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Minimal annuli with and without slits

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0. Introduction.

In this paper we bound the oscillation of the unit normal of minimal annuli with and without slits. Our estimates are independent of the ratio of the inner and outer radii. Hence, we recover standard removable singularity results as the inner radius goes to zero. The estimate for annuli with slits is important in proving a removable singularities theorem for minimal limit laminations.

Proposition 1.3 shows that if a minimal annulus Σ in $\mathbf{R}^3$ has $\int_{\partial \Sigma} |A| < \pi/8$ and $\int_\Sigma K \geq -\pi$, then Σ is a graph. Proposition 1.12 extends this to surfaces with quasi-conformal Gauss map and shows that if $\int_{\partial\Sigma} |A|$ is actually small, then Σ is Lipschitz close to a plane. These results, as well as the rest of the results of this paper, can be easily extended to hold locally in any 3-manifold.

In Section 3 we extend this to what we call "minimal annuli with slits". These are multi-valued minimal graphs over an annulus in the plane. A particular example is a rotation of a rescaled half-helicoid:
(0.1)
$$\{\epsilon\,\delta/(2\pi)\,(s\,\cos t, s\,\sin t, t) \mid 2\,\pi/\epsilon \leq s \leq 2\,\pi\,R/(\epsilon\,\delta) \text{ and } 0 \leq t \leq 2\pi\}\,.$$

Since $\mathbf{n}(\epsilon\,\delta/(2\pi)\,(s\,\cos t, s\,\sin t, t)) = (\sin t, -\cos t, s)/(1+s^2)^{1/2}$ and $s \geq 2\,\pi/\epsilon$,

$$|\mathbf{n} - (0,0,1)| < \epsilon \text{ in } (0.1)\,, \tag{0.2}$$

independent of δ and R. In Theorem 3.36, we obtain a similar bound in general for minimal annuli with slits satisfying certain boundary conditions. This bound implies that there is a fixed plane which these are Lipschitz

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close to on every scale. This consequence will be used elsewhere to prove the removability of singularities for minimal limit laminations.

The results given here should be compared with Rado's theorem (see for instance [CM]) which states that a minimal surface in $\mathbf{R}^3$ whose boundary is a circle which is a graph over the boundary of a convex set in a plane is itself a graph (and in fact a disk).

Throughout, $\Sigma \subset \mathbf{R}^3$ is a compact connected oriented immersed surface; $A, \nabla_\Sigma, \Delta_\Sigma, \mathbf{n}, K$ are the second fundamental form, covariant derivative, Laplacian, unit normal, and sectional curvature. Given $x \in \mathbf{R}^3$ and $s > 0$, $B_s(x)$ is the extrinsic ball of radius s centered at x. Likewise $D_s(z)$ will be the disk in the plane centered at z and with radius s. Finally, $\mathcal{B}_s(a)$ is the intrinsic ball in $\mathbf{S}^2$ of radius s centered at $a \in \mathbf{S}^2$.

1. Minimal annuli.

In this section, if $a \in \mathbf{S}^2$, $a^\perp$ denotes $\{x \in \mathbf{R}^3 \,|\, \langle x, a\rangle = 0\}$. For $a, b \in \mathbf{S}^2$, $\text{Angle}(a,b)$ is the angle between $a^\perp, b^\perp$; i.e., $\text{Angle}(a,b) = \text{dist}_{\mathbf{S}^2}(a, \{b, -b\})$.

Let f be harmonic on Σ^2 with critical points $\{y_i\}$ with multiplicities $\{m_i\}$. Suppose that none of the y_i's lie on $\partial \Sigma$. The Bochner formula on $\Sigma \setminus \{y_i\}$ gives

$$\Delta_\Sigma \log |\nabla_\Sigma f|^2 = 2 \frac{|\text{Hess}_f|^2}{|\nabla_\Sigma f|^2} + 2\,K - \frac{|\nabla_\Sigma |\nabla_\Sigma f|^2|^2}{|\nabla_\Sigma f|^4} = 2\,K\,. \tag{1.1}$$

Here we used that since $\Delta_\Sigma f = 0$, then $2\,|\text{Hess}_f|^2\,|\nabla_\Sigma f|^2 = |\nabla_\Sigma |\nabla_\Sigma f|^2|^2$. Hence, by Stokes' theorem

$$\begin{aligned} \int_{\partial \Sigma} \frac{d \log |\nabla_\Sigma f|^2}{dn} &= \int_{\Sigma \setminus \{y_i\}} \Delta_\Sigma \log |\nabla_\Sigma f|^2 + 4\,\pi \sum_i m_i \\ &= 2 \int_\Sigma K + 4\,\pi \sum_i m_i\,. \end{aligned} \tag{1.2}$$

Proposition 1.3. *If Σ is connected and minimal with boundaries σ_1 and σ_2, $\int_{\sigma_1 \cup \sigma_2} |A| < \pi/8$, and $\int_\Sigma K \geq -\pi$, then Σ is graphical.*

Proof. Fix $q_i \in \sigma_i$. Since $|\nabla_\Sigma \text{dist}_{\mathbf{S}^2}(\mathbf{n}(q_i), \mathbf{n}(\cdot))| \leq |A|$, the assumption on $\partial \Sigma$ gives

$$\sum_i \sup_{z_i \in \sigma_i} \text{dist}_{\mathbf{S}^2}(\mathbf{n}(q_i), \mathbf{n}(z_i)) \leq \sum_i \int_{\sigma_i} |A| < \pi/8\,. \tag{1.4}$$

Choose $b \in \mathbf{S}^2$ with $\text{Angle}(\mathbf{n}(q_i), b) \leq \pi/4$ for $i = 1, 2$. We will show that Σ is graphical over the plane $b^\perp$. By the triangle inequality and (1.4), for $i = 1, 2$,

$$\sup_{z_i \in \sigma_i} \text{Angle}(b, \mathbf{n}(z_i)) \leq \pi/4 + \sup_{z_i \in \sigma_i} \text{dist}_{\mathbf{S}^2}(\mathbf{n}(q_i), \mathbf{n}(z_i)) < 3\,\pi/8\,. \tag{1.5}$$

Rotate coordinates so that $b = (0, 0, 1)$ and $b^\perp$ is the x_1-x_2-plane. Fix θ and set $f = x_1 \cos\theta + x_2 \sin\theta$. Given $x \in \Sigma$,

$$|\nabla_\Sigma f|^2(x) = 1 - \langle(\cos\theta, \sin\theta, 0), \mathbf{n}(x)\rangle^2 \geq \langle b, \mathbf{n}(x)\rangle^2\,. \tag{1.6}$$

On $\partial\Sigma = \sigma_1 \cup \sigma_2$, (1.5) and (1.6) imply that

$$\inf_{\partial\Sigma} |\nabla_\Sigma f| \geq \inf_{\partial\Sigma} |\langle b, \mathbf{n}(x)\rangle| > \cos(3\,\pi/8) > 1/3\,. \tag{1.7}$$

Since $|\nabla_\Sigma |\nabla_\Sigma f|| \leq |\text{Hess}_f| \leq |A|$, (1.7) gives on $\partial\Sigma$

$$|\nabla_\Sigma \log |\nabla_\Sigma f|^2| = 2\,|\nabla_\Sigma |\nabla_\Sigma f||\,/\,|\nabla_\Sigma f| \leq 6\,|A|\,. \tag{1.8}$$

Integrating (1.8), we get

$$\int_{\partial\Sigma} \left| \frac{d\,\log|\nabla_\Sigma f|^2}{dn} \right| \leq 6 \int_{\partial\Sigma} |A| < 3\,\pi/4\,. \tag{1.9}$$

Since Σ is minimal, $\Delta_\Sigma f = 0$. Substituting (1.9) into (1.2),

$$4\,\pi \sum_i m_i = \int_{\partial\Sigma} \frac{d\,\log|\nabla_\Sigma f|^2}{dn} - 2\int_\Sigma K < 3\,\pi/4 + 2\pi < 4\,\pi\,; \tag{1.10}$$

hence, f has no critical points. Since this is true for any θ, Σ is graphical over $b^\perp$. □

We next generalize Proposition 1.3 to Σ with $\mathbf{n}$ quasi-conformal. We will use that if $\Omega \subset \mathbf{S}^2$ is connected and $\text{diam}_{\mathbf{S}^2}(\Omega) < \pi/2$, then by the maximum principle

$$\text{diam}_{\mathbf{S}^2}(\Omega) = \text{diam}_{\mathbf{S}^2}(\partial\Omega)\,; \tag{1.11}$$

here we used that $\text{dist}^2_{\mathbf{S}^2}(z, \cdot)$ is convex on $\mathcal{B}_{\pi/2}(z) \subset \mathbf{S}^2$.

Proposition 1.12. *If $\Sigma \subset \mathbf{R}^3$ is connected, $\int_\Sigma |K| \leq \pi$, $|A|^2 \leq C\,|K|$, and $\partial\Sigma$ has components $\{\sigma_i\}_{1\leq i\leq n}$ with*

$$\sum_{i=1}^{n} \inf_{a\in\mathbf{S}^2} \sup_{x\in\sigma_i} \{\mathrm{dist}_{\mathbf{S}^2}(\mathbf{n}(x), a)\} < \epsilon < \pi/8\,, \tag{1.13}$$

then $\mathbf{n}(\Sigma) \subset \mathcal{B}_{2\,\epsilon}(a)$ for some $a \in \mathbf{S}^2$ and Σ is the graph of u over $a^\perp$ with $|\nabla u| \leq 4\,\epsilon$.

Proof. Choose $a_i \in \mathbf{S}^2$ and $\epsilon_i > 0$ so that $\mathbf{n}(\sigma_i) \subset \mathcal{B}_{\epsilon_i}(a_i) \subset \mathbf{S}^2$ for each i, $\sum_i \epsilon_i < \epsilon$, $\mathrm{dist}_{\mathbf{S}^2}(a_i, a_j) \neq \epsilon_i + \epsilon_j$, and $\mathrm{dist}_{\mathbf{S}^2}(a_i, a_j) \neq |\epsilon_i - \epsilon_j|$ for $i \neq j$ (i.e., the $\partial\mathcal{B}_{\epsilon_i}(a_i)$'s are transverse). Let $\Omega_1, \dots, \Omega_m$ be the connected components of $\cup_{i=1}^n \overline{\mathcal{B}_{\epsilon_i}}(a_i)$, so that

$$\sum_{i=1}^{m} \mathrm{diam}_{\mathbf{S}^2}(\Omega_i) \leq \sum_{i=1}^{n} \mathrm{diam}_{\mathbf{S}^2}(\mathcal{B}_{\epsilon_i}(a_i)) < 2\,\epsilon < \pi/4\,. \tag{1.14}$$

Since the $\partial\mathcal{B}_{\epsilon_i}(a_i)$'s are transverse, each $\partial\Omega_i$ is a finite union of transverse circular arcs. Hence, there are (closed) disks Ω_i^0 with $\Omega_i \subset \Omega_i^0$, $\partial\Omega_i^0 \subset \partial\Omega_i$, and $\mathrm{diam}_{\mathbf{S}^2}(\Omega_i^0) = \mathrm{diam}_{\mathbf{S}^2}(\Omega_i)$ (using (1.11)). Since $\partial\Omega_i^0 \cap \partial\Omega_j^0 = \emptyset$ if $i \neq j$, a disjoint subset (after reordering) $\Omega_1^0, \dots, \Omega_q^0$ has

$$\cup_{i=1}^n \mathbf{n}(\sigma_i) \subset \cup_{i=1}^m \Omega_i \subset \cup_{i=1}^q \Omega_i^0\,. \tag{1.15}$$

Since $\mathrm{diam}_{\mathbf{S}^2}(\Omega_i^0) = \mathrm{diam}_{\mathbf{S}^2}(\Omega_i)$, (1.14) and the volume comparison imply that

$$\sum_{i=1}^{q} \mathrm{Area}\,(\Omega_i^0) < \pi \sum_{i=1}^{q} \mathrm{diam}^2_{\mathbf{S}^2}(\Omega_i^0) \leq \pi \sum_{i=1}^{n} \mathrm{diam}^2_{\mathbf{S}^2}(\Omega_i) < \pi(\pi/4)^2 < \pi\,. \tag{1.16}$$

Since the Ω_i^0's are disjoint disks,

$$\Omega = \mathbf{S}^2 \setminus \cup_{i=1}^q \Omega_i^0 \tag{1.17}$$

is a connected open set with q boundary components. We show next that $q = 1$. If $q > 1$, then $\mathcal{H}^1(\mathbf{n}(\Sigma) \cap \Omega) \geq \mathrm{dist}_{\mathbf{S}^2}(\Omega_1^0, \Omega_2^0) > 0$ since Σ is connected and $\mathbf{n}(\Sigma) \cap \Omega_i^0 \neq \emptyset$ for each i. Here $\mathcal{H}^1$ is the one-dimensional Hausdorff measure. Hence, since $\overline{\Omega} \cap \mathbf{n}(\partial\Sigma) = \emptyset$, Lemma A.1 implies that $\Omega \subset \mathbf{n}(\Sigma)$. However, together with (1.16), this would imply that

$$3\,\pi < 4\,\pi - \mathrm{Area}\,(\cup_{i=1}^q \Omega_i^0) = \mathrm{Area}\,(\Omega) \leq \mathrm{Area}\,(\mathbf{n}(\Sigma)) \leq \int_\Sigma |K| \leq \pi\,, \tag{1.18}$$

so that we must have $q = 1$. Hence, $\mathbf{n}(\Sigma) \subset \Omega_1^0$ and the claim follows from (1.14). $\square$

2. Holomorphic functions on annuli.

We will now obtain $C^{0,1}$ estimates for holomorphic functions on annuli as a model for the Gauss map of a minimal annulus with a slit. The next section extends these arguments to that case.

Lemma 2.1. *If $f : D_R \setminus D_\delta \to \mathbf{C}$ is holomorphic and $\int_{\partial D_R \cup \partial D_\delta} |\nabla f| \leq \epsilon$, then*

$$\min_{c \in \mathbf{C}} \max_z |f(z) - c| \leq \epsilon \, . \tag{2.2}$$

Proof. For $\delta \leq s \leq R$, set

$$I(s) = (2\,\pi\,s)^{-1} \int_{\partial D_s} f = (2\,\pi)^{-1} \int_0^{2\,\pi} f(s\,\mathrm{e}^{i\,\theta})\, d\theta \, , \tag{2.3}$$

and $c = I(\delta)$. Differentiating (2.3), we have

$$\begin{aligned} 2\,\pi\,s\,I'(s) &= \int_{\partial D_s} \frac{\partial f}{\partial r} = -i\,s^{-1} \int_{\partial D_s} \frac{\partial f}{\partial \theta} = -i \int_0^{2\,\pi} \frac{\partial f}{\partial \theta}\, d\theta \\ &= -i\,[f(s\,\mathrm{e}^{i\,2\,\pi}) - f(s\,\mathrm{e}^{0})] = 0 \, , \end{aligned} \tag{2.4}$$

where we used that $\frac{\partial f}{\partial r} = -i\,r^{-1}\,\frac{\partial f}{\partial \theta}$ since f is holomorphic. In particular, $I(R) = c$. Since $I(s)$ is the average of f over ∂D_s, there exist $y_1, y_2 \in \partial D_R$ with

$$c = \mathrm{Re}(f(y_1)) + i\,\mathrm{Im}(f(y_2)) \, . \tag{2.5}$$

Combining (2.5) with $\int_{\partial D_R \cup \partial D_\delta} |\nabla f| \leq \epsilon$,

$$\max_{y \in \partial D_R} |\mathrm{Re}(f(y) - c)| \leq \epsilon/2 \text{ and } \max_{y \in \partial D_R} |\mathrm{Im}(f(y) - c)| \leq \epsilon/2 \, , \tag{2.6}$$

so that $|f - c| \leq \epsilon$ on $\partial(D_R \setminus D_\delta)$. The maximum principle then gives (2.2). $\square$

Note that Lemma 2.1 does not hold for harmonic functions (in particular, (2.4)); e.g., take $\epsilon \log r/(4\,\pi)$ and $R > \mathrm{e}^{8\,\pi}\,\delta$. Clearly, Lemma 2.1 holds for the real part of a holomorphic function. However, on an annulus, not every harmonic function can be written this way.

It is not hard to see that $|f'(z_0)| \leq \epsilon\,(1/R + \delta\,|z_0|^{-2})/\pi$ for $z_0 \in D_{R/2} \setminus D_{2\delta}$. It is not clear whether a corresponding estimate holds when there is a slit.

3. Minimal annuli with slits.

We will now bound the oscillation of the Gauss map of a double-valued minimal graph over an annulus with a slit. Double-valued on the slit. Fix $0 < \alpha < 1$, $0 < \delta < R/4$, $0 < \epsilon$. Suppose $\Omega \subset \{x_3 = 0\}$ is a topological annulus,

$$\Sigma \subset B_R \setminus B_\delta \text{ is a minimal graph of } u \text{ over } \Omega \text{ with } |\nabla u| < 1/6\,, \tag{3.1}$$

$$\partial \Sigma = \gamma_\delta \cup \gamma_R \cup \gamma_+ \cup \gamma_- \text{ with } \gamma_\delta = \partial B_\delta \cap \Sigma, \gamma_R = \partial B_R \cap \Sigma\,, \tag{3.2}$$

$$\gamma_\pm = \{(t, 0, u_\pm(t))\,|\,\delta_\pm \leq t \leq R_\pm\}\,, u_-(\delta) = 0,\ \mathbf{n}_-(\delta, 0, 0) = (0, 0, 1)\,, \tag{3.3}$$

$$\delta_- = \delta\,, 5\delta/6 < \delta_+ \leq \delta,\ 5R/6 < R_\pm \leq R\,, \tag{3.4}$$

$$|u_+(t) - u_-(t)| + t\,|\nabla u_+(t) - \nabla u_-(t)| \leq \epsilon\,\delta\,(t/\delta)^\alpha \text{ for } \delta \leq t \leq \min\{R_\pm\}\,, \tag{3.5}$$

$$|A|(x) \leq \epsilon/r \text{ for } x \in \Sigma\,, \tag{3.6}$$

where $r^2 = x_1^2 + x_2^2 + x_3^2$. In Lemma 3.9, we will see that Ω is given in polar coordinates by

$$\{(\rho, \theta)\,|\,\rho_\delta(\theta) \leq \rho \leq \rho_R(\theta) \text{ and } 0 \leq \theta \leq 2\,\pi\}\,, \tag{3.7}$$

where ρ_δ and ρ_R are functions of θ and, by convention, $\theta = 0$ and $\theta = 2\,\pi$ correspond to u_- and u_+, respectively. Hence, u is double-valued over the x_1-axis from δ to $\min\{R_\pm\}$ (the "slit") with values $u_\pm$ and $\nabla u_\pm$ for u and $\nabla u = (\partial_1 u, \partial_2 u)$, respectively, and $\partial\Omega$ has corners (e.g., if $\rho_\delta(0) \neq \rho_\delta(2\,\pi)$). Let $\Pi : \mathbf{S}^2 \setminus \{(0, 0, -1)\} \to \mathbf{C}$ be the stereographic projection so $\Pi(x) = (x_1 + i\,x_2)/(1 + x_3)$ and if $a, a' \in \mathcal{B}_{\pi/4}(0, 0, 1)$, then

$$|\Pi(a) - \Pi(a')| \leq |a - a'| \leq 2\,|\Pi(a) - \Pi(a')|\,. \tag{3.8}$$

(See, e.g., (28) on page 20 of [A] for (3.8)).

Our arguments will be modelled on those for holomorphic functions in Section 2. New error terms will arise both from the presence of the slit and from the fact that Σ is not flat. Set $\gamma_t = \partial B_t \cap \Sigma$ and let $\sigma_t \subset \{x_3 = 0\}$ be the orthogonal projection of γ_t.

Lemma 3.9. *If $\epsilon < 1/(54\,\pi^2)$, $\delta \leq t \leq R$, and Σ satisfies (3.1)-(3.6), then each σ_t is a radial graph of a function ρ_t of θ with $5\,t/6 \leq \rho_t \leq t$, and*

$$u^2(x_1,x_2) \leq (x_1^2+x_2^2)/36 \text{ for } (x_1,x_2) \in \Omega\,, \tag{3.10}$$

$$|\nabla u| \leq 3\,\pi\,\epsilon \text{ and } |u| \leq 5\,\pi^2\,\epsilon\,\delta \text{ on } \overline{D}_\delta \cap \Omega\,, \tag{3.11}$$

$$|\rho_t(2\,\pi) - \rho_t(0)| < \epsilon\,\delta\,(t/\delta)^\alpha\,. \tag{3.12}$$

Proof. By (3.1) and (3.6), we have

$$|\nabla|\nabla u||(x_1,x_2) \leq |A|(x_1,x_2,u)\,(1+|\nabla u|^2)^{3/2} < 9\,\epsilon\,(x_1^2+x_2^2)^{-1/2}/8\,. \tag{3.13}$$

Combining (3.3) and (3.13),

$$\max_{\partial D_\delta}|\nabla u| \leq |\nabla u_-(\delta)| + 2\,\pi\,\delta\,\max_{\partial D_\delta}|\nabla|\nabla u|| \leq 9\,\pi\,\epsilon/4\,, \tag{3.14}$$

$$\max_{\partial D_\delta}|u| \leq |u_-(\delta)| + 2\,\pi\,\delta\,\max_{\partial D_\delta}|\nabla u| \leq 9\,\pi^2\,\epsilon\,\delta/2 < \delta/12\,. \tag{3.15}$$

Integrating $|\nabla u| < 1/6$ along rays, (3.15) implies that

$$u^2(x_1,x_2) \leq (x_1^2+x_2^2)/36 \text{ for } (x_1,x_2) \in \Omega \cap \{(x_1^2+x_2^2)^{1/2} > 5\delta/6\}\,. \tag{3.16}$$

Since $25\,\delta^2(1+1/36)/36 < \delta^2$ and Ω is connected, it follows that $\Omega \subset \{(x_1^2+x_2^2)^{1/2} > 5\delta/6\}$ and (3.10) holds for all $(x_1,x_2) \in \Omega$. Define the function $r_u : \Omega \to \mathbf{R}$ by $r_u(x_1,x_2) = |(x_1,x_2,u(x_1,x_2))|$. Using $|\nabla u| < 1/6$ and (3.10), for any θ,

$$\partial r_u^2/\partial\rho = \partial/\partial\rho\left[\rho^2 + u^2(\rho\cos\theta,\rho\sin\theta)\right] > r_u > \rho > 0\,. \tag{3.17}$$

In particular, σ_t is a radial graph $\rho_t(\theta)(\cos\theta,\sin\theta)$ with $5t/6 \leq \rho_t(\theta) \leq t$. Integrating (3.13) along rays from ∂D_δ and using (3.14)–(3.15) gives (3.11). Using (3.5), the definition of ρ_t, and $|u_\pm|(t) \leq t/6$,

$$\begin{aligned}
&|(\rho_t^2(2\,\pi) + u_-^2(\rho_t(2\,\pi)))^{1/2} - t| \\
&\leq \left|\rho_t^2(2\,\pi) + u_-^2(\rho_t(2\,\pi)) - t^2\right|/t \\
&= \left|u_-^2(\rho_t(2\,\pi)) - u_+^2(\rho_t(2\,\pi))\right|/t \\
&\leq |u_-(\rho_t(2\,\pi)) - u_+(\rho_t(2\,\pi))|/3 \leq \epsilon\,\delta\,(t/\delta)^\alpha/3\,.
\end{aligned} \tag{3.18}$$

Finally, since $\frac{d}{dt}(t^2+u_-^2(t))^{1/2} > 1/2$ (by (3.17)), (3.18) implies that

$$|\rho_t(2\,\pi) - \rho_t(0)| < 2\,|(\rho_t^2(2\,\pi) + u_-^2(\rho_t(2\,\pi)))^{1/2} - t| < \epsilon\,\delta\,(t/\delta)^\alpha\,. \tag{3.19}$$

□

Lemma 3.20. *If $\epsilon < 1/(54\,\pi^2)$, $\delta \leq t \leq R$, and Σ satisfies (3.1)-(3.6), then*

$$2\,\pi - C_0\,\epsilon \leq \delta^{-1} \int_{\gamma_\delta} |\nabla_\Sigma r|\,, \tag{3.21}$$

$$t^{-1}\,\text{Length}(\gamma_t) \leq \min\{2\,\pi + C_0\,\epsilon, 3\,\pi\}\,. \tag{3.22}$$

Proof. By Lemma 3.9, each σ_t is a radial graph of ρ_t. Hence,

$$(\rho_t(\theta))^2 + u^2(\rho_t(\theta)\cos\theta, \rho_t(\theta)\sin\theta) = t^2\,. \tag{3.23}$$

Differentiating (3.23) (with respect to θ), we get

$$\rho_t\,\partial_\theta\rho_t + u\,\langle\nabla u, (\cos\theta, \sin\theta)\rangle\,\partial_\theta\rho_t = -u\,\rho_t\langle\nabla u, (-\sin\theta, \cos\theta)\rangle\,. \tag{3.24}$$

Substituting $5t/6 \leq \rho_t \leq t$, $|\nabla u| < 1/6$, and $|u| \leq t/6$ on σ_t,

$$|\partial_\theta\rho_t| \leq |u|\,t\,|\nabla u|/[\rho_t - |u|\,|\nabla u|] \leq 3\,|u|\,|\nabla u|/2 \leq t/24\,. \tag{3.25}$$

By (3.25) (and since $\rho_t \leq t$), $|d\sigma_t/d\theta| = |(\cos\theta, \sin\theta)\,\partial_\theta\rho_t + \rho_t(-\sin\theta, \cos\theta)| < 25\,t/24$. Combining this with $|\nabla u| < 1/6$ implies that

$$\text{Length}(\gamma_t) \leq \int_0^{2\,\pi} \left|\frac{d\sigma_t}{d\theta}\right|\,(1 + |\nabla u|^2)^{1/2}\,d\theta < 5\,\pi\,t/2\,. \tag{3.26}$$

The triangle inequality and (3.11) imply that $\rho_\delta \geq \delta - \max_{\sigma_\delta}|u| \geq (1 - 5\,\pi^2\,\epsilon)\,\delta$, so

$$\text{Length}(\gamma_\delta) \geq \text{Length}(\sigma_\delta) \geq 2\pi\,\min\rho_\delta \geq (2\pi - C_1\,\epsilon)\delta\,. \tag{3.27}$$

Given $x = (x_1, x_2, x_3) \in \gamma_\delta$, then $(x_1, x_2) \in \sigma_\delta \subset \overline{D}_\delta$ and (3.11) give

$$|\langle\mathbf{n}(x), x/\delta\rangle| = \delta^{-1}\,(1 + |\nabla u|^2)^{-1/2}\,|u - x_1\partial_1 u - x_2\partial_2 u| < C_2\,\epsilon\,. \tag{3.28}$$

Consequently,

$$|\nabla_\Sigma r| > 1 - C_2\,\epsilon \text{ on } \gamma_\delta\,. \tag{3.29}$$

Combining (3.27) and (3.29) gives (3.21) for any $C_0 \geq C_1 + 2\,\pi\,C_2$.

Below, we will need that the endpoints of γ_t are close. By (3.12), $|\rho_t(2\,\pi) - \rho_t(0)| < \epsilon\,\delta\,(t/\delta)^\alpha < \epsilon\,t$. Therefore, by the triangle inequality, $|\nabla u| < 1/6$, and (3.5), we get

$$\begin{aligned} &|u_+(\rho_t(2\pi)) - u_-(\rho_t(0))| \\ &\leq |u_+(\rho_t(2\pi)) - u_+(\rho_t(0))| + |u_+(\rho_t(0)) - u_-(\rho_t(0))| \\ &\leq \epsilon\,t/6 + \epsilon\,t = 7\,\epsilon\,t/6\,. \end{aligned} \tag{3.30}$$

We conclude that the distance between the endpoints of γ_t is at most $2\,\epsilon\, t$.

Let P_t be the plane orthogonal to $\mathbf{n}(\gamma_t \cap \gamma_-)$ through $\gamma_t \cap \gamma_-$. Let $\hat{\sigma}_t$ be the (orthogonal) projection of γ_t to P_t. Since $|\nabla u| < 1/6$, Σ is a multi-valued graph over (a subset of) P_t of a function w. To complete the proof, we will argue as above to bound the length of $\hat{\sigma}_t$ and $|\nabla w|$ on $\hat{\sigma}_t$. Using (3.26), $\text{Length}(\hat{\sigma}_t) < 5\,\pi\, t/2$. Hence, as in (3.13)–(3.15), we get

$$\max_{\hat{\sigma}_t} |\nabla w| \leq C_3\, \epsilon\,, \tag{3.31}$$

$$\max_{\hat{\sigma}_t} |w| \leq 5\,\pi\, C_3\, \epsilon\, t/2\,. \tag{3.32}$$

It remains to bound the length of $\hat{\sigma}_t$ by $2\pi\, t$ plus a multiple of $\epsilon\, t$. Using $|\nabla u| < 1/6$ and (3.10), $B_t \cap P_t \subset \{|x_3| < t/2\}$. In particular, ∂B_t intersects P_t with slope at least $\sqrt{3}$ (in absolute value) over $\{x_3 = 0\}$. Since the slope of P_t is less than $1/6$ over $\{x_3 = 0\}$, this implies that $|\nabla_{P_t} r| \geq C_4 > 0$ on $\partial B_t \cap P_t$. Choose polar coordinates $\hat{\rho}, \hat{\theta}$ on P_t so $\partial B_t \cap P_t$ is a circle $\hat{\rho} = \text{Const} \leq t$. Arguing as in (3.17), we get, for any $\hat{\theta}$,

$$\partial/\partial\hat{\rho} \left[\hat{\rho}^2 + w^2(\hat{\rho}\cos\hat{\theta}, \hat{\rho}\sin\hat{\theta})\right] > C_5\, \hat{\rho} > 0\,. \tag{3.33}$$

Hence, $\hat{\sigma}_t$, like σ_t, is a connected (multi-valued) radial graph of a function $\hat{\rho}_t$ of $\hat{\theta}$ satisfying (3.23)–(3.25). From (3.25), (3.31), and (3.32), we get that $|\partial_{\hat{\theta}} \hat{\rho}_t| \leq C_6\, \epsilon\, t$. Inserting this and (3.31) into (3.26) (with w in place of u) gives (3.22) since the domain of $\hat{\rho}_t$ is contained in $\{\hat{\theta} \mid -C\,\epsilon \leq \hat{\theta} \leq 2\pi + C\,\epsilon\}$. This followed since the distance between the endpoints of γ_t is at most $2\,\epsilon\, t$ (and orthogonal projection is distance nonincreasing). □

The key point in the next theorem, our main result, is that the constants are independent of δ and R. The proof follows Lemma 2.1. If Σ is as in (3.1), then the Gauss map $\mathbf{n} : \Sigma \to \mathcal{B}_{\pi/4}(0,0,1) \subset \mathbf{S}^2$ is conformal. Composing $\mathbf{n}$ with the stereographic projection Π and using (3.8), we get a conformal map

$$f = \Pi(\mathbf{n}(\cdot)) : \Sigma \to D_1(0) \subset \mathbf{C} \text{ with } |\nabla_\Sigma f| \leq |A|\,. \tag{3.34}$$

Note that, if $x, y \in \Omega$, then

$$\begin{aligned} |f(x,u(x)) - f(y,u(y))| &= \left| \frac{\nabla u(x)}{1 + (1 + |\nabla u(x)|^2)^{1/2}} - \frac{\nabla u(y)}{1 + (1 + |\nabla u(y)|^2)^{1/2}} \right| \\ &\leq 2\, |\nabla u(x) - \nabla u(y)|\,. \end{aligned} \tag{3.35}$$

Theorem 3.36. *There are $C(\alpha), \epsilon_0(\alpha) > 0$ so that if $\epsilon < \epsilon_0$ and Σ satisfies (3.1)-(3.6), then*

$$\max_{z \in \Sigma} |\mathbf{n}(z) - (0,0,1)| \leq C\, \epsilon\,. \tag{3.37}$$

Proof. We will prove (3.37) by bounding $|f|$. For $\delta \leq s \leq R$, set

$$I(s) = s^{-1} \int_{\gamma_s} f\, |\nabla_\Sigma r| = \int_{\gamma_s} f\, \frac{d\, \log r}{dn}\,. \tag{3.38}$$

We will show that $I(s)$ is almost constant. By Stokes' theorem and the co-area formula,

$$\begin{aligned} I(s) - I(\delta) &= \int_{B_s \cap \Sigma} \langle \nabla_\Sigma f, \nabla_\Sigma \log r \rangle + \int_{B_s \cap \Sigma} f\, \Delta_\Sigma \log r \\ &\quad - \int_{B_s \cap (\gamma_+ \cup \gamma_-)} f\, \frac{d\, \log r}{dn} \\ &= \int_\delta^s t^{-1} \int_{\gamma_t} |\nabla_\Sigma r|^{-1} \langle \nabla_\Sigma f, \nabla_\Sigma r \rangle\, dt \\ &\quad + \int_{B_s \cap \Sigma} f\, \Delta_\Sigma \log r - \int_\delta^{s_0} [f_+ F_+ - f_- F_-](t)\, dt \\ &\quad - \int_{\delta_+}^{\delta} f_+(t)\, F_+(t)\, dt - \int_{s_0}^{s_+} f_+(t)\, F_+(t)\, dt \\ &\quad + \int_{s_0}^{s_-} f_-(t)\, F_-(t)\, dt\,. \end{aligned} \tag{3.39}$$

Here $s_- = \rho_s(0)$, $s_+ = \rho_s(2\,\pi)$ (so that $s^2 = s_\pm^2 + u_\pm^2(s_\pm)$), $s_0 = \min\{s_+, s_-\}$, and

$$\begin{aligned} f_\pm(t) &= f(t, 0, u_\pm(t))\,, \\ F_\pm(t) &= \left\langle \frac{(-\partial_1 u_\pm \partial_2 u_\pm, 1 + (\partial_1 u_\pm)^2, \partial_2 u_\pm)}{(|\partial_1 u_\pm\, \partial_2 u_\pm|^2 + (1 + |\partial_1 u_\pm|^2)^2 + |\partial_2 u_\pm|^2)^{1/2}}, \frac{(t, 0, u_\pm)}{t^2 + u_\pm^2} \right\rangle \\ &\quad \cdot (1 + |\partial_1 u_\pm|^2)^{1/2} \end{aligned} \tag{3.40}$$

$$= \frac{\partial_2 u_\pm}{(1 + |\nabla u_\pm|^2)^{1/2}} \left[\frac{-t\, \partial_1 u_\pm + u_\pm}{t^2 + u_\pm^2} \right]. \tag{3.41}$$

In (3.41), $\partial_i u_\pm(t)$ denote the values of $\partial_i u(t, 0)$. When Σ is an annulus in the plane, the the last five terms in (3.39) don't appear and the first term in

(3.39) vanishes since f is conformal (see (2.4)); we will bound these terms in general in Lemma 3.45 below.

Lemmas 3.9, 3.20 give bounds for $|\nabla u|$ along γ_δ and the length of γ_δ; together this bounds $|I(\delta)|$. Combining this with the bound on $I(s) - I(\delta)$ from Lemma 3.45, we get

$$|I(s)| \leq |I(\delta)| + |I(s) - I(\delta)| \leq C_3' \, \epsilon \,. \tag{3.42}$$

By (3.47) and (3.42) (and since γ_s is connected), there exist $y_s \in \gamma_s$ with

$$|\mathrm{Re}(f(y_s))| \leq \left| \int_{\gamma_s} f \, |\nabla_\Sigma r| \right| / \int_{\gamma_s} |\nabla_\Sigma r| \leq |I(s)|/(2\,\pi - C_5\,\epsilon) \leq C_4' \, \epsilon \,, \tag{3.43}$$

so long as $\epsilon < 2\,\pi/C_5$. Combining (3.43) with the gradient bound (3.34), the length bound on γ_s from Lemma 3.20, and (3.6), we get

$$\max_{y \in \gamma_s} |\mathrm{Re}(f(y))| \leq C_4' \, \epsilon + 3\,\pi \, s \sup_{\gamma_s} |A| \leq C_4 \, \epsilon \,. \tag{3.44}$$

Repeating this, (3.44) holds also for $\max_{\gamma_s} |\mathrm{Im}(f)|$ so $|f| \leq \sqrt{2} \, C_4 \, \epsilon$ on Σ, giving (3.37). $\square$

We will now show that $I(s)$ defined in (3.38) is almost constant.

Lemma 3.45. *With the notation as in Theorem* 3.36 *and its proof* (*see* (3.34) *and* (3.38))

$$|I(s) - I(\delta)| \leq C_3 \, \epsilon \,, \tag{3.46}$$

$$2\,\pi - C_5 \, \epsilon \leq s^{-1} \int_{\gamma_s} |\nabla_\Sigma r| \,. \tag{3.47}$$

Proof. To get (3.46), it suffices to bound the six terms in (3.39). We begin with the last three. Since $|\nabla u| < 1/6$ and $|u_\pm(t)| \leq t/6$ (by (3.10)), we get $|F_\pm(t)| \leq |\nabla u_\pm| \,/(3\,t) < 1/(18t)$. By this, $|f| < 1$, and $|\nabla u_+(t)| \leq 3\,\pi\,\epsilon$ for $t \leq \delta$ (by (3.11)), we get

$$\int_{\delta_+}^{\delta} |f_+(t) \, F_+(t)| \, dt \leq \int_{\delta_+}^{\delta} |F_+(t)| \, dt \leq \int_{\delta_+}^{\delta} \frac{|\nabla u_+|}{3\,t} \, dt < \epsilon \,, \tag{3.48}$$

where we also used that $\delta_+ > 5\delta/6$. By (3.12), $|\rho_s(2\,\pi) - \rho_s(0)| < \epsilon \, \delta \, (s/\delta)^\alpha$. Therefore, since $|f| < 1$ and $|F_\pm(t)| \leq |\nabla u_\pm| \,/(3\,t) < 1/(18t)$,

$$\int_{s_0}^{s_\pm} |f_\pm \, F_\pm|(t) \, dt \leq \int_{s_0}^{s_\pm} |F_\pm(t)| \, dt < \frac{\epsilon \, \delta}{18 \, s_0} \left(\frac{s}{\delta}\right)^\alpha < \epsilon \left(\frac{\delta}{s}\right)^{1-\alpha} \leq \epsilon \,. \tag{3.49}$$

Let γ_t' be the unit tangent to γ_t. For the first term in (3.39), since f is conformal and $\nabla_\Sigma r/|\nabla_\Sigma r|$ is the unit normal to γ_t,

$$\int_{\gamma_t} |\nabla_\Sigma r|^{-1} \langle \nabla_\Sigma f, \nabla_\Sigma r\rangle = i \int_{\gamma_t} \langle \nabla_\Sigma f, \gamma_t'\rangle = i\,[f_+(t_+) - f_-(t_-)]\,, \tag{3.50}$$

where $t^2 = t_\pm^2 + u_\pm^2(t_\pm)$ as before. By (3.5), (3.6), (3.34), (3.35), and (3.12),

$$\begin{aligned} |f_+(t_+) - f_-(t_-)| &\leq |f_+(t_+) - f_-(t_+)| + |f_-(t_+) - f_-(t_-)| \\ &\leq 2\,\epsilon\,(\delta/t_+)^{1-\alpha} + \epsilon\,|t_+ - t_-|/t_0 < 4\,\epsilon\,(\delta/t)^{1-\alpha}\,, \end{aligned} \tag{3.51}$$

where $t_0 = \min\{t_+, t_-\} > 5t/6$. Hence, (3.50) and (3.51) give

$$\left| \int_\delta^s t^{-1} \int_{\gamma_t} |\nabla_\Sigma r|^{-1} \langle \nabla_\Sigma f, \nabla_\Sigma r\rangle \, dt \right| \leq 4 \int_\delta^s t^{-1}\,\epsilon\,(\delta/t)^{1-\alpha}\,dt \leq \frac{4\,\epsilon}{1-\alpha}\,. \tag{3.52}$$

Note that, since Σ is minimal,

$$\Delta_\Sigma \log r = 2\,(1 - |\nabla_\Sigma r|^2)\,r^{-2} \geq 0\,. \tag{3.53}$$

Using $|f| < 1$, (3.53), and Stokes' theorem, the second term in (3.39) is bounded by

$$\begin{aligned} \left| \int_{B_s\cap\Sigma} f\,\Delta_\Sigma \log r \right| &\leq \int_{B_s\cap\Sigma} \Delta_\Sigma \log r \leq s^{-1} \int_{\gamma_s} |\nabla_\Sigma r| \\ &\quad - \delta^{-1} \int_{\gamma_\delta} |\nabla_\Sigma r| + \left| \int_{B_s\cap(\gamma_+\cup\gamma_-)} \frac{d\,\log r}{dn} \right| \\ &\leq s^{-1} \int_{\gamma_s} |\nabla_\Sigma r| - \delta^{-1} \int_{\gamma_\delta} |\nabla_\Sigma r| \\ &\quad + \int_\delta^{s_0} |F_+(t) - F_-(t)|\,dt + \int_{\delta_+}^{\delta} |F_+(t)|\,dt \\ &\quad + \int_{s_0}^{s_+} |F_+(t)|\,dt + \int_{s_0}^{s_-} |F_-(t)|\,dt\,. \end{aligned} \tag{3.54}$$

We next bound $|F_+(t) - F_-(t)|$. Given $p, t \in \mathbf{R}$ with $|p| \leq t/6$ and $q \in D_{1/6}(0) \subset \mathbf{R}^2$, define $F_1(q) = q_2\,(1 + q_1^2 + q_2^2)^{-1/2}$, and $F_2(t,p,q) = (p - q_1\,t)\,(t^2 + p^2)^{-1}$, so that

$$F_\pm(t) = F_1(\nabla u_\pm)\;F_2(t, u_\pm, \nabla u_\pm)\,. \tag{3.55}$$

Keeping in mind that $|p| \leq t/6$ and $|q| < 1/6$, it is easy to see that

$$
\begin{aligned}
|\nabla F_1|^2 &= \left(q_1^2\, q_2^2 + (1+q_1^2)^2\right)/(1+|q|^2)^3 \\
&\leq (1+q_1^2)(1+|q|^2)/(1+|q|^2)^3 \leq 1\,, \\
|\partial_p F_2| &= [t^2 - p^2 + 2p\, q_1\, t]/(t^2+p^2)^2 \\
&\leq (t^2 + q_1^2\, t^2)/(t^2+p^2)^2 \leq 2/t^2\,, \\
|\partial_{q_1} F_2| &= t/(t^2+p^2) \leq 1/t\,, \text{ and } \partial_{q_2} F_2 \equiv 0\,.
\end{aligned}
\tag{3.56}
$$

Since $|F_1| < 1/6$ and $|F_2| \leq 1/(3t)$, (3.5), (3.55), and (3.56) imply that

$$
\begin{aligned}
|F_+(t) - F_-(t)| &\leq |F_1(\nabla u_+)\, F_2(t,u_+,\nabla u_+) - F_1(\nabla u_-)\, F_2(t,u_+,\nabla u_+)| \\
&\quad + |F_1(\nabla u_-)\, F_2(t,u_+,\nabla u_+) - F_1(\nabla u_-)\, F_2(t,u_+,\nabla u_-)| \\
&\quad + |F_1(\nabla u_-)\, F_2(t,u_+,\nabla u_-) - F_1(\nabla u_-)\, F_2(t,u_-,\nabla u_-)| \\
&\leq |\nabla u_+(t) - \nabla u_-(t)|\ (\max|\nabla F_1|/(3t) + \max|\nabla_q F_2|/6) \\
&\quad + |u_+(t) - u_-(t)|\ \max|\partial_p F_2|/6 \\
&\leq \epsilon\, \delta^{1-\alpha}\, t^{\alpha-2}\,.
\end{aligned}
\tag{3.57}
$$

Combining Lemma 3.20, (3.48), (3.49), and (3.57), (3.54) becomes

$$
\left| \int_{B_s \cap \Sigma} f\, \Delta_\Sigma \log r \right| \leq 2\, C_0\, \epsilon + 3\, \epsilon + \epsilon\, \delta^{1-\alpha} \int_\delta^{s_0} t^{\alpha-2}\, dt \leq C_5\, \epsilon\,. \tag{3.58}
$$

As in (3.58), Lemma 3.20, Stokes' theorem, (3.53), (3.48), (3.49), and (3.57) imply that

$$
s^{-1} \int_{\gamma_s} |\nabla_\Sigma r| \geq \delta^{-1} \int_{\gamma_\delta} |\nabla_\Sigma r| - \left| \int_{B_s \cap (\gamma_+ \cup \gamma_-)} \frac{d\, \log r}{dn} \right| \geq 2\,\pi - C_5\, \epsilon\,, \tag{3.59}
$$

giving (3.47).

Combining $|f| < 1$, $|F_\pm(t)| \leq 1/(18t)$, (3.5), (3.35), and (3.57), we have

$$
\begin{aligned}
|f_+\, F_+ - f_-\, F_-|\,(t) &\leq |f_+\, F_+ - f_+\, F_-|\,(t) + |f_+\, F_- - f_-\, F_-|\,(t) \\
&< \epsilon\, \delta^{1-\alpha}\, t^{\alpha-2} + 1/(9t)\, \epsilon\, \delta^{1-\alpha}\, t^{\alpha-1}\,.
\end{aligned}
\tag{3.60}
$$

For the remaining term in (3.39), we integrate (3.60) to get

$$
\int_\delta^{s_0} |f_+\, F_+ - f_-\, F_-|(t)\, dt < 2\, \epsilon\, \delta^{1-\alpha} \int_\delta^{s_0} t^{\alpha-2}\, dt \leq 2\, \epsilon/(1-\alpha)\,. \tag{3.61}
$$

Finally, substituting (3.48), (3.49), (3.52), (3.58), and (3.61) into (3.39), we get (3.46). □

Appendix A. A fact about qausi-conformal maps.

The following lemma is a special case of the fact that quasi-conformal maps are open (see, e.g., [V]); we include for completeness a proof of this lemma.

Lemma A.1. *If* Σ *is compact, connected,* $|A|^2 \leq C\,|K|$, $\Omega \subset \mathbf{S}^2$ *is a connected open set with* $\partial\Omega$ *piecewise smooth and compact,* $\overline{\Omega} \cap \mathbf{n}(\partial\Sigma) = \emptyset$, *and* $\mathcal{H}^1(\Omega \cap \mathbf{n}(\Sigma)) \neq 0$, *then* $\Omega \subset \mathbf{n}(\Sigma)$.

Proof. Set $J = \{y \in \Sigma \,|\, K(y) = 0\}$. $\mathbf{n}(J)$ is closed and $\Omega \cap \mathbf{n}(\Sigma) \setminus \mathbf{n}(J)$ is (relatively) closed in $\Omega \setminus \mathbf{n}(J)$. Since $|A|^2 \leq C\,|K|$, the differential of $\mathbf{n}$ vanishes on J. By the general Morse-Sard-Federer theorem, $\mathcal{H}^1(\mathbf{n}(J)) = 0$ (see [F] theorem 3.4.3). Hence, $\Omega \setminus \mathbf{n}(J)$ is connected (and open) and $\Omega \cap \mathbf{n}(\Sigma) \setminus \mathbf{n}(J) \neq \emptyset$. Since the implicit function theorem implies that $\Omega \cap \mathbf{n}(\Sigma) \setminus \mathbf{n}(J)$ is also open, $\Omega \setminus \mathbf{n}(J) \subset \mathbf{n}(\Sigma)$, giving the claim. □

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