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On the finiteness of co-associated primes of local homology modules

By

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Abstract

We show that if a linearly compact $R-$module M and local homology modules $H_i^I(M)$ satisfy the finite condition for co-associsted primes for all $i < s$, then the set $\mathrm{Coass}_R(H_s^I(M))$ is finite.

1. Introduction

Throughout this paper R is a commutative noetherian ring with non-zero identity and has a topological structure. From the Huneke's question in [13]: If M is finitely generated, is the number of associated primes of local cohomology modules $H_I^i(M)$ always finite?, there are some papers studying about that such as: Brodmann-Faghani [3], Divaani-Aazar-Mafi [9], [10], Cuong-Nam [6], [16] .... We know that duality of the Grothendieck's theory of local cohomology is the theory of local homology studied by many mathematicians: Greenlees-May [12], Tarrio-Lopez-Lipman [1], Cuong-Nam [5], [6], [7], [8] .... There is a similar question: when is the set of co-associated primes of local homology modules finite? Note that the finiteness of co-associated and associated primes of local homology and cohomology modules is closely related to the local-global-principle for finiteness dimensions of Faltings [11]. In [6, 4.5] Cuong-Nam showed that if M is a semi-discrete linearly compact $R-$module and the local homology $R-$modules $H_0^I(M), H_1^I(M), \ldots, H_{s-1}^I(M)$ are artinian, then the set $\mathrm{Coass}_R(H_s^I(M))$ is finite. Next, in [16, 3.1] we proved that the set of co-associated primes of the local homology module $H_s^I(M)$ is finite in either of the following cases: (i) The $R-$modules $H_i^I(M)$ are finitely generated for all $i < s$; (ii) $I \subseteq \mathrm{Rad}(\mathrm{Ann}_R(H_i^I(M)))$ for all $i < s$.

The purpose of this paper is to show a more general result for the finiteness of co-associated primes of local homology modules. The main result is Theorem 3.1 which says that if a linearly compact $R-$module M and local homology modules $H_i^I(M)$ satisfy the finite condition for co-associsted primes for all

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$i < s$, then $\mathrm{Coass}_R(H_s^I(M))$ is finite. From Theorem 3.1 we get back the results in [6, 4.6] and [16, 3.1] (Corollary 3.3).

2. Premilinaries

We begin by recalling the concept of *linearly compact* modules by terminology of Macdonald [14]. Let M be a topological $R-$module. M is said to be *linearly topologized* if M has a base of neighborhoods of the zero element $\mathcal{M}$ consisting of submodules. M is called *Hausdorff* if the intersection of all the neighborhoods of the zero element is 0. A Hausdorff linearly topologized $R-$module M is said to be *linearly compact* if $\mathcal{F}$ is a family of closed cosets (i.e., cosets of closed submodules) in M which has the finite intersection property, then the cosets in $\mathcal{F}$ have a non-empty intersection.

It is clear that artinian $R-$modules are linearly compact and discrete. If (R, m) is a complete ring, then finitely generated $R-$modules are also linearly compact and discrete.

A Hausdorff linearly topologized $R-$module M is called *semi-discrete* if every submodule of M is closed. Thus a discrete $R-$module is semi-discrete. The class of semi-discrete linearly compact $R-$modules contains all artinian $R-$modules. Moreover, it also contains all finite modules in case R is a complete local ring ([14, 7.3]).

Let I be an ideal of the ring R and M an $R-$module. De note by $\Lambda_I(M) = \varprojlim_t M/I^tM$ the $I-$adic completion of M. We have the functor Λ_I which is neither left nor right exact, since the tensor functor is not left exact and the inverse limit is not right exact. Let $L_i\Lambda_I$ the $i-$th left derived functor of Λ_I.

Theorem 2.1 ([15, 15] or [2, 1.3.1]). *Let M be an $R-$module. Then $\Lambda_I(M)/I\Lambda_I(M) \cong M/IM$.*

The *i-th local homology module* $H_i^I(M)$ of M with respect to I can be defined by $H_i^I(M) = \varprojlim_t \mathrm{Tor}_i^R(R/I^t; M)$ ([5, 3.1]). In particular $H_0^I(M) \cong \Lambda_I(M)$. It should be noted that this definition of local homology modules is in some sense dual to the definition of local homology of Grothendieck and coincident with the definition of J. P. C. Greenlees and J. P. May [12] in the case of linearly compact modules. Note that every short exact sequence of linearly compact $R-$modules induces a long exact sequence of local homology modules ([8, 4.6]). We have some basic properties of local homology modules.

Lemma 2.1 ([8, 3.3]). *If M is a linearly compact $R-$module, then for all $i \geq 0$, $H_i^I(M)$ is a linearly compact $R-$module.*

Lemma 2.2 ([8, 3.5]). *Let M be a linearly compact $R-$module. Then*

$$H_i^I(M) \cong L_i\Lambda_I(M)$$

for all $i \geq 0$.

A module is called *cocyclic* if it is a submodule of $E(R/m)$ for some maximal ideal m of R, where $E(R/m)$ is the injective envelope of R/m. The *co-support* $\mathrm{Cosupp}_R(M)$ of an $R-$module M is the set of primes p such that there exists a cocyclic homomorphic image L of M with $\mathrm{Ann}(L) \subseteq p$. If $0 \longrightarrow N \longrightarrow M \longrightarrow K \longrightarrow 0$ is an exact sequence of $R-$modules, then $\mathrm{Cosupp}_R(M) = \mathrm{Cosupp}_R(N) \cup \mathrm{Cosupp}_R(K)$ ([19, 2.7]).

Lemma 2.3 ([6, 3.8] and [17, 3.8]). *Let M be a linearly compact $R-$module. Then*

$$\mathrm{Cosupp}_R(H_i^I(M)) \subseteq \mathrm{Cosupp}_R(M) \bigcap V(I)$$

for all $i \geq 0$.

3. Co-associated primes of local homology modules

We first recall the concept of *co-associated primes* ([4], [20], [19], ...). A prime ideal p is called *co-associated* to a non-zero $R-$module M if there is a cocyclic homomorphic image L of M such that $p = \mathrm{Ann}_R(L)$. The set of co-associated primes of M is denoted by $\mathrm{Coass}_R(M)$. Note that, $\mathrm{Coass}_R(M) \subseteq \mathrm{Cosupp}_R(M)$ and every minimal element of $\mathrm{Cosupp}_R(M)$ belongs to $\mathrm{Coass}_R(M)$ ([19, 2.2,2.6]). If M is a semi-discrete linearly compact $R-$module, then the set $\mathrm{Coass}_R(M)$ is finite ([20, 1(L4)]). If $0 \longrightarrow N \longrightarrow M \longrightarrow K \longrightarrow 0$ is an exact sequence of $R-$modules, then $\mathrm{Coass}_R(K) \subseteq \mathrm{Coass}_R(M) \subseteq \mathrm{Coass}_R(N) \cup \mathrm{Coass}_R(K)$ ([19, 1.10]). Thus let N be a submodule of M and $\mathrm{Coass}_R(M)$ is finite, it is not sure that $\mathrm{Coass}_R(N)$ is finite. We suggest the following definition which is in some sense dual to the definition of weakly Lashkerian modules in [9].

Definition 3.1. We say that an $R-$module M satisfies the finite condition for co-associsted primes if the set of co-associsted primes of any submodule of M is finite.

Remark 1. (i) It is clear that if M is an artinian $R-$module, then M satisfies the finite condition for co-associsted primes. More generally, any semi-discrete linearly compact $R-$module satisfies the finite condition for co-associsted primes.

(ii) If an $R-$module M satisfies the finite condition for co-associsted primes, then any subquotient of M satisfies the finite condition for co-associsted primes. Therefore if N is a finitely generated $R-$module, then $\mathrm{Tor}_i^R(N, M)$ satisfies the finite condition for co-associsted primes (for a proof, see 3.2).

We have the following theorem for the finiteness of co-associated primes of local homology modules.

Theorem 3.1. *Let M be a linearly compact $R-$module and s a non-negative integer. If M and $H_i^I(M)$ satisfy the finite condition for co-associsted primes for all $i < s$, then $\mathrm{Coass}_R(H_s^I(M))$ is finite.*

To prove Theorem 3.1 we need following lemmas.

Lemma 3.1. *Let M be a linearly compact $R-$module. We have the following statements*:

(i) *There is a Grothendieck spectral sequence*

$$E^2_{p,q} = \mathrm{Tor}^R_p(R/I, H^I_q(M)) \underset{p}{\Rightarrow} \mathrm{Tor}^R_{p+q}(R/I, M);$$

(ii) $\mathrm{Coass}_R(H^I_q(M)) \subseteq \left(\bigcup_{i=2}^{q+1} \mathrm{Coass}_R(E^i_{i,q+1-i})\right) \bigcup \mathrm{Coass}_R(E^\infty_{0,q})$.

Proof. (i) Let us consider functors $F = R/I \otimes_R -$ and $G = \Lambda_I$. The functor F is obviously right exact and a projective module P implies $\Lambda_I(P)$ is flat by [2, 1.4.7]. Combining [18, 11.39] with 2.1 yields a Grothendieck spectral sequence

$$E^2_{p,q} = \mathrm{Tor}^R_p(R/I, L_q\Lambda_I(M)) \underset{p}{\Rightarrow} \mathrm{Tor}^R_{p+q}(R/I, M).$$

By 2.2 we get the Grothendieck spectral sequence as required.

(ii) We have the Grothendieck spectral sequence by (i)

$$E^2_{p,q} = \mathrm{Tor}^R_p(R/I, H^I_q(M)) \underset{p}{\Rightarrow} \mathrm{Tor}^R_{p+q}(R/I, M).$$

For all $i \geq 2$ we consider the following homomorphisms of the spectral sequence

$$E^i_{i,q-i+1} \xrightarrow{d^i_{i,q-i+1}} E^i_{0,q} \xrightarrow{d^i_{0,q}} E^i_{-i,q-1+i}.$$

Since $E^i_{-i,q-1+i} = 0$, $\ker d^i_{0,q} = E^i_{0,q}$. It follows an exact sequence

$$E^i_{i,q+1-i} \longrightarrow E^i_{0,q} \longrightarrow E^{i+1}_{0,q} \longrightarrow 0.$$

Then

$$\mathrm{Coass}_R(E^i_{0,q}) \subseteq \mathrm{Coass}_R(E^i_{i,q+1-i}) \cup \mathrm{Coass}_R(E^{i+1}_{0,q})$$

by [19, 1.10]. By iterating this for all $i = 2, \ldots, q$, note that $E^i_{0,q} = E^\infty_{0,q}$ for all $i \geq q+2$, we get

$$\mathrm{Coass}_R(E^2_{0,q}) \subseteq \left(\bigcup_{i=2}^{q+1} \mathrm{Coass}_R(E^i_{i,q+1-i})\right) \bigcup \mathrm{Coass}_R(E^\infty_{0,q}).$$

Finally, combining 2.3 and [19, 1.21] we have $\mathrm{Coass}_R(H^I_q(M)) = \mathrm{Coass}_R(E^2_{0,q})$. The proof is complete. □

Lemma 3.2. *Let M be a linearly compact $R-$module.*

(i) *If M satisfies the finite condition for co-associsted primes, then $E^\infty_{p,q}$ also satisfies the finite condition for co-associsted primes*;

(ii) *If M is semi-discrete linearly compact, then $E^\infty_{p,q}$ is also semi-discrete linearly compact.*

Proof. From 3.1 we have the Grothendieck spectral sequence

$$E^2_{p,q} = \operatorname{Tor}^R_p(R/I, H^I_q(M)) \underset{p}{\Rightarrow} \operatorname{Tor}^R_{p+q}(R/I, M).$$

Then there is a filtration Φ of $H_n = \operatorname{Tor}^R_n(R/I, M)$ with

$$0 = \Phi^{-1}H_n \subseteq \ldots \subseteq \Phi^n H_n = H_n \text{ and } \Phi^p H_n/\Phi^{p-1}H_n \cong E^\infty_{p,q}$$

where $n = p+q$. As R is the noetherian ring, there is a free resolution $\mathbf{F}_\bullet$ of R/I with the finitely generated free $R-$modules and $\operatorname{Tor}^R_n(R/I, M) \cong H_n(\mathbf{F}_\bullet \otimes_R M)$. Thus if M satisfies the finite condition for co-associsted primes (or is semi-discrete linearly compact), then $\operatorname{Tor}^R_n(R/I, M)$ satisfies the finite condition for co-associsted primes (or is semi-discrete linearly compact). It follows $E^\infty_{p,q}$ also satisfies the finite condition for co-associsted primes (or is semi-discrete linearly compact). □

We now can prove Theorem 3.1.

Proof of Theorem 3.1. We have by 3.1(ii),

$$\operatorname{Coass}_R(H^I_s(M)) \subseteq \left(\bigcup_{i=2}^{s+1} \operatorname{Coass}_R(E^i_{i,s+1-i})\right) \bigcup \operatorname{Coass}_R(E^\infty_{0,s}).$$

It follows from 3.2 (i) that $\operatorname{Coass}_R(E^\infty_{0,s})$ is finite. On the other hand, $E^i_{i,s+1-i}$ is a subquotient of $E^2_{i,s+1-i} = \operatorname{Tor}^R_i(R/I, H^I_{s+1-i}(M))$. For all $i = 2, \ldots, s+1$ as $H^I_{s+1-i}(M)$ satisfies the finite condition for co-associsted primes, so is $E^2_{i,s+1-i}$. Thus $\operatorname{Coass}_R(E^i_{i,s+1-i})$ is finite for all $i = 2, \ldots, s+1$. It follows $\operatorname{Coass}_R(H^I_s(M))$ is finite. □

Corollary 3.1. *Let M be a linearly compact $R-$module and s a non-negative integer. If M satisfies the finite condition for co-associsted primes and* $\operatorname{Cosupp}_R(H^I_i(M))$ *is finite for all $i < s$, then* $\operatorname{Coass}_R(H^I_s(M)$ *is finite.*

Proof. It follows from the hypothesis and [19, 2.2] that $H^I_i(M)$ satisfies the finite condition for co-associsted primes for all $i < s$. Thus the result follows from 3.1. □

Corollary 3.2. *Let M be a semi-discrete linearly compact $R-$module and s a non-negative integer. If $H^I_i(M)$ is a semi-discrete $R-$module for all $i < s$, then* $\operatorname{Coass}_R(H^I_s(M)$ *is finite.*

Proof. It should be noted that if M is a semi-discrete linearly compact $R-$module, then $\operatorname{Coass}_R(M)$ is finite by [20, 1(L3, L4)]. Moreover, if M is a semi-discrete linearly compact $R-$module, then any submodule of M is a semi-discrete linearly compact $R-$module. Thus M satisfies the finite condition for co-associsted primes. On the other hand, in virtue of 2.1, $H^I_i(M)$ is semi-discrete linearly compact and then satisfies the finite condition for co-associsted primes. Therefore, $\operatorname{Coass}_R(H^I_s(M)$ is finite by 3.1. □

We now get back the results for the finiteness of the set $\operatorname{Coass}_R(H^I_s(M)$ in [6, 4.6] and [16, 3.1].

Corollary 3.3 ([6, 4.6] and [16, 3.1]). *Let M be a semi-discrete linearly compact $R-$module and s a non-negative integer. The set of co-associated primes of the local homology $R-$module $H_s^I(M)$ is finite in either of the following cases*:

(i) *The $R-$modules $H_i^I(M)$ are artinian for all $i < s$;*

(ii) *$I \subseteq \mathrm{Rad}(\mathrm{Ann}_R(H_i^I(M)))$ for all $i < s$.*

Proof. (i) follows immediately from 3.2, as any artinian module is semi-discrete linearly compact.

(ii) Let us prove that $H_i^I(M)$ is semi-discrete linearly compact for all $i < s$. When $i = 0$, $H_0^I(M)$ is clearly semi-discrete linearly compact. If $i > 0$, denote by $L(M)$ the sum of all artinian submodules of M, by [20, 1(L5)], $L(M)$ is artinian. From [8, 5.2] we have the isomorphism $H_i^I(M) \cong H_i^I(L(M))$ for all $i > 0$ and the short exact sequence

$$0 \longrightarrow H_0^I(L(M)) \longrightarrow H_0^I(M) \longrightarrow H_0^I(M/L(M)) \longrightarrow 0.$$

Thus for all $i < s$ we have $I \subseteq \mathrm{Rad}(\mathrm{Ann}_R(H_i^I(L(M))))$, hence $H_i^I(L(M))$ are artinian by [5, 4.7]. Therefore $H_i^I(M)$ is semi-discrete linearly compact for all $i < s$ and (ii) follows from 3.2. □

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