Here the limit is pointwise convergence over $\mathcal{T}$. In fact, it follows from the results in [A3] that if the limit is zero at some point in $\mathcal{T}$ then it holds for all points in $\mathcal{T}$ and in fact the convergence is uniform on compact subsets of $\mathcal{T}$.

Using these two conditions, we see immediately how to determine the Nielsen-Thurston classification of a mapping class:

1. Given a mapping class ϕ, we first determine if it is finite order or not by checking if $\rho_k(\phi)$ is finite order bounded in k.

2. If not, we check through the non-trivial homotopy classes γ of simple closed curves to see if there exist an integer M with the property that $T^{(k)}_{h_\gamma}$ asymptotically commutes with $\rho_k(\phi^M)$:

$$\lim_{k \to \infty} ||[\rho_k(\phi^M), T^{(k)}_{h_\gamma}]|| = 0.$$

If so, ϕ is reducible.

3. If not, then ϕ is Pseudo-Anosov.

We expect the following should be true.

Conjecture 1 (AMU). For a mapping class $\phi \in \Gamma$ we have that ϕ is Pseudo-Anosov or reducible with Pseudo-Anosov pieces if and only if $\rho_k(\phi)$ is infinite order for large enough k.

Evidence for this conjecture is provided by [M1] and [AMU].

This paper is organized as follows. In Section 2 we recall the needed gauge theory setup. In the process we prove that the free homotopy class of a simple closed curve on a surface is determined by its holonomy function on the $SU(n)$ moduli space (for each $n \geq 2$). In Section 3 we recall the main analytic estimate from [A3], which states that the Toeplitz operators are asymptotically flat with respect to Hitchin's connection. The proof of Theorem 4 is given in Section 4. In the last section we translate the formulation of the results of this paper in the BHMV skein model for these TQFT's.

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2. The gauge theory construction of the quantum $SU(n)$ representations

Let us now very briefly recall the construction of the quantum $SU(n)$ representations. Only the details needed in this paper will be given. We refer e.g. to [H2], [A3] and [A5] for further details. As in the introduction we let Σ be a closed oriented surface of genus $g \geq 2$ and $p \in \Sigma$. Let P be a principal $SU(n)$-bundle over Σ. Clearly, all such P are trivializable. As above let $d \in \mathbb{Z}/n\mathbb{Z} \cong Z_{SU(n)}$. Throughout the rest of this paper we will assume that n and d are coprime, although in the case $g = 2$ we also allow $(n, d) = (2, 0)$. Let $M^{(d)}$ be the moduli space of flat $SU(n)$-connections in $P|_{\Sigma - p}$ with holonomy d around p. We can identify

$$M^{(d)} = \mathrm{Hom}_d(\tilde{\pi}_1(\Sigma), SU(n))/SU(n).$$

Here $\tilde{\pi}_1(\Sigma)$ is the universal central extension

$$0 \to \mathbb{Z} \to \tilde{\pi}_1(\Sigma) \to \pi_1(\Sigma) \to 1$$

as discussed in [H2] and in [AB] and Hom_d means the space of homomorphisms from $\tilde{\pi}_1(\Sigma)$ to $SU(n)$ which send the image of $1 \in \mathbb{Z}$ in $\tilde{\pi}_1(\Sigma)$ to d (see [H2]). When n and d are coprime, $M^{(d)}$ is a compact smooth manifold of dimension $m = (n^2 - 1)(g - 1)$. In general, when n and d are not coprime $M^{(d)}$ is not smooth, except in the case where $g = 2$, $n = 2$ and $d = 0$. In this case $M^{(d)}$ is in fact diffeomorphic to $\mathbb{CP}^3$. There is a natural homomorphism from the mapping class group to the outer automorphisms of $\tilde{\pi}_1(\Sigma)$, hence Γ acts on $M^{(d)}$.

We have that $M^{(d)}$ is a component of the real slice in $\mathcal{M}^{(d)}$, the moduli space of flat $SL(n, \mathbb{C})$-connections on $\Sigma - p$, whose holonomy around p equals $A_d = \exp(2\pi\sqrt{-1}\frac{d}{n}) \times \mathrm{Id}$. See [H1] for a thorough discussion of the spaces $\mathcal{M}^{(d)}$ from the point of view of Higgs bundles and for a proof that these spaces are connected.

Suppose γ is a closed oriented curve on $\Sigma - p$. When we choose an orientation on γ, we have the algebraic holonomy function h_γ defined on the variety $\mathcal{M}^{(d)}$. We are in particular interested in the restriction of h_γ to $M^{(d)}$. We observe that this restriction map is injective, since $M^{(d)}$ is a real slice in $\mathcal{M}^{(d)}$.

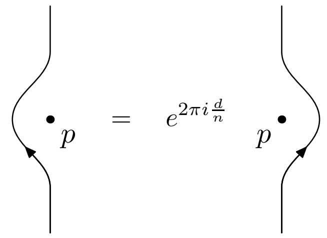
Proposition 1. *If γ is a simple closed curve on $\Sigma - p$, then the holonomy function h_γ (for any choice of orientation of γ) restricted to $M^{(d)}$ determines γ up to free homotopy on Σ.*

We will use this proposition in the proof of Theorem 4 below.

Proof. In the proof we will need the following results of Sikora [Si].

Let $\mathbb{A}_n(\Sigma)$ and $\mathbb{A}_n(\Sigma - p)$ be the n-th skein algebra of Σ respectively $\Sigma - p$ as defined in Definition 3.2 in [Si]. Let $S_n(\Sigma) = \mathbb{A}_n(\Sigma)/\sqrt{0}$ and $S_n(\Sigma - p) = \mathbb{A}_n(\Sigma - p)/\sqrt{0}$ be the quotients of the n-th skein algebras by their nilradicals. One of the main results on [Si] is that $S_n(\Sigma)$ and $S_n(\Sigma - p)$ are isomorphic to the algebra of algebraic functions on $\mathcal{M}^{(0)}$ respectively $\mathcal{M}$, the moduli space of all flat $SL(n,\mathbb{C})$-connections on $\Sigma - p$. These two isomorphisms, which we will simply also denote h, are induced by associating to an oriented closed curve γ its holonomy function h_γ. This is the content of Corollary 6.1 in[Si].

Consider the following local relation in a neighborhood of p:



Let R_d be the radical of the ideal in $S_n(\Sigma - p)$ generated by this relation. It is clear that $h(R_d)$ vanishes along $\mathcal{M}_d$ by construction.

Claim 1. The algebra homomorphism h from $S_n(\Sigma - p)/R_d$ to the algebra of algebraic function on $\mathcal{M}^{(d)}$ is an isomorphism.

Proof. Pick a point q on the boundary of an embedded disk D in Σ center at p. Choose a set of generators $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ of $\pi_1(\Sigma - p, q)$ such that $\prod_{i=1}^g [\alpha_i, \beta_i]$ is represented by the oriented curve ∂D based at q.

The choice of generators induces an algebra isomorphism (by definition)

$$\mathcal{O}(\mathcal{M}) \cong \mathcal{O}(SL(n,\mathbb{C})^{\times 2g})^{SL(n,\mathbb{C})}.$$

Consider the map

$$\mu : SL(n,\mathbb{C})^{\times 2g} \to SL(n,\mathbb{C})$$

given by

$$\mu(A_i, B_i) = \prod_{i=1}^{g} [A_i, B_i].$$

Then the generators (α_i, β_i) induces an algebra isomorphism

$$\mathcal{O}(\mathcal{M}_d) \cong \mathcal{O}(SL(n,\mathbb{C})^{\times 2g})^{SL(n,\mathbb{C})} / \mu^* I_d$$

where I_d is the ideal in the algebra of $SL(n,\mathbb{C})$ invariant algebraic functions on $SL(n,\mathbb{C})$, which vanishes at A_d.

We observe that $h(R_d) \subset \mu^*(I_d)$, since A_d is central and

$$\operatorname{Tr}(A_d X) = \exp(2\pi i \frac{d}{n}) \operatorname{Tr}(X)$$

for all $X \in SL(n,\mathbb{C})$. By general invariant theory (see e.g. [Pr]) all $SL(n,\mathbb{C})$ invariant algebraic functions on $SL(n,\mathbb{C})$ is of the form

$$f_P(X) = P(\operatorname{Tr}(X), \operatorname{Tr}(X^2), \dots, \operatorname{Tr}(X^{n-1})),$$

where P is an arbitrary complex polynomial in $n-1$ variables.

Assume now $f_P(A_d) = 0$. Let $\tau_i^{(d)} = \mathrm{Tr}(A_d^i)$ for $i = 1, \dots, n-1$. Then $P(\tau_1^{(d)}, \dots, \tau_{n-1}^{(d)}) = 0$. But then there exists a polynomial Q with zero constant term such that

$$P(t_1, \dots, t_{n-1}) = Q(t_1 - \tau_1^{(d)}, \dots, t_{n-1} - \tau_{n-1}^{(d)})$$

so

$$f_p(X) = Q(\mathrm{Tr}(X) - \mathrm{Tr}(A_d), \dots, \mathrm{Tr}(X^{n-1}) - \mathrm{Tr}(A_d^{n-1})).$$

Let $g_i^{(d)}(X) = \mathrm{Tr}(X^i) - \mathrm{Tr}(A_d^i)$. It is immediate that $\mu^* g_i^{(d)} \in h(R_d)$. But then $f_P \in h(R_d)$, which completes the proof of the claim.

Returning to the proof of the proposition, suppose we now have two simple oriented closed curves γ_1 and γ_2, such that $h_{\gamma_1} = h_{\gamma_2}$ on $M^{(d)}$ (for some choice of orientation of γ_1 and γ_2). Then we also have that $h_{\gamma_1} = h_{\gamma_2}$ on $\mathcal{M}^{(d)}$ since $M^{(d)}$ is a real slice in $\mathcal{M}^{(d)}$ and $\mathcal{M}^{(d)}$ is connected. But then we get that $[\gamma_1] = [\gamma_2]$ in $S_n(\Sigma - p)/R_d$. This implies therefore that $[\gamma_1]^n = [\gamma_2]^n$ in $S_n(\Sigma - p)/R_0$. Hence $h^n_{\gamma_1} = h^n_{\gamma_2}$ on $\mathcal{M}^{(0)}$.

Now recall that the natural map from $\mathrm{End}(\mathbb{C}^2)$ to $\mathrm{End}(S^{n-1}(\mathbb{C}^2))$ induces a group homomorphism from $SL(2, C)$ to $SL(n, C)$, which gives an embedding of Teichmüller space of Σ into $\mathcal{M}^{(0)}$. But then from this we get that $l_{\gamma_1} = l_{\gamma_2}$, where l_{γ_i} is the length function of γ_i.

If the geometric intersection number of the homotopy classes of the two curves is positive, we see that the two length functions cannot possibly be identical: Consider a sequence of metrics where the length of say γ_1 goes to zero. Then the length of γ_2 most go to infinity, so we get a contradiction.

Now suppose the geometric intersection number of the homotopy classes of the two curves is zero. Then we can vary the length of the two curves completely independently, if they are not freely homotopic on Σ. Hence they must be freely homotopic on Σ. □

Continuing towards the gauge theory definition of the quantum $SU(n)$ representations, we now choose an invariant bilinear form $\{\cdot, \cdot\}$ on $\mathfrak{g} = \mathrm{Lie}(SU(n))$, normalized such that $-\frac{1}{6}\{\vartheta \wedge [\vartheta \wedge \vartheta]\}$ is a generator of the image of the integer cohomology in the real cohomology in degree 3 of $SU(n)$, where ϑ is the $\mathfrak{g}$-valued Maurer-Cartan 1-form on $SU(n)$.

This bilinear form induces a symplectic form on $M^{(d)}$. In fact

$$T_{[A]} M^{(d)} \cong H^1(\Sigma, d_A),$$

where A is any flat connection in P representing a point in $M^{(d)}$ and d_A is the induced covariant derivative in the associated adjoint bundle. Using this identification, the symplectic form on $M^{(d)}$ is:

$$\omega(\varphi_1, \varphi_2) = \int_\Sigma \{\varphi_1 \wedge \varphi_2\},$$

where φ_i are d_A-closed 1-forms on Σ with values in $\operatorname{ad} P$. See e.g. [H2] for further details on this. The natural action of Γ on $M^{(d)}$ is symplectic.

Let $\mathcal{L}$ be the Hermitian line bundle over $M^{(d)}$ and ∇ the compatible connection in $\mathcal{L}$ constructed by Freed in [Fr]. This is the content of Corollary 5.22, Proposition 5.24 and equation (5.26) in [Fr] (see also the work of Ramadas, Singer and Weitsman [RSW]). By Proposition 5.27 in [Fr] we have that the curvature of ∇ is $\frac{\sqrt{-1}}{2\pi}\omega$. We will also use the notation ∇ for the induced connection in $\mathcal{L}^k$, where k is any integer.

By an almost identical construction, we can lift the action of Γ on $M^{(d)}$ to act on $\mathcal{L}$ such that the Hermitian connection is preserved (See e.g. [A1]). In fact, since $H^2(M^{(d)}, \mathbb{Z}) \cong \mathbb{Z}$ and $H^1(M^{(d)}, \mathbb{Z}) = 0$, it is clear that the action of Γ leaves the isomorphism class of $(\mathcal{L}, \nabla)$ invariant, thus alone from this one can conclude that a central extension of Γ acts on $(\mathcal{L}, \nabla)$ covering the Γ action on $M^{(d)}$. This is actually all we need in this paper, since we are only interested in the projectivized action.

Let now $\sigma \in \mathcal{T}$ be a complex structure on Σ. Let us review how σ induces a complex structure on $M^{(d)}$ which is compatible with the symplectic structure on this moduli space. The complex structure σ induces a $*$-operator on 1-forms on Σ and via Hodge theory we get that

$$H^1(\Sigma, d_A) \cong \ker(d_A + *d_A*).$$

On this kernel, consisting of the harmonic 1-forms with values in $\operatorname{ad} P$, the $*$-operator acts and its square is -1, hence we get an almost complex structure on $M^{(d)}$ by letting $I = I_\sigma = -*$. It is a classical result by Narasimhan and Seshadri (see [NS1]), that this actually makes $M^{(d)}$ a smooth Kähler manifold, which as such, we denote $M_\sigma^{(d)}$. By using the $(0,1)$ part of ∇ in $\mathcal{L}$, we get an induced holomorphic structure in the bundle $\mathcal{L}$. The resulting holomorphic line bundle will be denoted $\mathcal{L}_\sigma$. See also [H2] for further details on this.

From a more algebraic geometric point of view, we consider the moduli space of S-equivalence classes of semi-stable bundles of rank n and determinant isomorphic to the line bundle $\mathcal{O}(d[p])$. By using Mumford's Geometric Invariant Theory, Narasimhan and Seshadri (see [NS2]) showed that this moduli space is a smooth complex algebraic projective variety which is isomorphic as a Kähler manifold to $M_\sigma^{(d)}$. Referring to [DN] we recall that

Theorem 5 (Drezet & Narasimhan). *The Picard group of $M_\sigma^{(d)}$ is generated by the holomorphic line bundle $\mathcal{L}_\sigma$ over $M_\sigma^{(d)}$ constructed above*:

$$\operatorname{Pic}(M_\sigma^{(d)}) = \langle \mathcal{L}_\sigma \rangle.$$

Definition 1. The bundle $\mathcal{H}^{(k)}$ over Teichmüller space is by definition the bundle whose fiber over $\sigma \in \mathcal{T}$ is $H^0(M_\sigma^{(d)}, \mathcal{L}_\sigma^k)$, where k is a positive integer.

There is a projective flat connection $\hat{\nabla}$ in the bundle $\mathcal{H}^{(k)}$ over Teichmüller space, which is Γ-invariant due to Axelrod, Della Pietra and Witten's and

Hitchin's. We refer to [H2] and [ADW] for further details. See also [A3] and [A5]. Faltings has established this theorem in the case where one replaces $SU(n)$ with a general semi-simple Lie group (see [Fal]). We further remark about genus 2, that [ADW] covers this case, but [H2] excludes this case, however, the work of Van Geemen and De Jong [vGdJ] extends Hitchin's approach to the genus 2 case.

Definition 2. The quantum $SU(n)$ projective representation ρ_k at level k of the mapping class group is given by Γ's action on the covariant constant sections of $(\mathbb{P}(\mathcal{H}^{(k)}), \hat{\nabla})$. We use the notation $\mathbb{P}(Z^{(k)}(\Sigma))$ for this projective space of covariant constant sections.

We denote the induced flat connection in the endomorphism bundle $\mathrm{End}(\mathcal{H}^{(k)})$ by $\hat{\nabla}^e$.

Definition 3. Let $\mathcal{E}^{(k)}(\Sigma)$ be the finite dimensional algebra of covariant constant sections of $(\mathrm{End}(\mathcal{H}^{(k)}), \hat{\nabla}^e)$ over $\mathcal{T}$ and let

$$\rho_k^e : \Gamma \to \mathrm{Aut}(\mathcal{E}^{(k)})$$

be the corresponding representation.

3. Toeplitz operators and their asymptotic flatness

On $C^\infty(M^{(d)}, \mathcal{L}^k)$ we have the L_2-inner product:

$$\langle s_1, s_2 \rangle = \frac{1}{m!} \int_{M^{(d)}} (s_1, s_2) \omega^m$$

where $s_1, s_2 \in C^\infty(M^{(d)}, \mathcal{L}^k)$ and $(\cdot, \cdot)$ is the fiberwise Hermitian structure in $\mathcal{L}^k$.

Now let $\sigma \in \mathcal{T}$. Then this L_2-inner product gives the orthogonal projection

$$\pi_\sigma^{(k)} : C^\infty(M^{(d)}, \mathcal{L}^k) \to H^0(M_\sigma^{(d)}, \mathcal{L}_\sigma^k).$$

For each $f \in C^\infty(M^{(d)})$ consider the associated *Toeplitz operator* $T_{f,\sigma}^{(k)}$ given as the composition of the multiplication operator

$$M_f : H^0(M_\sigma^{(d)}, \mathcal{L}_\sigma^k) \to C^\infty(M^{(d)}, \mathcal{L}^k)$$

with the orthogonal projection $\pi_\sigma^{(k)}$:

$$T_{f,\sigma}^{(k)}(s) = \pi_\sigma^{(k)}(fs).$$

Then $T_{f,\sigma}^{(k)} \in \mathrm{End}(H^0(M_\sigma^{(d)}, \mathcal{L}_\sigma^k))$, and we get a smooth section

$$T_f^{(k)} \in C^\infty(\mathcal{T}, \mathrm{End}(\mathcal{H}^{(k)}))$$

by letting $T_f^{(k)}(\sigma) = T_{f,\sigma}^{(k)}$ (see [A3]).

The L_2-inner product on $C^\infty(M^{(d)}, \mathcal{L}^k)$ induces an inner product on $H^0(M_\sigma^{(d)}, \mathcal{L}_\sigma^k)$, which in turn induces the operator norm $\|\cdot\|$ on $\mathrm{End}(H^0(M_\sigma^{(d)}, \mathcal{L}_\sigma^k))$.

We need the following Theorem on Toeplitz operators due to Bordemann, Meinrenken and Schlichenmaier (see [BMS], [Sch], [Sch1] and [Sch2]).

Theorem 6 (Bordemann, Meinrenken and Schlichenmaier)**.** *For any $f \in C^\infty(M^{(d)})$ we have that*

$$\lim_{k\to\infty} \|T_{f,\sigma}^{(k)}\| = \sup_{x\in M^{(d)}} |f(x)|.$$

Since the association of the sequence of Toeplitz operators $T_{f,\sigma}^k$, $k \in \mathbb{Z}_+$ is linear in f, we see from this Theorem, that this association is faithful.

Theorem 10 in [A3] tells us that the Toeplitz operators are asymptotically flat with respect to $\hat{\nabla}^e$:

Theorem 7. *Let σ_0 and σ_1 be two points in Teichmüller space and P_{σ_0,σ_1} be the parallel transport in the flat bundle $(\mathrm{End}(\mathcal{H}^{(k)}), \hat{\nabla}^e)$ from σ_0 to σ_1. Then*

$$\|P_{\sigma_0,\sigma_1} T_{f,\sigma_0}^{(k)} - T_{f,\sigma_1}^{(k)}\| = O(k^{-1}),$$

where $\|\cdot\|$ is the operator norm on $H^0(M_{\sigma_1}^{(d)}, \mathcal{L}_{\sigma_1}^k)$.

4. Proof of Theorem 4

First we establish the following Proposition.

Proposition 2. *For any $\phi \in \Gamma$, $f \in C^\infty(M^{(d)})$ and $\sigma \in \mathcal{T}$, we have that*

$$\lim_{k\to\infty} \|T_{f,\sigma}^{(k)} - \rho_k(\phi) T_{f,\sigma}^{(k)} \rho_k(\phi^{-1})\| = \lim_{k\to\infty} \|T_{(f - f\circ\phi),\sigma}^{(k)}\|.$$

Proof. Suppose we have a $\phi \in \Gamma$. Then ϕ induces a symplectomorphism of $M^{(d)}$ which we also just denote ϕ and we get the following commutative diagram for any $f \in C^\infty(M^{(d)})$

$$\begin{tikzcd}
H^0(M_\sigma^{(d)}, \mathcal{L}_\sigma^k) \arrow[r, "\phi^*"] \arrow[d, "T_{f,\sigma}^{(k)}"'] & H^0(M_{\phi(\sigma)}^{(d)}, \mathcal{L}_{\phi(\sigma)}^k) \arrow[r, "P_{\phi(\sigma),\sigma}"] \arrow[d, "T_{f\circ\phi,\phi(\sigma)}^{(k)}"'] & H^0(M_\sigma^{(d)}, \mathcal{L}_\sigma^k) \arrow[d, "P_{\phi(\sigma),\sigma} T_{f\circ\phi,\phi(\sigma)}^{(k)}"] \\
H^0(M_\sigma^{(d)}, \mathcal{L}_\sigma^k) \arrow[r, "\phi^*"] & H^0(M_{\phi(\sigma)}^{(d)}, \mathcal{L}_{\phi(\sigma)}^k) \arrow[r, "P_{\phi(\sigma),\sigma}"] & H^0(M_\sigma^{(d)}, \mathcal{L}_\sigma^k),
\end{tikzcd}$$

where $P_{\phi(\sigma),\sigma} : H^0(M_{\phi(\sigma)}^{(d)}, \mathcal{L}_{\phi(\sigma)}^k) \to H^0(M_\sigma^{(d)}, \mathcal{L}_\sigma^k)$ on the horizontal arrows refer to parallel transport in the bundle $\mathcal{H}^{(k)}$, whereas $P_{\phi(\sigma),\sigma}$ refers to the parallel transport in the endomorphism bundle $\mathrm{End}(\mathcal{H}^{(k)})$ in the last vertical arrow. By the definition of ρ_k, we see that

$$\rho_k(\phi)T^{(k)}_{f,\sigma}\rho_k(\phi^{-1}) = P_{\phi(\sigma),\sigma}T^{(k)}_{f\circ\phi,\phi(\sigma)}.$$

By Theorem 7 we get that

$$\begin{aligned}\lim_{k\to\infty}\|T^{(k)}_{(f-f\circ\phi),\sigma}\| &= \lim_{k\to\infty}\|T^{(k)}_{f,\sigma} - T^{(k)}_{f\circ\phi,\sigma}\|\\ &= \lim_{k\to\infty}\|T^{(k)}_{f,\sigma} - P_{\phi(\sigma),\sigma}T^{(k)}_{f\circ\phi,\phi(\sigma)}\|\\ &= \lim_{k\to\infty}\|T^{(k)}_{f,\sigma} - \rho_k(\phi)T^{(k)}_{f,\sigma}\rho_k(\phi^{-1})\|.\end{aligned}$$

□

Proof of Theorem 4. Let $\phi \in \Gamma$ and assume that there exists γ non nulhomotopic simple closed curve on Σ and $\sigma \in \mathcal{T}$, such that

$$\lim_{k\to\infty}\|[\rho_k(\phi), T^{(k)}_{h_\gamma,\sigma}]\| = 0.$$

We have that

$$\begin{aligned}\|T^{(k)}_{h_\gamma,\sigma} - \rho_k(\phi^{-1})T^{(k)}_{h_\gamma,\sigma}\rho_k(\phi)\| &= \|\rho_k(\phi^{-1})\rho_k(\phi)(T^{(k)}_{h_\gamma,\sigma} - \rho_k(\phi^{-1})T^{(k)}_{h_\gamma,\sigma}\rho_k(\phi))\|\\ &\leq \|\rho_k(\phi^{-1})\|\|[\rho_k(\phi), T^{(k)}_{h_\gamma,\sigma}]\|.\end{aligned}$$

Lemma 1 and Proposition 2 in [A3] gives a uniform bound on $\|\rho_k(\phi^{-1})\|$. Hence Proposition 2 implies that

$$\lim_{k\to\infty}\|T^{(k)}_{(h_\gamma - h_\gamma\circ\phi^{-1}),\sigma}\| = 0.$$

Then by Bordemann, Meinrenken and Schlichenmaier's Theorem 6, we must have that $h_\gamma = h_\gamma \circ \phi^{-1}$. But so by Proposition 1, we must have that ϕ preserved the homotopy class of γ.

The converse follows directly from Proposition 2, since $h_\gamma = h_\gamma \circ \phi$, if $\phi(\gamma) = \gamma$. □

5. The formulation in the BHMV-skein model

In this section we provide a translation of our conditions for separating the Nielsen-Thurston types of mapping classes to the BHMV-skein model. The determination of the finite order elements, Theorem 3 needs no translation, since it follows directly from the asymptotic faithfulness property. We just need to translate the condition in Theorem 4 which determines when a mapping class is reducible.

Let V_p be the TQFT defined by Blanchet, Habegger, Masbaum and Vogel in [BHMV1] and [BHMV2], where $p = 2r$, r being an integer and we choose the $2p$ root of 1 to be $A = e^{i\pi/2r}$. In particular there is also a projective representation

$$V_p : \Gamma \to \mathrm{Aut}(\mathbb{P}(V_p(\Sigma)))$$

for each p.

Let $\mathcal{S}(\Sigma)$ be the free $\mathbb{C}$-vector space generated by isotopy classes of one-dimensional sub-manifolds of Σ. We note that $\mathcal{S}(\Sigma)$ is a subspace of the free $\mathbb{C}$-vector space generated by links in $\Sigma \times [0,1]$.

Since $V_p(\partial(\Sigma \times [0,1])) = \mathrm{End}(V_p(\Sigma))$, we get by the very construction of the BHMV-skein model of this TQFT a sequence of linear maps

$$V_p : \mathcal{S}(\Sigma) \to \mathrm{End}(V_p(\Sigma)).$$

Theorem 8. *For any mapping class $\phi \in \Gamma$ and any non-trivial homotopy class γ of a simple closed curve on Σ we have that ϕ is reducible along γ, i.e.*

$$\phi(\gamma) = \gamma$$

if and only if

$$[V_p(\phi), V_p(\gamma)] = 0$$

for all $p = 2r$.

Our joint work with K. Ueno [AU1], [AU2] and [AU3], combined with the work of Laszlo [La1] gives the following result for the $(n,d) = (2,0)$ theory:

Theorem 9. *There is a projective linear isomorphism of representations of Γ*

$$I_k : \mathbb{P}(V_{2k+4}(\Sigma)) \to \mathbb{P}(Z^{(k)}(\Sigma)).$$

The projective linear isomorphism I_k induces a linear isomorphism of representations of Γ

$$I_k^e : \mathrm{End}(V_{2k+4}(\Sigma)) \to \mathcal{E}^{(k)}(\Sigma).$$

In [A6] we prove that

Theorem 10. *For any non-trivial homotopy class γ of a simple closed curve on Σ we have that*

$$\lim_{k \to \infty} \| I_k^e(V_{2k+4}(\gamma)) - T_{h_\gamma}^{(k)} \| = 0.$$

Proof of Theorem 8. Let $\phi \in \Gamma$ be a mapping class. Assume there exists γ non nul-homotopic simple closed curve on Σ such that

$$[V_{2k+4}(\phi), V_{2k+4}(\gamma)] = 0$$

for all $k \in \mathbb{N}$. Then it follows immediately that

$$\lim_{k \to \infty} \| [I_k^e(V_{2k+4}(\phi)), T_{h_\gamma}^{(k)}] \| = 0$$

by Theorem 10. But since $I_k^e(V_{2k+4}(\phi)) = \rho_k(\phi)$ by Theorem 9, we get by Theorem 4 that ϕ is reducible. The converse statement is trivial. $\square$

After the completion of this work we were made aware of [MN]. One can translate our proof of Theorem 4 into a BHMV-skein model proof of Theorem 8 using the result of [MN].

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