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Locally nilpotent derivations on modules

By

Mikiya Tanaka

Abstract

We extend a locally nilpotent derivation δ on a ring B to B-modules M and investigate the properties of such B-modules M. It turns out that the various geometric properties of the associated G_a-action on Spec B can be reflected on the properties of M.

1. Introduction

An algebraic action of the additive group scheme G_a on an affine scheme Spec B is described equivalently in terms of a locally nilpotent derivation δ on the coordinate ring B. The G_a-invariant subring A and the quotient morphism $\pi : \text{Spec}\, B \to \text{Spec}\, A$ are given by Ker δ and the inclusion $A \to B$. Contrary to this simple description, A and π have properties which are not controlled within the frameworks of ordinary algebraic geometry, e.g., the existence of counterexamples to the fourteenth problem of Hilbert, the non-surjectivity of the quotient morphism π, etc. (see Bonnet [1], Freudenburg [4]). In this article, we introduce the notion of a (B, δ)-module (abbreviated as a δ-module), which is a B-module with a module derivation $\delta_M : M \to M$ such that δ_M is compatible with δ and is locally nilpotent. Prototype examples are δ-ideals of B and the residue rings of B with respect to δ-ideals. It seems that this notion has not been considered except for scattered introduction of the notions similar to this but confined in the limited circumstances. Hence we begin with developing some standard theory of δ-modules and then shift to the study of a δ_M-invariant submodule $M_0 = \text{Ker}\, \delta_M$, which is an A-module.

Some of central problems concerning M_0 include (1) the generation of M by M_0 as a B-module, and (2) the finite generation of M_0 as an A-module. We can only give partial solutions to these problems. The problem (1) is affirmatively answered if M is a free B-module of rank one (Proposition 3.14) or if δ on B has a slice (Lemma 3.3, (2)) or if some geometric conditions are satisfied (Proposition 3.18). Meanwhile, (1) is negatively answered in the case where M is a projective B-module of rank one (Example 8.1). The problem (2) is affirmatively answered if M is a free B-module of rank one (Proposition

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3.14) or if δ has a slice (Lemma 3.3, (2)) or if $B = A[y]$, $\delta \in \mathrm{LND}_A(B)$, and $\delta(y) \in A - (0)$ (Proposition 4.1).

Difficulty on δ-modules lies also in the complexity of δ itself when the dimension of B is higher. But it seems reasonable that we assume δ to be fixed-point free in order to state the properties which are linked to the geometric structure of B and π. Suppose B is one of the following rings: (i) a polynomial ring in one, two or three variables over a field of characteristic zero, (ii) a PID, and (iii) an affine algebra of dimension one over a field of characteristic zero. Then the fixed-point freeness of δ implies that δ has a slice and hence both (1) and (2) above hold true (Corollary 3.4, Corollary 3.6, and Corollary 3.9). Noteworthy is the result that if B is a normal affine $\mathbb{C}$-domain of dimension two with a fixed-point free G_a-action, then any torsion-free, finitely generated (B, δ)-module is a projective module and projectively triangulable (Theorem 7.7).

Finally, we here add that the primary decomposition of a δ-module is treated in Takata [10].

Throughout this paper, we assume that all rings are commutative and have the identity element. We also assume that all rings and fields have characteristic zero. If A is an integral domain, we denote the quotient field of A by $Q(A)$. Given any ring B, we denote the set of units of B by B^*.

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2. Basic properties of locally nilpotent derivations

In this section, we summarize the basic properties of locally nilpotent derivations on rings that we make use of below. We refer the readers to [7], [4] as basic references.

Definition 2.1. Let B be a ring and let A_0 be a subring of B. An A_0-derivation δ of B is said to be locally nilpotent if, for each $b \in B$, there exists an integer N such that $\delta^N(b) = 0$. The set of all A_0-derivations of B is denoted by $\mathrm{Der}_{A_0}(B)$, and the set of all locally nilpotent A_0-derivations of B is denoted by $\mathrm{LND}_{A_0}(B)$. When we say simply that δ is a derivation of B, we mean that δ is a $\mathbb{Z}$-derivation. We write $\mathrm{Der}(B)$ and $\mathrm{LND}(B)$ instead of $\mathrm{Der}_{\mathbb{Z}}(B)$ and $\mathrm{LND}_{\mathbb{Z}}(B)$ respectively.

Given $\delta \in \mathrm{LND}_{A_0} B$, the kernel A of δ is a subring of B such that $A \supset A_0$ and satisfies the following properties.

Lemma 2.2. *Let B be an integral domain containing $\mathbb{Q}$, $\delta \in \mathrm{LND}(B)$, and $A = \mathrm{Ker}\,\delta$. Then we have*:

(1) *A is a subring of B containing B^*, and hence we have $B^* = A^*$.*

(2) *A is factorially closed in B, i.e., if $bb' \in A$ with $b, b' \in B - (0)$, then $b \in A$ and $b' \in A$. Hence if B is a UFD, then so is A.*

(3) *A is algebraically closed in B. Hence if B is integrally closed, then so is A.*

(4) *The derivation* δ *extends uniquely to a derivation* $\delta_{Q(B)}$ *on* $Q(B)$ *and we have* $Q(A) = \operatorname{Ker} \delta_{Q(B)}$.

(5) *If* $\delta(b) \in bB$ *with* $b \in B$, *then* $b \in A$.

A derivation $\delta \in \operatorname{LND}(B)$ defines an A-algebra homomorphism φ_t from B to $B[t]$ and A-algebra automorphisms φ_α of B as follows.

Lemma 2.3. *Let* B *be an algebra over a field* k *of characteristic zero,* $\delta \in \operatorname{LND}(B)$, *and* $A = \operatorname{Ker} \delta$. *Let* $\varphi_t : B \to B[t]$ *be defined by*

$$\varphi_t(b) = \sum_{i=0}^{\infty} \frac{1}{i!} \delta^i(b) t^i,$$

where $B[t]$ *is a polynomial ring over* B. *Then* φ_t *is an* A-*algebra homomorphism. For each* $\alpha \in k$, *let* $\varphi_\alpha : B \to B$ *be defined by*

$$\varphi_\alpha(b) = \sum_{i=0}^{\infty} \frac{1}{i!} \delta^i(b) \alpha^i.$$

Then φ_α *is an* A-*algebra automorphism satisfying* $\varphi_0 = \text{id.}$ *and* $\varphi_{\alpha+\beta} = \varphi_\alpha \circ \varphi_\beta$.

Next we define a slice and summarize the property of a slice.

Definition 2.4. Let B be a ring and $\delta \in \operatorname{LND}(B)$. An element u of B is called a slice of δ if $\delta(u) = 1$.

Lemma 2.5. *Let* B *be a ring containing* $\mathbb{Q}$, $\delta \in \operatorname{LND}(B)$ *and* $A = \operatorname{Ker}\delta$. *Suppose that* δ *has a slice* $u \in B$. *Then we have*:

(1) $B = A[u]$ *and* u *is transcendental over* B.

(2) *Define an* A-*algebra endomorphism* $\varphi_{-u} : B \to B$ *by*

$$\varphi_{-u}(b) = \sum_{i=0}^{\infty} \frac{1}{i!} \delta^i(b)(-u)^i.$$

Then $A = \varphi_{-u}(B)$. *In particular, if* B *is a* k-*algebra generated by* $b_1, \dots, b_r \in B$, *i.e.,* $B = k[b_1, \dots, b_r]$, *where* k *is a field of characteristic zero, then* $A = k[\varphi_{-u}(b_1), \dots, \varphi_{-u}(b_r)]$.

Next we consider a ring of fractions of B.

Lemma 2.6. *Let* $\delta \in \operatorname{LND}(B)$ *and let* S *be a multiplicatively closed subset of* $\operatorname{Ker} \delta$. *Let* $S^{-1}\delta : S^{-1}B \to S^{-1}B$ *be defined by*

$$S^{-1}\delta\left(\frac{b}{s}\right) = \frac{\delta(b)}{s}.$$

Then we have $S^{-1}\delta \in \operatorname{LND}(S^{-1}B)$ *and* $\operatorname{Ker}(S^{-1}\delta) = S^{-1}(\operatorname{Ker} \delta)$.

We define a δ-ideal and summarize the properties of a δ-ideal.

Definition 2.7. Let $\delta \in \mathrm{Der}(B)$. An ideal I of B is called a δ-ideal if $\delta(I) \subset I$. Then we can define $\overline{\delta} \in \mathrm{LND}(B/I)$ in a natural way.

Lemma 2.8. *Let B be an algebra over a field k of characteristic zero, $\delta \in \mathrm{LND}(B)$, and $A = \mathrm{Ker}\,\delta$. With the notation of Lemma 2.3, an ideal I of B is a δ-ideal if and only if $\varphi_\alpha(I) \subset I$ for any $\alpha \in k$.*

Lemma 2.9. *Let B be a noetherian domain containing $\mathbb{Q}$, $\delta \in \mathrm{LND}(B)$, and I a δ-ideal. Then we have*:
(1) *Every prime divisor of I is a δ-ideal.*
(2) *The radical $\sqrt{I}$ is a δ-ideal.*

Next we introduce the notion of fixed-point freeness.

Definition 2.10. Let $\delta \in \mathrm{LND}(B)$. We say that δ is fixed-point free if the ideal of B generated by $\delta(B)$ is a unit ideal.

The fixed-point freeness is characterized as follows.

Lemma 2.11. *Let B be a ring containing $\mathbb{Q}$. Then $\delta \in \mathrm{LND}(B)$ is fixed-point free if and only if there is no maximal ideal of B which is a δ-ideal.*

To end this section, we consider some specialized cases.

Lemma 2.12. *Let $B = k[x]$ be a polynomial ring over a field k of characteristic zero and let $\delta \in \mathrm{LND}(B)$. Then $\delta = a\frac{d}{dx}$ for some $a \in k$.*

Lemma 2.13. *Let B be a polynomial ring in two variables over a field k of characteristic zero and let $\delta \in \mathrm{LND}(B)$. Then there exist $x, y \in B$ and $f \in k[x]$ such that $B = k[x, y]$ and $\delta = f\frac{\partial}{\partial y}$.*

The above lemma is due to Rentschler (see [9]). We can easily deduce the following corollary.

Corollary 2.14. *Let B, δ be as in Theorem 2.13. Suppose that δ is nonzero and fixed-point free. Then δ has a slice.*

The above corollary was generalized by Kaliman [3] in dimension 3. His result states:

Lemma 2.15. *Let B be a polynomial ring in three variables over a field of characteristic zero and let $\delta \in \mathrm{LND}(B)$. Suppose that δ is nonzero and fixed-point free. Then δ has a slice.*

3. δ-modules

Let B be a ring, $\delta \in \mathrm{LND}(B)$ and $A = \mathrm{Ker}\,\delta$. Generalizing δ-ideals and the residue rings of B by δ-ideals, we shall introduce the notion of δ-modules.

Definition 3.1. Let M be a B-module with an A-module endomorphism $\delta_M : M \to M$. A pair (M, δ_M) is called a *pre-(B, δ)-module* (abbreviated as a *pre-δ-module*) if for any $b \in B$ and $m \in M$,

$$\delta_M(bm) = \delta(b)m + b\delta_M(m).$$

Then we say that δ_M is a module derivation. A pre-δ-module (M, δ_M) is called a *(B, δ)-module* (abbreviated as a *δ-module*) if δ_M is locally nilpotent, i.e., for each $m \in M$, there exists an integer n such that ${\delta_M}^n(m) = 0$. If there is no fear of confusion, we denote δ_M simply by δ. Whenever we consider a pre-δ-module structure on a B-module, the derivation δ on B is the same and fixed once for all.

Let (M, δ_M) be a δ-module. A B-submodule N of M is called a δ-submodule of (M, δ_M) if $\delta_M(N) \subset N$. Then (N, δ_N) is a δ-module, where δ_N is the restriction of δ_M. The quotient module M/N is also a δ-module with $\delta_{M/N}$ defined in a natural fashion.

If (M, δ_M) is a δ-module, then $M_0 := \mathrm{Ker}\,\delta_M = \{m \in M \mid \delta_M(m) = 0\}$ is an A-module. We retain below the notations A, M_0, etc. unless otherwise specified.

We can regard module derivations on modules as homogeneous locally nilpotent derivations on graded rings as follows. Let $\delta \in \mathrm{LND}(B)$ and let M be a δ-module. We define B-modules $T_i(M)$ $(i \in \mathbb{N})$ inductively by $T_0(M) = B$ and $T_i(M) = M \otimes_B T_{i-1}(M)$. Then the $T_i(M)$ are δ-modules in a natural fashion by (1) of Lemma 5.3 below. Hence $T(M) := \bigoplus_{i=0}^{\infty} T_i(M)$ is a δ-module. Let I be the two-sided ideal of $T(M)$ generated by $\{m \otimes n - n \otimes m \mid m, n \in M\}$. Since I is a δ-submodule of $T(M)$, it follows that $S(M) := T(M)/I$ is a δ-module. Then $\delta_{S(M)}$ is a homogeneous locally nilpotent derivation on a graded ring $S(M) = \bigoplus_{i=0}^{\infty} T_i(M)/(I \cap T_i(M))$. Moreover, we have $T_1(M)/(I \cap T_1(M)) = T_1(M) = M$ and $\delta_{S(M)}|_M : M \to M$ is equal to δ_M. Conversely, let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded ring such that $R_0 = B$ and let $\widetilde{\delta}$ be a homogeneous locally nilpotent derivation on R. Then $\delta := \widetilde{\delta}|_{R_0} \in \mathrm{LND}(B)$. Moreover, R_i is a (B, δ)-module for each i. Hence considering module derivations on modules correspond equivalently to homogeneous locally nilpotent derivations on graded rings $R = \bigoplus_{i=0}^{\infty} R_i$ when R_1 generates R over R_0.

From this point of view, we can obtain the assertions concerning δ-modules in this section from those for homogeneous locally nilpotent derivations on graded rings. Since we are interested in the module derivations below, we rewrite the known properties in terms of module derivations.

Lemma 3.2. *For $\delta \in \mathrm{LND}(B)$, we have the following assertions.*

(1) *Let δ_M be a module derivation on a B-module M. Then the Leibniz rule holds. Namely, for any $b \in B, m \in M$, and for any positive integer n, we have*

$${\delta_M}^n(bm) = \sum_{i=0}^{n} \binom{n}{i} \delta^{n-i}(b){\delta_M}^i(m).$$

(2) *Let M be a pre-δ-module. Suppose that a subset T of M generates M as a B-module and suppose that, for each $m \in T$, there exists an integer n such that ${\delta_M}^n(m) = 0$. Then M is a δ-module.*

(3) *Given an A-module M_0, let $M = M_0 \otimes_A B$ and define $\delta_M : M \to M$ by $\delta_M(m \otimes b) = m \otimes \delta(b)$. Then (M, δ_M) is a δ-module.*

(4) *Let M be a pre-δ-module and let S be a multiplicatively closed subset of B. The module derivation δ_M extends uniquely to a module derivation $S^{-1}\delta_M$ on $S^{-1}M$ by*

$$S^{-1}\delta_M\left(\frac{m}{s}\right) = \frac{s\delta_M(m) - \delta(s)m}{s^2}.$$

Hence $S^{-1}M$ is a pre-δ-module. In particular, if $S \subset A$ and M is a δ-module, then $S^{-1}M$ is a $(S^{-1}B, S^{-1}\delta)$-module and we have $\mathrm{Ker}\,(S^{-1}\delta_M) = S^{-1}(\mathrm{Ker}\,\delta_M)$.

We have the results similar to Lemma 2.3 and Lemma 2.5 for a module derivation.

Lemma 3.3. *Suppose that $B \supset \mathbb{Q}$ and define $\varphi_t : B \to B[t]$ as in Lemma 2.3. Let M be a δ-module. Assume further in the assertions* (2) *and* (3) *that δ has a slice $u \in B$. Then we have the following assertions*:

(1) *Define $\theta_t : M \to M[t]$ by*

$$\theta_t(m) = \sum_{i=0}^{\infty} \frac{1}{i!}{\delta_M}^i(m)t^i,$$

where $M[t] = M \otimes_B B[t]$. Then θ_t is an A-module homomorphism satisfying $\theta_t(bm) = \varphi_t(b)\theta_t(m)$ for any $b \in B$, $m \in M$.

(2) *We have $M = M_0[u] = M_0 \otimes_A B$. Hence M is a finitely generated B-module if and only if M_0 is a finitely generated A-module. Moreover, M is a free B-module if and only if M_0 is a free A-module.*

(3) *We define an A-module endomorphism $\theta_{-u} : M \to M$ by*

$$\theta_{-u}(m) = \sum_{i=0}^{\infty} \frac{1}{i!}{\delta_M}^i(m)(-u)^i.$$

Then we have $M_0 = \mathrm{Ker}\,\delta_M = \theta_{-u}(M)$. In particular, if M is generated by $m_1, \ldots, m_r$ as a B-module, then $M_0 = A\theta_{-u}(m_1) + \cdots + A\theta_{-u}(m_r)$.

We shall apply Lemma 3.3, (2) to the special cases. From Lemma 2.12, Corollary 2.14, and Lemma 2.15, we can easily deduce the following corollary.

Corollary 3.4. *Let $\delta \in \mathrm{LND}(B)$, and M a δ-module. Suppose that δ is nonzero and fixed-point free. If B is a polynomial ring in one, two or three variables over a field of characteristic zero, then we have $M = M_0[u] = M_0 \otimes_A B$ for some $u \in B$.*

We can also apply Lemma 3.3, (2) to the case where B is a PID (Corollary 3.6). In order to show it, we need the following lemma.

Lemma 3.5. *Let B be a PID and let $\delta \in \mathrm{LND}(B)$. Suppose that δ is nonzero and fixed-point free. Then $A = \mathrm{Ker}\,\delta$ is a field and $B = A[x]$ is a polynomial ring in one variable.*

Proof. Since B is a PID, A is a UFD. We shall first show that A is a field. Assume the contrary, and let f be a nonzero non-unit element of A. We can take f to be irreducible in A. Then f is irreducible in B as well. Hence fB is a prime ideal. Since B is a PID, fB must be a maximal ideal. Meanwhile fB is a δ-ideal, which contradicts the fixed-point freeness of δ (Lemma 2.11). Now it is clear that $B = A[x]$ with $\delta(x) = 1$. □

Corollary 3.6. *Let B, δ be as in Lemma 3.5. Let M be a δ-module. Then M is a free B-module with a basis contained in M_0.*

Proof. Since A is a field, M_0 is a free A-module. Since δ has a slice, it follows from Lemma 3.3, (2) that $M = M_0 \otimes_A B$, which concludes the proof. □

We can also apply Lemma 3.3, (2) to the case where B is an affine algebra of dimension one over a field of characteristic zero (Corollary 3.9). We need the following lemma (see Miyanishi [8, Theorem2.1]).

Lemma 3.7. *Let B be an affine algebra of dimension one over a field of characteristic zero, $\delta \in \mathrm{LND}(B)$, and $A = \mathrm{Ker}\,\delta$. Suppose that δ is nonzero and fixed-point free. Then there exists $u \in B$ such that $B = A[u]$, u is transcendental over A and $1 - \delta(u)$ is nilpotent. Furthermore, A is an Artin ring.*

In fact, we have the following result.

Proposition 3.8. *Let B, δ be as in Lemma 3.7. Then δ has a slice.*

Proof. Since $q = 1 - \delta(u)$ is nilpotent, we have $q^r = 0$ for some integer r. Let $f \in B$ and write $f = a_0 + a_1 u + \cdots + a_n u^n$ with $a_i \in A$. Then $\delta(f) = (a_1 + 2a_2 u + \cdots + n a_n u^{n-1})(1 - q)$. If we can take f so that $a_1 + 2a_2 u + \cdots + n a_n u^{n-1} = 1 + q + \cdots + q^{r-1}$, then we have $\delta(f) = 1$. The existence of the a_i satisfying this equality is obvious. □

Applying Lemma 3.3, (2), we deduce:

Corollary 3.9. *Let B, δ be as in Lemma 3.7. Let M be a δ-module. Then we have $M = M_0[u] = M_0 \otimes_A B$ for some $u \in B$.*

We also obtain the following properties of δ-modules.

Lemma 3.10. *Let M be a δ-module. Then we have the following assertions*:

(1) $\mathrm{Ann}(M)$ *is a δ-ideal of B.*

(2) *If B is a reduced ring, then the torsion part M_{tor} of M is a δ-submodule of M.*

Proof. (1) Take any $b \in \mathrm{Ann}\,(M)$. Then $bm = 0$ for any $m \in M$. Thus we have $0 = \delta_M(bm) = \delta(b)m + b\delta_M(m) = \delta(b)m$ for any $m \in M$. This implies that $\delta(b) \in \mathrm{Ann}\,(M)$.

(2) Let $m \in M_{\mathrm{tor}}$. Then there exist a nonzero element $b \in B$ such that $bm = 0$. We have $0 = \delta_M(bm) = \delta(b)m + b\delta_M(m)$. Multiplying b, we have $b^2\delta_M(m) = 0$. Since B is reduced, $b^2 \neq 0$ and hence $\delta_M(m) \in M_{\mathrm{tor}}$. □

Definition 3.11. For $\delta \in \mathrm{LND}(B)$, the degree of $\varphi_t(b) \in B[t]$, which we denote by $\nu(b)$, is called the δ-degree of $b \in B$. Similarly for a δ-module M, the degree of $\theta_t(m) \in M[t]$, which is equal to the integer r such that $\delta^r(m) \neq 0$ and $\delta^{r+1}(m) = 0$ if $m \neq 0$, is called the *δ-degree* of $m \in M$. We define the degree of the zero element of $M[t]$ to be $-\infty$. For $m \in M$, we denote the δ-degree of $m \in M$ by $\nu_M(m)$.

We have the following properties of δ-degrees.

Lemma 3.12. *Let B be a ring containing $\mathbb{Q}$, $\delta \in \mathrm{LND}(B)$ and M a δ-module. Then we have the following assertions*:

(1) *For any $b \in B$ and $m \in M$, we have $\nu_M(bm) \leq \nu(b) + \nu_M(m)$. The equality holds if M is B-torsion-free.*

(2) $\nu_M(m + m') \leq \max(\nu_M(m), \nu_M(m'))$ *for any $m, m' \in M$.*

Proof. (1) is easily shown by the Leibniz rule (Lemma 3.2). (2) is obvious. □

The following result is an analogue of Lemma 2.2, (5).

Lemma 3.13. *Let B be a ring containing $\mathbb{Q}$, $\delta \in \mathrm{LND}(B)$ and M a δ-module. Suppose that M is B-torsion-free and that $m \in M$ satisfies $\delta_M(m) \in Bm$. Then $\delta_M(m) = 0$.*

Proof. Suppose that $m \neq 0$ and $\delta_M(m) \neq 0$. Then $\delta_M(m) = bm$ for some nonzero element $b \in B$. We have

$$\nu(b) + \nu_M(m) = \nu_M(bm) = \nu_M(\delta_M(m)) = \nu_M(m) - 1.$$

Hence $\nu(b) = -1$, which is absurd. □

Lemma 3.13 implies the following result.

Proposition 3.14. *Let B be a ring containing $\mathbb{Q}$, $\delta \in \mathrm{LND}(B)$, and $M = Be$ a free B-module of rank one. Suppose that M is a δ-module. Then $\delta_M(e) = 0$. Hence $M_0 = Ae$ and $M = B \otimes_A M_0$.*

Proposition 3.14 implies that M is identified with B as δ-modules. The equality $M = BM_0$ does not hold for a projective (B, δ)-module of rank one (see Example 8.1). We can generalize Proposition 3.14 as follows.

Proposition 3.15. *Let B be a ring containing $\mathbb{Q}$, $\delta \in \mathrm{LND}(B)$, M a B-module, and $M' = M \oplus Be$ a direct sum of M and a free B-module of rank one. Suppose that M' is a δ-module and M is a δ-submodule of M'. Then $\delta_{M'}(e) \in M$.*

Proof. Since M is a δ-submodule of M', it follows that $M'/M = Be$ is a δ-module. By Proposition 3.14, we have $\delta_{M'}(e) \in M$. □

The following result is an analogue of Lemma 2.2, (2).

Lemma 3.16. *Let B be a ring containing $\mathbb{Q}$, $\delta \in \mathrm{LND}(B)$ and M a δ-module. Suppose that M is B-torsion-free. Then if $\delta_M(bm) = 0$ with $b \in B-(0)$ and $m \in M-(0)$, then $\delta(b) = 0$ and $\delta_M(m) = 0$.*

Proof. If $\delta_M(bm) = 0$, then $\nu_M(bm) = \nu(b) + \nu_M(m) = 0$ and hence $\nu(b) = 0$ and $\nu_M(m) = 0$, which implies that $\delta(b) = 0$ and $\delta_M(m) = 0$. □

In proving the above results by using δ-degrees, we have to assume that M is B-torsion-free. We shall show that this assumption is equivalent to assuming that M_0 is A-torsion-free.

Lemma 3.17. *Let B be a ring containing $\mathbb{Q}$, $\delta \in \mathrm{LND}(B)$, and M a δ-module. Then M is B-torsion-free if and only if M_0 is A-torsion-free.*

Proof. "Only if" part is clear. We prove "if" part. Suppose that M_0 is A-torsion-free. For $b \in B$ and $m \in M$, suppose that $bm = 0$, $b \neq 0$ and $m \neq 0$. Let $r = \nu(b)$, $s = \nu_M(m)$. Then we have

$$0 = \delta^{r+s}(bm) = \sum_{i+j=r+s} \binom{r+s}{i} \delta^i(b)\delta^j(m) = \binom{r+s}{r} \delta^r(b)\delta^s(m).$$

Since $\delta^r(b) \in A$, $\delta^s(m) \in M_0$ and M_0 is A-torsion-free, either $\delta^r(b)$ or $\delta^s(m)$ must be zero, which is absurd. □

If δ has a slice or if M is a free B-module of rank one, then it holds that $M = M_0 \otimes_A B$. This is also the case with more geometric conditions.

Proposition 3.18. *Let B be an affine normal ring over an algebraically closed field k of characteristic zero, $\delta \in \mathrm{LND}(B)$, and $\mathfrak{a} = \delta(B) \cap A$, which is an ideal of A. Let $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, and $\pi : X \to Y$ the natural quotient morphism. Assume that* $\operatorname{codim}_X(\pi^{-1}(V(\mathfrak{a})) \geq 2$. *Let M be a finitely generated (B,δ)-module. Assume that* $\operatorname{depth}(M) \geq 2$ *and* $\operatorname{depth}(M_0 \otimes_A B) \geq 2$. *Then $M = M_0 \otimes_A B$.*

Proof. Let $\mathcal{F}, \mathcal{G}$ be the $\mathcal{O}_X$-Module associated to $M, M_0 \otimes_A B$ respectively. Let $U = \pi^{-1}(Y - V(\mathfrak{a}))$ and let $j : U \to X$ be the natural open immersion. By the next lemma, we have $j_*(j^*\mathcal{F}) \cong \mathcal{F}$ and $j_*(j^*\mathcal{G}) \cong \mathcal{G}$. Note that $j^*\mathcal{F} = \mathcal{F}|_U$ and $j^*\mathcal{G} = \mathcal{G}|_U$. There is a natural morphism $f : \mathcal{G} \to \mathcal{F}$. In order to show that f is an isomorphism, it suffices to show that $f|_U : \mathcal{G}|_U \to \mathcal{F}|_U$ is an isomorphism, i.e., $f_x : \mathcal{G}_x \to \mathcal{F}_x$ is an isomorphism for every $x \in U$. If we show that $M_{\mathfrak{p}} \cong (M_0 \otimes_A B)_{\mathfrak{p}}$ for any $\mathfrak{p} \in Y - V(\mathfrak{a})$, then for any $\mathfrak{P} \in U$ with $\mathfrak{p} = \pi(\mathfrak{P})$, we have

$$\begin{aligned} M_{\mathfrak{P}} &= (B - \mathfrak{P})^{-1}M = (B - \mathfrak{P})^{-1}(A - \mathfrak{p})^{-1}M \\ &= (B - \mathfrak{P})^{-1}(A - \mathfrak{p})^{-1}(M_0 \otimes_A B) = (M_0 \otimes_A B)_{\mathfrak{P}}. \end{aligned}$$

Hence it is enough to prove that $M_{\mathfrak{p}} \cong (M_0 \otimes_A B)_{\mathfrak{p}}$ for any $\mathfrak{p} \in Y - V(\mathfrak{a})$. If $\mathfrak{p} \not\supset \mathfrak{a}$, then there exists $a \in \mathfrak{a} - \mathfrak{p}$. Since $a \in \mathfrak{a}$, we have $\delta(u) = a$ for some $u \in B$. Then u/a is a slice of the derivation on $B[a^{-1}]$ induced by δ. Hence we have

$$\begin{aligned} M[a^{-1}] &= (M[a^{-1}])_0[u/a] = M_0[a^{-1}][u/a] \\ &= M_0 \otimes_A A[a^{-1}] \otimes_{A[a^{-1}]} A[a^{-1}][u/a] \\ &= M_0 \otimes_A B \otimes_A A[a^{-1}] = (M_0 \otimes_A B)[a^{-1}]. \end{aligned}$$

Since $a \notin \mathfrak{p}$, we have $M_{\mathfrak{p}} = (M_0 \otimes_A B)_{\mathfrak{p}}$. □

In the proof of Proposition 3.18, we use the following lemma, which is Corollary 5.10.6 of Chapter IV of EGA [5].

Lemma 3.19. *Let X be a normal scheme, U an open subset of X, and $j : U \to X$ the natural open immersion. Assume that* $\operatorname{codim}_X(X - U) \geq 2$. *Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module with* $\operatorname{depth}_{X-U}\mathcal{F} \geq 2$. *Then $j_*(j^*(\mathcal{F})) \cong \mathcal{F}$.*

We give an example of a δ-module M of depth one which does not satisfy the equality $BM_0 = M$ (cf. Bonnet [1]).

Example 3.20. Let $B = \mathbb{C}[x, x', y, y']$ be a polynomial ring and let $\delta \in \mathrm{LND}(B)$ be defined by $\delta(x) = \delta(x') = 0$, $\delta(y) = x$ and $\delta(y') = x'$. Then we can easily show that $A = \mathbb{C}[x, x', xy' - x'y]$ as follows. Indeed let $S = \{1, x, x^2, \dots\}$ be a multiplicatively closed subset of B. Since $S^{-1}\delta(y/x) = 1$, we have $\operatorname{Ker} S^{-1}\delta = \mathbb{C}[x, 1/x, xy' - x'y, x']$ by using Lemma 2.5, (2). Since $A = \operatorname{Ker} S^{-1}\delta \cap B$, we have $A = \mathbb{C}[x, x', xy' - x'y]$. Let M be the ideal of B generated by x, y, x', y'. Then M is a δ-ideal and hence a δ-module. We claim that $\operatorname{depth} M = 1$. Indeed x is not a zero divisor on M. For any $f \in M$, we have $f\overline{x} = 0$ and $\overline{x} \neq 0$ in M/xM, where $\overline{x}$ is the residue class of $x \in M$ in M/xM. Hence $\operatorname{depth} M = 1$. We have $M_0 = A \cap M = xA + x'A + (xy' - x'y)A$ and therefore $BM_0 = xB + x'B + (xy' - x'y)B = xB + x'B$, which is not equal to M. Note that $\mathfrak{a} = xA + x'A$ and $V(\mathfrak{a})$ contracts to a point $(0, 0, 0)$ by the quotient morphism $\pi : \operatorname{Spec} B \to \operatorname{Spec} A$.

Let M be a B-module generated by $m_1, \dots, m_r \in M$. Suppose that M is a pre-δ-module. Then the pre-δ-module structure is determined by $\delta_M(m_i)$, $1 \leq i \leq r$. Further, if ${\delta_M}^N(m_i) = 0$ for some $N > 0$ $(1 \leq i \leq r)$, then M is a δ-module (see Lemma 3.2, (2)). With this remark in mind, we define the triangulability of δ_M for a free (B, δ)-module M of finite rank.

Definition 3.21. Let M be a pre-(B,δ)-module which is a free B-module of finite rank. A module derivation δ_M is determined by the $\delta_M(e_i)$ for a free basis $\{e_1,\ldots,e_r\}$ of M. We say that a module derivation δ_M is *triangulable* if there exists a free basis $\{e_1,\ldots,e_r\}$ such that $\delta_M(e_i)\in Be_1\oplus\cdots\oplus Be_{i-1}$ $(2\le i\le r)$ and $\delta_M(e_1)=0$. Clearly, a triangulable module derivation is locally nilpotent.

To end this section, we raise the following three questions for a (B,δ)-module M, which we consider in the subsequent sections.

(1) Is M generated by M_0 as a B-module?

(2) Is M_0 a finitely generated A-module provided M is a finitely generated B-module?

(3) Is a module derivation δ_M on a free B-module M triangulable?

4. The case where $B=A[y]$

In this section, we consider the case where $B=A[y]$ is a polynomial ring in one variable over a noetherian ring $A\supset\mathbb{Q}$. Suppose that $\delta\in\mathrm{LND}(B)$ is nonzero. We assume that $\delta\in\mathrm{LND}_A(B)$ is defined by $\delta(y)=a\in A-(0)$ and $\operatorname{Ker}\delta=A$. This condition is automatically satisfied if A is an integral domain. The case where B is a polynomial ring in two variables over a field k of characteristic zero is a special case. Indeed, if B is a polynomial ring in two variables over a field k of characteristic zero, then there exist $x,y\in B$ such that $B=k[x,y]$, $\delta(x)=0$ and $\delta(y)=a\in k[x]-(0)$ by Lemma 2.13. Then $A=\operatorname{Ker}\delta=k[x]$ is a PID and $B=A[y]$. We shall investigate the structure of (B,δ)-modules.

Proposition 4.1. *Let $B=A[y]$ be a polynomial ring in one variable over a noetherian ring $A\supset\mathbb{Q}$, δ a nonzero locally nilpotent A-derivation on B defined by $\delta(y)=a\in A-(0)$ with* $\operatorname{Ker}\delta=A$, *$M$ a finitely generated torsion-free (B,δ)-module, $M_0=\operatorname{Ker}\delta_M$, and M_1 the B-submodule generated by M_0. Then $M_1=M_0\otimes_A B$ and M_0 is a finitely generated A-module. Furthermore, we have $S^{-1}M=S^{-1}M_1$, where $S=\{1,a,a^2,\ldots\}$.*

Proof. First we show that $M_1=M_0\otimes_A B$. Indeed, we have $M_1=BM_0=\sum_{i=0}^{\infty}M_0y^i=\bigoplus_{i=0}^{\infty}M_0y^i\cong M_0\otimes_A A[y]=M_0\otimes_A B$, where the equality $\sum_{i=0}^{\infty}M_0y^i=\bigoplus M_0y^i$ is proved as follows. Take any element $m=m_0+m_1y+\cdots+m_ry^r\in\sum_{i=0}^{\infty}M_0y^i$ and suppose that $m=0$. Then $\delta^r(m)=r!a^rm_r=0$. Since M is B-torsion-free, we have $m_r=0$. By induction on r, we have $m_i=0$ for all i. Next we show that M_0 is a finitely generated A-module. Note that B is a noetherian ring. Since M_1 is a B-submodule of the finitely generated B-module M, it follows that M_1 is a finitely generated B-module. Write $M_1=Bm_1+\cdots+Bm_n$ with $m_i\in M_1$. Since $M_1=BM_0$, we may take each m_i as an element of M_0. Then we have $M_0=M_1/yM_1=Am_1+\cdots+Am_n$. Finally we show that $S^{-1}M=S^{-1}M_1$. Since the derivation $S^{-1}\delta$ on $S^{-1}B$ has a slice y/a, we have $S^{-1}M=(S^{-1}M)_0[y/a]=S^{-1}M_0[y/a]=S^{-1}(M_0[y])=S^{-1}M_1$. □

The following is an immediate consequence of the above proposition.

Corollary 4.2. *Let B, A, δ, M, M_0, and M_1 be as in Proposition* 4.1. *Suppose that M_0 is a free A-module. This condition is automatically satisfied if A is a PID. If $\{f_1, \dots, f_n\}$ is a free A-basis of M_0, then it is also a free B-basis of M_1 and hence a free $S^{-1}B$-basis of $S^{-1}M = S^{-1}M_1$.*

Meanwhile, if M is not B-torsion-free, then M_0 is not necessarily a finitely generated A-module. The following is a counter example.

Example 4.3. Let $B = k[x, y]$ be a polynomial ring in two variables over a field k of characteristic zero. Let $\delta \in \mathrm{LND}(B)$ be defined by $\delta(x) = 0$ and $\delta(y) = x$. Let $M = B/xB$. Then $A = k[x]$ and M is a δ-module. We have $M_0 = M$. It is obvious that $M_0 = \sum_{i=0}^{\infty} Ay^i$ is not finitely generated over A.

Hence we pose the following conjecture.

Conjecture 4.4. *Let B be an affine domain over a field of characteristic zero. We suppose that $\delta \in \mathrm{LND}(B)$ is fixed-point free. Let M be a torsion-free, finitely generated (B, δ)-module. Then M_0 is a finitely generated A-module.*

For the sake of simplicity, we denote all of δ, δ_M, $S^{-1}\delta$, and $S^{-1}\delta_M$ by the same symbol δ if there is no fear of confusion. We extend δ to a derivation on the matrix rings $\mathrm{M}(n, B)$ and $\mathrm{M}(n, S^{-1}B)$ in a natural fashion, which is also denoted by δ. We have the following assertion.

Lemma 4.5. *Let $B = A[y]$ be a polynomial ring in one variable over a PID A and let $\delta \in \mathrm{LND}_A(B)$ be defined by $\delta(y) = a \in A - (0)$ with* $\mathrm{Ker}\,\delta = A$. *Let $S = \{1, a, a^2, \dots\}$. Let M be a free (B, δ)-module with a free basis $\{e_1, \dots, e_n\}$. Then δ_M is defined by*

$$\begin{pmatrix} \delta(e_1) \\ \vdots \\ \delta(e_n) \end{pmatrix} = \delta(D^{-1})D \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix},$$

where D satisfies (1) $D \in \mathrm{M}(n, B)$, (2) $D^{-1} \in \mathrm{M}(n, S^{-1}B)$ *and* (3) $\delta(D^{-1})D \in \mathrm{M}(n, B)$. *In addition, the components of any row vector of D have no common factors in B.*

Proof. Let $M_1 = BM_0$. By Corollary 4.2, there exists a free basis $\{f_1, \dots, f_n\}$ of $S^{-1}M_1 = S^{-1}M$ such that $f_i \in M_0$. Since $f_i \in M$, we have

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = D \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

with $D \in \mathrm{M}(n, B)$. Since both $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$ are free bases of the free $S^{-1}B$-module $S^{-1}M$, we have

$$\begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = D^{-1} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

and $D^{-1} \in \mathrm{M}(n, S^{-1}B)$. Moreover, since the f_i are in M_0, we have

$$\begin{pmatrix} \delta(e_1) \\ \vdots \\ \delta(e_n) \end{pmatrix} = \delta(D^{-1}) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \delta(D^{-1})D \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}.$$

Hence $\delta(D^{-1})D \in \mathrm{M}(n, B)$. Finally we claim that the components of any row vector of D have no common factors in B, i.e., we have $\gcd(d_{i1}, \ldots, d_{in}) = 1$ for all i, where we denote $D = (d_{ij})$. Indeed, suppose that $d_{i1} = dd'_{i1}, \ldots, d_{in} = dd'_{in}$. Then $f_i = df'_i$ for some $f'_i \in M$. Since $f_i \in M_0$, we have $d \in A = \operatorname{Ker}\delta$ and $f'_i \in M_0$ by Lemma 3.16. Since $\{f_1, \ldots, f_n\}$ is a free basis of the A-module M_0, there exist $a_1, \ldots, a_n \in A$ such that $f'_i = a_1 f_1 + \cdots + a_n f_n$. Then $f_i = da_1 f_1 + \cdots + da_i f_i + \cdots + da_n f_n$ and hence $1 = da_i$. Thus $d \in A^*$. □

We have the converse of Lemma 4.5.

Lemma 4.6. *Let B, A, δ and S be as in Lemma 4.5. Let $M = Be_1 \oplus \cdots \oplus Be_n$ be a free B-module. If an $n \times n$ matrix D satisfies* (1) $D \in \mathrm{M}(n, B)$, (2) $D^{-1} \in \mathrm{M}(n, S^{-1}B)$ *and* (3) $\delta(D^{-1})D \in \mathrm{M}(n, B)$, *then a module derivation on M defined by*

$$\begin{pmatrix} \delta(e_1) \\ \vdots \\ \delta(e_n) \end{pmatrix} = \delta(D^{-1})D \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

is locally nilpotent. In addition, we can take D so that the components of any row vector of D have no common factors.

Proof. Let

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = D \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}.$$

Since $0 = \delta(E) = \delta(DD^{-1}) = \delta(D)D^{-1} + D\delta(D^{-1})$, it follows that $\delta(D) + D\delta(D^{-1})D = 0$. Hence we have

$$\begin{pmatrix} \delta(f_1) \\ \vdots \\ \delta(f_n) \end{pmatrix} = \delta(D) \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} + D \begin{pmatrix} \delta(e_1) \\ \vdots \\ \delta(e_n) \end{pmatrix}$$

$$= (\delta(D) + D\delta(D^{-1})D) \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus we have

$$\begin{pmatrix} \delta^N(e_1) \\ \vdots \\ \delta^N(e_n) \end{pmatrix} = \delta^N(D^{-1}) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

for a sufficiently large integer N since the derivation $S^{-1}\delta$ on $S^{-1}B$ is locally nilpotent. Next, we claim that we can take $D = (d_{ij})$ so that $d_{i1}, \dots, d_{in}$ have no common factors for each i. Suppose that $d_{11} = dd'_{11}, \dots, d_{1n} = dd'_{1n}$ with $d, d'_{1j} \in B$. Then $f_1 = df'_1 \in \operatorname{Ker} \delta_M = M_0$ with $f'_1 \in M$. Hence $d \in \operatorname{Ker} \delta = A$ by Lemma 3.16. Let

$$D' = \begin{pmatrix} d'_{11} & d'_{12} & \dots & d'_{1n} \\ d_{21} & d_{22} & \dots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \dots & d_{nn} \end{pmatrix}.$$

Then $D = E_d D'$ with

$$E_d := \begin{pmatrix} d & & & \text{\huge 0} \\ & 1 & & \\ & & \ddots & \\ \text{\huge 0} & & & 1 \end{pmatrix}.$$

Since $d \in A$, we have

$$\delta(D^{-1})D = \delta({D'}^{-1}E_{1/d})E_d D' = \delta({D'}^{-1})E_{1/d}E_d D' = \delta({D'}^{-1})D'$$

and ${D'}^{-1} = D^{-1}E_d \in \mathrm{M}(n, S^{-1}B)$. Hence D' and D define the same module derivation on M, which completes the proof. □

We also have the following.

Lemma 4.7. *Let the $f_i \in M_0$ be as in the proof of Lemma* 4.6. *Then the f_i generate $S^{-1}M_0$ as an $S^{-1}A$-module.*

Proof. It suffices to show that $M_0 \subset S^{-1}Af_1 + \cdots + S^{-1}Af_n$. Let $m = b_1e_1 + \cdots + b_ne_n \in M_0$ with $b_i \in B$. Then if we write

$$m = \begin{pmatrix} b_1, \dots, b_n \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix},$$

we have

$$0 = \delta(m) = \left(\delta\begin{pmatrix} b_1, \dots, b_n \end{pmatrix} + \begin{pmatrix} b_1, \dots, b_n \end{pmatrix}\delta(D^{-1})D\right) \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}.$$

Hence $\delta(b_1, \dots, b_n) + (b_1, \dots, b_n)\delta(D^{-1})D = (0, \dots, 0)$. On the other hand, we have

$$m = (b_1, \dots, b_n) \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = (b_1, \dots, b_n)D^{-1} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

and

$$\begin{aligned}\delta((b_1,\dots,b_n)D^{-1}) &= (\delta(b_1,\dots,b_n)+(b_1,\dots,b_n)\delta(D^{-1})D)D^{-1}\\ &= (0,\dots,0),\end{aligned}$$

which implies that any component of the row vector $(b_1,\dots,b_n)D^{-1}$ is in $S^{-1}A$. Thus $m \in S^{-1}Af_1+\cdots+S^{-1}Af_n$. □

We redefine the triangulability of a module derivation on a free (B,δ)-module in a matrix form.

Definition 4.8. Let $M = Be_1 \oplus \cdots \oplus Be_n$ be a free (B,δ)-module. Suppose that δ_M is defined by

$$\begin{pmatrix}\delta_M(e_1)\\ \vdots \\ \delta_M(e_n)\end{pmatrix} = C\begin{pmatrix}e_1\\ \vdots\\ e_n\end{pmatrix}$$

with $C \in \mathrm{M}(n,B)$. We say that δ_M is triangulable if there exist an invertible matrix P in $\mathrm{M}(n,B)$ and $b_{ij} \in B$ such that

$$(\delta(P)+PC)P^{-1} = \begin{pmatrix} 0 & 0 & \dots & \dots & 0\\ b_{21} & 0 & \dots & \dots & 0\\ \vdots & \ddots & \ddots & & \vdots\\ \vdots & & \ddots & \ddots & \vdots\\ b_{n1} & b_{n2} & \dots & b_{n,n-1} & 0\end{pmatrix}.$$

In fact, set

$$\begin{pmatrix}e'_1\\ \vdots\\ e'_n\end{pmatrix} = P\begin{pmatrix}e_1\\ \vdots\\ e_n\end{pmatrix}.$$

Then $\{e'_1,\dots,e'_n\}$ is a free basis of M, and we have

$$\begin{aligned}\begin{pmatrix}\delta(e'_1)\\ \vdots\\ \delta(e'_n)\end{pmatrix} &= \delta(P)\begin{pmatrix}e_1\\ \vdots\\ e_n\end{pmatrix} + P\begin{pmatrix}\delta(e_1)\\ \vdots\\ \delta(e_n)\end{pmatrix}\\ &= (\delta(P)+PC)P^{-1}\begin{pmatrix}e'_1\\ \vdots\\ e'_n\end{pmatrix}.\end{aligned}$$

Hence δ_M satisfies $\delta_M(e'_1) = 0, \delta_M(e'_2) \in Be'_1,\dots,\delta_M(e'_n) \in Be'_1 \oplus \cdots \oplus Be'_{n-1}$.

We give an example of a triangulable module derivation in the case where $B = k[x,y]$.

Example 4.9. Let $B = k[x, y]$ be a polynomial ring in two variables over a field k of characteristic zero. Let $\delta \in \mathrm{LND}(B)$ be defined by $\delta(x) = 0$ and $\delta(y) = x$. Let $S = \{1, x, x^2, \dots\}$ be a multiplicatively closed subset of B. Let $M = Be_1 \oplus Be_2$ be a free B-module and let

$$D = \begin{pmatrix} y+1 & y \\ y^2+y+x & y^2+x \end{pmatrix} \in \mathrm{M}(2, B).$$

Then we can define a locally nilpotent module derivation δ_M on M by

$$\begin{aligned} \begin{pmatrix} \delta_M(e_1) \\ \delta_M(e_2) \end{pmatrix} &= \delta(D^{-1})D \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \\ &= \begin{pmatrix} y^2+y-x & y^2-x \\ -y^2-2y+x-1 & -y^2-y+x \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}. \end{aligned}$$

Let

$$P = \begin{pmatrix} -y-1 & -y \\ 1 & 1 \end{pmatrix}$$

and let

$$\begin{pmatrix} e'_1 \\ e'_2 \end{pmatrix} = P \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

Then it is easy to see that $\{e'_1, e'_2\}$ is a free basis of the B-module M such that $\delta_M(e'_1) = 0$ and $\delta_M(e'_2) = e'_1$.

Next we give an example of a non-triangulable module derivation.

Example 4.10. Let B, δ be the same as in Example 4.9. Let $M = Be_1 \oplus Be_2$ be a free B-module and let

$$D = \begin{pmatrix} y^2+1 & xy \\ y & x \end{pmatrix}.$$

Define a module derivation δ_M by

$$\begin{pmatrix} \delta_M(e_1) \\ \delta_M(e_2) \end{pmatrix} = \delta(D^{-1})D \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} -xy & -x^2 \\ y^2-1 & xy \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

Then (M, δ_M) is a δ-module. First we claim that M_0 is a free A-module with a basis $\{f_1, f_2\}$, where f_1, f_2 is defined by

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = D \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

We can show as before that $f_1, f_2 \in M_0$ and hence $Af_1 + Af_2 \subset M_0$. We prove the other inclusion. Take any element $m \in M_0$. Then $m \in S^{-1}M_0 = S^{-1}Af_1 + S^{-1}Af_2$ by Lemma 4.7. Hence $x^l m = g_1 f_1 + g_2 f_2$ with $l \in \mathbb{N} \cup \{0\}$ and $g_1, g_2 \in A = k[x]$. We take l to be the smallest integer. Then we have

$$\begin{aligned} x^l m = g_1 f_1 + g_2 f_2 &= g_1((y^2+1)e_1 + xye_2) + g_2(ye_1 + xe_2) \\ &= (g_1 y^2 + g_2 y + g_1)e_1 + (g_1 xy + g_2 x)e_2. \end{aligned}$$

If $l > 0$, then x divides $g_1y^2 + g_2y + g_1$, i.e., $g_1(0)y^2 + g_2(0)y + g_1(0) = 0$ and therefore $g_1(0) = g_2(0) = 0$, which contradicts the choice of l. Thus $l = 0$ and hence $m \in Af_1 + Af_2$ as desired. Next we claim that δ_M is not triangulable. For this purpose, we may assume that k is algebraically closed. Suppose that there exists a B-basis $\{e'_1, e'_2\}$ of M such that $\delta_M(e'_1) = 0$ and $\delta_M(e'_2) \in Be'_1$. Then there exist $g_1, g_2 \in A = k[x]$ such that

$$\begin{aligned} e'_1 &= g_1f_1 + g_2f_2 = g_1((y^2+1)e_1 + xye_2) + g_2(ye_1 + xe_2) \\ &= (g_1y^2 + g_2y + g_1)e_1 + (g_1xy + g_2x)e_2. \end{aligned}$$

Since both $\{e_1, e_2\}$ and $\{e'_1, e'_2\}$ are B-bases of M, the B-ideal generated by $g_1y^2+g_2y+g_1$ and g_1xy+g_2x is a unit ideal, i.e., the closed set $V(g_1y^2+g_2y+g_1, g_1xy+g_2x)$ of $\operatorname{Spec} B$ is the empty set. Namely, there is no solution of the system of equations

$$\begin{cases} g_1y^2 + g_2y + g_1 = 0 \\ g_1xy + g_2x = 0 \end{cases}.$$

Here $x = 0$ is a solution of the second equation. Then it is easy to see that the equation $g_1(0)y^2 + g_2(0)y + g_1(0) = 0$ has at least one solution of y. This is a contradiction.

5. The homological property of δ-modules

In this section, we describe the homological property of δ-modules.

Definition 5.1. Let $\delta \in \mathrm{LND}(B)$. Let M, N be a pre-(B,δ)-modules and let $f : M \to N$ be a B-module homomorphism. We say that f is a (B,δ)-homomorphism (abbreviated as a δ-homomorphism) if $f(\delta_M(m)) = \delta_N(f(m))$ for any $m \in M$. We denote the set of all δ-homomorphisms from M to N by $\mathrm{Hom}_\delta(M, N)$.

Lemma 5.2. *Let M, N be pre-δ-modules (resp. δ-modules), and $f : M \to N$ a δ-homomorphism. Then* $\operatorname{Ker} f$, $\operatorname{Im} f$ *and* $\operatorname{Coker} f$ *are also pre-δ-modules (resp. δ-modules).*

Proof. Straightforward. □

Lemma 5.3. *We have the following assertions*:

(1) *Let M, N be δ-modules. Define a module derivation $\delta_{M\otimes N}$ on the tensor product $M \otimes_B N$ by*

$$\delta_{M\otimes N}(m \otimes n) = \delta_M(m) \otimes n + m \otimes \delta_N(n).$$

Then $(M \otimes_B N, \delta_{M\otimes N})$ is a δ-module.

(2) *Let M, N be δ-modules. Define a module derivation on the B-module* $\mathrm{Hom}_B(M, N)$ *of B-module homomorphisms from M to N by*

$$\delta(f)(m) = \delta_N(f(m)) - f(\delta_M(m))$$

for $f \in \mathrm{Hom}_B(M,N)$, $m \in M$. *Then* $\mathrm{Hom}_B(M,N)$ *is a pre-*δ*-module. Furthermore, if* M *is a finitely generated* B*-module, then* $\mathrm{Hom}_B(M,N)$ *is a* δ*-module. Here the kernel of* $\delta_{\mathrm{Hom}_B(M,N)}$ *consists of* (B,δ)*-homomorphisms from* M *to* N.

(3) *Let* L, M, N *be* δ*-modules. Suppose that* L *and* M *are finitely generated* B*-modules. Define a* B*-module isomorphism*

$$\sigma : \mathrm{Hom}_B(L \otimes_B M, N) \to \mathrm{Hom}_B(L, \mathrm{Hom}_B(M,N))$$

by $\sigma(f)(l)(m) = f(l \otimes m)$ *for* $l \in L$ *and* $m \in M$. *Then* σ *is a* (B,δ)*-isomorphism.*

(4) *Suppose that* $(M_\lambda, \delta_\lambda)$ *is a* δ*-module for each* $\lambda \in \Lambda$. *Then* $\bigoplus_{\lambda \in \Lambda} M_\lambda$ *is a* δ*-module and* $\prod_{\lambda \in \Lambda} M_\lambda$ *is a pre-*δ*-module in a natural fashion.*

Proof. (1) It is easy to show that $\delta_{M \otimes N}$ is a well-defined A-module endomorphism. For any $m \otimes n \in M \otimes_B N$ and $b \in B$, we have

$$\begin{aligned}\delta(b(m \otimes n)) &= \delta((bm) \otimes n) = \delta_M(bm) \otimes n + bm \otimes \delta_N(n) \\ &= (\delta(b)m + b\delta_M(m)) \otimes n + bm \otimes \delta_N(n) \\ &= \delta(b)(m \otimes n) + b\delta(m \otimes n).\end{aligned}$$

For each positive integer l, we have an equality

$$\delta^l(m \otimes n) = \sum_{i+j=l} \binom{l}{i} \delta^i(m) \otimes \delta^j(n).$$

Hence $\delta_{M \otimes N}$ is locally nilpotent.

(2) First we show that $\delta(f) : M \to N$ is a B-module homomorphism. For any $f \in \mathrm{Hom}_B(M,N)$ and $b \in B$ and $m \in M$, we have

$$\begin{aligned}\delta(f)(bm) &= \delta_N(f(bm)) - f(\delta_M(bm)) \\ &= \delta_N(bf(m)) - f(\delta(b)m + b\delta_M(m)) \\ &= \delta(b)f(m) + b\delta_N(f(m)) - \delta(b)f(m) - bf(\delta_M(m)) \\ &= b\delta_N(f(m)) - bf(\delta_M(m)) \\ &= b\delta(f)(m).\end{aligned}$$

It is easily verified that $\delta(f)(m+n) = \delta(f)(m) + \delta(f)(n)$. Next we show that $\mathrm{Hom}_B(M,N)$ is a pre-δ-module. For any $f, g \in \mathrm{Hom}_B(M,N)$, it is clear that $\delta(f+g) = \delta(f) + \delta(g)$. For any $f \in \mathrm{Hom}_B(M,N)$ and $b \in B$, we have

$$\begin{aligned}\delta(bf)(m) &= \delta_N(bf(m)) - bf(\delta_M(m)) \\ &= \delta(b)f(m) + b\delta_N(f(m)) - bf(\delta_M(m)) \\ &= (\delta(b)f + b\delta(f))(m),\end{aligned}$$

and hence $\delta(bf) = \delta(b)f + b\delta(f)$. Finally we claim that $\delta_{\mathrm{Hom}_B(M,N)}$ is locally nilpotent provided M is a finitely generated B-module. First, we can prove by induction on l that for any positive integer l,

$$\delta^l(f)(m) = \left(\sum_{i=0}^{l}(-1)^i \binom{l}{i} {\delta_N}^{l-i} f {\delta_M}^{i}\right)(m).$$

Suppose that $M = Bm_1 + \cdots + Bm_r$. Let $f \in \mathrm{Hom}_B(M, N)$. In order to show that $\delta^l(f) = 0$ for some integer l, it is enough to show that there exists an integer l such that $\delta^l(f)(m_j) = 0$ for all j. For each m_j, there exists an integer l_j such that ${\delta_M}^{l_j}(m_j) = 0$. Then ${\delta_N}^{l_j - i} f {\delta_M}^{i}(m_j) = 0$ for any $i \geq l_j$. Since $f(m_j), f\delta_M(m_j), \ldots, f{\delta_M}^{l_j-1}(m_j)$ are finitely many elements of N, it follows that there exists an integer $l'_j > l_j$ such that ${\delta_N}^{l'_j - i} f {\delta_M}^{i}(m_j) = 0$ for $0 \leq i \leq l_j - 1$. Hence we have

$$\delta^{l'_j}(f)(m_j) = \left(\sum_{i=0}^{l'_j}(-1)^i \binom{l'_j}{i} {\delta_N}^{l'_j-i} f {\delta_M}^{i}\right)(m_j) = 0.$$

Let $l' = \max(l'_1, \ldots, l'_r)$. Then $\delta^{l'}(f)(m_j) = 0$ for all j.

(3) It is well known that σ is an isomorphism of B-modules and it is easy to show that $\delta\sigma(f) = \sigma\delta(f)$ for any B-module homomorphism $f : L \otimes_B M \to N$.

(4) Obvious. □

Definition 5.4. Let L, N be δ-modules. A short exact sequence of δ-modules

$$0 \longrightarrow L \xrightarrow{\ f\ } M \xrightarrow{\ g\ } N \longrightarrow 0$$

is denoted by (M, f, g) once L and N are fixed, and called an *extension* of N by L. We say that extensions (M, f, g) and (M', f', g') are isomorphic if there exists a δ-homomorphism $\theta : M \to M'$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\ f\ } & M & \xrightarrow{\ g\ } & N & \longrightarrow & 0 \\ & & \| & & {\scriptstyle\theta}\downarrow & & \| & & \\ 0 & \longrightarrow & L & \xrightarrow{\ f'\ } & M & \xrightarrow{\ g'\ } & N & \longrightarrow & 0 \end{array}$$

The isomorphism classes of extensions of N by L form an A-module, which we denote by $\mathrm{Ext}^1_\delta(N, L)$ (see [6, p. 68, Theorem 2.1]). We refer to [6] also for the following result (see [6, p. 73, Theorem 3.2, p. 74, Theorem 3.4]).

Lemma 5.5. *For a finitely generated (B, δ)-module N and an exact sequence of δ-modules*

$$0 \longrightarrow L_1 \xrightarrow{\ \alpha\ } L_2 \xrightarrow{\ \beta\ } L_3 \longrightarrow 0$$

we have the following natural exact sequence:

$$\begin{array}{lccccc} 0 \longrightarrow & \mathrm{Hom}_\delta(N, L_1) & \xrightarrow{\ \alpha_*\ } & \mathrm{Hom}_\delta(N, L_2) & \xrightarrow{\ \beta_*\ } & \mathrm{Hom}_\delta(N, L_3) \\ \quad\longrightarrow & \mathrm{Ext}^1_\delta(N, L_1) & \xrightarrow{\ \alpha_*\ } & \mathrm{Ext}^1_\delta(N, L_2) & \xrightarrow{\ \beta_*\ } & \mathrm{Ext}^1_\delta(N, L_3) \end{array} \tag{5.1}$$

For a δ-module L and an exact sequence of finitely generated (B,δ)-modules

$$0 \longrightarrow N_1 \xrightarrow{\alpha} N_2 \xrightarrow{\beta} N_3 \longrightarrow 0$$

we have the following natural exact sequence:

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_\delta(N_3, L) &\xrightarrow{\beta^*} \mathrm{Hom}_\delta(N_2, L) \xrightarrow{\alpha^*} \mathrm{Hom}_\delta(N_1, L) \\ \longrightarrow \mathrm{Ext}^1_\delta(N_3, L) &\xrightarrow{\beta^*} \mathrm{Ext}^1_\delta(N_2, L) \xrightarrow{\alpha^*} \mathrm{Ext}^1_\delta(N_1, L) \end{aligned} \tag{5.2}$$

For the next result (Proposition 5.7), we need the following lemma.

Lemma 5.6. *Let B be a noetherian ring, $\delta \in \mathrm{LND}(B)$ and S a multiplicatively closed subset of B. Let M, N be a pre-δ-modules. Define a module derivation δ_L on $L := \mathrm{Hom}_{S^{-1}B}(S^{-1}M, S^{-1}N)$ by*

$$\delta_L(f)(w) = \delta_{S^{-1}N}(f(w)) - f(\delta_{S^{-1}M}(w))$$

for $f \in \mathrm{Hom}_{S^{-1}B}(S^{-1}M, S^{-1}N)$ and $w \in S^{-1}M$. Then (L, δ_L) is a pre-δ-module. In addition, suppose that M is a finitely generated B-module and define a B-module isomorphism $\sigma : S^{-1}(\mathrm{Hom}_B(M,N)) \to \mathrm{Hom}_{S^{-1}B}(S^{-1}M, S^{-1}N)$ by

$$\sigma\left(\frac{f}{s}\right)\left(\frac{m}{t}\right) = \frac{f(m)}{st}$$

for $f \in \mathrm{Hom}_B(M,N)$, $m \in M$ and $s, t \in S$. Then σ is a δ-isomorphism.

Proof. Straightforward. □

Proposition 5.7. *Let B be a noetherian ring and N a finitely generated projective (B,δ)-module of rank one. Then the following assertions hold*:

(1) $\mathrm{Hom}_\delta(N, L) \cong (L \otimes_B N^{-1})_0$ *and* $\mathrm{Ext}^1_\delta(N, L) \cong L \otimes_B N^{-1}/\delta(L \otimes_B N^{-1})$ *with $N^{-1} = \mathrm{Hom}_B(N, B)$, which is a δ-module.*

(2) *In the exact sequence (5.1), $\beta_* : \mathrm{Ext}^1_\delta(N, L_2) \to \mathrm{Ext}^1_\delta(N, L_3)$ is surjective.*

(3) *In the exact sequence (5.2), $\alpha^* : \mathrm{Ext}^1_\delta(N_2, L) \to \mathrm{Ext}^1_\delta(N_1, L)$ is surjective if N_1, N_2 and N_3 are finitely generated projective (B,δ)-modules of rank one.*

Proof. (1) First we shall show that $\mathrm{Hom}_B(N, L) \cong L \otimes_B N^{-1}$ as B-modules. Since N is a projective B-module of rank one, there exists an affine open covering $\{D(s_i)\}_i$ of $\mathrm{Spec}\, B$ such that $N[{s_i}^{-1}] = B[{s_i}^{-1}]e_i$ with $e_i \in N$, $s_i \in B$. Then $N^{-1}[{s_i}^{-1}] = B[{s_i}^{-1}]e_i^*$ with $e_i^*(e_i) = 1$. For any i, j, there exists $f_{ji} \in B[{s_i}^{-1}, {s_j}^{-1}]$ such that $e_j = f_{ji}e_i$ holds in $N[{s_i}^{-1}, {s_j}^{-1}]$. Then $e_i^* = f_{ji}e_j^*$ holds in $N^{-1}[{s_i}^{-1}, {s_j}^{-1}]$. Take any element $\varphi \in \mathrm{Hom}_B(N, L)$. Then $\varphi(e_i) = l_i \in L[{s_i}^{-1}]$. For any i, j, we have $l_j = \varphi(e_j) = \varphi(f_{ji}e_i) = f_{ji}l_i$ in $L[{s_i}^{-1}, {s_j}^{-1}]$ and therefore $l_i \otimes e_i^* = l_i \otimes f_{ji}e_j^* = l_j \otimes e_j^*$ in $L \otimes_B N^{-1}[{s_i}^{-1}, {s_j}^{-1}]$. Hence the $l_i \otimes e_i^*$ correspond to an element of $L \otimes_B N^{-1}$. Conversely, any element of $L \otimes_B N^{-1}$ corresponds to the $l_i \otimes e_i^*$ and these elements define $\varphi \in \mathrm{Hom}_B(N, L)$ by $\varphi(e_i) = l_i \in L[{s_i}^{-1}]$.

Next we show that $\varphi\delta = \delta\varphi$ if and only if $\delta(l_i \otimes e_i^*) = 0$ with the above notation. Write $\delta(e_i) = c_i e_i$ with $c_i \in B[{s_i}^{-1}]$. Then we have

$$\delta(e_i^*)(e_i) = \delta(e_i^*(e_i)) - e_i^*(\delta(e_i)) = -e_i^*(c_i e_i) = -c_i$$

in $B[{s_i}^{-1}]$ and therefore $\delta(e_i^*) = -c_i e_i^*$ in $N^{-1}[{s_i}^{-1}]$. We also have $\varphi(\delta(e_i)) = \varphi(c_i e_i) = c_i l_i$ and $\delta(\varphi(e_i)) = \delta(l_i)$. Suppose that $\varphi\delta = \delta\varphi$. Then $\delta(l_i) = c_i l_i$ holds in $L[{s_i}^{-1}]$. Hence we have

$$\delta(l_i \otimes e_i^*) = \delta(l_i) \otimes e_i^* + l_i \otimes \delta(e_i^*) = c_i l_i \otimes e_i^* + l_i \otimes (-c_i e_i^*) = 0$$

in $L \otimes N^{-1}[{s_i}^{-1}]$ as desired. Conversely, the condition $\delta(l_i \otimes e_i^*) = 0$ for all i implies that $\delta(l_i) = c_i l_i$ and hence $\varphi\delta(e_i) = \delta\varphi(e_i)$ for all i, which implies that $\varphi\delta = \delta\varphi$. Hence we conclude that $\mathrm{Hom}_\delta(N, L) \cong (L \otimes_B N^{-1})_0$ as A-modules.

It is easy to see that $\mathrm{Ext}^1_\delta(N, L) \cong \mathrm{Ext}^1_\delta(B, L \otimes_B N^{-1})$ via the correspondence of $(M, f, g) \in \mathrm{Ext}_\delta(N, L)$ to $(M \otimes_B N^{-1}, f \otimes 1, g \otimes 1) \in \mathrm{Ext}_\delta(B, L \otimes_B N^{-1})$. Let $M' \in \mathrm{Ext}^1_\delta(B, L \otimes_B N^{-1})$. Then $M' = (L \otimes_B N^{-1}) \oplus Be'$ and $\delta(e') \in L \otimes N^{-1}$. Suppose that $M'' = (L \otimes_B N^{-1}) \oplus Be''$ is isomorphic to M' by a (B, δ)-isomorphism $\theta : M' \to M''$ such that $\theta(e') = e'' + l'$ and $\theta|_{L \otimes N^{-1}} = \mathrm{id.}$, where $l' \in L \otimes_B N^{-1}$. We have $\delta\theta(e') = \delta(e'' + l') = \delta(e'') + \delta(l')$ and $\theta\delta(e') = \delta(e')$. Since $\delta\theta = \theta\delta$, we have $\delta(e') = \delta(e'') + \delta(l')$. Since $\delta(e')$ determines the δ-module structure on M', we conclude that $\mathrm{Ext}^1_\delta(B, L \otimes_B N^{-1}) \cong L \otimes_B N^{-1}/\delta(L \otimes_B N^{-1})$ as A-modules.

(2) The exact sequence (5.1) coincides with

$$\begin{aligned} 0 &\longrightarrow (L_1 \otimes_B N^{-1})_0 \longrightarrow (L_2 \otimes_B N^{-1})_0 \longrightarrow (L_3 \otimes_B N^{-1})_0 \\ &\longrightarrow L_1 \otimes_B N^{-1}/\delta(L_1 \otimes_B N^{-1}) \longrightarrow L_2 \otimes_B N^{-1}/\delta(L_2 \otimes_B N^{-1}) \\ &\longrightarrow L_3 \otimes_B N^{-1}/\delta(L_3 \otimes_B N^{-1}) \longrightarrow 0 \end{aligned}$$

which follows by the snake lemma applied to the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 \otimes_B N^{-1} & \longrightarrow & L_2 \otimes_B N^{-1} & \longrightarrow & L_3 \otimes_B N^{-1} & \longrightarrow & 0 \\ & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\ 0 & \longrightarrow & L_1 \otimes_B N^{-1} & \longrightarrow & L_2 \otimes_B N^{-1} & \longrightarrow & L_3 \otimes_B N^{-1} & \longrightarrow & 0 \end{array}$$

(3) We have an exact sequence of δ-modules

$$0 \longrightarrow {N_3}^{-1} \longrightarrow {N_2}^{-1} \longrightarrow {N_1}^{-1} \longrightarrow 0$$

and hence an exact sequence of δ-modules

$$0 \longrightarrow L \otimes_B {N_3}^{-1} \longrightarrow L \otimes_B {N_2}^{-1} \longrightarrow L \otimes_B {N_1}^{-1} \longrightarrow 0.$$

Applying the snake lemma to this exact sequence of δ-modules, we obtain the result. $\square$

6. The relation between $\delta \in \mathrm{LND}(B)$ and δ-modules

In this section, we consider the relation between $\delta \in \mathrm{LND}(B)$ and δ-modules. First, the existence of a slice can be characterized as follows.

Lemma 6.1. *For $\delta \in \mathrm{LND}(B)$, the following conditions are equivalent:*
(1) *δ has a slice.*
(2) $B = \delta(B)$
(3) *For any δ-module M, we have $M = \delta(M)$.*
(4) *For any δ-module M, we have $\mathrm{Ext}^1_\delta(B, M) = (0)$.*

Proof. The conditions (3) and (4) are equivalent in view of (1) of Proposition 5.7.

(2)⇒(1) is clear. (1)⇒(2). If δ has a slice $u \in B$, then $B = A[u]$. Hence any $b \in B$ is written as $b = f(u) \in A[u]$. Then $b = \delta(b')$ with $b' = \int f(u)du$.

(3)⇒(2) is clear. (2)⇒(3). For any δ-module M, we have $M = M_0 \otimes_A B$ (Lemma 3.3, (2)). Hence any element $m \in M$ is written as $m = \sum_i b_i m_i$ with $m_i \in M_0$ and $b_i \in B$. Let $m' = \sum_i b'_i m_i$ with $\delta(b'_i) = b_i$. Then we have $m = \delta(m')$ and hence $M = \delta(M)$. □

Even if there exists a δ-module M satisfying $\delta(M) = M$, it does not guarantee $\delta(B) = B$, which is shown by the following example.

Example 6.2. Let $B = k[x, y]$ be a polynomial ring over a field k of characteristic zero. Let $\delta \in \mathrm{LND}(B)$ be defined by $\delta(x) = 0$ and $\delta(y) = 1 + x$. Let $M = B/xB$. Then M is a δ-module. Since $\delta_M \in \mathrm{LND}(B/xB)$ has a slice $\overline{y}$, we have $\delta_M(M) = M$. On the other hand, we have $\delta(B) = (1 + x)B \neq B$.

However, we have the following result.

Lemma 6.3. *Let $\delta \in \mathrm{LND}(B)$, and M a δ-module. If there exists an exact sequence of δ-modules*

$$0 \longrightarrow N \longrightarrow M \longrightarrow B \longrightarrow 0,$$

then $\delta(M) = M$ implies that $\delta(B) = B$. Hence, if M is a free B-module and δ_M is triangulable, then $\delta(M) = M$ implies that $\delta(B) = B$.

Proof. By the snake lemma applied to the above exact sequence of δ-modules, we have an exact sequence of A-modules

$$\begin{aligned} 0 \longrightarrow N_0 \longrightarrow M_0 \longrightarrow A \longrightarrow N/\delta(N) \\ \longrightarrow M/\delta(M) \longrightarrow B/\delta(B) \longrightarrow 0. \end{aligned}$$

Hence $\delta(M) = M$ implies that $\delta(B) = B$. □

The fixed-point freeness of $\delta \in \mathrm{LND}(B)$ can be characterized as follows.

Lemma 6.4. *For $\delta \in \mathrm{LND}(B)$, the following conditions are equivalent:*

(1) *δ is fixed-point free.*
(2) *For any pre-δ-modue M, we have $M = (0)$ if $\delta(M) = (0)$.*
(3) *For any δ-module M, we have $M = (0)$ if $\delta(M) = (0)$.*
(4) *Any pre-δ-module M is generated by $\delta(M)$ as a B-module.*
(5) *Any δ-module M is generated by $\delta(M)$ as a B-module.*

Proof. (1) $\Rightarrow$ (2). For $b \in B$ and $m \in M$, we have $\delta(b)m = \delta(bm) - b\delta(m) = 0$ by the assumption that $\delta(M) = (0)$. On the other hand, the ideal of B generated by $\delta(B)$ is a unit ideal because of the fixed-point freeness. Namely, there exist elements b_i and b'_i of B for $1 \le i \le r$ such that $\sum_{i=1}^r b'_i\delta(b_i) = 1$. Then $m = (\sum_{i=1}^r b'_i\delta(b_i))m = 0$. Hence $M = (0)$.

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are clear.

(3) $\Rightarrow$ (4). Let $\overline{M} = M/B\delta(M)$. Then δ_M induces a module derivation $\delta_{\overline{M}}$ on $\overline{M}$, which is zero. It follows from (3) that $\overline{M} = (0)$. This implies that M is generated as a B-module by $\delta(M)$.

(5) $\Rightarrow$ (1). Obviously B is a δ-module and therefore (5) implies that the ideal of B generated by $\delta(B)$ is a unit ideal. ☐

We have the following assertion.

Lemma 6.5. *Let M be a nonzero finitely generated torsion-free (B, δ)-module. Suppose that $M = BM_0$ and $\delta(M) = M$. Then the derivation δ on B is fixed-point free.*

Proof. Since $M = BM_0$, there exist $m_1, \ldots, m_r \in M_0$ such that $M = \sum_{i=1}^r Bm_i$. Further, since $\delta(M) = M$, there exist $b_{ij} \in B$ such that $m_i = \delta(\sum_{j=1}^r b_{ij}m_i) = \sum_{j=1}^r \delta(b_{ij})m_i$ for $1 \le i \le r$. These relations are written as

$$P \begin{pmatrix} m_1 \\ \vdots \\ m_r \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

where

$$P = \begin{pmatrix} \delta(b_{11}) - 1 & \delta(b_{12}) & \ldots & \delta(b_{1r}) \\ \delta(b_{21}) & \delta(b_{22}) - 1 & \ldots & \delta(b_{2r}) \\ \vdots & \vdots & \ddots & \vdots \\ \delta(b_{r1}) & \delta(b_{r2}) & \ldots & \delta(b_{rr}) - 1 \end{pmatrix}.$$

Multiplying the adjoint matrix P^* of P, we have $(\det P) \cdot m_i = 0$ for all i. Since M is nonzero and B-torsion-free, we have $\det P = 0$, which implies that $1 \in B\delta(B)$. Hence the derivation δ on B is fixed-point free. ☐

7. Projective triangulability

In this section, we shall extend the notion of triangulability of δ_M on a free B-module M in the case where M is a finitely generated, torsion-free (B, δ)-module with the restriction that B is an affine normal $\mathbb{C}$-domain of dimension two.

Definition 7.1. Let B be a domain and let M be a finitely generated, torsion-free (B,δ)-module. Let r be the rank of M, which is by definition equal to $\dim_{Q(B)} M \otimes_B Q(B)$. We say that δ_M is *projectively triangulable* if there exists a sequence of projective (B,δ)-submodules $0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_r = M$ such that for each $1 \le i \le r$, M_{i+1}/M_i is a projective (B,δ)-module of rank one.

Note that the triangulability implies the projective triangulability for a module derivation on a free module. Furthermore, note that if M_i/M_{i-1} is a free B-module of rank one for any i, then the projective triangulability implies the triangulability on a free module.

We consider the case where B is a normal affine $\mathbb{C}$-domain of dimension two, $\delta \in \mathrm{LND}(B)$ is fixed-point free, and M is a finitely generated, torsion-free (B,δ)-module. Then we show that δ_M is projectively triangulable (Theorem 7.7).

Definition 7.2. Let M be a finitely generated B-module. We call $M^* = \mathrm{Hom}_B(M,B)$ the dual of M and $M^{**} = \mathrm{Hom}_B(M^*,B)$ the bidual of M. Define a B-module homomorphism $\alpha : M \to M^{**}$ by $\alpha(m) = \widehat{m}$, where $\widehat{m}$ is defined by $\widehat{m}(f) = f(m)$ for $f \in M^*$. We say that M is reflexive if $\alpha : M \to M^{**}$ is an isomorphism. If B is a noetherian ring, then M^* and M^{**} are also finitely generated. Hence if B is a noetherian ring, $\delta \in \mathrm{LND}(B)$, and M is a finitely generated (B,δ)-module, then M^* and M^{**} are also finitely generated (B,δ)-modules.

Lemma 7.3. *Let B be an integral domain and let M be a finitely generated reflexive B-module. Then M is torsion-free.*

Proof. We have an exact sequence $F \to M \to 0$ for some free module F. The sequence $0 \to M^* \to F^*$ is exact. Since F^* is isomorphic to F, it follows that M^* is torsion-free. In a similar fashion, M^{**} is torsion-free. Hence M is torsion-free if M is reflexive. □

In the subsequent arguments, we need the following result, which is well known but we give a short proof for the lack of suitable references.

Lemma 7.4. *Let B be a regular affine $\mathbb{C}$-domain of dimension two. Then any finitely generated reflexive B-module M is a projective B-module.*

Proof. We have only to show that $M_\mathfrak{p}$ is a free $B_\mathfrak{p}$-module for every $\mathfrak{p} \in \mathrm{Spec}\, B$. If $\mathrm{ht}\,\mathfrak{p} = 1$, then $M_\mathfrak{p}$ is a free $B_\mathfrak{p}$-module since $B_\mathfrak{p}$ is a discrete valuation ring and $M_\mathfrak{p}$ is $B_\mathfrak{p}$-torsion-free by Lemma 7.3. In the case $\mathrm{ht}\,\mathfrak{p} = 2$, we have an equality

$$\mathrm{proj.dim}\, M_\mathfrak{p} + \mathrm{depth}\, M_\mathfrak{p} = \mathrm{depth}\, B_\mathfrak{p}$$

by a theorem of Auslander-Buchsbaum (see [2, p. 17, Theorem 1.3.3]), where we note that $\mathrm{proj.dim}\, M_\mathfrak{p} < \infty$ since B is regular (see [2, p. 66, Theorem 2.2.7]). Meanwhile, since $R_\mathfrak{p}$ is regular, we have $\mathrm{depth}\, R_\mathfrak{p} = \dim R_\mathfrak{p} = 2$. On the other hand, we know that depth $M_\mathfrak{p}$ = depth $M_\mathfrak{p}^{**} \geq \min(2, \text{depth } R_\mathfrak{p}) = 2$ (see [2, p. 26, Exercise 1.4.19]). Hence we have proj.dim $M_\mathfrak{p} = 0$, and therefore $M_\mathfrak{p}$ is a free $B_\mathfrak{p}$-module. □

Lemma 7.5. *Let B be as in Lemma* 7.4, $\delta \in \mathrm{LND}(B)$ *and M a finitely generated, torsion-free (B, δ)-module. Suppose that δ is fixed-point free. Then M is a reflexive B-module and hence a projective B-module.*

Proof. Let $\alpha : M \to M^{**}$ be a B-module homomorphism as in Definition 7.2. We claim that α is a δ-isomorphism. First, we shall show that α is a δ-homomorphism, i.e., $\alpha\delta(m) = \delta\alpha(m)$ for any $m \in M$. Indeed, for any $m \in M$ and $f \in M^*$, we have

$$\begin{aligned}
\alpha(\delta(m))(f) &= \widehat{\delta(m)}(f) = f(\delta(m)),\\
\delta(\alpha(m))(f) &= \delta(\alpha(m)(f)) - \alpha(m)(\delta(f)) = \delta(f(m)) - (\delta f)(m)\\
&= \delta(f(m)) - \delta(f(m)) + f(\delta(m)) = f(\delta(m)).
\end{aligned}$$

Next we shall show that α is an isomorphism. We consider an exact sequence of δ-modules

$$0 \longrightarrow K \longrightarrow M \xrightarrow{\ \alpha\ } M^{**} \longrightarrow C \longrightarrow 0,$$

where $K = \mathrm{Ker}\,\alpha$ and $C = \mathrm{Coker}\,\alpha$. Since both K and C are finitely generated B-modules, we have $\mathrm{Supp}\,K = V(\mathrm{Ann}\,K)$ and $\mathrm{Supp}\,C = V(\mathrm{Ann}\,C)$. We have only to show that $K = C = (0)$. For this, it suffices to show that $K_\mathfrak{p} = C_\mathfrak{p} = (0)$ for any prime ideal $\mathfrak{p}$ of B. Suppose first that $\mathrm{ht}\,\mathfrak{p} = 1$. Then $M_\mathfrak{p}$ is a free $B_\mathfrak{p}$-module since $B_\mathfrak{p}$ is a discrete valuation ring and $M_\mathfrak{p}$ is a finitely generated, torsion-free $B_\mathfrak{p}$-module. Hence $\alpha_\mathfrak{p} : M_\mathfrak{p} \to M_\mathfrak{p}^{**}$ is an isomorphism, and therefore $K_\mathfrak{p} = C_\mathfrak{p} = (0)$. Hence if $\mathrm{Supp}\,K \neq \emptyset$, then any $\mathfrak{p} \in \mathrm{Supp}\,K$ has height 2 and therefore we have a minimal prime decomposition $\sqrt{\mathrm{Ann}\,K} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r$ with maximal ideals $\mathfrak{m}_i$. Since $\mathrm{Ann}\,K$ is a δ-ideal (Lemma 3.10, (1)), the $\mathfrak{m}_i$ are also δ-ideals (Lemma 2.9, (1)), which contradicts the fixed-point freeness of δ (Lemma 2.11). Hence $\mathrm{Supp}\,K = \emptyset$ and $K = (0)$. In a similar fashion, we have $C = (0)$. □

In Theorem 7.7, we need the following result, which is obvious because the singular locus is a finite G_a-stable set for a two-dimensional normal variety.

Lemma 7.6. *Let B be a normal affine $\mathbb{C}$-domain of dimension two and let $\delta \in \mathrm{LND}(B)$. Suppose that δ is fixed-point free. Then B is a regular ring.*

Theorem 7.7. *Let B, δ be as in Lemma* 7.6, *and let M be a finitely generated, torsion-free (B, δ)-module. Then δ_M is projectively triangulable.*

Proof. Let $\{m_1 \otimes 1, \ldots, m_r \otimes 1\}$ be a basis of the $Q(B)$-vector space $M \otimes_B Q(B)$. Let $S = A - (0)$. Then we have

$$\begin{aligned}
M \otimes_B Q(B) &= M \otimes_B S^{-1}B \otimes_{S^{-1}B} Q(B)\\
&= S^{-1}M_0 \otimes_{Q(A)} S^{-1}B \otimes_{S^{-1}B} Q(B) = S^{-1}M_0 \otimes_{Q(A)} Q(B)\\
&= M_0 \otimes_A Q(A) \otimes_{Q(A)} Q(B) = M_0 \otimes_A Q(B).
\end{aligned}$$

Hence we can take all the m_i in M_0. Let $N = Bm_1 + \cdots + Bm_{r-1}$. Then clearly N is a δ-module and hence $L := (M/N)/(M/N)_{\mathrm{tor}}$ is a δ-module by (2) of Lemma 3.10, where $(M/N)_{\mathrm{tor}}$ is the torsion part of M/N. In addition, L is a finitely generated, torsion-free B-module. Hence L is a projective B-module by Lemma 7.5, (2). Then we have an exact sequence of (B, δ)-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow L \longrightarrow 0,$$

where M' is the kernel of the natural homomorhpism from M to L. Since rank $M = r$ and rank $L = 1$, we have rank $M' = r - 1$. If we use induction on rank M, we conclude that δ_M is projectively triangulable. $\square$

8. Invertible δ-modules

In this section, we consider an *invertible* (B, δ)-module M, i.e., a δ-module which is a projective B-module of rank one. If $M = Be$ is a free B-module of rank one, then $\delta_M(e) = 0$ and hence $M = BM_0$ (Proposition 3.14). The general case is not so simple. We give an example of an invertible (B, δ)-module M such that $M \neq BM_0$.

Example 8.1. Let $\widetilde{B} = \mathbb{C}[x, y, z]/(z^2 - 2xy - 2)$. Define $\widetilde{\delta} \in \mathrm{LND}(\widetilde{B})$ by $\widetilde{\delta}(x) = 0$, $\widetilde{\delta}(y) = z$ and $\widetilde{\delta}(z) = x$. Then it is easily verified that $\widetilde{A} = \mathrm{Ker}\,\widetilde{\delta} = \mathbb{C}[x]$, and the quotient morphism $\widetilde{\pi} : \mathrm{Spec}\,\widetilde{B} \to \mathrm{Spec}\,\widetilde{A}$ has a unique singular fiber over the point $x = 0$, which is a disjoint sum of two affine lines $\mathrm{Spec}\,(\widetilde{B}/(x, z + \sqrt{2})\widetilde{B})$, $\mathrm{Spec}\,(\widetilde{B}/(x, z - \sqrt{2})\widetilde{B})$. Let $\widetilde{M} = x\widetilde{B} + (z - \sqrt{2})\widetilde{B}$, which is a $\widetilde{\delta}$-ideal of $\widetilde{B}$. It is easy to show that $\widetilde{M}$ is a non-free, projective $\widetilde{B}$-module of rank one. We can also show easily that $\widetilde{M_0} = \widetilde{M} \cap \widetilde{A} = x\widetilde{A}$ and $\widetilde{M} \neq \widetilde{B}\widetilde{M_0}$.

Now consider an involution σ on $\widetilde{B}$ defined by

$$\sigma(x, y, z) = (-x, -y, -z).$$

Let $B = \widetilde{B}^{\sigma} = \mathbb{C}[x^2, y^2, xy, xz, yz]$, where we denote the residue classes of x, y, z by x, y, z respectively for simplicity. Write $X = x^2$, $Y = y^2$, $Z = xy$, $U = xz$ and $V = yz$. Then $\widetilde{\delta}$ descends to a locally nilpotent derivation δ on B such that $\delta(X) = 0$, $\delta(Y) = 2V$, $\delta(Z) = U$, $\delta(U) = X$ and $\delta(V) = 2 + 3Z$. Furthermore, it is easy to see that $A = \mathrm{Ker}\delta = \mathbb{C}[X]$ and the quotient morphism $\pi : \mathrm{Spec}\,B \to \mathrm{Spec}\,A$ has a unique singular multiple fiber over the point $X = 0$, which is a non-reduced irreducible scheme. Let $\mathfrak{P}$ be the ideal of B generated by X, Z, U. Set $M = \mathfrak{P}$, which is an invertible δ-module with $M_0 = M \cap A = XA$. Hence $M \neq BM_0$.

The above two examples imply that an invertible (B, δ)-module M is related to irreducible components of the singular fibers of the quotient morphism $\pi : \mathrm{Spec}\,B \to \mathrm{Spec}\,A$. The following result shows, in the case of dimension two with some additional conditions, that this is true.

Proposition 8.2. *Suppose that B is a locally factorial, normal affine $\mathbb{C}$-domain. Let $X = \operatorname{Spec} B$. Suppose further that $B^* = \mathbb{C}^*$ and $\operatorname{Pic}(X)$ is a discrete group, hence every element of $\operatorname{Pic}(X)$ is G_a-invariant. Then every invertible sheaf $\mathcal{L} \in \operatorname{Pic}(X)$ is represented by a Cartier divisor $D = \sum_i n_i Y_i$ such that Y_i is a G_a-stable irreducible subvariety of codimension one for each i.*

Proof. It suffices to show the case where $D \geq 0$. For each $\lambda \in \mathbb{C}$, since ${}^{\lambda}D \sim D$, there exists f_λ such that ${}^{\lambda}D = D + (f_\lambda)$. Then for each $\lambda, \mu \in \mathbb{C}$, we have

$$\begin{aligned} {}^{\lambda+\mu}D &= {}^{\mu}({}^{\lambda}D) = {}^{\mu}(D + (f_\lambda)) = {}^{\mu}D + ({}^{\mu}f_\lambda) \\ &= D + (f_\mu) + ({}^{\mu}f_\lambda) = D + (f_\mu \cdot {}^{\mu}f_\lambda). \end{aligned}$$

Hence $(f_{\lambda+\mu}) = (f_\mu \cdot {}^{\mu}f_\lambda)$ on X and there exists $c(\lambda,\mu) \in \Gamma(X, \mathcal{O}_X{}^*) = \mathbb{C}^*$ such that $f_{\lambda+\mu} = c(\lambda,\mu) f_\mu \cdot {}^{\mu}f_\lambda$. Note that the morphism $c : \mathbb{A}^2 \to \mathbb{C}^* = \mathbb{A}^1_*$ is constant. Hence there exists $c \in \mathbb{C}^*$ such that $f_{\lambda+\mu} = c f_\mu \cdot {}^{\mu}f_\lambda$. Then $c f_{\lambda+\mu} = (c f_\mu) \cdot {}^{\mu}(c f_\lambda)$ and therefore we may assume that

$$f_{\lambda+\mu} = f_\mu \cdot {}^{\mu}f_\lambda. \tag{8.1}$$

If we write $D = \sum_i n_i Y_i$, then each Y_i corresponds to an ideal I_{Y_i} of B. Since ${}^{\lambda}D = D + (f_\lambda)$, we have $\otimes_i {}^{\lambda}(I_{Y_i})^{n_i} = f_\lambda \otimes_i I_{Y_i}{}^{n_i}$. This implies that $f_\lambda = F(\lambda)/b$ for some $F(\lambda), b \in B$. Write $F(t) = b_0 + b_1 t + \cdots + b_m t^m \in B[t]$ with $b_i \in B$. Then the equality (8.1) implies the relation

$$\frac{F(t+u)}{b} = \frac{F(u)}{b} \cdot \frac{\varphi_u(F(t))}{\varphi_u(b)}$$

in $Q(B[t,u])$, where $\varphi_u : B[t] \to B[t,u]$ is defined by

$$\varphi_u\left(\sum_{i=0}^{\infty} b_i t^i\right) = \sum_{i=0}^{\infty} \varphi_u(b_i) t^i.$$

This gives rise to

$$\begin{aligned} &\varphi_u(b)(b_0 + b_1(t+u) + \cdots + b_m(t+u)^m) \\ &= (b_0 + b_1 u + \cdots + b_m u^m)(\varphi_u(b_0) + \varphi_u(b_1) + \cdots + \varphi_u(b_m) t^m). \end{aligned}$$

The comparison of the constant terms implies $b_0 = b$. Now compare the top t-terms:

$$\varphi_u(b) b_m = (b + b_1 u + \cdots + b_m u^m) \varphi_u(b_m).$$

Hence we have

$$\frac{\varphi_u(b)}{\varphi_u(b_m)} = \frac{b}{b_m} + \frac{b_1}{b_m} u + \cdots + \frac{b_m}{b_m} u^m.$$

On the other hand, we have

$$\frac{\varphi_u(b)}{\varphi_u(b_m)} = \varphi_u\left(\frac{b}{b_m}\right) = \sum_{i=0}^{\infty} \frac{1}{i!}\delta^i\left(\frac{b}{b_m}\right)u^i.$$

Hence we deduce

$$\frac{b_i}{b_m} = \frac{1}{i!}\delta^i\left(\frac{b}{b_m}\right)$$

for $0 \le i \le m$. Then $F(t) = b + b_1 t + \cdots + b_m t^m = b_m \varphi_t(b/b_m)$ and therefore

$$\frac{F(t)}{b} = \frac{\varphi_t\left(\frac{b}{b_m}\right)}{\frac{b}{b_m}}$$

Thus ${}^{\lambda}D - D = (F(\lambda)/b) = (\varphi_\lambda(b/b_m)) - (b/b_m)$. This implies that ${}^{\lambda}(D - (b/b_m)) = D - (b/b_m)$. $\square$

School of Science and Technology
Kwansei Gakuin University 2-1 Gakuen
Sanda-shi, Hyogo 669-1337
Japan
e-mail: mtanaka@kwansei.ac.jp

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