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Stability and genericity for spde's driven by spatially correlated noise

By

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Abstract

We consider stochastic partial differential equations, on $\mathbb{R}^d$ $(d \geq 1)$, driven by a Gaussian noise white in time and colored in space. Assuming pathwise uniqueness holds, we establish various strong stability results. As consequence, we give an application to the convergence of the Picard successive approximation. Finally, we show that in the sense of Baire category, almost all stochastic partial differential equations with continuous and bounded coefficients have the properties of existence and pathwise uniqueness of solutions as well as the continuous dependence on the coefficients.

1. Introduction and general framework

The paper is concerned with stochastic partial differential equations (spde's) of the form

$$\begin{cases} Lu(t,x) = \sigma(t,x,u(t,x))\dot{F}(t,x) + b(t,x,u(t,x)), \\ \\ u(0,x) = 0, \ \dfrac{\partial u}{\partial t}(0,x) = 0, \end{cases} \tag{1.1}$$

where, $t \in [0,T]$ for some fixed $T > 0$, $x \in \mathbb{R}^d, d \geq 1$ and L is a second order partial differential operator. The coefficients σ and $b : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ are given measurable functions. Our spde's include, for instance, the stochastic heat and wave equations in spatial dimension $d \geq 1$. For the simplicity, we will assume that the initial condition is null. However, the result can be properly extended to cover smooth initial conditions.
Let $\mathcal{D}(\mathbb{R}^{d+1})$ be the space of all infinitely differentiable functions with compact support. On a probability space $(\Omega, \mathcal{G}, P)$, the noise $F = \{F(\varphi), \varphi \in \mathcal{D}(\mathbb{R}^{d+1})\}$ is assumed to be an $L^2(\Omega, \mathcal{G}, P)$–valued Gaussian process with mean zero and covariance functional given by

$$J(\varphi,\psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \Gamma(dx) \left(\varphi(s,\cdot) * \widetilde{\psi}(s,\cdot)\right)(x),$$

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where $\widetilde{\psi}(s,x) = \psi(s,-x)$ and Γ is a non–negative and non–negative definite tempered measure, therefore symmetric. Let μ denote the spectral measure of Γ, which is also a tempered measure.
Denote by $\mathcal{F}\varphi$ the Fourier transform of φ. Clearly $\mu = \mathcal{F}^{-1}(\Gamma)$. This gives

$$J(\varphi,\psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \mu(d\xi)\mathcal{F}\varphi(s,\cdot)(\xi)\overline{\mathcal{F}\psi(s,\cdot)}(\xi),$$

where $\overline{z}$ is the complex conjugate of z.
Following the same approach of [7], the Gaussian process F can be extended to a worthy martingale measure $M = \{M(t,A) := F([0,t]\times A) : t \geq 0,\ A \in \mathcal{B}_b(\mathbb{R}^d)\}$ which shall acts as integrator, in the Walsh sense [23]. Here $\mathcal{B}_b(\mathbb{R}^d)$ denotes the bounded Borel subsets of $\mathbb{R}^d$. Let $\mathcal{G}_t$ be the completion of the σ–field generated by the random variables $\{M(s,A),\ 0 \leq s \leq t,\ A \in \mathcal{B}_b(\mathbb{R}^d)\}$. The properties of F ensure that the process $M = \{M(t,A),\ t \geq 0,\ A \in \mathcal{B}_b(\mathbb{R}^d)\}$, is a martingale with respect to the filtration $\{\mathcal{G}_t : t \geq 0\}$.

One can give a rigorous meaning to solution of equation (1.1), by means of a jointly measurable and $\mathcal{G}_t$–adapted process $\{u(t,x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ satisfying, for each $t \geq 0$ and $x \in \mathbb{R}^d$, a.s. the following evolution equation:

$$\begin{aligned} u(t,x) = &\int_0^t \int_{\mathbb{R}^d} S(t-s,x,y)\sigma(s,y,u(s,y))M(ds,dy) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} dyS(t-s,x,y)b(s,y,u(s,y)) \end{aligned} \tag{1.2}$$

where $S(t,x,y)$ denotes for the fundamental solution of $Lu = 0$ with boundary conditions specified before. More developments on this kind of spde's and generalized ones can be found in [7], [14], [8], [20] and the references therein.

In the sequel, we shall refer the equation (1.2) as Eq(σ,b). To simplify the notation, we shall write

$$b(u)(s,y) := b(s,y,u(s,y)) \text{ and } \sigma(u)(s,y) := \sigma(s,y,u(s,y)).$$

In this paper, we establish strong stability, as well as some generic property (in the Baire category sense) for the solution of spde's of the form (1.2).

We first prove that pathwise uniqueness property implies the L^p−stability of the solutions. It is then quite natural to raise the question whether the set of all "*nice*" functions (σ,b) for which the pathwise uniqueness holds for the stochastic partial differential equation Eq(σ,b) is larger than its complement, in a Baire category sense. To make the question meaningful let us recall what we mean by the generic property. Let $\mathcal{E}$ be a Baire space. A subset $\mathcal{A}$ of $\mathcal{E}$ is said to be meager (or a first category set in the Baire sense), if it is contained in a countable union of closed nowhere dense subsets of $\mathcal{E}$. The complement of a meager set is called a comeager (or residual or a second category set). A property $\mathcal{P}$ is said to be generic in a Baire space $\mathcal{E}$ (see Bourbaki [5]) if $\mathcal{P}$ holds in $\mathcal{E}\setminus\mathcal{A}$, where $\mathcal{A}$ is a set of first category (in the sense of Baire) in $\mathcal{E}$. Results on generic properties for ordinary differential equations seems to go back to an earlier paper of Orlicz [19] and later developed by Lasota and Yorke [16]. The investigation of such questions for stochastic differential equations is carried out in Heunis [13] and, Alibert and Bahlali [1]. In this note, we study the corresponding problem for spde's like Eq(σ, b). We show that the subset of continuous and bounded coefficients for which existence and pathwise uniqueness hold (for equation Eq(σ, b)) is a residual set. The proof is based essentially on Theorem 3.2. Moreover it does not use the oscillation function introduced by Lasota and Yorke in [16] for ordinary differential equations and used for stochastic differential equations by Heunis [13] and in further development for the generic property of stochastic differential equations by Bahlali *et al.* [3]. See also Bahlali *et al.* [4] for backward stochastic differential equations.

More precisely, we will examine the following four points,

1. the L^p-stability of the solution under Lipschitz conditions on the coefficients
2. the L^p-stability of the solution with respect to the driving process under pathwise uniqueness of solution.
3. The relations between pathwise uniqueness and convergence of the Picard successive approximation
4. the genericity of the existence, pathwise uniqueness and continuous dependence on the coefficients

The paper is organized as follows. In Section 2 we establish a Hölder regularity of the solution of (1.2). Section 3 is devoted to prove some results of stability of solutions of (1.1), first on the coefficients and next on the driving process. In section 4, we give, under pathwise uniqueness, a necessary and sufficient conditions which ensure the convergence of the Picard successive approximation associated to the equation (1.2). Section 5 shows that, the existence, pathwise uniqueness and the continuous dependence on the coefficients are generic properties (in the sense of Baire category) for spde's with bounded continuous coefficients. In section 6, we study some examples of spde's of kind (1.1). Finally, an appendix gathers some technical lemmas which are used through the work. We always assume that all constants will be denoted by c or C independently of its value.

2. Definitions and Hölder regularity of the solution

Definition 2.1. Let $p \geq 2$, a stochastic process u defined on $\Omega \times \mathbb{R}_+ \times \mathbb{R}^d$, which is jointly measurable and $\mathcal{G}_t$–adapted, is said to be a solution to the spde's (1.1), if it is an $\mathbb{R}$–valued fields which satisfies (1.2) and $\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} E|u(t,x)|^p < +\infty$.

We consider the following assumptions,

(L) There exists is a constant c such that

$$|\sigma(t,x,r) - \sigma(t,x,v)| + |b(t,x,r) - b(t,x,v)| \leq c|r-v| \quad \text{for all } t,x,r,v,$$

(LG) There exists is a constant c such that

$$|\sigma(t,x,r)| + |b(t,x,r)| \leq c\,(1+|r|) \qquad \text{for all } t,x,r.$$

(R.1) For any $T>0$, $\displaystyle\int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi)\,|\mathcal{F}S(s,\cdot)(\xi)|^2 < +\infty.$

(R.2) (i) There exist constants $c>0$ and $\delta_1>0$ such that for $0\leq t_1\leq t_2\leq T$,

$$\sup_{x\in\mathbb{R}^d}\int_{t_1}^{t_2} ds \int_{\mathbb{R}^d} \mu(d\xi)\,|\mathcal{F}S(t_2-s,x,\cdot)(\xi)|^2 \leq c\,|t_2-t_1|^{2\delta_1}.$$

(ii) For any compact subset $K\subset\mathbb{R}^d$ there exist constants $c>0$ and $\delta_2>0$ such that

$$\int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi)\,|\mathcal{F}S(s,x+z,\cdot)(\xi) - \mathcal{F}S(s,x,\cdot)(\xi)|^2 \leq c\,\|z\|^{2\delta_2},$$

for any $x\in\mathbb{R}^d$ and $z\in K$.

It is proved in Dalang [7] that, the assumption **(R.1)**, **(L)** and **(LG)** ensure the existence and uniqueness of the solution of (1.2) in the case of stochastic heat equation in any space dimension $d\geq 1$ and for stochastic wave equation in dimension $d=1,2$.

Let us recall some recent results on the regularity of $u(t,x)$, which has been proved by Sanz–Solé and Sarrà [21] (see also [20] and [9]).

Let $\gamma=(\gamma_1,\gamma_2)$ such that $\gamma_1,\gamma_2>0$ and let K be a compact subset of $\mathbb{R}^d$. We denote by $\mathcal{C}^\gamma([0,T]\times K;\mathbb{R})$ the set of γ–Hölder continuous functions equipped with the norm defined by:

$$\|f\|_{\gamma,T,K} = \sup_{(t,x)\in[0,T]\times K}|f(t,x)| + \sup_{s\neq t\in[0,T]}\ \sup_{x\neq y\in K}\frac{|f(t,x)-f(s,y)|}{|t-s|^{\gamma_1}+\|x-y\|^{\gamma_2}}. \tag{2.1}$$

Theorem 2.1. *Assume that* **(*LG*)**, **(*R.1*)** *and* **(*R.2*)** *hold. Let u be a solution to equation $Eq(\sigma,b)$. Then,*

(*i*) *u belongs to $\mathcal{C}^\gamma([0,T]\times K;\mathbb{R})$ a.s. for any $\gamma_i<\delta_i$, $i=1,2$ and for any compact subset K of $\mathbb{R}^d$.*

(*ii*) *$E\left(\|u\|^p_{\gamma,T,K}\right)<+\infty$ for any $p\geq 2$.*

Proof. The proof of (i) follows using Kolmogorov's criterium. For more details (see [21]).

To prove (ii), we use the Garcia–Rodemich–Rumsey Lemma (see [18]), to obtain

$$E(\|u\|^p_{\gamma,T,K}) \leq c_{p,\gamma,T,d}.$$

□

3. Stability of equation (1.1)

Let f be a function defined on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$. For $T > 0$, we set

$$\|f\|_{T,\infty} := \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \sup_{r\in\mathbb{R}} |f(t,x,r)| \,. \tag{3.1}$$

3.1. Stability under Lipschitz conditions on the coefficients

Assume that **(R.1)** and **(R.2)** hold. Let $(\sigma_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences of functions on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$ which satisfy **(L)** and **(LG)** uniformly in n.

Denote by $\{u_n(t,x), t \geq 0, x \in \mathbb{R}^d\}$ the unique solution of equation $\mathrm{Eq}(\sigma_n, b_n)$ i.e.

$$\begin{aligned} u_n(t,x) = &\int_0^t \int_{\mathbb{R}^d} S(t-s,x,y)\sigma_n(u_n)(s,y)M(ds,dy) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} dy S(t-s,x,y) b_n(u_n)(s,y). \end{aligned} \tag{3.2}$$

Then, we have the following theorem:

Theorem 3.1. *Assume that $(\sigma_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ converge respectively to σ and b uniformly on compact sets of $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$. Then for any $p \geq 2$*

$$\lim_{n\to+\infty} E\left(\|u_n - u\|^p_{\gamma,T,K}\right) = 0,$$

where u is the unique solution of $Eq(\sigma, b)$.

The proof of this Theorem is a consequence of Theorem 2.1 and the following lemma.

Lemma 3.1. *Assume that there exist real valued functions σ and b defined on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$ such that*

$$\lim_{n\to+\infty} \left(\|\sigma_n - \sigma\|_{T,\infty} + \|b_n - b\|_{T,\infty}\right) = 0. \tag{3.3}$$

Then, for any $p \geq 2$

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} E\left(|u_n(t,x) - u(t,x)|^p\right) = 0,$$

where u is the unique solution of $Eq(\sigma, b)$.

Proof. Without loss of generality we assume that $b \equiv 0$. For, $p \geq 2$, set

$$\varphi_n(t,x) = E\left|u_n(t,x) - u(t,x)\right|^p$$

and $\phi_n(t) = \sup_{x\in\mathbb{R}} \varphi_n(t,x)$. Clearly,

$$\begin{aligned} \varphi_n(t,x) \leq c_p E &\left| \int_0^t \int_{\mathbb{R}^d} S(t-s,x,y)\left[\sigma_n(u_n)(s,y) - \sigma(u_n)(s,y)\right] M(ds,dy) \right|^p \\ &+ c_p E \left| \int_0^t \int_{\mathbb{R}^d} S(t-s,x,y)\left[\sigma(u_n)(s,y) - \sigma(u)(s,y)\right] M(ds,dy) \right|^p . \end{aligned}$$

We use, Burkholder's inequality, Hölder's inequalities and assumption (**L**) on σ_n to get,

$$\begin{aligned}\phi_n(t) &\leq c_p \left\|\sigma_n - \sigma\right\|_{T,\infty}^p \nu_t^{\frac{p}{2}} + c_p \nu_t^{\frac{p}{2}-1} \int_0^t J(t-s)\phi_n(s)ds \\ &\leq c_{p,T}\left(\left\|\sigma_n - \sigma\right\|_{T,\infty}^p + \int_0^t J(t-s)\phi_n(s)ds \right)\end{aligned}$$

where

$$J(s) = \int_{\mathbb{R}^d} \mu(d\xi) \left|\mathcal{F}S(s,\cdot)(\xi)\right|^2 \quad \text{and} \quad \nu_t = \int_0^t J(t-s)ds. \tag{3.4}$$

Therefore, assumption **(R.1)** and Lemma 7.1 yield

$$\sup_{t\in[0,T]} \phi_n(t) \leq c_{p,T} \left\|\sigma_n - \sigma\right\|_{T,\infty}^p .$$

which proves the lemma. □

Proof of Theorem 3.1. It is enough to prove that the sequence $(u_n - u)_{n\geq 1}$ satisfies the properties ($\mathbf{P}_1$) and ($\mathbf{P}_2$) of Lemma 7.2. Clearly, by Theorem 2.1, $(u_n - u)_{n\geq 1}$ satisfy the property ($\mathbf{P}_1$) of Lemma 7.2. The property ($\mathbf{P}_2$) is given by Lemma 3.1. Therefore the proof of Theorem 3.1 follows from the above properties. □

Definition 3.1. We say that the pathwise uniqueness (**PU**) holds for equation (1.2) if whenever $(u, M, (\Omega, \mathcal{G}, P), \mathcal{G}_t)$ and $(u', M', (\Omega, \mathcal{G}, P), \mathcal{G}'_t)$ are two weak solutions of equation (1.2) such that $M \equiv M'$ P–a.s., then $u \equiv u'$ P–a.s.

We will establish a variant of Theorem 3.1. Consider, for $\lambda \in \mathbb{R}$, the following sequence of stochastic partial differential equation,

$$\begin{aligned} u^\lambda(t,x) &= \varphi(\lambda) + \int_0^t \int_{\mathbb{R}^d} S(t-s,x,y)\sigma_\lambda(u^\lambda)(s,y)M(ds,dy) \\ &\qquad + \int_0^t ds \int_{\mathbb{R}^d} dy S(t-s,x,y) b_\lambda(u^\lambda)(s,y), \\ u^\lambda(0,x) &= \varphi(\lambda) \ \text{ for } x \in \mathbb{R}^d, \ \text{ where } \varphi \text{ is a given function.}\end{aligned} \tag{3.5}$$

We then have the following theorem:

Theorem 3.2. *Assume that $\sigma_\lambda(t,x,r)$ and $b_\lambda(t,x,r)$ are continuous with respect to their arguments. Further, suppose that φ is continuous at $\lambda_0 \in \mathbb{R}$, and for each $T > 0$ and each compact subset K of $\mathbb{R}^d$ there exists a constant $c > 0$ such that for all $r \in \mathbb{R}$*

$$\sup_{\lambda\in\mathbb{R}} \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \left(|\sigma_\lambda(t,x,r)| + |b_\lambda(t,x,r)|\right) \leq c\left(1 + |r|\right)$$

and

$$\lim_{\lambda\to\lambda_0}\left(\|\sigma_\lambda-\sigma_{\lambda_0}\|_{T,\infty}+\|b_\lambda-b_{\lambda_0}\|_{T,\infty}\right)=0,$$

where $\|\cdot\|_{T,\infty}$ *has been defined in* (3.1). *Then, under* (**PU**) *for the equation* (3.5) *at* λ_0 *we have*:

$$\lim_{\lambda\to\lambda_0} E\left[\left\|u^\lambda-u^{\lambda_0}\right\|_{T,\infty}^2\right]=0 \text{ for every } T\geq 0.$$

Proof. Similar to the proof of Theorem 3.1. □

3.2. Stability with respect to the driving process under (PU)

In this subsection, we consider spde's driven by spatially correlated noise. We prove the continuity of solutions with respect to the driving processes, when the pathwise uniqueness of solutions holds.
Let $\{M^n\}_{n\geq 0}$ be a sequence of continuous $(\mathcal{G}_t,P)$–martingale measure, with $M^0=M$ and $\sigma,b:\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}\longrightarrow\mathbb{R}$ be continuous functions satisfying **(LG)**. Define the sequence

$$\begin{aligned}u^n(t,x)=&\int_0^t\int_{\mathbb{R}^d}S(t-s,x,y)\sigma\left(u^n\right)(s,y)M^n(ds,dy)\\&+\int_0^t ds\int_{\mathbb{R}^d}dyS(t-s,x,y)b\left(u^n\right)(s,y).\end{aligned}$$

Suppose that $\{M^n\}_{n\geq 0}$ satisfy the following conditions:

(**H.1**) The family $\{M^n\}_{n\geq 0}$ is bounded in probability in $\mathcal{C}\left([0,T]\times K\right)$.

(**H.2**) $M^n-M^0\longrightarrow_{n\to+\infty}0$ in probability on $\mathcal{C}\left([0,T]\times K\right)$.

Then, we have the following theorem:

Theorem 3.3. *Suppose that* **(R.1)**, **(R.2)**, **(H.1)**, **(H.2)** *and* **(PU)** *hold and that* $Eq(\sigma,b)$ *is non–degenerate. Assume, moreover, that* $\frac{\partial\sigma}{\partial r}$ *is a locally bounded functions of* (t,x,r) *and that it is Lipschitz continuous in* $r\in\mathbb{R}$. *Then for any* $\varepsilon>0$,

$$\lim_{n\to+\infty}P\left(\|u^n-u\|_{\gamma,T,K}>\varepsilon\right)=0.$$

The main tool used in the proofs is the Skorokhod representation theorem given by the following:

Lemma 3.2 ([22]). *Let* $(\mathcal{X},\rho)$ *be a complete separable metric space,* $\{P_n:n\geq 1\}$ *and* P *be probability measures on* $(\mathcal{X},\mathcal{B}(\mathcal{X}))$ *such that* $P_n\to_{n\to+\infty}P$. *Then, on a probability space* $(\widehat{\Omega},\widehat{\mathcal{G}},\widehat{P})$, *we can construct* $\mathcal{X}$*–valued random variables* $\{u_n:n\geq 1\}$ *and* u *such that*:

(*i*) $P_n=\widehat{P}_{u_n}$, $n=1,2,\ldots$ *and* $P=\widehat{P}_u$.

(*ii*) u_n *converges to* u $\widehat{P}$*–a.s.*

The following lemma gives criteria which allow us to apply Lemma 3.2 to sequences of laws associated to continuous processes.

Lemma 3.3 ([22]). *Let $\{u_n(t,x) : n \geq 1\}$ be a sequence of real valued continuous processes satisfying the following two conditions*:

(i) *There exist positive constants C and q such that*

$$\sup_{n\geq 1} E\left[|u_n(0,x_0)|^q\right] \leq C \text{ for some given } x_0.$$

(ii) *There exist positive constants p, β_1, β_2, C_T such that*:

$$\sup_{n\geq 1} E\left[|u_n(t,x) - u_n(s,y)|^p\right] \leq C_T \left(|t-s|^{1+\beta_1} + \|x-y\|^{d+\beta_2} \right)$$

for every $s,t \in [0,T]$ and $x,y \in \mathbb{R}^d$.
Then, there exist a subsequence $(n_k)_{k\geq 1}$, a probability space $(\widehat{\Omega}, \widehat{\mathcal{G}}, \widehat{P})$ and real valued continuous processes $\widehat{u}_{n_k}$, $k = 1, 2, \ldots$ and $\widehat{u}$ defined on it such that:

1. *The two random field $\widehat{u}_{n_k}$ and u_{n_k} have the same law.*
2. *$\widehat{u}_{n_k}(t,x)$ converge to $\widehat{u}(t,x)$ uniformly on every compact subset of $\mathbb{R}_+ \times \mathbb{R}^d$ $\widehat{P}$–a.s.*

The following lemma is a variant of the Lemma 4.3 in ([12, p. 744–745]) in which we replace the Brownian sheet with a martingale measure.

Lemma 3.4. *For every $n \geq 0$ let $\{z^n(t,x) : t \in \mathbb{R}_+ \times \mathbb{R}^d\}$ be a family of continuous $\mathcal{G}_t^n$–adapted random field and let M^n be a Gaussian martingale measure carried by some filtered probability space $(\Omega, \mathcal{G}, \mathcal{G}_t^n, P)$. Assume that for every $\varepsilon > 0$, $T > 0$ and K a compact subset of $\mathbb{R}^d$*:

$$\lim_{n\to+\infty} P\left(\left\| z^n - z^0 \right\|_{\gamma,T,K} + \left\| M^n - M^0 \right\|_{T,\infty} > \varepsilon \right) = 0.$$

Let $h(t,x,r)$ be a bounded Borel function of $(t,x,r) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$ and define,

$$I_n(t,x) := \int_0^t \int_{\mathbb{R}^d} S(t-s,x,y) h(s,y,z^n(s,y)) dy ds$$

and

$$J_n(t,x) := \int_0^t \int_{\mathbb{R}^d} S(t-s,x,y) h(s,y,z^n(s,y)) M^n(ds,dy)$$

Then the following assertions hold:

(i) *If h is continuous in $r \in \mathbb{R}$, then*

$$\lim_{n\to+\infty} I_n(t,x) = I_0(t,x) \quad \text{and} \quad \lim_{n\to+\infty} J_n(t,x) = J_0(t,x) \tag{3.6}$$

in probability for every $t \geq 0$ and every $x \in \mathbb{R}^d$.

(ii) If for almost every $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ *the law* $Q^n_{t,x}$ *of* $z^n(t,x)$ *is absolutely continuous with respect to the Lebesgue measure* λ *on* $\mathbb{R}$ *and the density* $p^n_{t,x} = \frac{dQ^n_{t,x}}{d\lambda}$ *satisfies for some* $\alpha > 1$

$$\sup_{n\geq 0} \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}} \left(p^n_{t,x}(r)\right)^\alpha drdxdt < +\infty,$$

then (3.6) *also hold in probability for every* $t \geq 0$ *and every* $x \in \mathbb{R}^d$.

Proof of Theorem 3.3. Suppose that the conclusion of our theorem is false. Then there exists $\varepsilon > 0$ and a subsequence $(n_k)_{k\geq 0}$ such that

$$\inf_{n_k} P\left(\|u^{n_k} - u\|_{\gamma,T,K} > \varepsilon\right) \geq \varepsilon.$$

Under conditions **(R.1), (R.2)** and **(H.1)** one can show that the family $Z^n = (u^n, u, M^n, M)$ is tight in $[\mathcal{C}([0,T] \times \mathbb{R}^d, \mathbb{R})]^4$. Then, by Skorokhod's representation theorem see Lemma 3.2, there exist a probability space $(\widehat{\Omega}, \widehat{\mathcal{G}}, \widehat{P})$ and $\widehat{Z}^{n_k} = (\widehat{u}^{n_k}, \widetilde{u}^{n_k}, \widehat{M}^{n_k}, \widetilde{M}^{n_k})$ which satisfy:

$i)$ $Law(Z^{n_k}) = Law(\widehat{Z}^{n_k})$

$ii)$ There exists a subsequence $(\widehat{Z}^{n_k})_k$ also denoted by $(\widehat{Z}^{n_k})_k$ which converges $\widehat{P}$–a.s. in $[\mathcal{C}([0,T] \times \mathbb{R}^d, \mathbb{R})]^4$ to $\widehat{Z} = (\widehat{u}, \widetilde{u}, \widehat{M}, \widetilde{M})$.

Let $\mathcal{G}^n_t$ denotes the completion of the σ–algebra generated by $\{\widehat{Z}^n_s : s \leq t\}$ and set $\widehat{\mathcal{G}}^n_t = \cap_{s>t} \mathcal{G}^n_s$.

Similarly, we define the σ–algebra $\{\widehat{\mathcal{F}}_t : t \in [0,T]\}$ for the limiting process $\widehat{Z}$. Then $(\widehat{\Omega}, \widehat{\mathcal{G}}, \widehat{\mathcal{G}}^n_t, \widehat{P})$ (resp. $(\widehat{\Omega}, \widehat{\mathcal{G}}, \widehat{\mathcal{G}}_t, \widehat{P})$) are stochastic basis and $\widehat{M}^n$, $\widetilde{M}^n$ (resp. $\widehat{M}$, $\widetilde{M}$) are $\widehat{\mathcal{G}}^n_t$ (resp. $\widehat{\mathcal{G}}_t$)–continuous martingale measures. Moreover the two random fields $\widehat{u}^n$ and $\widetilde{u}^n$ satisfy the following spde:

$$\begin{aligned} \widehat{u}^n(t,x) &= \int_0^t \int_{\mathbb{R}^d} S(t-s,x,y)\sigma\left(\widehat{u}^n\right)(s,y)\widehat{M}^n(ds,dy) \\ &\quad + \int_0^t ds \int_{\mathbb{R}^d} dy S(t-s,x,y) b\left(\widehat{u}^n\right)(s,y). \end{aligned}$$

and

$$\begin{aligned} \widetilde{u}^n(t,x) &= \int_0^t \int_{\mathbb{R}^d} S(t-s,x,y)\sigma\left(\widetilde{u}^n\right)(s,y)\widetilde{M}^n(ds,dy) \\ &\quad + \int_0^t ds \int_{\mathbb{R}^d} dy S(t-s,x,y) b\left(\widetilde{u}^n\right)(s,y) \end{aligned}$$

on the same stochastic basis $(\widehat{\Omega}, \widehat{\mathcal{G}}, \widehat{\mathcal{G}}^n_t, \widehat{P})$.

Using the Lemma 3.4, we show that the limiting processes satisfy the following equations:

$$\begin{aligned}\widehat{u}(t,x) = &\int_0^t \int_{\mathbb{R}^d} S(t-s,x,y)\sigma\left(\widehat{u}\right)(s,y)\widehat{M}(ds,dy)\\ &+\int_0^t ds \int_{\mathbb{R}^d} dy S(t-s,x,y) b\left(\widehat{u}\right)(s,y).\end{aligned}$$

and

$$\begin{aligned}\widetilde{u}(t,x) = &\int_0^t \int_{\mathbb{R}^d} S(t-s,x,y)\sigma\left(\widetilde{u}\right)(s,y)\widetilde{M}(ds,dy)\\ &+\int_0^t ds \int_{\mathbb{R}^d} dy S(t-s,x,y) b\left(\widetilde{u}\right)(s,y).\end{aligned}$$

Due to **(H.2)** it is easy to see that $\widehat{M} = \widetilde{M}$, $\widehat{P}$–a.s.

Hence, by the pathwise uniqueness, $\widehat{u}$ and $\widetilde{u}$ are indistinguishable. This contradicts our assumption. Therefore u^n converges to the unique solution u. $\square$

4. Pathwise uniqueness and Picard's successive approximation

Throughout this section, we assume that σ and b are continuous and satisfy assumption **(LG)**. We define the sequence of the Picard successive approximation associated to (1.2) as follows:

$$\left\{\begin{aligned} &u^0 := 0\\ &u^{n+1}(t,x) := \int_0^t \int_{\mathbb{R}^d} S(t-s,x,y)\sigma\left(u^n\right)(s,y)M(ds,dy)\\ &\qquad\qquad + \int_0^t \int_{\mathbb{R}^d} S(t-s,x,y) b\left(u^n\right)(s,y)dyds.\end{aligned}\right. \tag{4.1}$$

It is well known that if the coefficients σ and b satisfy the condition **(L)**, then the sequence $(u^n)_{n\geq 0}$ converges in $L^p(\Omega)$ (as $n \to \infty$). Moreover, this scheme gives an effective way for construction of a (unique) solution to equation (1.2) (see for instance [7]).

Now, if we drop the Lipschitz condition on the coefficients and assume only that equation (1.2) admits a pathwise unique solution, does the sequence $(u^n)_{n\geq 0}$ converge to u ? The answer is negative even in the deterministic case, (see [10, p. 114–124]).

Assuming pathwise uniqueness, we then give a necessary and sufficient condition which ensures the convergence of the Picard successive approximation. More precisely, we have the following,

Theorem 4.1. *Let σ and b be continuous functions satisfying* **(LG)**. *Let $p \geq 2$. Assume further that* **(R.1)**, **(R.2)** *and* **(PU)** *hold for equation*

(1.2). *Then, for any* $\gamma = (\gamma_1, \gamma_2)$ *with* $\gamma_1 < \delta_1$ *and* $\gamma_2 < \delta_2$*, the sequence* $(u^n)_{n\geq 0}$ *converges in* $L^p(\Omega; \mathcal{C}^\gamma([0,T] \times K, \mathbb{R}))$*, to the unique solution of* (1.2) *if and only if* $\left\| u^{n+1} - u^n \right\|_{\gamma,T,K}$ *converges to* 0 *in* $L^p(\Omega)$.

To prove this theorem, we need the following lemma.

Lemma 4.1. *Under assumptions of the previous theorem, the sequence* $(u^n)_{n\geq 0}$ *defined by* (4.1) *is tight in* $\mathcal{C}([0,T] \times \mathbb{R}^d, \mathbb{R})$. *Moreover,* $\sup_{n\geq 0} E[\|u^n\|^p_{\gamma,T,K}] < +\infty$, *for any* $p \geq 2$ *and any* $\gamma = (\gamma_1, \gamma_2)$ *such that* $\gamma_1 < \delta_1$ *and* $\gamma_2 < \delta_2$.

Proof. For all $t > 0$ and $n > 1$, we have

$$
\begin{aligned}
|u^n(t,x)|^p \leq c_p & \left| \int_0^t \int_{\mathbb{R}^d} S(t-s,x,y)\sigma(u^{n-1})(s,y) M(ds,dy) \right|^p \\
& + c_p \left| \int_0^t ds \int_{\mathbb{R}^d} dy S(t-s,x,y) b(u^{n-1})(s,y) \right|^p .
\end{aligned}
$$

Burkholder's and Hölder's inequalities provide the following estimate

$$
E\,|u^n(t,x)|^p \leq c_p \nu_t^{\frac{p}{2}-1} \int_0^t \left(1 + \sup_{y \in \mathbb{R}^d} E\left| u^{n-1}(s,y) \right|^p \right) J(t-s) ds.
$$

Set $\phi_n(t) := \sup_{x\in\mathbb{R}^d} E\,|u^n(t,x)|^p$. We have,

$$
\begin{aligned}
\phi_n(t) &\leq c_{p,T} \int_0^t \left(1 + \phi_{n-1}(s)\right) J(t-s) ds \\
&\leq c_{p,T} + c_{p,T} \int_0^t \phi_{n-1}(s) J(t-s) ds.
\end{aligned}
$$

Using **(R.1)** and Lemma 15 p. 22 in Dalang (1999), we show that

$$
\sup_{n\geq 0} \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} E\,|u^n(t,x)|^p = \sup_{n\geq 0} \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \phi_n(t) \leq c_{p,T}. \tag{4.2}
$$

We shall prove the tightness of the sequence u^n. we have

$$
\begin{aligned}
u^n(t_2,x_2) - u^n(t_1,x_1) &= \int_0^{t_2} \int_{\mathbb{R}^d} S(t_2 - s, x_2, y)\sigma(u^{n-1})(s,y) M(ds,dy) \\
&\quad - \int_0^{t_1} \int_{\mathbb{R}^d} S(t_1 - s, x_1, y)\sigma(u^{n-1})(s,y) M(ds,dy) \\
&= \int_0^{t_1} \int_{\mathbb{R}^d} \Lambda(t_1,t_2,x_1,x_2,s,y)\sigma(u^{n-1})(s,y) M(ds,dy) \\
&\quad + \int_{t_1}^{t_2} \int_{\mathbb{R}^d} S(t_2 - s, x_2, y)\sigma(u^{n-1})(s,y) M(ds,dy).
\end{aligned}
$$

Passing to expectations, we get

$$
\begin{aligned}
&E\left|u^n(t_2,x_2)-u^n(t_1,x_1)\right|^p \\
&\quad \le c_pE\left|\int_0^{t_1}\int_{\mathbb{R}^d}\Lambda(t_1,t_2,x_1,x_2,s,y)\sigma(u^{n-1})(s,y)M(ds,dy)\right|^p \\
&\qquad + c_pE\left|\int_{t_1}^{t_2}\int_{\mathbb{R}^d}S(t_2-s,x_2,y)\sigma(u^{n-1})(s,y)M(ds,dy)\right|^p \\
&\quad \le c_p\left[\left(\int_0^{t_1}ds\int_{\mathbb{R}^d}\mu(d\xi)\left|\mathcal{F}\Lambda(t_1,t_2,x_1,x_2,s,\cdot)(\xi)\right|^2\right)^{\frac{p}{2}-1}\right. \\
&\qquad \left.\times\int_0^{t_1}ds\,(1+\phi_{n-1}(s))\int_{\mathbb{R}^d}\mu(d\xi)\left|\mathcal{F}\Lambda(t_1,t_2,x_1,x_2,s,\cdot)(\xi)\right|^2\right] \\
&\qquad + c_p\left[\left(\int_{t_1}^{t_2}ds\int_{\mathbb{R}^d}\mu(d\xi)\left|\mathcal{F}S(t_2-s,x_2,\cdot)(\xi)\right|^2\right)^{\frac{p}{2}-1}\right. \\
&\qquad \left.\times\int_{t_1}^{t_2}ds\,(1+\phi_{n-1}(s))\int_{\mathbb{R}^d}\mu(d\xi)\left|\mathcal{F}S(t_2-s,x_2,\cdot)(\xi)\right|^2\right].
\end{aligned}
$$

Using (4.2), **(R.1)** and **(R.2)**, we obtain

$$
E\left|u^n(t_2,x_2)-u^n(t_1,x_1)\right|^p \le c_p\left(\left|t_2-t_1\right|^{\delta_1 p}+\left\|x_2-x_1\right\|^{\delta_2 p}\right).
$$

Therefore, since $\sup_{n\ge 0}E[\|u^n\|^p_{\gamma,T,K}]<+\infty$ (theorem 2.1 (ii)), u^n is tight in $\mathcal{C}([0,T]\times\mathbb{R}^d,\mathbb{R})$. □

Proof of Theorem 4.1. Suppose that $\left\|u^{n+1}-u^n\right\|_{\gamma,T,K}$ converges to 0, as $n\to\infty$, in $L^p(\Omega)$, $(p\ge 2)$ and that there is some $\varepsilon>0$, and a sequence $(n_k)_k$ such that:

$$
\inf_{n_k}E\left(\|u^{n_k}-u\|^p_{\gamma,T,K}\right)\ge\varepsilon.
$$

According to Lemma 4.1, the family $\left(u^n,u^{n+1},u,M\right)$ satisfies conditions (i) and (ii) of Lemma 3.3. Then by the Skorokhod representation theorem, there exists some probability space $(\widehat{\Omega},\widehat{\mathcal{G}},\widehat{P})$ carrying a sequence of stochastic processes $(\widehat{u}^n,\widetilde{u}^{n+1},\overline{u}^n,\widehat{M}^n)$ which satisfies the following properties:

(α) For each $n\in\mathbb{N}$, the two random field $(\widehat{u}^n,\widetilde{u}^{n+1},\overline{u}^n,\widehat{M}^n)$ and $\left(u^n,u^{n+1},u,M\right)$ have the same law for each $n\in\mathbb{N}$.

(β) There exists a subsequence $(n_k)_{k\ge0}$ such that $(\widehat{u}^{n_k},\widetilde{u}^{n_k+1},\overline{u}^{n_k},\widehat{M}^{n_k})$ converges to $(\widehat{u},\widetilde{u},\overline{u},\widehat{M})$ uniformly on every compact subset of $\mathbb{R}_+\times\mathbb{R}^d$ $\widehat{P}$–a.s.

Since $u^{n+1}-u^n$ converges to 0, one can show easily that $\widehat{u}=\widetilde{u}$, $\widehat{P}$–a.s. If we denote

$$
\widehat{\mathcal{G}}^n_t=\sigma\left(\widehat{u}^n(s,y),\ \overline{u}^n(s,y),\ \widehat{M}^n(s,y)\ ;\ s\le t,\ y\in K\right)
$$

and

$$\widehat{\mathcal{G}}_t = \sigma\left(\widehat{u}(s,y),\ \overline{u}(s,y),\ \widehat{M}(s,y)\ ;\ s \le t,\ y \in K\right),$$

then $(\widehat{M}^n, \widehat{\mathcal{G}}_t^n)$ and $(\widehat{M}, \widehat{\mathcal{G}}_t)$ are gaussian processes (even martingales measures) which have the same law as M.
The property (α) and the fact that u^n and u satisfy respectively (4.1) and (1.2) allow us to prove that $\widehat{u}^n$ satisfies the following stochastic integral equation

$$\begin{aligned} \widehat{u}^{n+1}(t,x) &= \int_0^t \int_{\mathbb{R}^d} S(t-s,x,y)\sigma\left(\widehat{u}^n\right)(s,y)\widehat{M}^n(ds,dy) \\ &\quad + \int_0^t ds \int_{\mathbb{R}^d} dy S(t-s,x,y) b\left(\widehat{u}^n\right)(s,y). \end{aligned} \tag{4.3}$$

Similarly, $\overline{u}^n$ satisfies

$$\begin{aligned} \overline{u}^n(t,x) &= \int_0^t \int_{\mathbb{R}^d} S(t-s,x,y)\sigma\left(\overline{u}^n\right)(s,y)\widehat{M}^n(ds,dy) \\ &\quad + \int_0^t ds \int_{\mathbb{R}^d} dy S(t-s,x,y) b\left(\overline{u}^n\right)(s,y). \end{aligned}$$

Using the property (β) and a limit theorem of Skorokhod [22, p. 32], it holds that

$$\int_0^t \int_{\mathbb{R}^d} S(t-s,x,y)\sigma\left(\widehat{u}^{n_k}\right)(s,y)\widehat{M}^{n_k}(ds,dy)$$

and

$$\int_0^t ds \int_{\mathbb{R}^d} dy S(t-s,x,y) b\left(\widehat{u}^{n_k}\right)(s,y)$$

converge, in probability (as $k \to \infty$), respectively to

$$\int_0^t \int_{\mathbb{R}^d} S(t-s,x,y)\sigma\left(\widehat{u}\right)(s,y)\widehat{M}(ds,dy)$$

and

$$\int_0^t ds \int_{\mathbb{R}^d} dy S(t-s,x,y) b\left(\widehat{u}\right)(s,y).$$

Therefore $\widehat{u}$ and $\overline{u}$ satisfy the same stochastic partial differential equation (1.2), on the space $(\widehat{\Omega}, \widehat{\mathcal{G}}, \widehat{P})$, with the same gaussian noise $\widehat{M}$ and the same initial condition. Then, by the pathwise uniqueness property, we conclude that $\widehat{u}(t,x) = \overline{u}(t,x)$, $\forall\, t, x\ \widehat{P}$–a.s.
By the uniform integrability, it holds that:

$$\begin{aligned} \varepsilon &\le \liminf_{n\in\mathbb{N}} E(\|u^n - u\|^p_{\gamma,T,K}) \\ &\le \liminf_{k\in\mathbb{N}} \widehat{E}\left(\|\widehat{u}^{n_k} - \overline{u}^{n_k}\|^p_{\gamma,T,K}\right) = \widehat{E}\left(\|\widehat{u} - \overline{u}\|^p_{\gamma,T,K}\right), \end{aligned}$$

which is a contradiction. $\square$

Remark 1. Note that under **(R.1)**, **(R.2)** and **(PU)**, the series $\sum_{n\geq 0}\left(u^{n+1}-u^n\right)$ converges in $L^p(\Omega;\mathcal{C}^\gamma([0,T]\times K,\mathbb{R}))$, $(p\geq 2)$ if and only if $\left(u^{n+1}-u^n\right)_{n\geq 0}$ converges to 0 in $L^p(\Omega;\mathcal{C}^\gamma([0,T]\times K,\mathbb{R}))$.

5. Genericity of the existence and uniqueness

As we have seen in previous sections, the pathwise uniqueness property plays a key role in the proof of many stability results. It is then quite natural to raise the question whether the set of all "*nice*" functions (σ,b) for which the pathwise uniqueness holds for the stochastic partial differential equation $\mathrm{Eq}(\sigma,b)$ is larger than its complement.

In this section we improve the result obtained in [2] by considering small spaces which contain the solutions of spde's in any space dimension but driven by a spatially correlated noise.

Let $\mathcal{E}$ be a Baire space. A subset $\mathcal{A}$ of $\mathcal{E}$ is said to be meager (or a first category set in the Baire sense), if it is contained in a countable union of closed nowhere dense subsets of $\mathcal{E}$. The complement of a meager set is called a comeager (or residual or a second category set).
Let us introduce some notations.
For $p\geq 2$, we define $\mathcal{M}^p:=\{u:\Omega\times\mathbb{R}_+\times\mathbb{R}^d\longrightarrow\mathbb{R}$, jointly continuous in time and space such that for all $T>0$ and a compact subset K of $\mathbb{R}^d$, $E\left\|u\right\|^p_{\gamma,T,K}<+\infty\}$.

The space $\mathcal{M}^p$ is equipped with the topology of uniform convergence on compact sets of $\mathbb{R}_+\times\mathbb{R}^d$, that is: (u_n) converges to u if for every $T>0$ and every compact set K,

$$d\left(u_n,u\right):=\left(E\left\|u_n-u\right\|^p_{\gamma,T,K}\right)^{\frac{1}{p}}\longrightarrow 0\quad\text{as}\quad n\longrightarrow\infty.$$

Endowed with this topology $\mathcal{M}^p$ is complete. It is worth noting that this topology allows to define a metric d on $\mathcal{M}^p$ so that $(\mathcal{M}^p,d)$ is complete.

Let $\kappa>0$ and $\mathcal{C}_1:=\{\ b:\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}\longrightarrow\mathbb{R}$ which are continuous and bounded by $\kappa\}$. Define the metric ρ_1 on $\mathcal{C}_1$ as follows:

$$\rho_1\left(b_1,b_2\right)=\sum_{n=1}^{+\infty}\frac{1}{2^n}\cdot\frac{\rho_{1,n}(b_1-b_2)}{1+\rho_{1,n}(b_1-b_2)},$$

where

$$\rho_{1,n}(h)=\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d}\sup_{|r|\leq n}|h(t,x,r)|$$

Note that the metric ρ_1 is compatible with the topology of uniform convergence on compact subsets of $\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}$.

Let $\mathcal{C}_2$ the set of continuous κ–bounded functions $\sigma:\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}\longrightarrow\mathbb{R}$. Define the metric ρ_2 on $\mathcal{C}_2$ by:

$$\rho_2\left(\sigma_1,\sigma_2\right)=\sum_{n=1}^{+\infty}\frac{1}{2^n}\cdot\frac{\rho_{2,n}(\sigma_1-\sigma_2)}{1+\rho_{2,n}(\sigma_1-\sigma_2)},$$

where

$$\rho_{2,n}(h) = \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \sup_{|r|\le n} |h(t,x,r)|$$

Clearly the space $\mathcal{R} = \mathcal{C}_1 \times \mathcal{C}_2$ endowed with the product metric $\rho := \rho_1 \times \rho_2$ is a complete metric space. Hence a Baire space.
Let $\mathcal{R}_{Lip}$ be the subset of $\mathcal{R}$ consisting of functions which are continuous and satisfy **(L)** and **(LG)**.

Proposition 5.1. *The space $\mathcal{R}_{Lip}$ is a dense subset in $\mathcal{R}$.*

Proof. By truncation and regularization arguments. □

5.1. The existence property is generic

We denote by $\mathcal{R}_e$ the subset of functions σ, b in $\mathcal{R}$ for which equation (1.1) has a, not necessarily unique, solution and by $\mathcal{R}_u$ the subset of $\mathcal{R}$ which consists to all functions σ, b for which equation (1.1) has a pathwise unique solution.

Theorem 5.1. *$\mathcal{R}_u$ is a residual set in the Baire space $(\mathcal{R}, \rho)$.*

To prove this Theorem we need some lemmas.

Lemma 5.1. *Let (σ, b) be an element of $\mathcal{R}_{Lip}$. Denote by $u^{\sigma,b}$ the unique solution corresponding to (σ, b). Let u^{σ_n,b_n} be an arbitrary solution of the spde* (1.1) *with coefficients (σ_n, b_n). We assume that*

$$\rho\left[(\sigma_n, b_n), (\sigma, b)\right] \longrightarrow 0 \quad as \quad n \longrightarrow \infty.$$

Then, under **(R.1)** *and* **(R.2)**, *u^{σ_n,b_n} converges to $u^{\sigma,b}$ in $(\mathcal{M}^p, d)$ as $n \to +\infty$.*

Proof. To simplify the notation we write u for $u^{\sigma,b}$ and u_n for u^{σ_n,b_n}. □

Step 1. Let us first prove that $\sup_{n\ge0} \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} E|u_n(t,x)|^p$ is finite for $p \ge 2$. Clearly,

$$\begin{aligned} E\left|u_n(t,x)\right|^p \le c_p E &\left|\int_0^t \int_{\mathbb{R}^d} S(t-s,x,y)\sigma_n(u_n)(s,y)M(ds,dy)\right|^p \\ &+ c_p E \left|\int_0^t \int_{\mathbb{R}^d} S(t-s,x,y) b_n(u_n)(s,y)dyds\right|^p. \end{aligned}$$

We use, Burkholder's inequality, Hölder's inequalities and the boundedness of σ_n and b_n to get,

$$E\left|u_n(t,x)\right|^p \le c_p \left(\|\sigma_n\|_{T,\infty}^p + \|b_n\|_{T,\infty}^p\right) \nu_t^{\frac{p}{2}}$$

where

$$J(s) = \int_{\mathbb{R}^d} \mu(d\xi) \left|\mathcal{F}S(s,\cdot)(\xi)\right|^2 \quad \text{and} \quad \nu_t = \int_0^t J(t-s)ds. \tag{5.1}$$

Therefore, assumption **(R.1)** yields

$$\sup_{n\geq 0}\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d} E|u_n(t,x)|^p \leq c_{p,T}\kappa^p.$$

Step 2. It is clear that $u_n - u$ satisfies the property (P_1) of Lemma 7.2 (appendix). Let us put $\varphi_n(t,x) = E|u_n(t,x) - u(t,x)|^p$ and $\phi_n(t) = \sup_{x\in\mathbb{R}^d}\varphi_n(t,x)$. Hence

$$\varphi_n(t,x) \leq c_p(\nu_t)^{\frac{p}{2}-1}\left[\int_0^t ds\left(\sup_{y\in\mathbb{R}^d} E|\sigma_n(u_n)(s,y) - \sigma(u)(s,y)|^p\right) J(t-s),\right.$$
$$\left. + \int_0^t ds\left(\sup_{y\in\mathbb{R}^d} E|b_n(u_n)(s,y) - b(u)(s,y)|^p\right) J(t-s)\right],$$

where J and ν are defined by (5.1).
Let $T>0$ and $N\in\mathbb{N}^*$. We have,

$$\begin{aligned}
&\sup_{y\in\mathbb{R}^d} E|\sigma_n(u_n)(s,y) - \sigma(u)(s,y)|^p \\
&\quad\leq c_p \sup_{y\in\mathbb{R}^d} E\left|\sigma_n(u_n)(s,y) - \sigma(u_n)(s,y)\right|^p \\
&\qquad + c_p \sup_{y\in\mathbb{R}^d} E\left|\sigma(u_n)(s,y) - \sigma(u)(s,y)\right|^p \\
&\quad\leq c_p\left(\sup_{y\in\mathbb{R}^d} E\left[\left|\sigma_n(u_n)(s,y) - \sigma(u_n)(s,y)\right|^p \mathbf{1}_{\{|u_n(s,y)|\leq N\}}\right]\right. \\
&\qquad + \sup_{y\in\mathbb{R}^d} E\left[\left|\sigma_n(u_n)(s,y) - \sigma(u_n)(s,y)\right|^p \mathbf{1}_{\{|u_n(s,y)|> N\}}\right] \\
&\qquad \left. + \sup_{y\in\mathbb{R}^d} E\left|\sigma(u_n)(s,y) - \sigma(u)(s,y)\right|^p\right) \\
&\quad\leq c_p \sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d}\sup_{|r|\leq N} |\sigma_n(t,x,r) - \sigma(t,x,r)|^p \\
&\qquad + c_p\frac{1}{N^p}\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d} E|u_n(t,x)|^p + c_p\phi_n(s) \\
&\quad\leq c_p\left(\rho_{2,N}^p(\sigma_n - \sigma) + \frac{1}{N^p}\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d} E|u_n(t,x)|^p + \phi_n(s)\right),
\end{aligned}$$

where we have used the Lipschitz property of σ.

Similar arguments give

$$\begin{aligned}
&\sup_{y\in\mathbb{R}^d} E|b_n(u_n)(s,y) - b(u)(s,y)|^p \\
&\quad\leq c_p\left(\rho_{1,N}^p(b_n - b) + \frac{1}{N^p}\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d} E|u_n(t,x)|^p + \phi_n(s)\right).
\end{aligned}$$

Set $\nu_T^* = \sup_{t\le T}\nu_t$. Then for $0 \le t \le T$, we have

$$\begin{aligned}
\phi_n(t) \le c_p\,\nu_t^{\frac{p}{2}-1}&\Bigg[\int_0^t \Bigg(\rho_{2,N}^p(\sigma_n-\sigma)+\rho_{1,N}^p(b_n-b)\\
&\quad + \frac{1}{N^p}\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d} E|u_n(t,x)|^p + \phi_n(s)\Bigg) J(t-s)ds\Bigg]\\
\le c_p\,\nu_t^{\frac{p}{2}-1}&\Bigg[\nu_t\left(\rho_{2,N}^p(\sigma_n-\sigma)+\rho_{1,N}^p(b_n-b)\right)\\
&\quad + \nu_t\frac{1}{N^p}\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d} E|u_n(t,x)|^p + \int_0^t \phi_n(s)J(t-s)ds\Bigg]\\
\le c_p\,(\nu_T^*)^{\frac{p}{2}-1}&\Bigg[\nu_T^*(\rho_{2,N}^p(\sigma_n-\sigma)+\rho_{1,N}^p(b_n-b))\\
&\quad + \nu_T^*\frac{1}{N^p}\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d} E|u_n(t,x)|^p + \int_0^t \phi_n(s)J(t-s)ds\Bigg].
\end{aligned}$$

Hence, using Lemma 7.1 (appendix) and *Step* 1, we get

$$\sup_{0\le t\le T}\phi_n(t) \le c_{p,T}\left(\rho_{2,N}^p(\sigma_n-\sigma)+\rho_{1,N}^p(b_n-b)+\frac{c_{p,T}\kappa^p}{N^p}\right).$$

In view of the convergence of σ_n, b_n to σ, b it follows that

$$\lim_{n\to+\infty}\left[\rho_{2,N}^p(\sigma_n-\sigma)+\rho_{1,N}^p(b_n-b)\right]=0.$$

Moreover, using the previous inequality, it follows by *Step* 1 that

$$\lim_{n\to+\infty}\sup_{t\le T}\phi_n(t)\le \frac{c_{p,T}\kappa^p}{N^p}.$$

Now letting N goes to infinity the Lemma 5.1 follows.

Let $\theta : \mathcal{R} \longrightarrow \mathbb{R}_+$ be the oscillation function defined by

$$\begin{aligned}
\theta(\sigma,b) := \lim_{\delta\to0}\big[\sup\big\{&d(u^{\sigma_1,b_1},u^{\sigma_2,b_2}) \text{ s.t. } (\sigma_i,b_i)\in\mathcal{R}_{Lip},\ i=1,2\\
&\text{and }\max\{\rho[(\sigma,b);(\sigma_1,b_1)],\rho[(\sigma,b);(\sigma_2,b_2)]\}<\delta\big\}\big]
\end{aligned}$$

Then, we then have the following properties:

Lemma 5.2. *Assume that* **(R.1)** *and* **(R.2)** *hold.*
(i) If (σ,b) belongs to $\mathcal{R}_{Lip}$ then $\theta(\sigma,b)=0$.
(ii) The function θ is upper semicontinuous on $\mathcal{R}$.
(iii) If $\theta(\sigma,b)=0$ for σ,b in $\mathcal{R}$, then equation (1.2) *has at least one solution in $\mathcal{M}^p$.*

Remark 2. (j) Lemma 5.2 is similar to that given in [13]. Note that this semicontinuity property is essential in the sequel of the proof. However, the authors couldn't understand the proof given in [13] for the semicontinuity property.

(jj) The assertion *(iii)* of Lemma 5.2 is a sufficient condition to ensure existence of solutions of equation (1.2).

Proof of Lemma 5.2. The assertion *(i)* is a consequence of Lemma 5.1.
Proof of *(ii)*. By the definition of θ we can write:

$$\theta(\sigma,b) = \lim_{\delta\to 0} \theta_\delta(\sigma,b)$$

where

$$\begin{aligned}\theta_\delta(\sigma,b) = \big[\sup\big\{&d(u^{\sigma_1,b_1},u^{\sigma_2,b_2}) \text{ s.t. } (\sigma_i,b_i)\in\mathcal{R}_{Lip},\ i=1,2 \\ &\text{and } \max\left\{\rho\left[(\sigma,b);(\sigma_1,b_1)\right],\rho\left[(\sigma,b);(\sigma_2,b_2)\right]\right\}<\delta\big\}\big].\end{aligned}$$

Let $\eta > 0$ and (σ,b) in $\mathcal{R}$. It is not difficult to see that for every $\delta > 0$, $\theta_\delta(\sigma,b) \le \theta_{\delta+\eta}(\sigma_1,b_1)$ for each (σ_1,b_1) in $\mathcal{R}$ such that $\rho[(\sigma,b);(\sigma_1,b_1)] < \eta$. Which implies that the mapping θ is upper semicontinuous.
We now turn to the proof of *(iii)*. Let $(\sigma,b)\in\mathcal{R}$. Since $\theta(\sigma,b)=0$, then there exists a decreasing sequence of strictly positive numbers δ_n ($\delta_n \searrow 0$) such that

$$\begin{aligned}&\sup\big\{d(u^1,u^2) \text{ such that } (\sigma_i,b_i))\in\mathcal{R}_{Lip},\ i=1,2 \\ &\quad\text{and } \max\left\{\rho\left[(\sigma,b);(\sigma_1,b_1)\right];\ \rho\left[(\sigma,b);(\sigma_2,b_2)\right]\right\}<\delta\big\}<\frac{1}{n}.\end{aligned}\tag{5.2}$$

But, by Proposition 5.1, for each $n\in\mathbb{N}^*$, there exists a $(\sigma^n,b^n)\in\mathcal{R}_{Lip}$ such that $\rho[(\sigma^n,b^n);(\sigma,b)] < \delta_n$. Since δ_n decreases, it follows from (5.2) that $d(u_n,u_m) < \max(\frac{1}{m},\frac{1}{n})$. Hence, $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $(\mathcal{M}^p,d)$. Thus, there exists $u\in\mathcal{M}^p$ such that

$$\lim_{n\to\infty} d(u_n,u)=0.\tag{5.3}$$

Let us now check that u satisfies equation (1.2). From (5.3), there exists a subsequence n_k such that

$$u_{n_k}(t,x) \quad\text{converges to}\quad u(t,x) \quad P\text{–a.s.}\ \text{ as } k\to+\infty.\tag{5.4}$$

It remains to show that for each $t\in\mathbb{R}_+$,

$$I_n(t,x) := \int_0^t ds \int_{\mathbb{R}^d} dy S(t-s,x,y)\left[b^n(u_n)(s,y)-b(u)(s,y)\right]$$

and

$$J_n(t,x) := \int_0^t\int_{\mathbb{R}^d} S(t-s,x,y)\left[\sigma^n(u_n)(s,y)-\sigma(u)(s,y)\right]M(ds,dy).$$

converge to 0 in Probability.

We have

$$
\begin{aligned}
E\left|J_n(t,x)\right|^p &\leq c_p\nu_t^{\frac{p}{2}-1}\int_0^t ds\left(\sup_{y\in\mathbb{R}^d} E|\sigma^n(u_n)(s,y)-\sigma(u)(s,y)|^2\right)J(t-s)\\
&\leq \frac{c_p\nu_t^{\frac{p}{2}}}{N^2}+c_p\rho_{2,N}^p(\sigma^n-\sigma)\nu_t^{\frac{p}{2}}\\
&\quad + c_p\nu_t^{\frac{p}{2}-1}\int_0^t ds\left(\sup_{y\in\mathbb{R}^d} E|\sigma(u_n)(s,y)-\sigma(u)(s,y)|^p\right)J(t-s).
\end{aligned}
$$

Lemma 5.1 implies that

$$
\lim_{n,N\to+\infty}\frac{c_p}{N^2}\nu_T^{\frac{p}{2}}+c_p\rho_{2,N}^p(\sigma^n-\sigma)\nu_T^{\frac{p}{2}}=0.
$$

On the other hand, since $\sigma\in\mathcal{R}_{Lip}$, (5.4) implies that $\sigma(u_n)$ converges to $\sigma(u)$ $dP\times dt\times dx$–a.e. Hence, the Lebesgue dominated convergence theorem allows us to deduce that

$$
\lim_{n\to\infty}\int_0^t ds\left(\sup_{y\in\mathbb{R}^d} E|\sigma(u_n)(s,y)-\sigma(u)(s,y)|^p\right)J(t-s)=0.
$$

Similar arguments allow us to prove that $I_n(t,x)$ converge to 0 in probability. which complete the proof of assertion (iii). Lemma 5.2 is proved. □

Proof of Theorem 5.1. Lemma 5.1 and Lemma 5.2 imply that for each natural number n, the set $\mathcal{H}_n=\{(\sigma,b)\in\mathcal{R}:\theta(\sigma,b)<\frac{1}{n}\}$ is a dense open subset of $(\mathcal{R},\rho)$. Then, by the Baire category theorem, the set $\mathcal{H}=\cap_{n\in\mathbb{N}^*}\mathcal{H}_n$ is a dense G_δ subset of the Baire space $(\mathcal{R},\rho)$. Moreover, if $(\sigma,b)\in\mathcal{H}$ then Lemma 5.2 (iii) implies that the corresponding equation (1.2) has one solution. Hence $\mathcal{H}\subset\mathcal{R}_e$. This implies that $\mathcal{R}_e$ is a residual subset in $(\mathcal{R},\rho)$.
To prove that $\mathcal{R}_u$ is residual, we define the function $\widetilde{\theta}:\mathcal{H}\longrightarrow\mathbb{R}_+$ as follows,

$$
\widetilde{\theta}(\sigma,b)=\sup\left\{d(u_1^{\sigma,b},u_2^{\sigma,b}):u_i^{\sigma,b}\text{ is a solution to equation (1.2)},\ i=1,2\right\}
$$

and for each $n\in\mathbb{N}^*$ we put $\overline{\mathcal{H}}_n=\{(\sigma,b)\in\mathcal{H}:\widetilde{\theta}(\sigma,b)<\frac{1}{n}\}$. By using Lemma 5.1 we see, as in the proof of Lemma 5.2 (ii), that the function $\widetilde{\theta}$ is upper semi-continuous on $\mathcal{R}$. This implies that each $\overline{\mathcal{G}}_n$ contains the intersection of $\mathcal{H}$ and a dense open subset of $(\mathcal{R},\rho)$. Thus the set $\overline{\mathcal{H}}=\cap_{n\in\mathbb{N}^*}\overline{\mathcal{H}}_n$ contains a dense G_δ subset of the Baire space $(\mathcal{R},\rho)$. Hence it is residual in $(\mathcal{R},\rho)$. Finally, if $(\sigma,b)\in\overline{\mathcal{H}}$ then the corresponding equation (1.2) has a unique solution. Thus $\overline{\mathcal{H}}\subset\mathcal{R}_u$. Theorem 5.1 follows. □

5.2. Continuous dependence on the coefficients

For a given $\sigma,b\in\mathcal{R}$ we denote by $\Phi(\sigma,b)=u^{\sigma,b}$ the solution of Eq(σ,b) when it exists.

Theorem 5.2. *There exists a second category set $\mathcal{R}_2$ such that the map $\Phi : \mathcal{R}_2 \longrightarrow \mathcal{M}^p$ given by $\Phi(\sigma, b) = u^{\sigma,b}$ is well defined and continuous at each point of $\mathcal{R}_2$.*

Proof. We shall show that Φ is continuous on $\overline{\mathcal{H}}$ (the dense G_δ set which has been defined in the proof of Theorem 5.1). Suppose the contrary. Then there exist $\sigma \in \overline{\mathcal{H}}$, $\varepsilon > 0$ and a sequence $(\sigma^p)_p \subset \overline{\mathcal{H}}$ such that,

$$\lim_{p\to+\infty} \rho\left[(\sigma^p, b^p); (\sigma, b)\right] = 0 \text{ and } d\left[\Phi(\sigma^p, b^p); \Phi(\sigma, b)\right] \geq \varepsilon \text{ for each } p. \tag{5.5}$$

Fix $n \in \mathbb{N}$ such that $\varepsilon < \frac{1}{n}$. Since $\overline{\mathcal{H}} \subset \mathcal{H}$ then there exists a decreasing sequence of strictly positive numbers δ_n ($\delta_n \searrow 0$) and a sequence of functions $\sigma^n, b^n \in \mathcal{R}_{Lip}$ such that,

$$\rho\left[(\sigma^n, b^n); (\sigma, b)\right] < \delta_n \text{ and } d\left[\Phi(\sigma^n, b^n), \Phi(\sigma, b)\right] < \frac{1}{n}. \tag{5.6}$$

We choose p large enough to have $\rho((\sigma^p, b^p); (\sigma, b)] < \delta_n - \rho[(\sigma^n, b^n); (\sigma, b)]$ then we use (5.6) to obtain $\rho[(\sigma^p, b^p); (\sigma^n, b^n)] < \delta_n$. Hence by Lemma 5.1 we have $d[[\Phi(\sigma^p, b^p); \Phi(\sigma^n, b^n)] < \frac{1}{n}$. Thus

$$\begin{aligned} d[\Phi(\sigma^p, b^p); \Phi(\sigma, b)] &\leq d[\Phi(\sigma^p, b^p); \Phi(\sigma^n, b^n)] \\ &\quad + d[\Phi(\sigma^n, b^n), \Phi(\sigma, b)] \\ &< \frac{2}{n}, \end{aligned}$$

which contradicts (5.5). Theorem 5.2 is then proved. ☐

5.3. The Uniqueness property is generic

The main result of this section is the following

Theorem 5.3. *The subset $\mathcal{R}_{pu}$ of $\mathcal{R}$ consisting of those (σ, b) for which the property* **(PU)** *holds for $Eq(\sigma, b)$ is a residual set.*

Lemma 5.3. *For each $(\sigma_1, b_1) \in \mathcal{R}_{Lip}$ and $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\forall$ $(\sigma, b) \in B((\sigma_1, b_1), \delta)$ and for every pair of solutions u, v of $Eq(\sigma, b)$ (defined on the same probability space, with the same martingale measure), we have $d(u, v) < \varepsilon$.*

Proof. Let w be the unique strong solution of the spde's Eq(σ_1, b_1) defined on the same probability space, and with respect to the same martingale measure M. We have

$$d(u, v) \leq d(u, w) + d(w, v),$$

the result follows from the continuity of w with respect to the coefficients. ☐

Proof of Theorem 5.3. We put $\mathcal{K} = \cap_{k\geq 1} \cup_{(\sigma,b)\in\mathcal{R}_{Lip}} B((\sigma, b), \delta(\frac{1}{k}))$, the subset $\mathcal{K}$ is a G_δ dense subset in the Baire space $(\mathcal{R}, \rho)$, and for every $(\sigma, b) \in \mathcal{K}$, the pathwise uniqueness holds for the spde's Eq(σ, b). It follows that $\mathcal{R}_{pu}$ is a residual subset in $\mathcal{R}$. ☐

Remark 3. Gyöngy (2001) has shown that $\mathcal{R} \setminus \mathcal{K}$ is not empty.

Remark 4. Roughly speaking, we can summarize our results as follows. Assume that condition **(R.1)**, **(R.2)** hold and the coefficients are bounded. Then we have three situations.

1) If condition **(L)** is satisfied by (σ, b), then $\mathrm{Eq}(\sigma, b)$ has a solution which is pathwise unique and the stability holds.

2) Drooping condition **(L)**. If pathwise uniqueness holds for $\mathrm{Eq}(\sigma, b)$, then the stability holds also.

3) If neither conditions **(L)** nor pathwise are satisfied, then the set for which existence and pathwise uniqueness as well as the stability of solutions hold, is a $\mathcal{G}_\delta$ dense in the space of continuous functions. That is, Existence uniqueness and the stability of solutions are generic properties for spde's with continuous coefficients.

6. Applications

This section is devoted to study two examples of stochastic partial differential equations for which Theorems 3.1, 4.1, 5.1, 5.2 and 5.3 can be applied.

6.1. Stochastic heat equation

Consider the following spde

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\Delta u(t,x) + \sigma(u(t,x))\dot{M}(t,x) + b(u(t,x)) \\ \\ u(0,x) = 0, \ t \in [0,T], \ x \in \mathbb{R}^d, \ d \geq 1. \end{cases}$$

Recall that for simplicity, we consider null initial conditions. The integral equation is given by

$$\begin{aligned} u(t,x) = &\int_0^t \int_{\mathbb{R}^d} S_h^d(t-s, x-y)\sigma(u(s,y))M(ds,dy) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} dy S_h^d(t-s,x-y) b(u(s,y)), \end{aligned} \tag{6.1}$$

where $S_h^d(t,x) = (2\pi t)^{-d/2}\exp(\frac{-|x|^2}{2t})$ is the fundamental solution to heat equation, with d–dimensional spatial parameter.

6.2. Stochastic wave equation

In this part, we deal with the spde's

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2}(t,x) = \Delta v(t,x) + b(v)(t,x) + \sigma(v)(t,x)\dot{M}(t,x) \\ v(0,x) = v_0(x) \ \text{ and } \dfrac{\partial v}{\partial t}(0,x) = 0, \ t \geq 0, x \in \mathbb{R}^d \ \text{ and } d \in \{1,2\}. \end{cases}$$

Its integral equation takes the form

$$(6.2)\qquad \begin{aligned} v(t,x) &= \int_{\mathbb{R}^d} W_t^d(x-y)v_0(y)dy + \int_0^t\int_{\mathbb{R}^d} W_{t-s}^d(x-y)b(v)(s,y)dyds \\ &+ \int_0^t\int_{\mathbb{R}^d} W_{t-s}^d(x-y)\sigma(v)(s,y)M(ds,dy) \end{aligned}$$

where $W_t^d(x)$ is the fundamental solution to the wave equation with d–dimensional spatial parameter. In particular if $d=1,2$ the wake kernel is given by

$$\begin{cases} W_t^1(x) = \dfrac{1}{2}\mathbf{1}_{\{|x|\le t\}} & \text{for } \ t\ge 0,\ x\in\mathbb{R} \\ \\ W_t^2(x) = \dfrac{1}{2\pi}\dfrac{1}{\sqrt{t^2-|x|^2}}\mathbf{1}_{\{|x|<t\}} & \text{for } \ t\ge 0,\ x\in\mathbb{R}^2. \end{cases}$$

Let us now consider the stochastic wave equation in dimension 3 whose integral equation is given by

$$(6.3)\qquad \begin{aligned} v(t,x) &= \int_{\mathbb{R}^3} W_t^3(dy)v_0(x-y) + \int_0^t ds\int_{\mathbb{R}^3} b(v(t-s,x-y))W_s^3(dy) \\ &+ \int_0^t\int_{\mathbb{R}^3} W_{t-s}^3(x-y)\sigma(v(s,y))M(ds,dy), \end{aligned}$$

where $W_s^3(dy)$ is the fundamental solution of the wave equation in $\mathbb{R}^3$. More precisely for each $s\in[0,T]$ $W_s^3=\frac{\Sigma_s}{4\pi s}$, where Σ_s denotes the uniform surface measure, with total mass $4\pi s^2$, on the sphere of radius s.
Let us now verify that Assumptions **(R.1)** and **(R.2)** are satisfied for the heat and wave kernels. We have $\mathcal{F}S_h^d(t,\cdot)(\xi)=\exp(-2\pi t|\xi|^2)$, and

$$\frac{c_1}{1+|\xi|^2} \le \int_0^T ds\left|\mathcal{F}S_h^d(s,\cdot)(\xi)\right|^2 \le \frac{c_2}{1+|\xi|^2} < +\infty,$$

and $\mathcal{F}W_t^d(\cdot)(\xi)=\frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}$, therefore

$$\frac{c_3}{1+|\xi|^2} \le \int_0^T ds\left|\mathcal{F}W_t^d(s,\cdot)(\xi)\right|^2 \le \frac{c_4}{1+|\xi|^2} < +\infty.$$

By integrating this inequality over $\mathbb{R}$ we observe that the condition **(R.1)** is equivalent to $(\mathbf{A}_1)$ where for $\eta\in]0,1]$, $(\mathbf{A}_\eta)$:= $\int_{\mathbb{R}^d}\frac{\mu(d\xi)}{(1+|\xi|^2)^\eta}<+\infty$.

From this inequalities and some easy computations we deduce that $(\mathbf{A}_\eta)$ for some η implies that **(R.1)** and **(R.2)** are satisfied.
It is proved by Dalang and Sanz–Solé [9] that under the following conditions:
(R.3)

1. The covariance measure is absolutely continuous with respect to Lebesgue measure with density given by $\Gamma(dx)=\varphi(x)|x|^{-\beta}dx$, where $\varphi\in\mathcal{C}^1(\mathbb{R}^3)$ bounded and positive and $\nabla\varphi\in\mathcal{C}_b^\delta(\mathbb{R}^3)$ for some $\beta\in]0,2[$ and $\delta\in]0,1[$.

2. The support of the function v_0 is contained in the ball $B_{r_0}(0)$, for some $r_0 > 0$.

3. The initial value function v_0 belongs to $\mathcal{C}^2(\mathbb{R}^3)$ and Δv_0 is γ–Hölder continuous with $\gamma \in]0,1[$.

The solution v of the equation (6.3) is α–Hölder continuous with $\alpha \in]0, \gamma \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2}[$ jointly in (t,x).

We can conclude our application as follows. Assume that the spectral measure μ of the noise satisfy $(\mathbf{A}_1)$. If the coefficient b and σ satisfy **(L)** and **(LG)** then equations (6.1), (6.2) and (6.3) have unique solutions.

Assume that **(R.1)** and **(R.2)** hold. Let $(\sigma_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences of functions on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$ which satisfy **(L)** and **(LG)** uniformly in n. Denote by $\{u_n(t,x), t \geq 0, x \in \mathbb{R}^d\}$ the unique solution of equation (6.1) [resp. (6.2)] with coefficients (σ_n, b_n) instead of (σ, b). If σ_n and b_n converge uniformly on compact sets of $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$ tot σ and b respectively. Then, for any $p \geq 2$

$$\lim_{n\to+\infty} E\left(\|u_n - u\|^p_{\gamma,T,K}\right) = 0,$$

where u is the unique solution of equation (6.1) [resp. (6.2)] and K is a compact set of $\mathbb{R}$. This means that we have the stability for this class of equations. Remark also that hypothesis **(R.3)** implies **(R.1)** and **(R.2)**. Therefore, the wave equation (6.3) has unique solution and the stability theorem holds true. In general Theorems 3.1, 4.1, 5.1, 5.2 and 5.3 are applicable for equations (6.1), (6.2) and (6.3).

7. Appendix

Lemma 7.1 ([6, p. 226])**.** *Let f be a positive function on $[0,T]$ such that*

$$f(t) \leq h(t) + \int_0^t g(t-s)f(s)ds, \quad t \in [0,T],$$

with $g \in L^1([0,T])$ and $h \in L^p([0,T])$, $p \geq 1$ both positive. Then

$$f(t) \leq h(t) + \sum_{n=1}^{+\infty} (G^n h)(t) \quad \text{for all} \quad t \in [0,T],$$

where, for $n \geq 1$, $G^n(h) = \int_0^t g_n(t-s)h(s)ds$ and $g_n(t) = \int_0^t g(t-s)g_{n-1}(s)ds$, $g_1(t) = g(t)$.
In particular, if $h \equiv 0$ then $f \equiv 0$.

Lemma 7.2. *Let $\{Y_n(t,x)\}_{n\geq 1}$ be a sequence of processes indexed by $[0,T] \times \mathbb{R}^d$ such that*
$(\mathbf{P}_1)$ *For any $p \geq 2$ there exist $c_p, \delta_1, \delta_2 > 0$ such that for any $t_1, t_2 \in [0,T]$ and $x_1, x_2 \in \mathbb{R}^d$*

$$\sup_n E\left[|Y_n(t_2,x_2) - Y_n(t_1,x_1)|^p\right] \leq c_p \left(|t_2 - t_1|^{\delta_1 p} + \|x_2 - x_1\|^{\delta_2 p}\right).$$

(P$_2$) *For every* $(t,x) \in [0,T] \times \mathbb{R}^d$ *and* $p \geq 2$

$$\lim_{n\to+\infty} E\left[|Y_n(t,x)|^p\right] = 0.$$

Then for any $\gamma_1 \in (0,\delta_1)$ *and* $\gamma_2 \in (0,\delta_2)$

$$\lim_{n\to+\infty} E\left[\|Y_n\|^p_{\gamma,T,K}\right] = 0,$$

where K *is a compact subset of* $\mathbb{R}^d$.

Proof. The proof of this lemma can be found in Millet and Sanz–Solé [17]. □

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